Ann. Inst. Fourier, Grenoble
1, 0 (-1) 000-000

DYNAMICAL ZETA FUNCTIONS FOR BILLIARDS

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ABSTRACT. — Let $D \subset \mathbb{R}^d$, $d \geqslant 2$, be the union of a finite collection of pairwise disjoint strictly convex compact obstacles. Let $\mu_j \in \mathbb{C}$, $\operatorname{Im} \mu_j > 0$ be the resonances of the Laplacian in the exterior of D with Neumann or Dirichlet boundary condition on ∂D. For d odd, $u(t) = \sum_j \mathrm{e}^{i|t|\mu_j}$ is a distribution in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ and the Laplace transforms of the leading singularities of $u(t)$ yield the dynamical zeta functions η_{N}, η_{D} for Neumann and Dirichlet boundary conditions, respectively. These zeta functions play a crucial role in the analysis of the distribution of the resonances. Under a non-eclipse condition, for every $d \geqslant 2$ we show that η_{N} and η_{D} admit a meromorphic continuation to the whole complex plane. In the particular case when the boundary ∂D is real analytic, by using a result of Fried [17], we prove that the function η_{D} cannot be entire. Following Ikawa [29], this implies the existence of a strip $\{z \in \mathbb{C} :\ 0 < \operatorname{Im} z \leqslant \alpha\}$ containing an infinite number of resonances μ_j for the Dirichlet problem. Moreover, for $\alpha \gg 1$ we obtain a lower bound for the resonances lying in this strip.

RÉSUMÉ. — Soit $d \geqslant 2$, et $D \subset \mathbb{R}^d$ une union finie d'obstacles strictement convexes, compacts et deux à deux disjoints. Soient $\mu_j \in \mathbb{C}$, $\operatorname{Im} \mu_j > 0$, les résonances du Laplacien à l'extérieur de D avec conditions aux limites de Neumann ou de Dirichlet sur ∂D. Pour d impair, la formule $u(t) = \sum_j \mathrm{e}^{i|t|\mu_j}$ définit une distribution de $\mathcal{D}'(\mathbb{R} \setminus \{0\})$. Les transformées de Laplace des singularités principales de $u(t)$ s'expriment comme des fonctions zêta dynamiques η_{N} et η_{D}, associées aux conditions aux limites de Neumann et Dirichlet, respectivement. Ces fonctions zêta jouent un rôle crucial dans l'analyse de la distribution des résonances. Sous une condition de non-éclipse, pour $d \geqslant 2$ quelconque, nous montrons que η_{N} et η_{D} admettent un prolongement méromorphe à tout le plan complexe. Dans le cas particulier où la frontière ∂D est analytique réelle, en utilisant un résultat de Fried [17], nous prouvons que la fonction η_{D} ne peut pas être entière. Ceci implique, d'après un résultat de Ikawa [29], l'existence d'une bande $\{z \in \mathbb{C} :\ 0 < \operatorname{Im} z \leqslant \alpha\}$ contenant un nombre infini de résonances μ_j pour le problème de Dirichlet. De plus, pour $\alpha \gg 1$, nous obtenons une borne inférieure sur le nombre de résonances se trouvant dans cette bande.

Keywords: Dynamical zeta function, Billiard flow, Scattering Resonances, Modified Lax–Phillips conjecture.
2020 *Mathematics Subject Classification:* 37D20, 37D05, 37D40, 81U24.
(*) During the project the first author was supported by the European Research Council (ERC) under the European Unions Horizon 2020 research and innovation programme with agreement No. 725967.

1. Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with smooth boundary and let $D = \bigcup_{j=1}^r D_j$. We assume that every D_j has non-empty interior and throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \operatorname{convex\,hull}(D_i \cup D_j) = \emptyset, \tag{1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all periodic trajectories for the billiard flow in $\Omega = \mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangential intersections to the boundary of D. Notice that if (1.1) is not satisfied, for generic perturbations of ∂D all periodic reflecting trajectories in Ω have no tangential intersections to ∂D (see Theorem 6.3.1 in [47]). We consider the (non-grazing) billiard flow φ_t (see Section 2.2 for a precise definition). In this paper the periodic trajectories will be called periodic rays and we refer to Chapter 2 in [47] for basic definitions. For any periodic trajectory γ, denote by $\tau(\gamma) > 0$ its period, by $\tau^\sharp(\gamma) > 0$ its primitive period, and by $m(\gamma)$ the number of reflections of γ at the obstacles. Denote by P_γ the associated linearized Poincaré map (see Section 2.3 in [47] and Appendix A for the definition). Let $\mathcal{P}$ be the set of all periodic rays. The counting function of the lengths of periodic rays satisfies

$$\sharp\{\gamma \in \mathcal{P} :\ \tau^\sharp(\gamma) \leqslant x\} \sim \frac{\mathrm{e}^{ax}}{ax}, \quad x \longrightarrow +\infty, \tag{1.2}$$

for some $a > 0$ (see for instance, [42, Theorem 6.5] for weak-mixing suspension symbolic flow and [31, 41]). In contrast to the case $r = 2$, for $r \geqslant 3$ there exists an infinite number of primitive periodic trajectories and we have (see Corollary 2.2.5 in [47]) the estimate

$$\sharp\{\gamma \in \mathcal{P} :\ \tau(\gamma) \leqslant x\} \leqslant \mathrm{e}^{a_1 x},\ x > 0$$

with $a_1 > a$. Moreover, for some positive constants C_1, b_1, b_2 we have (see for instance [44, Appendix])

$$C_1 \mathrm{e}^{b_1 \tau(\gamma)} \leqslant |\det(\mathrm{Id} - P_\gamma)| \leqslant \mathrm{e}^{b_2 \tau(\gamma)}, \quad \gamma \in \mathcal{P}.$$

Using these estimates, we may define for $\mathrm{Re}(s) \gg 1$ two Dirichlet series

$$\eta_{\mathrm{N}}(s) = \sum_{\gamma \in \mathcal{P}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \quad \eta_{\mathrm{D}}(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},$$

where the sums run over all the oriented periodic rays. Notice that some periodic rays have only one orientation, while others admits two (see

Section 2.3). On the other hand, the length $\tau^\sharp(\gamma)$, the period $\tau(\gamma)$ and $|\det(\mathrm{Id} - P_\gamma)|^{1/2}$ are independent of the orientation of γ.

The series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$ are related to the resonances of the self-adjoint operators $-\Delta_{\mathrm{b}}$, $\mathrm{b} = \mathrm{N}, \mathrm{D}$, acting on domains $\mathcal{D}_{\mathrm{b}} \subset \mathcal{H} = L^2(\mathbb{R}^d \setminus D)$, with Neumann and Dirichlet boundary conditions on ∂D, respectively. To explain this relation, consider the resolvents

$$\mathcal{R}_{\mathrm{b}}(\mu) = \left(-\Delta_{\mathrm{b}} - \mu^2\right)^{-1} : \mathcal{H} \longrightarrow \mathcal{D}_{\mathrm{b}},$$

which are analytic in $\{\mu \in \mathbb{C} : \operatorname{Im} \mu < 0\}$. Then $\mathcal{R}_{\mathrm{b}}(\mu) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{b,loc}}$ has a meromorphic continuation to $\mu \in \mathbb{C}$ if d is odd, and to the logarithmic covering of $\mathbb{C} \setminus \{0\}$ if d is even (see [37, Chapter 5] for d odd and [15, Chapter 4]). These resolvents have poles in $\{z \in \mathbb{C} : \ \operatorname{Im} z > 0\}$ and the poles μ_j are called *resonances*. Introduce the distribution $u \in \mathcal{D}'(\mathbb{R})$ by the formula

$$\langle u, \varphi \rangle = 2 \ \mathrm{tr}_{L^2(\mathbb{R}^d)} \int_{\mathbb{R}} \Big(\cos(t\sqrt{-\Delta_{\mathrm{b}}}) \oplus 0 - \cos(t\sqrt{-\Delta_0})\Big)\varphi(t)\mathrm{d}t, \qquad \varphi \in C_c^\infty(\mathbb{R}).$$

Here Δ_0 is the free Laplacian in $\mathbb{R}^d$ and writing $L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d \setminus D) \oplus L^2(D)$, the operator $\cos(t\sqrt{-\Delta_{\mathrm{b}}}) \oplus 0$ acts as 0 on $L^2(D)$. Then for d odd, Melrose [40] (see also [3] for a slightly weaker result) proved that $u|_{\mathbb{R}\setminus\{0\}}$ is a distribution in $\mathcal{D}'(\mathbb{R} \setminus \{0\})$ having the representation

$$u(t) = \sum_j m(\mu_j) \mathrm{e}^{i|t|\mu_j},$$

where $m(\mu_j)$ is the multiplicity of μ_j. In the notation we omitted the dependence on the boundary conditions. The above series converges in the sense of distributions since we have a bound $\sharp\{\mu_j : \ |\mu_j| \leqslant r\} \leqslant Cr^d$ for all $r > 0$ (see Section 4.3 in [15]) and we may express the action $\langle u, \varphi \rangle$ on functions $\varphi \in C_c^\infty(\mathbb{R}^+)$ by the derivatives of φ (see Lemma B.1 in Appendix B). The reader may consult [57] and [15] for the form of the singularity of $u(t)$ at $t = 0$, though it is not important for our exposition.

For d even, the situation is more complicated since the resonances are defined in a logarithmic covering $\exp^{-1}(\mathbb{C}\setminus\{0\})$ of $\mathbb{C}\setminus\{0\}$ and the arguments of the resonances are not bounded (see [54, 55]). Let $\Lambda = \mathbb{C} \setminus \mathrm{e}^{i\frac{\pi}{2}\overline{\mathbb{R}^+}}$ and for $0 < \omega < \frac{\pi}{2}$ introduce

$$\Lambda_\omega = \{\mu \in \Lambda : \ 0 < \operatorname{Im} \mu \leqslant \omega |\operatorname{Re} \mu|, \ 0 < \arg \mu < \pi\}.$$

Choose a function ψ in $C_c^\infty(\mathbb{R}; [0,1])$ equal to 1 in a neighborhood of 0 and denote by $\sigma_{\mathrm{b}}(\lambda) := \frac{i}{2\pi} \log \det S_{\mathrm{b}}(\lambda)$ the scattering phase related to $-\Delta_{\mathrm{b}}$, where $S_{\mathrm{b}}(\lambda)$ is the scattering matrix (see Definition 4.25 in [58] for $S_{\mathrm{b}}(\lambda)$).

Following the work of Zworski (Theorem 1 in [58]), there exists a function $v_{\omega,\psi} \in C^\infty(\mathbb{R} \setminus \{0\})$ such that for even dimension d one has in the sense of distributions $\mathcal{D}'(\mathbb{R} \setminus \{0\})$

$$u(t) = \sum_{\mu_j \in \Lambda_\omega} m(\mu_j) \mathrm{e}^{i\mu_j |t|} + m(0) + 2\int_0^\infty \psi(\lambda) \frac{\mathrm{d}\sigma_{\mathrm{b}}}{\mathrm{d}\lambda}(\lambda) \cos(t\lambda) \mathrm{d}\lambda + v_{\omega,\psi}(t), \tag{1.3}$$

where $m(0)$ is a constant and

$$\partial_t^k v_{\omega,\psi}(t) = \mathcal{O}(|t|^{-N}), \quad \forall\, k,\ \forall\, N,\ |t| \longrightarrow \infty.$$

The reader may consult [50] for a local trace formula involving the resonances. Concerning the singularities of the distribution $u(t) \in \mathcal{D}'(\mathbb{R} \setminus \{0\})$, from [3] it follows that

$$\operatorname{sing\,supp} u \subset \{\pm\tau(\gamma) : \gamma \in \mathcal{P}\}.$$

Under the condition (1.1), every periodic trajectory γ with period $T = \tau(\gamma)$ is an ordinary reflecting ray and the singularity of u at $t = T$ was described by Guillemin and Melrose [22]. More precisely, the singularity at T has the form

$$\sum_{\gamma \in \mathcal{P}, \tau(\gamma) = T} (-1)^{m(\gamma)} \tau^\sharp(\gamma) |\det(\mathrm{Id} - P_\gamma)|^{-1/2} \delta(t - T) + L^1_{\mathrm{loc}}(\mathbb{R}) \tag{1.4}$$

(see for instance, Corollary 4.3.4 in [47]), where for the Neumann problem the factor $(-1)^{m(\gamma)}$ must be omitted. Taking the sum of the Laplace transforms of the singularities of $u(t)|_{\mathbb{R}^+}$ at $\tau(\gamma)$, $\gamma \in \mathcal{P}$, we obtain the Dirichlet series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$.

The poles of $\eta_{\mathrm{N}}(s)$ and $\eta_{\mathrm{D}}(s)$ are important for the analysis of the distribution of the resonances (see [30, 31, 32, 33, 46, 52] and the papers cited there). By using the Ruelle transfer operator and symbolic dynamics (see [32, 41, 44, 52]), a meromorphic continuation of $s \mapsto \eta_{\mathrm{N}}(s), \eta_{\mathrm{D}}(s)$ has been proved in a domain $s_0 - \epsilon \leqslant \operatorname{Re} s$ with a suitable $\epsilon > 0$, where s_0 is the abscissa of absolute convergence of the Dirichlet series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$. In particular, these results imply the asymptotic (1.2). Recently, a meromorphic continuation to $\mathbb{C}$ of the series

$$\sum_{\gamma \in \mathcal{P}} \frac{\tau^\sharp(\gamma) \mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|}, \quad \operatorname{Re}(s) \gg 1, \tag{1.5}$$

has been proved by Delarue–Schütte–Weich (see Theorem 5.8 in [12]). We refer also to [49] for results concerning weighted zeta functions. On the other hand, a meromorphic continuation to the whole complex plane of the semi-classical zeta function for contact Anosov flows was established by Faure–Tsujii [16]. Their zeta function is similar to the function $\zeta_{\mathrm{N}}(s)$ defined in (1.6) below. The meromorphic continuation of the Ruelle zeta function $\prod_{\gamma\in\mathcal{P}}(1-\mathrm{e}^{-s\tau(\gamma)})^{-1}$ for general Anosov flows was established by Giulietti–Liverani–Pollicott [18] (see also the work of Dyatlov–Zworski [14] for another proof based on microlocal analysis). In this paper the series $\eta_{\mathrm{N}}(s)$, $\eta_{\mathrm{D}}(s)$ are simply called dynamical zeta functions following previous works [44, 46] and we refer to the book of Baladi [1] for more references concerning zeta functions for hyperbolic dynamical systems.

Our main result is the following

Theorem 1.1. — *Let $d \geqslant 2$ and let the obstacles $D_j, j = 1,\dots,r$, satisfy the condition (1.1). Then the series $\eta_{\mathrm{N}}(s)$ and $\eta_{\mathrm{D}}(s)$ admit a meromorphic continuation to the whole complex plane with simple poles and integer residues.*

One may also consider the zeta functions $\zeta_{\mathrm{b}}(s)$ associated to the boundary conditions $\mathrm{b} = \mathrm{D}, \mathrm{N}$, defined for $\operatorname{Re} s$ large enough by

$$\zeta_{\mathrm{b}}(s) = \exp\left(-\sum_{\gamma\in\mathcal{P}}(-1)^{m(\gamma)\varepsilon(\mathrm{b})}\frac{\mathrm{e}^{-s\tau(\gamma)}}{\mu(\gamma)|\det(\mathrm{Id}-P_\gamma)|^{1/2}}\right), \tag{1.6}$$

where $\varepsilon(\mathrm{D}) = 1$, $\varepsilon(\mathrm{N}) = 0$ and $\tau(\gamma) = \mu(\gamma)\tau^\sharp(\gamma)$; $\mu(\gamma) \in \mathbb{N}$ is the repetition number. Notice that we have

$$\frac{\zeta_{\mathrm{b}}'(s)}{\zeta_{\mathrm{b}}(s)} = \eta_{\mathrm{b}}(s), \quad \mathrm{b} = \mathrm{D}, \mathrm{N}, \quad \operatorname{Re} s \gg 1. \tag{1.7}$$

In particular, since by the above theorem $\eta_{\mathrm{b}}(s)$ has simple poles with integer residues, it follows by a classical argument of complex analysis that we have the following

Corollary 1.2. — *Under the assumptions of Theorem 1.1 for* $\mathrm{b} = \mathrm{D}, \mathrm{N}$, *the function $s \mapsto \zeta_{\mathrm{b}}(s)$ extends meromorphically to the whole complex plane.*

In fact, we will prove a slightly more general result. For $q \in \mathbb{N}$, $q \geqslant 2$, consider the Dirichlet series

$$\eta_q(s) = \sum_{\gamma\in\mathcal{P},\, m(\gamma)\in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}}, \quad \operatorname{Re}(s) \gg 1,$$

where the sum runs over all periodic rays γ with $m(\gamma) \in q\mathbb{N}$. We will show that $\eta_q(s)$ admits a meromorphic continuation to the whole complex plane, with simple poles and residues valued in $\mathbb{Z}/q$ (see Theorem 4.1). In particular, considering the function $\zeta_q(s)$ defined by

$$\zeta_q(s) = \exp\left(-\sum_{\gamma\in\mathcal{P},\, m(\gamma)\in q\mathbb{N}} \frac{\mathrm{e}^{-s\tau(\gamma)}}{\mu(\gamma)|\det(\mathrm{Id}-P_\gamma)|^{1/2}}\right), \quad \mathrm{Re}\, s \gg 1,$$

one gets $q\zeta_q'/\zeta_q = q\eta_q$. Thus the function $s \mapsto \zeta_q(s)^q$ extends meromorphically to the whole complex plane since its logarithmic derivative is $q\eta_q$ and by Theorem 4.1 the function $q\eta_q$ has simple poles with integer residues. One reason for which it is interesting to study these functions is the relation

$$\eta_{\mathrm{D}}(s) = \frac{\mathrm{d}}{\mathrm{d}s}\log\frac{\zeta_2(s)^2}{\zeta_{\mathrm{N}}(s)} = 2\eta_2(s) - \eta_{\mathrm{N}}(s), \tag{1.8}$$

showing that $\eta_{\mathrm{D}}(s)$ for $\mathrm{Re}\, s \gg 1$ is expressed as the difference of two Dirichlet series with positive coefficients. In particular, to show that $\eta_{\mathrm{D}}(s)$ has a meromorphic extension to $\mathbb{C}$, it is sufficient to prove that both series $\eta_{\mathrm{N}}(s)$ and $\eta_2(s)$ have this property.

The distribution of the resonances μ_j in $\mathbb{C}$ depends on the geometry of the obstacles and for trapping obstacles it was conjectured by Lax and Phillips [37, p. 158] that there exists a sequence of resonances μ_j with $\mathrm{Im}\,\mu_j \searrow 0$. For two disjoint strictly convex obstacles this conjecture is false since there exists a strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z \leqslant a\}$ without resonances (see [26]). Ikawa [31, p. 212] conjectured that for *trapping obstacles* and d odd there exists $\alpha > 0$ such that

$$N_{0,\alpha} = \sharp\{\mu_j \in \mathbb{C} : 0 < \mathrm{Im}\,\mu_j \leqslant \alpha\} = \infty. \tag{1.9}$$

For d even we must consider

$$N_{0,\alpha} = \sharp\{\mu_j \in \exp^{-1}(\mathbb{C}\setminus\{0\}) : 0 < \mathrm{Im}\,\mu_j \leqslant \alpha,\ 0 < \arg\mu_j < \pi\} \tag{1.10}$$

since a meromorphic extension of $\mathcal{R}_{\mathrm{D}}(\mu)$ is possible to the logarithmic covering $\exp^{-1}(\mathbb{C}\setminus\{0\})$ of $\mathbb{C}\setminus\{0\}$ (see [54, 55] for the counting function of the number of resonances μ_j when $|\mu_j| \leqslant r$ and $|\arg\mu_j| \to \infty$). Ikawa called this conjecture modified Lax–Phillips conjecture (MLPC). In this direction, for d odd, Ikawa [29, 31] proved for strictly convex disjoint obstacles satisfying (1.1) that if $\eta_{\mathrm{N}}(s)$ or $\eta_{\mathrm{D}}(s)$ cannot be prolonged as *entire functions* to $\mathbb{C}$, then there exists $\alpha > 0$ for which (1.9) holds for the Neumann or Dirichlet boundary problem. Notice that the value $\alpha > 0$ in [31] is related to the singularity of $\eta_{\mathrm{D}}(s)$ and to some dynamical characteristics. The proof in [31] can be modified to cover also the case d even, applying the trace formula of Zworski (1.3) and the results of Vodev [54, 55] (see Appendix B). It is important to note that the meromorphic continuation of

$\eta_{\rm D}(s)$ to $\mathbb{C}$ was not established previously and to apply the result of Ikawa we need to show that some (analytic) singularity exists. The existence of a such singularity is trivial for the Neumann problem since $\eta_{\rm N}(s)$ is a Dirichlet series with positive coefficients, and by the theorem of Landau (see for instance, [5, Théorème 1, Chapitre IV]), $\eta_{\rm N}(s)$ must have a singularity at $s_0 \in \mathbb{R}$, where s_0 is the abscissa of absolute convergence of $\eta_{\rm N}(s)$. Moreover, for d odd it was proved (see [45]) that there are constants $c_0 > 0$, $\varepsilon_0 > 0$ such that for every $0 < \varepsilon \leqslant \varepsilon_0$ we have a lower bound

$$\sharp \left\{ \mu_j \in \mathbb{C} :\ 0 < \operatorname{Im} \mu_j \leqslant \frac{c_0}{\varepsilon},\ |\mu_j| \leqslant r \right\} \geqslant C_\varepsilon r^{1-\varepsilon},\ r \longrightarrow \infty.$$

The situation for the Dirichlet problem is more complicated since $\eta_{\rm D}(s)$ is analytic for $\operatorname{Re} s \geqslant s_0$, s_0 being the abscissa of absolute convergence [44]. Moreover, for $d = 2$ [51] and for $d \geqslant 3$ under some conditions [53] Stoyanov proved that there exists $\varepsilon > 0$ such that $\eta_{\rm D}(s)$ is analytic for $\operatorname{Re} s \geqslant s_0 - \varepsilon$. The reason of this cancellation of singularities is related to the change of signs in the Dirichlet series defining $\eta_{\rm D}(s)$, as it is emphasised by the relation (1.8). Despite many works in the physical literature concerning the n-disk problem (see for example [4, 11, 38, 48, 56] and the references cited there), a rigorous proof of the (MLPC) was established only for sufficiently small balls [32] and for obstacles with sufficiently small diameters [52].

In this direction we prove the following

Theorem 1.3. — *Assume the boundary ∂D real analytic. Under the assumptions of Theorem 1.1, the function $\eta_{\rm D}$ has at least one pole and the* (MLPC) *is satisfied for the Dirichlet problem. Moreover, for every* $0 < \delta < 1$ *there exists $\alpha_\delta > 0$ such that for $\alpha > \alpha_\delta$ and d odd we have*

$$\sharp\{\mu_j \in \mathbb{C} :\ 0 < \operatorname{Im} \mu_j \leqslant \alpha,\ |\mu_j| \leqslant r\} \neq \mathcal{O}(r^\delta), \tag{1.11}$$

while for d even we have

$$\sharp\{\mu_j \in \Lambda_\omega :\ 0 < \operatorname{Im} \mu_j \leqslant \alpha,\ 0 < \arg \mu_j < \pi,\ |\mu_j| \leqslant r\} \neq \mathcal{O}(r^\delta). \tag{1.12}$$

Thus for the resonances of Dirichlet problem we obtain the analog of the result concerning the Neumann problem mentioned above. More precisely, in Appendix B (see Proposition B.2 and Theorem B.3) we show that there exists $a > 0$ depending on the singularity of η_D and the dynamical characteristics of D such that for any $0 < \delta < 1$, if we choose

$$\alpha = \frac{a}{1-\delta},$$

then for d odd and any constant $0 < C < \infty$ the estimate

$$\sharp\{\mu_j \in \mathbb{C} :\ 0 < \operatorname{Im} \mu_j \leqslant \alpha,\ |\mu_j| \leqslant r\} \leqslant C r^\delta, \quad r \geqslant 1,$$

does not hold. For similar results and reference concerning the Pollicott–Ruelle resonances we refer to [35, Theorem 2] and [34, Theorem 4.1].

Our paper relies heavily on the works [12, 13] and we provide specific references in the text. For convenience of the reader we explain briefly the general idea of the proofs of Theorems 1.1 and 1.3. First, in Section 2 we make some geometric preparations. The non-grazing billiard flow φ_t acts on $M = B/\sim$, where

$$B = S\mathbb{R}^d \setminus (\pi^{-1}(\mathring{D}) \cup \mathcal{D}_g),$$

$\pi : S\mathbb{R}^d \to \mathbb{R}^d$ is the natural projection, $\mathcal{D}_{\mathrm{g}} = \pi^{-1}(\partial D) \cap T(\partial D)$ is the grazing part and $(x, v) \sim (y, w)$ if and only if $(x, v) = (y, w)$ or $x = y \in \partial D$ and w is equal to the reflected direction of v at $x \in \partial D$. By using this equivalence relation, the flow φ_t is continuous in M. However, to apply the Dyatlov–Guillarmou theory [13] in order to study the spectral properties of φ_t which are related to the dynamical zeta functions, we need to work with a *smooth flow*. For this reason we use a special *smooth structure* on M defined by flow-coordinates introduced in the recent work of Delarue–Schütte–Weich [12] (see Section 2.2). In this smooth model, the flow φ_t is smooth, and it is uniformly hyperbolic when restricted to the compact trapped set K of φ_t (see Section 2.4). The periodic points are dense in K and for any $z \in K$ the tangent space T_zM has the decomposition $T_zM = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z)$ with unstable and stable spaces $E_u(z)$, $E_s(z)$, where X is the generator of φ_t. A meromorphic continuation of the cut-off resolvent $\chi(X + s)^{-1}\chi$ with $\chi \in C_c^\infty(M)$ supported near K has been established in [13] in a general setting. As in [14] and [13], the estimates on the wavefront set of the resolvent $\chi(X+s)^{-1}\chi$ allow to define its flat trace which is related to the series (1.5). This implies a meromorphic continuation of this series to $\mathbb{C}$ (see [12]).

To prove a meromorphic continuation of the series $\eta_{\mathrm{N}}(s)$ which involves factors $|\det(\mathrm{Id} - P_\gamma)|^{-1/2}$ instead of $|\det(\mathrm{Id} - P_\gamma)|^{-1}$, a natural approach would consist to study the Lie derivative $\mathcal{L}_X$ acting on sections of the unstable bundle E_u (see for example [16, pp. 6–8]). However, in general, $E_u(z)$ is not smooth with respect to z, but only Hölder continuous. Thus we are led to change the geometrical setting as in the work of Faure–Tsujii [16] (notice that the Grassmannian bundle introduced below also appears in [7] and [19]). Consider the Grassmannian bundle $\pi_G : G \to V$ over a neighborhood V of K; for every $z \in V$ the fiber $\pi_G^{-1}(z)$ is formed by all $(d-1)$-dimensional planes of T_zV. Define the trapped set

$$\widetilde{K}_u = \{(z, E_u(z)) :\ z \in K\} \subset G$$

and introduce the natural lifted smooth flow $\widetilde{\varphi}_t$ on G (see Section 2.5). Then according to [7, Lemma A.3], the set $\widetilde{K}_u$ is hyperbolic for $\widetilde{\varphi}_t$. We introduce the tautological bundle $\mathcal{E} \to G$ by setting

$$\mathcal{E} = \{(\omega, v) \in \pi_G^*(TV) \,:\, \omega \in G,\ v \in [\omega]\},$$

where $[\omega]$ denotes the subspace of $T_{\pi_G(\omega)}V$ that $\omega \in G$ represents, and $\pi_G^*(TV)$ is the pull-back of the tangent bundle $TV \to V$ by π_G. Next, we define the vector bundle $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG \,:\, \mathrm{d}\pi_G(w) \cdot W = 0\}$$

which is the "vertical subbundle" of the bundle $TG \to G$. Finally, set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$. We define a suitable flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ as well as a transfer operator (see Section 2.6 for the notations)

$$\Phi_{-t}^{k,\ell,*} u(\omega) = \Phi_t^{k,\ell}[\mathbf{u}(\widetilde{\varphi}_{-t}(\omega)], \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

For a periodic orbit $\gamma(t)$ of φ_t, this geometrical setting allows to express the term $|\det(\mathrm{Id} - P_\gamma)|^{-1/2}$ as a finite sum involving the traces $\mathrm{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})$ related to the periodic orbit $\widetilde{\gamma} = \{(\gamma(t), E_u(\gamma(t)) \,:\, t \in [0, \tau(\gamma)]\}$ of the flow $\widetilde{\varphi}_t$ (see Section 3.2 for the notation $\alpha_{\widetilde{\gamma}}^{k,\ell}$ and Lemma 3.1). This crucial argument explains the introduction of the bundles $\mathcal{E}_{k,\ell}$ and the related geometrical technical complications. In this context we may apply the Dyatlov–Guillarmou theory (see Theorem 1 in [13]) for the generators

$$\mathbf{P}_{k,\ell}\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_{-t}^{k,\ell,*}\mathbf{u}\Big)\Big|_{t=0}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell})$$

of the transfer operators $\Phi_{-t}^{k,\ell,*}$ (in fact, by using a smooth connexion, we introduce a new operator $\mathbf{Q}_{k,\ell}$ which coincides with $\mathbf{P}_{k,\ell}$ near $\widetilde{K}_u$ (see Section 2.8)). This leads to a meromophic continuation of the cut-off resolvent $\widetilde{\chi}(\mathbf{Q}_{k,\ell} + s)^{-1}\widetilde{\chi}$, where $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ is equal to 1 on $\widetilde{K}_u$ (see Section 2.8 for the notations). By applying the Guillemin flat trace formula [21] (see Appendix B in [14] and Section 3 in [49]), concerning

$$\mathrm{tr}^\flat\Big(\int_0^\infty \varrho(t)\widetilde{\chi}(\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u})\widetilde{\chi}\mathrm{d}t\Big), \quad \varrho \in C_c^\infty(0, \infty),$$

we obtain the meromorphic continuation of η_{N}. Finally, the meromorphic continuation of η_q is obtained in a similar way, by considering in addition a certain *q-reflection bundle* $\mathcal{R}_q \to G$ to which the flow $\widetilde{\varphi}_t$ can be lifted (see Section 4.1).

The strategy to prove Theorem 1.3 is the following. First, the representation (1.8) tells us that, if $\eta_{\mathrm{D}}(s)$ can be extended to an entire function, then the function $\zeta_2^2/\zeta_{\mathrm{N}}$ has neither zeros nor poles on the whole complex plane. For obstacles with real analytic boundary we may use real analytic charts near ∂D to define a real analytic structure on M which makes φ_t a real analytic flow. In this setting we may apply a result of Fried [17] to the non-grazing flow φ_t lifted to the Grassmannian bundle, and show that the entire functions ζ_2 and ζ_{N} have *finite order.* This crucial point implies that the meromorphic function $\zeta_2^2/\zeta_{\mathrm{N}}$ has also finite order. Finally, by using Hadamard's factorisation theorem, one concludes that we may write $\zeta_2(s)^2/\zeta_{\mathrm{N}}(s) = \mathrm{e}^{Q(s)}$ for some polynomial $Q(s)$. This leads to $\eta_{\mathrm{D}}(s) = -Q'(s)$. Since $\eta_{\mathrm{D}}(s) \to 0$ as $\mathrm{Re}\, s \to +\infty$, we obtain a contradiction and $\eta_D(s)$ is not entire. The existence of a singularity of $\eta_D(s)$ implies the lower bound (B.4) (see Appendix B) and we obtain (1.11) and (1.12). Notice that this argument works as soon as the entire functions ζ_2 and ζ_{N} have *finite order.* The recent work of Bonthonneau–Jézéquel [6] about Anosov flows suggests that this should be satisfied for obstacles with *Gevrey* regular boundary ∂D. In particular, the (MLPC) should be true for such obstacles. However in this paper we are not going to study this generalization.

The paper is organised as follows. In Section 2 one introduces the geometric setting of the billiard flow φ_t and its smooth model. We define the Grassmannian extension G and the bundles $\mathcal{E}, \mathcal{F}$, $\mathcal{E}_{k,l} = \Lambda^k \mathcal{E}^\star \otimes \Lambda^\ell \mathcal{F}$ over G. Next, we discuss the setting for which we apply the Dyatlov–Guillarmou theory [13] for some first order operator $\mathbf{Q}_{k,\ell}$ leading to a meromorphic continuation of the cut-off resolvent $\mathbf{R}_{k,\ell}(s) = \widetilde{\chi}(\mathbf{Q}_{k,\ell} + s)^{-1}\widetilde{\chi}$. In Section 3 we treat the flat trace of the resolvent $\mathbf{R}_{\varepsilon}^{k,\ell}(s) = \mathrm{e}^{-\varepsilon(\mathbf{Q}_{k,\ell}+s)}\mathbf{R}_{k,\ell}(s)$, $\varepsilon > 0$, and we obtain a meromorphic continuation of η_N. In Section 4 we study the dynamical zeta functions $\eta_q(s)$ for particular rays γ having number of reflections $m(\gamma) \in q\mathbb{N}$, $q \geqslant 2$. Applying the result for $\eta_2(s)$, we deduce the meromorphic continuation of η_D. Finally, in Section 5 we treat the modified Lax–Phillips conjecture for obstacles with real analytic boundary and we prove that the function η_{D} is not entire. In Appendix A we present a proof for $d \geqslant 2$ of the uniform hyperbolicity of the flow ϕ_t in the Euclidean metric in $\mathbb{R}^d$, while in Appendix B we discuss the modifications of the proof of Theorem 2.1 in [31] for even dimensions and we finish the proof of Theorem 1.3.

Acknowledgements

We would like to thank Frédéric Faure, Benjamin Delarue, Stéphane Nonnenmacher, Tobias Weich and Luchezar Stoyanov for very interesting discussions. Thanks are also due to Colin Guillarmou for pointing out to us that we could use Fried's result for the proof of Theorem 1.3. Finally, we would like to thank the anonymous referee for their comments and suggestions that helped to improve the manuscript.

2. Geometrical setting

2.1. The billiard flow

Let $D_1, \dots, D_r \subset \mathbb{R}^d$ be pairwise disjoint compact convex obstacles, satisfying the condition (1.1), where $r \geqslant 3$. We denote by $S\mathbb{R}^d$ the unit tangent bundle of $\mathbb{R}^d$ and by $\pi : S\mathbb{R}^d \to \mathbb{R}^d$ the natural projection. For $x \in \partial D_j$, we denote by $n_j(x)$ the *inward unit normal vector* to ∂D_j at the point x pointing into D_j. Set $D = \bigcup_{j=1}^r D_j$ and

$$\mathcal{D} = \{(x,v) \in S\mathbb{R}^d \,:\, x \in \partial D\}.$$

We will say that $(x,v) \in T_{\partial D_j}\mathbb{R}^d$ is incoming (resp. outgoing) if we have $\langle v, n_j(x)\rangle > 0$ (resp. $\langle v, n_j(x)\rangle < 0$). Introduce

$$\mathcal{D}_{\mathrm{in}} = \{(x,v) \in \mathcal{D} \,:\, (x,v) \text{ is incoming}\},$$
$$\mathcal{D}_{\mathrm{out}} = \{(x,v) \in \mathcal{D} \,:\, (x,v) \text{ is outgoing}\}.$$

We define the grazing set $\mathcal{D}_{\mathrm{g}} = T(\partial D) \cap \mathcal{D}$ and one gets

$$\mathcal{D} = \mathcal{D}_{\mathrm{g}} \sqcup \mathcal{D}_{\mathrm{in}} \sqcup \mathcal{D}_{\mathrm{out}}.$$

The billiard flow $(\phi_t)_{t\in\mathbb{R}}$ is the complete flow acting on $S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ which is defined as follows. For $(x,v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$ we set

$$\tau_{\pm}(x,v) = \pm \inf\{t \geqslant 0 : x \pm tv \in \partial D\}$$

and for $(x,v) \in \mathcal{D}_{\mathrm{in/out/g}}$ we denote by $v' \in \mathcal{D}_{\mathrm{out/in/g}}$ the image of v by the reflexion with respect to $T_x(\partial D)$ at $x \in \partial D$, that is

$$v' = v - 2\langle v, n_j(x)\rangle n_j(x), \quad v \in S_x\mathbb{R}^d, \quad x \in \partial D_j$$

(see Figure 2.1). By convention, we have $\tau_{\pm}(x,v) = \pm\infty$, if the ray $x \pm tv$ has no common point with ∂D for $\pm t > 0$. Then for $(x,v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{g}}$ we define

$$\phi_t(x,v) = (x + tv, v), \quad t \in [\tau_-(x,v), \tau_+(x,v)],$$

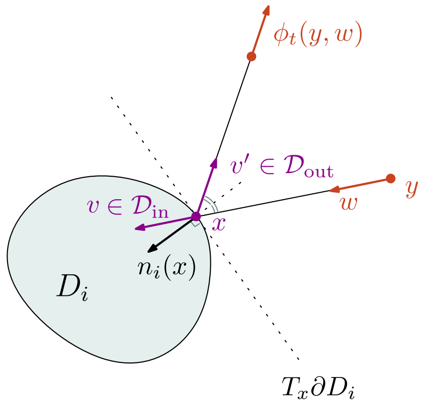

Figure 2.1. The billiard flow ϕ_t

while for $(x, v) \in \mathcal{D}_{\mathrm{in/out}}$, we set

$$\phi_t(x,v) = (x + tv, v) \quad \text{if} \quad \begin{cases} \quad\ (x,v) \in \mathcal{D}_{\mathrm{out}}, & t \in [0, \tau_+(x,v)[, \\ \text{or} \ \ (x,v) \in \mathcal{D}_{\mathrm{in}}, & t \in]\tau_-(x,v), 0], \end{cases}$$

and

$$\phi_t(x,v) = (x + tv', v') \quad \text{if} \quad \begin{cases} \quad\ (x,v') \in \mathcal{D}_{\mathrm{out}}, & t \in]0, \tau_+(x,v')[, \\ \text{or} \ \ (x,v') \in \mathcal{D}_{\mathrm{in}}, & t \in]\tau_-(x,v'), 0[. \end{cases}$$

Next we extend ϕ_t to a complete flow (which we still denote by ϕ_t) characterized by the property

$$\phi_{t+s}(x,v) = \phi_t(\phi_s(x,v)), \quad t, s \in \mathbb{R}, \quad (x,v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D}).$$

Strictly speaking, ϕ_t is not a flow, since the above flow property does not hold in full generality for $(x,v) \in \mathcal{D}_{\mathrm{in/out}}$. However, we can deal with this problem by considering an appropriate quotient space (see Section 2.2 below).

2.2. A smooth model for the non-grazing billiard flow

In this subsection, we briefly recall the construction of [12, Section 3] which allows to obtain a smooth model for the non-grazing billiard flow. First, we define the non-grazing billiard table M as

$$M = B/\sim, \quad B = S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right),$$

where $(x, v) \sim (y, w)$ if and only if $(x, v) = (y, w)$ or

$$x = y \in \partial D \quad \text{and} \quad w = v'.$$

The set M is endowed with the quotient topology. We will change the notation and pass from ϕ_t to the non-grazing flow φ_t, which is defined on M as follows. For $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\text{in}}$ we define

$$\varphi_t([(x, v)]) = [\phi_t(x, v)], \quad t \in]\tau_-^{\text{g}}(x, v), \tau_+^{\text{g}}(x, v)[,$$

where $[z]$ denotes the equivalence class of the vector $z \in B$ for the relation $\sim$, and

$$\tau_\pm^{\text{g}}(x, v) = \pm \inf\{t > 0 : \phi_{\pm t}(x, v) \in \mathcal{D}_{\text{g}}\}.$$

Clearly, we may have $\tau_\pm^{\text{g}}(x, v) = \pm\infty$. On other hand, it is important to note that $\tau_\pm^{\text{g}}(x, v) \neq 0$ for $(x, v) \in \mathcal{D}_{\text{in}}$. Note that this formula indeed defines a flow on M since each $(x, v) \in B$ has a unique representative in $(S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})) \cup \mathcal{D}_{\text{in}}$. Thus φ_t is continuous but the flow trajectory of the point (x, v) for times $t \notin]\tau_-^{\text{g}}(x, v), \tau_+^{\text{g}}(x, v)[$ is not defined.

Following [12, Section 3], we define smooth charts on $M = B/\sim$ as follows. Introduce the surjective map $\pi_M : B \to M$ by $\pi_M(x, v) = [(x, v)]$ and note that

$$\varphi_t \circ \pi_M = \pi_M \circ \phi_t. \tag{2.1}$$

Set $\mathring{B} := S\mathbb{R}^d \setminus \pi^{-1}(D)$. Then $\pi_M : \mathring{B} \to M$ is a homeomorphism onto its image $\mathcal{O}$. Let $\mathcal{G} = \pi_M(\mathcal{D}_{\text{in}})$ be the gluing region. We consider the map $\pi_M^{-1} : \mathcal{O} \to \mathring{B}$. Then we may pull back the smooth structure of $\mathring{B}$ to $\mathcal{O}$ and define the charts on $\mathcal{O}$ by using those of $\mathring{B}$. Next we wish to define charts in an open neighborhood of $\mathcal{G}$. For every point $z_\star = (x_\star, v_\star) \in \mathcal{D}_{\text{in}}$ let

$$F_{z_\star} : U_{z_\star} \times U_{z_\star} \longrightarrow \mathcal{D}_{\text{in}}$$

be a local smooth parametrization of $\mathcal{D}_{\text{in}}$, where $U_{z_\star}$ is an open small neighborhood of 0 in $\mathbb{R}^{d-1}$. For small $\varepsilon_{z_\star} > 0$, we may define the map

$$\Psi_{z_\star} :]-\varepsilon_{z_\star}, \varepsilon_{z_\star}[\times U_{z_\star} \times U_{z_\star} \longrightarrow M$$

by

$$\Psi_{z_\star}(t, y, w) = (\pi_M \circ \phi_t \circ F_{z_\star})(y, w). \tag{2.2}$$

Up to shrinking $U_{z_\star}$ and taking $\varepsilon_{z_\star}$ smaller, $\Psi_{z_\star}$ is a homeomorphism onto its image $\mathcal{O}_{z_\star} \subset M$, (see Corollary 4.3 in [12]). Indeed, repeating the argument of [12], to see that $\Psi_{z_\star}$ is injective, let $F_{z_\star}(y_k, w_k) = (x_k, v_k) \in \mathcal{D}_{\text{in}}$, $k = 1, 2$, and assume that $\pi_M \phi_{t_1}(x_1, v_1) = \pi_M \phi_{t_2}(x_2, v_2)$. Since the vectors in $\mathcal{D}_{\text{in}}$ are transversal to ∂D, we see that for each $z \in \mathcal{O}_{z_\star}$, there is a unique $t \in]-\varepsilon_{z_\star}, \varepsilon_{z_\star}[$ such that $\varphi_t(z) \in \mathcal{G}$. In particular, we have $t_1 = 0$

if and only if $t_2 = 0$. In this case, $(x_1, v_1) = (x_2, v_2)$ since $\pi_M : \mathcal{D}_{\text{in}} \to \mathcal{G}$ is injective. If $t_1 \neq 0, t_2 \neq 0$, then t_1 and t_2 have the same sign and by the injectivity of $\pi_M : \mathring{B} \to M$ and the definition of ϕ_t, we have

$$\begin{cases} (x_1 + t_1 v_1, v_1) = (x_2 + t_2 v_2, v_2) & \text{if } t_1, t_2 > 0, \\ (x_1 + t_1 v_1', v_1') = (x_2 + t_2 v_2', v_2') & \text{if } t_1, t_2 < 0, \end{cases}$$

where v_k' is the reflection of v_k with respect to $T_{x_k}\partial D$ for $k = 1, 2$. Thus one concludes that $(t_1, x_1, v_1) = (t_2, x_2, v_2)$. As mentioned above, the directions in $\mathcal{D}_{\text{in}}$ are transversal to the boundary ∂D. This implies that the maps $\Psi_{z_\star}$ are open. In particular, $\Psi_{z_\star}$ realises a homeomorphism onto its image $\mathcal{O}_{z_\star}$ and we declare the map $\Psi_{z_\star}^{-1} : \mathcal{O}_{z_\star} \to \,]-\varepsilon_{z_\star}, \varepsilon_{z_\star}[\times U_{z_\star} \times U_{z_\star}$ as a chart. Hence we obtain an open covering

$$\mathcal{G} \subset \bigcup_{z_\star \in \mathcal{D}_{\text{in}}} \mathcal{O}_{z_\star}.$$

Note that if $\mathcal{O} \cap \mathcal{O}_{z_\star} \neq \emptyset$ for any $z_\star$, clearly the map

$$(t, x, v) \longmapsto (\pi_M^{-1} \circ \Psi_{z_\star})(t, x, v) = (\phi_t \circ F_{z_\star})(x, v)$$

is smooth on $\Psi_{z_\star}^{-1}(\mathcal{O} \cap \mathcal{O}_{z_\star})$. On the other hand, assume that $\mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'} \neq \emptyset$ for some $z_\star, z_\star' \in \mathcal{D}_{\text{in}}$. If $\pi_M(\phi_t(F_{z_\star}(x, v))) = \pi_M(\phi_s(F_{z_\star'}(y, w))) \in \mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'}$, then as above this yields $t = s$, $F_{z_\star}(x, v) = F_{z_\star'}(y, w)$, and we conclude that

$$\begin{aligned} (\Psi_{z_\star}^{-1} \circ \Psi_{z_\star'})(t, y, w) &= \left(\Psi_{z_\star}^{-1} \circ \pi_M \circ \phi_t \circ F_{z_\star'}\right)(y, w) \\ &= \left(\Psi_{z_\star}^{-1} \circ \pi_M \circ \phi_t \circ F_{z_\star}\right)\Big((F_{z_\star}^{-1} \circ F_{z_\star'})(y, w)\Big) \\ &= \left(t, (F_{z_\star}^{-1} \circ F_{z_\star'})(y, w)\right). \end{aligned} \tag{2.3}$$

This shows that the change of coordinates $\Psi_{z_\star}^{-1} \circ \Psi_{z_\star'}$ is smooth on the set $\Psi_{z_\star'}^{-1}(\mathcal{O}_{z_\star} \cap \mathcal{O}_{z_\star'})$, and these charts endow M with a smooth structure. It is easy to see that with this differential structure the flow (φ_t) is smooth on M. Indeed, this is obvious far from the gluing region $\mathcal{G}$. Now let $z \in \mathcal{G}$ and $z_\star \in \mathcal{D}_{\text{in}}$ be such that $\pi_M(z_\star) = z$. Then for $s, t \in \mathbb{R}$, with $|t| + |s|$ small, and $(y, w) \in U_{z_\star} \times U_{z_\star}$, we have

$$\begin{aligned} \left(\Psi_{z_\star}^{-1} \circ \varphi_s \circ \Psi_{z_\star}\right)(t, y, w) &= \left(\Psi_{z_\star}^{-1} \circ \varphi_s \circ \pi_M \circ \phi_t \circ F_{z_\star}\right)(y, w) \\ &= \left(\Psi_{z_\star}^{-1} \circ \pi_M \circ \phi_{t+s} \circ F_{z_\star}\right)(y, w) \\ &= (s + t, y, w). \end{aligned}$$

Consequently, the flow (φ_t) is also smooth near $\mathcal{G}$ and we obtain a smooth non-complete flow on M.

2.3. Oriented periodic rays

A periodic point of the billiard flow is a pair (x, v) lying in $S\mathbb{R}^d$, together with a number $\tau > 0$, such that $\phi_\tau(x, v) = (x, v)$. The point (x, v) will be called τ- periodic. A *periodic trajectory* of ϕ_t, or equivalently an *oriented periodic ray*, is by definition an equivalence class of periodic points, where we identify two periodic points (x, v) and (y, w), if they are τ-periodic with the same τ and if there are $\tau_1, \tau_2 \in \mathbb{R}$ such that $\phi_{\tau_1}(x, v) = \phi_{\tau_2}(y, w)$. Of course, the map π_M induces a bijection between oriented periodic rays and periodic orbits of the non-grazing flow φ_t. For each periodic orbit γ, we will denote by $\tau(\gamma)$ its period. Also, we will often identify a periodic orbit with a parametrization $\gamma : [0, \tau(\gamma)] \to S\mathbb{R}^d$.

Note that every oriented periodic ray is determined by a sequence

$$\alpha_\gamma = (i_1, \ldots, i_k),$$

where $i_j \in \{1, \ldots, r\}$, with $i_k \neq i_1$ and $i_j \neq i_{j+1}$ for $j = 1, \ldots, k-1$, such that γ has *successive reflections* on $\partial D_{i_1}, \ldots, \partial D_{i_k}$. The sequence α_γ is well defined modulo cyclic permutation, and we say that the ray γ has type α_γ. The non-eclipse condition (1.1) implies that the reciprocal is true. More precisely, for any sequence $\alpha = (i_1, \ldots, i_k)$ with $i_j \neq i_{j+1}$ for $j = 1, \ldots, k-1$ and $i_k \neq i_1$, there exists a unique periodic ray γ such that $\alpha_\gamma = \alpha$ (see [47, Proposition 2.2.2 and Corollary 2.2.4]).

We conclude this paragraph by some remark on the oriented rays. For every oriented periodic ray γ generated by a periodic point $(x, v) \in \mathring{B}$ and period τ, one may consider the reversed ray $\overline{\gamma}$, generated by $(x, -v) \in \mathring{B}$ and τ. There are two possibilities. For most rays, γ and $\overline{\gamma}$ give rise to different oriented periodic rays, even if their projections in $\mathbb{R}^d$ are the same. However it might happen that $\overline{\gamma}$ coincides with γ. This is the case, for example, if the ray γ has type $\alpha = (1, 2)$ (modulo permutation).

2.4. Uniform hyperbolicity of the flow φ_t

From now on, we will work exclusively with the flow φ_t defined on $M = B/\sim$ by the smooth model described in Section 2.2. Let X be the generator of φ_t. The trapped set K of φ_t is defined as the set of points $z \in M$ which satisfy $-\tau_-^{\mathrm{g}}(z) = \tau_+^{\mathrm{g}}(z) = +\infty$ and

$$\sup A(z) = -\inf A(z) = +\infty, \quad \text{where} \quad A(z) = \{t \in \mathbb{R} \,:\, \pi(\varphi_t(z)) \in \partial D\}.$$

By definition, $\varphi_t(z)$ is defined for all $t \in \mathbb{R}$ whenever $z \in K$. The flow φ_t is called uniformly hyperbolic on K, if for each $z \in K$ there exists a decomposition

$$T_z M = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{2.4}$$

which is $\mathrm{d}\varphi_t$-invariant (in the sense that $\mathrm{d}\varphi_t(E_b(z)) = E_b(\varphi_t(z))$ where $b = u, s$), with $\dim E_s(z) = \dim E_u(z) = d-1$, such that for some constants $C > 0$, $\nu > 0$, independent of $z \in K$, and some smooth norm $\|\cdot\|$ on TM, we have

$$\|\mathrm{d}\varphi_t(z) \cdot v\| \leqslant \begin{cases} C\mathrm{e}^{-\nu t}\|v\|, & v \in E_s(z), \quad t \geqslant 0, \\ C\mathrm{e}^{-\nu |t|}\|v\|, & v \in E_u(z), \quad t \leqslant 0. \end{cases} \tag{2.5}$$

The spaces $E_s(z)$ and $E_u(z)$ depend continuously on z (see [23, Section 2]).

We may define the trapped set K_e for the flow ϕ_t in the Euclidean metric and note that $K = \pi_M(K_e)$. (Here we use the notation ϕ_t for the flow in the Euclidean metric to distinguish it with the flow φ_t definite on the smooth model). The uniform hyperbolicity on K_e of the flow ϕ_t in the Euclidean metric for $z \in \mathring{B} \cap K_e$ can be defined by the splitting of the tangent space $T_z(\mathring{B} \cap K_e)$ (see Definition 2.11 in [12] and Appendix A). Following this definition, one avoids the points $(x, v) \in K_e \cap \mathcal{D}_{in}$. To treat these points, denote $\overline{\mathcal{D}}_{\mathrm{in}} = \{(x, v) : x \in \partial D, |v| = 1, \langle v, n(x)\rangle \geqslant 0\}$ and define the *billiard ball map*

$$\mathbf{B} : \overline{\mathcal{D}}_{\mathrm{in}} \ni (x, v) \longmapsto (y, R_y w) \in \overline{\mathcal{D}}_{\mathrm{in}},$$

where $R_y : S_y\mathbb{R}^d \to S_y\mathbb{R}^d$ is the reflection with respect to $T_y(\partial D)$ and

$$(y, w) = \phi_{\tau_+(x,v)}(x, v),\ \tau_+(x, v) = \inf\{t > 0\ :\ \pi(\phi_t(x, v)) \in \partial D\}.$$

To see that $\mathbf{B}(x, v)$ is well defined we need $\tau_+(x, v) < \infty$ and this condition is satisfied for $(x, v) \in K_e \cap \overline{\mathcal{D}}_{\mathrm{in}}$. The map $\mathbf{B}$ is called *collision map* in [9], and it is smooth (see for instance, [36]). For $(x, v) \in K_e \cap \mathcal{D}_{\mathrm{in}}$ we can define $\mathrm{d}\mathbf{B}(x, v)$ and this is useful for the estimates of $\|\mathrm{d}\phi_t(x, v)\|$ for $(x, v) \in \mathring{B} \cap K_e$ (see [9, Section 4.4] and Appendix A).

The uniform hyperbolicity of ϕ_t in the Euclidean metric on $\mathring{B} \cap K_e$ implies the uniform hyperbolicity of φ_t in the smooth model (see [12, Proposition 3.7]). Thus, to obtain (2.5), we may apply the uniform hyperbolicity of ϕ_t in the Euclidean metric on $\mathring{B} \cap K_e$ established for $d = 2$ in [41] and [9, Section 4.4]. For $d \geqslant 3$, the same could perhaps be obtained by applying the results in [2, Section 4]. The hyperbolicity at the points $z = (x, v) \in K_e$ which are not periodic must be justified and the stable/unstable spaces $E_s(z)/E_u(z)$ must be well determined; for $d \geqslant 3$ this seems to be not sufficiently detailed in the literature. Since the hyperbolicity of φ_t is crucial for our exposition, and for the sake of completeness, we present in Appendix A a proof of the uniform hyperbolicity as well as a construction of $E_s(z)$ and $E_u(z)$ for all $z \in \mathring{B} \cap K_e$.

2.5. The Grassmannian extension

In what follows, we assume that the flow φ_t is hyperbolic on K and we will take a small neighborhood V of K in M, with smooth boundary. We embed V into a compact manifold without boundary N. For example, we may take the double manifold N of the closure of V. This means that $N = \overline{V} \times \{0,1\}/\sim$ and $(x,0) \sim (x,1)$ for all $x \in \partial V$. We arbitrarily extend X to obtain a smooth vector field on N, which we still denote by X. The associated flow is still denoted by φ_t (however note that this new flow φ_t is now complete).

For our exposition it is important to introduce the $(d-1)$-Grassmannian bundle

$$\pi_G : G \longrightarrow N$$

over N. More precisely, for every $z \in N$, the set $\pi_G^{-1}(z)$ consists of all $(d-1)$-dimensional planes of T_zN. Moreover, $\pi_G^{-1}(z)$ can be identified with the Grasmannian $G_{d-1}(\mathbb{R}^{2d-1})$ which is isomorphic to $O(2d-1)/(O(d-1)\times O(d))$, $O(k)$ being the space of $(k\times k)$ orthogonal matrices with entries in $\mathbb{R}$. The dimension of $O(k)$ is $k(k-1)/2$, hence the dimension of $\pi_G^{-1}(z)$ is $d(d-1)$. Note that G is a smooth compact manifold. We may lift the flow φ_t to a flow $\widetilde{\varphi}_t : G \to G$ which is simply defined by

$$\widetilde{\varphi}_t(z,E) = (\varphi_t(z), \mathrm{d}\varphi_t(z)(E)), \qquad z \in N, \quad E \subset T_zN, \quad \mathrm{d}\varphi_t(z)(E) \subset T_{\varphi_t(z)}N. \tag{2.6}$$

Introduce the set

$$\widetilde{K}_u = \{(z, E_u(z)) \,:\, z \in K\} \subset G.$$

Clearly, $\widetilde{K}_u$ is invariant under the action of $\widetilde{\varphi}_t$, since $\mathrm{d}\varphi_t(z)(E_u(z)) = E_u(\varphi_t(z))$. The set $\widetilde{K}_u$ will be seen as the trapped set of the restriction of $\widetilde{\varphi}_t$ to a neighborhood of $\widetilde{K}_u$. As K is a hyperbolic set, it follows from [7, Lemma A.3] that the set $\widetilde{K}_u$ is hyperbolic for $\widetilde{\varphi}_t$ and we have a decomposition

$$T_\omega G = \mathbb{R}\widetilde{X}(\omega) \oplus \widetilde{E}_u(\omega) \oplus \widetilde{E}_s(\omega), \quad \omega \in \widetilde{K}_u.$$

Here $\widetilde{X}$ is the generator of the flow $(\widetilde{\varphi}_t)$ and the spaces $\widetilde{E}_s(\omega)$ and $\widetilde{E}_u(\omega)$ are defined as follows. For small $\varepsilon > 0$, let

$$W_s(z,\varepsilon) = \{z' \in N : \mathrm{dist}(\varphi_t(z), \varphi_t(z')) \leqslant \varepsilon \text{ for every } t \geqslant 0\}$$

and

$$W_u(z,\varepsilon)=\{z'\in N:\operatorname{dist}(\varphi_{-t}(z),\varphi_{-t}(z'))\leqslant\varepsilon\text{ for every }t\geqslant 0\}$$

be the local stable and unstable manifolds at z of size ε, where dist is any smooth distance on N. It is known that the local stable and unstable manifolds are smooth (see for instance, [23, Section 2]). Moreover for any $b=s,u$ we have

$$T_z(W_b(z,\varepsilon))=E_b(z)$$

and for any $t\geqslant t_0>0$,

$$\varphi_t(W_s(z,\varepsilon))\subset W_s(\varphi_t(z),\varepsilon),\quad \varphi_{-t}(W_u(z,\varepsilon))\subset W_u(\varphi_{-t}(z),\varepsilon).$$

For $b=s,u$, we define

$$\widetilde{W}_b(z)=TW_b(z,\varepsilon)=\{(z',E_b(z'))\,:\,z'\in W_b(z,\varepsilon)\}\subset G.$$

Finally, for $\omega=(z,E_u(z))\in\widetilde{K}_u$, set

$$\widetilde{E}_u(\omega)=T_\omega(\widetilde{W}_u(z)),$$

and define $\widetilde{E}_s(\omega)$ as the tangent space at ω of the manifold

$$\widetilde{W}_{s,\mathrm{tot}}(z)=\left\{E\in\pi_G^{-1}(W_s(z,\varepsilon)):\operatorname{dist}(E_u(z),E)<\varepsilon\right\},$$

where dist is any smooth distance on the fibres of TN.

Lemma 2.1. — *For any $\omega=(z,E)\in G$ we have isomorphisms*

$$\widetilde{E}_u(\omega)\simeq E_u(z),\quad \widetilde{E}_s(\omega)\simeq E_s(z)\oplus\ker \mathrm{d}\pi_G(\omega).$$

Under these identifications, we have

$$\mathrm{d}\widetilde{\varphi}_t|_{\tilde{E}_u(\omega)}\simeq \mathrm{d}\varphi_t|_{E_u(z)},\quad \mathrm{d}\widetilde{\varphi}_t|_{\tilde{E}_s(\omega)}\simeq \mathrm{d}\varphi_t|_{E_s(z)}\oplus\mathrm{d}\widetilde{\varphi}_t|_{\ker\mathrm{d}\pi_G(\omega)}.$$

Proof. — Note that if $\omega=(z,E)\in G$, by (2.6) one has

$$\mathrm{d}\pi_G(\omega)\circ\mathrm{d}\widetilde{\varphi}_t(\omega)=\mathrm{d}(\pi_G\circ\widetilde{\varphi}_t)(\omega)=\mathrm{d}(\varphi_t\circ\pi_G)(\omega)=\mathrm{d}\varphi_t(z)\circ\mathrm{d}\pi_G(\omega). \tag{2.7}$$

This equality shows that $\mathrm{d}\widetilde{\varphi}_t$ preserves $\ker\mathrm{d}\pi_G$. Looking at the definitions of $\widetilde{W}_u(z)$ and $W_u(z,\varepsilon)$, we see that

$$\mathrm{d}\pi_G(\omega)|_{\tilde{E}_u(z)}:\widetilde{E}_u(z)\longrightarrow E_u(z)$$

realises an isomorphism. Then by (2.7), it is clear that $\mathrm{d}\pi_G(\omega)|_{T_\omega\tilde{W}_u(z)}$ realises a conjugation between $\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\tilde{E}_u(\omega)}$ and $\mathrm{d}\varphi_t(z)|_{E_u(z)}$. Similarly, $\mathrm{d}\pi_G|_{T_\omega\tilde{W}_s(\omega)}$ realises an isomorphism $T_\omega\widetilde{W}_s(\omega)\simeq E_s(z)$, which conjugates $\mathrm{d}\widetilde{\varphi}_t|_{\tilde{E}_s(\omega)}$ and $\mathrm{d}\varphi_t|_{E_s(z)}$. Thus the lemma will be proven if we show that we have the direct sum

$$\widetilde{E}_s(z)=T_\omega\widetilde{W}_{s,\mathrm{tot}}(z)=T_\omega\widetilde{W}_s(z)\oplus\ker\mathrm{d}\pi_G(\omega).$$

To see this, take a local trivialization $\widetilde{W}_{s,\mathrm{tot}}(z) \to W_s(z,\varepsilon) \times G_{d-1}(\mathbb{R}^{2d-1})$ sending $\omega \in G$ on (z, E_0) for some $E_0 \in G_{d-1}(\mathbb{R}^{2d-1})$ and such that $\widetilde{W}_s(z)$ is sent to $W_s(z,\varepsilon) \times \{E_0\}$. In these local coordinates one has the identifications

$$T_\omega \widetilde{W}_s(z) \simeq E_s(z) \oplus \{0\} \quad \text{and} \quad \ker \mathrm{d}\pi_G(\omega) \simeq \{0\} \oplus T_{E_0} G_{d-1}(\mathbb{R}^{2d-1}).$$

As $T_\omega \widetilde{W}_{s,\mathrm{tot}}(z)$ is identified with $E_s(z) \oplus T_{E_0} G_{d-1}(\mathbb{R}^{2d-1})$, the proof is complete. □

We conclude this paragraph by noting that for any $\omega = (z, E) \in \widetilde{K}_u$ we have

$$\begin{aligned} \dim \widetilde{E}_u(\omega) + \dim \widetilde{E}_s(\omega) &= \dim E_u(z) + \dim E_s(z) + \dim \ker \mathrm{d}\pi_G(\omega) \\ &= \dim N - 1 + \dim \pi_G^{-1}(z) \\ &= \dim G - 1, \end{aligned}$$

since $\dim G = \dim N + \dim \pi_G^{-1}(z)$.

2.6. Vector bundles

We define the tautological vector bundle $\mathcal{E} \to G$ by

$$\mathcal{E} = \{(\omega, u) \in \pi_G^*(TN) \,:\, \omega \in G,\ u \in [\omega]\},$$

where $[\omega] = E$ denotes the $(d-1)$ dimensional subspace of $T_{\pi_G(\omega)}N$ represented by $\omega = (z, E)$ and $\pi_G^*(TN)$ is the pullback bundle of TN. Also, we define the "vertical bundle" $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG \,:\, \mathrm{d}\pi_G(\omega) \cdot W = 0\}.$$

It is a subbundle of the bundle $TG \to G$. The dimensions of the fibres $\mathcal{E}_\omega$ and $\mathcal{F}_\omega$ of $\mathcal{E}$ and $\mathcal{F}$ over ω are given by

$$\dim \mathcal{E}_\omega = d - 1, \quad \dim \mathcal{F}_\omega = \dim \ker \mathrm{d}\pi_G(\omega) = \dim \pi_G^{-1}(z) = d^2 - d$$

for any $\omega \in G$ with $\pi_G(\omega) = z$. Finally, set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$, that is, we replace the fibre $\mathcal{E}_\omega$ by its dual space $\mathcal{E}_\omega^*$. We consider $\mathcal{E}^*$ and not $\mathcal{E}$ since the map $\mathrm{d}\varphi_t(\pi_G(\omega)) : \mathcal{E}_\omega \to \mathcal{E}_{\tilde{\varphi}_t(\omega)}$ is expanding for $\omega \in \widetilde{K}_u$ and $t \to +\infty$, whereas $\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top} : \mathcal{E}_\omega^* \to \mathcal{E}_{\tilde{\varphi}_t(\omega)}^*$ is contracting. Here ${}^{-\top}$ denotes the inverse transpose. Indeed, for

$\omega = (z, E_u(z)) \in \widetilde{K}_u$ and $u \in E_u(z)^*$ (here $E_u(z)^*$ is the dual vector space of $E_u(z)$) one has

$$\left\langle \mathrm{d}\varphi_t(z)^{-\top} u, v \right\rangle = \langle u, \mathrm{d}\varphi_{-t}(\varphi_t(z)) v\rangle$$

for any

$$v \in \mathrm{d}\varphi_t(z) E_u(z) = E_u(\varphi_t(z)) \in \mathcal{E}_{\tilde{\varphi}_t(\omega)},$$

where $\langle \cdot, \cdot \rangle$ is the pairing on $\mathcal{E}^*_{\tilde{\varphi}_t(\omega)} \times \mathcal{E}_{\tilde{\varphi}_t(\omega)}$. As a consequence, the map $\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}$ is contracting on $\mathcal{E}^*_\omega$ when $\omega \in \widetilde{K}_u$, since $\mathrm{d}\varphi_{-t}(\varphi_t(z))$ is contracting on $E_u(\varphi_t(z))$. This fact will be important for the proof of Lemma 3.1 below.

In what follows we use the notation $\omega = (z, \eta) \in G$ and $u \otimes v \in \mathcal{E}_{k,\ell}|_\omega$. By using the flow $\widetilde{\varphi}_t$, we introduce a flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ by setting

$$\begin{aligned} \Phi_t^{k,\ell}&(\omega, u \otimes v) \\ &= \Big(\widetilde{\varphi}_t(\omega),\ b_t(\omega) \cdot \Big[\left(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\right)^{\wedge k}(u) \otimes \mathrm{d}\widetilde{\varphi}_t(\omega)^{\wedge \ell}(v)\Big]\Big), \end{aligned} \tag{2.8}$$

where we set

$$b_t(\omega) = |\det \mathrm{d}\varphi_t(\pi_G(\omega))|_{[\omega]}|^{1/2} \cdot |\det\left(\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\ker \mathrm{d}\pi_G}\right)|^{-1}.$$

Here the determinants are taken with respect to any choice of smooth metrics g_N on N and the induced metrics g_G on G, as follows. If $\omega = (z, E) \in G$ and $t \in \mathbb{R}$, then the number $|\det \mathrm{d}\varphi_t(z)|_{[\omega]}|$ is defined as the absolute value of the ratio

$$\frac{(\mathrm{d}\varphi_t(z)|_{[\omega]})^{\wedge^{d-1}} \cdot \mu_{[\omega]}}{\mu_{[\tilde{\varphi}_t(\omega)]}},$$

where $\mu_{[\omega]} = e_{1,[\omega]} \wedge \cdots \wedge e_{d-1,[\omega]} \in \bigwedge^{d-1}[\omega]$ (resp. $\mu_{[\tilde{\varphi}_t(\omega))]} \in \bigwedge^{d-1}[\widetilde{\varphi}_t(\omega)]$) is a volume element given by any basis $e_{1,[\omega]}, \ldots, e_{d-1,[\omega]}$ of $[\omega]$ (resp. $[\widetilde{\varphi}_t(\omega)]$) which is orthonormal with respect to the scalar product induced by $g_N|_{[\omega]}$ (resp. $g_N|_{[\tilde{\varphi}_t(\omega)]}$). The number $|\det\left(\mathrm{d}\widetilde{\varphi}_t(\omega)|_{\ker \mathrm{d}\pi_G}\right)|$ is defined similarly. If we pass from one orthonormal basis to another one, we multiply the terms by the determinant of a unitary matrix and the absolute value of the above ratio is the same. On the other hand, for a periodic point $\omega_{\tilde{\gamma}} = \widetilde{\varphi}_{\tau(\gamma)}(\omega_{\tilde{\gamma}})$ this number is simply $|\det \mathrm{d}\varphi_{\tau(\gamma)}(\pi_G(\omega_{\tilde{\gamma}}))|_{[\omega_{\tilde{\gamma}}]}|$. Taking local trivializations of $\mathcal{E}^*$ and $\mathcal{F}$, we see that the action of $\Phi_t^{k,\ell}$ is smooth.

Thus we have the following diagram:

$$\begin{array}{ccc} \mathcal{E}_{k,\ell} & \xrightarrow{\Phi_t^{k,\ell}} & \mathcal{E}_{k,\ell} \\ \big\downarrow & & \big\downarrow \\ G & \xrightarrow{\tilde{\varphi}_t} & G \\ \big\downarrow{\scriptstyle \pi_G} & & \big\downarrow{\scriptstyle \pi_G} \\ N & \xrightarrow{\varphi_t} & N \end{array}$$

Now, consider the transfer operator

$$\Phi_{-t}^{k,\ell,*} : C^\infty(G, \mathcal{E}_{k,\ell}) \longrightarrow C^\infty(G, \mathcal{E}_{k,\ell})$$

defined by

$$\Phi_{-t}^{k,\ell,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell}\big[\mathbf{u}(\widetilde{\varphi}_{-t}(\omega))\big], \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{2.9}$$

Let $\mathbf{P}_{k,\ell} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$ be the generator of $\Phi_{-t}^{k,\ell,*}$, which is defined by

$$\mathbf{P}_{k,\ell}\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_{-t}^{k,\ell,*}\mathbf{u}\Big)\Big|_{t=0}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

Then we have the equality

$$\mathbf{P}_{k,\ell}(f\mathbf{u}) = (\widetilde{X}f)\mathbf{u} + f(\mathbf{P}_{k,\ell}\mathbf{u}), \quad f \in C^\infty(G), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{2.10}$$

Fix any norm on $\mathcal{E}_{k,\ell}$; this fixes a scalar product on $L^2(G, \mathcal{E}_{k,\ell})$. We also consider the transfer operator $\Phi_{-t}^{k,\ell,*}$ as a strongly continuous semigroup $e^{-t\mathbf{P}_{k,\ell}}$, $t \geqslant 0$ with generator $\mathbf{P}_{k,\ell}$ with domain in $L^2(G, \mathcal{E}_{k,\ell})$. The exponential bound of the derivatives of φ_{-t} implies an estimate

$$\|\mathrm{e}^{-t\mathbf{P}_{k,\ell}}\|_{L^2(G,\mathcal{E}_{k,\ell}) \to L^2(G,\mathcal{E}_{k,\ell})} \leqslant C\mathrm{e}^{\beta t}, \quad t \geqslant C_0 > 0,$$

for some constants $\beta > 0, C_0 > 0$. Next, we want to study the spectral properties of the operator $\mathbf{P}_{k,\ell}$ applying the work of Dyatlov–Guillarmou [13]. For this purpose, one needs to see $\widetilde{K}_u$ as the trapped set of the restriction of $\widetilde{\varphi}_t$ to some neighborhood $\widetilde{V}_u$ of $\widetilde{K}_u$ in G, so that $\partial\widetilde{V}_u$ has convexity properties with respect to $\widetilde{X}$ (see the condition (2.11) below with $\widetilde{Y}$ replaced by $\widetilde{X}$). These conditions are necessary if we wish to apply the results in [13]. However, it is not clear that such a neighborhood exists, and one needs to modify $\widetilde{X}$ slightly outside a neighborhood of $\widetilde{K}_u$ to obtain the desired properties. This is done in Section 2.7 below.

2.7. Isolating blocks

By [10, Theorem 1.5], there exists an arbitrarily small open neighborhood $\widetilde{V}_u$ of $\widetilde{K}_u$ in G such that the following holds.

(i) The boundary $\partial\widetilde{V}_u$ of $\widetilde{V}_u$ is smooth,

(ii) The set $\partial_0\widetilde{V}_u = \{z \in \partial\widetilde{V}_u : \widetilde{X}(z) \in T_z(\partial\widetilde{V}_u)\}$ is a smooth submanifold of codimension 1 of $\partial\widetilde{V}_u$,

(iii) There is $\varepsilon > 0$ such that for any $z \in \partial\widetilde{V}_u$ one has

$$\widetilde{X}(z) \in T_z(\partial\widetilde{V}_u) \quad\Longrightarrow\quad \widetilde{\varphi}_t(z) \notin \operatorname{clos} \widetilde{V}_u, \ \ t \in \,]-\varepsilon, \varepsilon[\, \setminus \{0\},$$

where clos A denotes the closure of a set A.

In what follows we denote

$$\Gamma_\pm(\widetilde{X}) = \big\{z \in \widetilde{V}_u : \widetilde{\varphi}_t(z) \in \widetilde{V}_u, \ \mp t > 0\big\}.$$

A function $\widetilde{\rho} \in C^\infty(\operatorname{clos} \widetilde{V}_u, \mathbb{R}_{\geqslant 0})$ will be called a boundary defining function for $\widetilde{V}_u$ if we have $\partial\widetilde{V}_u = \{z \in \operatorname{clos} \widetilde{V}_u : \widetilde{\rho}(z) = 0\}$ and $\mathrm{d}\widetilde{\rho}(z) \neq 0$ for any $z \in \partial\widetilde{V}_u$.

By [20, Lemma 2.3] (see also [12, Lemma 5.2]), we have the following result.

Lemma 2.2. — *For any small neighborhood $\widetilde{W}_0$ of $\partial\widetilde{V}_u$ in* clos $\widetilde{V}_u$*, we may find a vector field $\widetilde{Y}$ on* clos $\widetilde{V}_u$ *which is arbitrarily close to $\widetilde{X}$ in the C^∞-topology, such that the following holds.*

(1) $\operatorname{supp}(\widetilde{Y} - \widetilde{X}) \subset \widetilde{W}_0$,

(2) $\Gamma_\pm(\widetilde{X}) = \Gamma_\pm(\widetilde{Y})$, *where $\Gamma_\pm(\widetilde{Y})$ is defined as $\Gamma_\pm(\widetilde{X})$ by replacing the flow $(\widetilde{\varphi}_t)$ by the flow generated by $\widetilde{Y}$,*

(3) *For any defining function $\widetilde{\rho}$ of $\widetilde{V}_u$ and any $\omega \in \partial\widetilde{V}_u$ we have*

$$\widetilde{Y}\widetilde{\rho}(\omega) = 0 \quad\Longrightarrow\quad \widetilde{Y}^2\widetilde{\rho}(\omega) < 0. \tag{2.11}$$

From now on, we will fix $\widetilde{V}_u, \widetilde{W}_0$ and $\widetilde{Y}$ as above. Let $(\widetilde{\psi}_t)_{t\in\mathbb{R}}$ be the flow generated by $\widetilde{Y}$. By [13, Lemma 1.1] we may find a smooth extension of $\widetilde{Y}$ on G (still denoted by $\widetilde{Y}$) so that for every $\omega \in G$ and $t \geqslant 0$, we have

$$\omega, \widetilde{\psi}_t(\omega) \in \operatorname{clos} \widetilde{V}_u \quad\Longrightarrow\quad \widetilde{\psi}_\tau(\omega) \in \operatorname{clos} \widetilde{V}_u \text{ for every } \tau \in [0, t]. \tag{2.12}$$

Set $\widetilde{\Gamma}_\pm = \Gamma_\pm(\widetilde{Y})$ for simplicity. The extended unstable/stable bundles $\widetilde{E}_\pm^* \subset T^*\widetilde{V}_u$ over $\widetilde{\Gamma}_\pm$ are defined by

$$\widetilde{E}_\pm^*(\omega) = \{\eta \in T_\omega^*\widetilde{V}_u \ : \ \Psi_t(\eta) \longrightarrow_{t\to\pm\infty} 0\},$$

where Ψ_t is the symplectic lift of $\widetilde{\psi}_t$, that is

$$\Psi_t(\omega, \eta) = \Big(\widetilde{\psi}_t(\omega), \mathrm{d}\widetilde{\psi}_t(\omega)^{-\top} \cdot \eta\Big), \quad (\omega, \eta) \in T^*G, \quad t \in \mathbb{R},$$

and ${}^{-\top}$ denotes the inverse transpose. Then by [13, Lemma 1.10], the bundles $\widetilde{E}^*_\pm(\omega)$ depend continuously on $\omega \in \widetilde{\Gamma}_\pm$, and for any smooth norm $|\cdot|$ on T^*G with some constants $C > 0, \beta > 0$ independent of ω, η for $t \to \mp\infty$ we have

$$|\Psi_{\pm t}(\omega, \eta)| \leqslant C\mathrm{e}^{-\beta|t|}|\eta|, \quad \eta \in E^*_\pm(\omega).$$

2.8. Dyatlov–Guillarmou theory

Let $\nabla^{k,\ell}$ be any smooth connection on $\mathcal{E}_{k,\ell}$. Then by (2.10) we have

$$\mathbf{P}_{k,\ell} = \nabla^{k,\ell}_{\tilde{X}} + \mathbf{A}_{k,\ell}$$

for some $\mathbf{A}_{k,\ell} \in C^\infty(G, \mathrm{End}(\mathcal{E}_{k,\ell}))$. We define a new operator $\mathbf{Q}_{k,\ell}$ by setting

$$\mathbf{Q}_{k,\ell} = \nabla^{k,\ell}_{\tilde{Y}} + \mathbf{A}_{k,\ell} : C^\infty(G, \mathcal{E}_{k,\ell}) \longrightarrow C^\infty(G, \mathcal{E}_{k,\ell}).$$

Note that $\mathbf{Q}_{k,\ell}$ coincides with $\mathbf{P}_{k,\ell}$ near $\widetilde{K}_u$ since $\widetilde{Y}$ coincides with $\widetilde{X}$ near $\widetilde{K}_u$. Clearly, we have

$$\mathbf{Q}_{k,\ell}(f\mathbf{u}) = (\widetilde{Y}f)\mathbf{u} + f(\mathbf{Q}_{k,\ell}\mathbf{u}), \quad f \in C^\infty(G), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}). \tag{2.13}$$

Next, consider the transfer operator $\mathrm{e}^{-t\mathbf{Q}_{k,\ell}} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$ with generator $\mathbf{Q}_{k,\ell}$, that is,

$$\partial_t \mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u} = -\mathbf{Q}_{k,\ell}\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}), \quad t \geqslant 0.$$

As above, for some constant $C > 0$, we have

$$\|\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\|_{L^2(G,\mathcal{E}_{k,\ell}) \to L^2(G,\mathcal{E}_{k,\ell})} \leqslant C\mathrm{e}^{Ct}, \quad t \geqslant 0.$$

Then for $\mathrm{Re}(s) > C$, the resolvent $(\mathbf{Q}_{k,\ell} + s)^{-1}$ on $L^2(G, \mathcal{E}_{k,\ell})$ is given by

$$(\mathbf{Q}_{k,\ell} + s)^{-1} = \int_0^\infty \mathrm{e}^{-t(\mathbf{Q}_{k,\ell}+s)}\mathrm{d}t : L^2(G, \mathcal{E}_{k,\ell}) \longrightarrow L^2(G, \mathcal{E}_{k,\ell}). \tag{2.14}$$

Consider the operator

$$\mathbf{R}_{k,\ell}(s) = \mathbf{1}_{\tilde{V}_u}(\mathbf{Q}_{k,\ell} + s)^{-1}\mathbf{1}_{\tilde{V}_u}, \quad \mathrm{Re}(s) \gg 1,$$

from $C^\infty_c(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ to $\mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$, where $\mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ denotes the space of $\mathcal{E}_{k,\ell}$-valued distributions. Recall that $\widetilde{K}_u$ is the trapped set of $\widetilde{\varphi}_t$ when restricted to $\widetilde{V}_u$. Taking into account (2.11), (2.12) and (2.13), we see that the assumptions (A1)–(A5) in [13, Section 0] are satisfied. We are in position to apply [13, Theorem 1] in order to obtain a meromorphic extension of $\mathbf{R}_{k,\ell}(s)$ to the whole plane $\mathbb{C}$. Moreover, according to [13, Theorem 2], for every pole $s_0 \in \mathbb{C}$ in a small neighborhood of s_0 one has the representation

$$\mathbf{R}_{k,\ell}(s) = \mathbf{R}_{H,k,\ell}(s) + \sum_{j=1}^{J(s_0)} \frac{(-1)^{j-1}(\mathbf{Q}_{k,\ell} + s_0)^{j-1}\Pi_{s_0}^{k,\ell}}{(s-s_0)^j}, \tag{2.15}$$

where $\mathbf{R}_{H,k,\ell}(s) : C_c^\infty(\widetilde{V}_u, \mathcal{E}_{k,\ell}) \to \mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ is a holomorphic family of operators near $s = s_0$ and $\Pi_{s_0}^{k,\ell} : C_c^\infty(\widetilde{V}_u, \mathcal{E}_{k,\ell}) \to \mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ is a finite rank projector. Denote by $K_{\mathbf{R}_{H,k,\ell}(s)}$ and $K_{\Pi_{s_0}^{k,\ell}}$ the Schwartz kernels of the operators $\mathbf{R}_{H,k,\ell}(s)$ and $\Pi_{s_0}^{k,\ell}$, respectively. Recall the definition of the twisted wavefront set

$$\mathrm{WF}'(A) = \{(x,\xi,y,-\eta) : (x,\xi,y,\eta) \in \mathrm{WF}(K_A)\},$$

K_A being the distributional kernel of the operator A. By [13, Lemma 3.5], we have

$$\mathrm{WF}'(K_{\mathbf{R}_{H,k,\ell}}(s)) \subset \Delta(T^*\widetilde{V}_u) \cup \Upsilon_+ \cup (\widetilde{E}_+^* \times \widetilde{E}_-^*). \tag{2.16}$$

Here $\Delta(T^*\widetilde{V}_u)$ is the diagonal in $T^*(\widetilde{V}_u \times \widetilde{V}_u)$,

$$\Upsilon_+ = \{(\Psi_t(\omega,\Omega),\omega,\Omega) : (\omega,\Omega) \in T^*\widetilde{V}_u,\ t \geqslant 0,\ \langle \widetilde{Y}(\omega), \Omega\rangle = 0\},$$

while the bundles $\widetilde{E}_\pm^*$ and flow Ψ_t are defined in Section 2.7. Finally, we have

$$\mathrm{supp}(K_{\Pi_{s_0}^{k,\ell}}) \subset \Gamma_+ \times \Gamma_- \quad \text{and} \quad \mathrm{WF}'(K_{\Pi_{s_0}^{k,\ell}}(s)) \subset \widetilde{E}_+^* \times \widetilde{E}_-^*. \tag{2.17}$$

3. Dynamical zeta function for the Neumann problem

In this section we prove that the function η_{N} admits a meromorphic continuation to the whole complex plane, by relating $\eta_{\mathrm{N}}(s)$ to the flat trace of the cut-off resolvent $\mathbf{R}_{k,\ell}(s)$.

3.1. Flat trace

First, we recall the definition of the flat trace for operators acting on vector bundles. Consider a manifold V, a vector bundle $\mathcal{E}$ over V and a continuous operator $\mathbf{T} : C_c^\infty(V,\mathcal{E}) \to \mathcal{D}'(V,\mathcal{E})$. Fix a smooth density μ on V; this defines a pairing $\langle\cdot,\cdot\rangle$ on $C_c^\infty(V,\mathcal{E}) \times C_c^\infty(V,\mathcal{E}^*)$. Let

$$K_{\mathbf{T}} \in \mathcal{D}'(V \times V, \mathcal{E} \boxtimes \mathcal{E}^*)$$

be the Schwartz kernel of $\mathbf{T}$ with respect to this pairing, which is defined by

$$\langle K_{\mathbf{T}}, \pi_1^*\mathbf{u} \otimes \pi_2^*\mathbf{v}\rangle = \langle \mathbf{Tu}, \mathbf{v}\rangle, \quad \mathbf{u} \in C_c^\infty(V, \mathcal{E}), \quad \mathbf{v} \in C_c^\infty(V, \mathcal{E}^*),$$

where the pairing on $\mathcal{D}'(V \times V, \mathcal{E} \boxtimes \mathcal{E}^*) \times C_c^\infty(V \times V, \mathcal{E} \boxtimes \mathcal{E}^*)$ is taken with respect to $\mu \times \mu$. Here, the bundle

$$\mathcal{E} \boxtimes \mathcal{E}^* = \pi_1^*\mathcal{E} \otimes \pi_2^*\mathcal{E}^* \longrightarrow V$$

is given by the tensor product of the pullbacks $\pi_1^*\mathcal{E}$, and $\pi_2^*\mathcal{E}^*$, where the maps $\pi_1, \pi_2 : V \times V \to V$ denote the projections over the first and the second factor, respectively.

Denote by $\Delta = \{(x,x) : x \in V\} \subset V \times V$ the diagonal in $V \times V$ and consider the inclusion map $\iota_\Delta : \Delta \to V \times V$, $(x,x) \mapsto (x,x)$. Assume that

$$\mathrm{WF}'(K_{\mathbf{T}}) \cap \Delta(T^*V \setminus \{0\}) = \emptyset, \tag{3.1}$$

where $\Delta(T^*V \setminus \{0\})$ is the diagonal in $(T^*(V) \setminus \{0\}) \times (T^*(V) \setminus \{0\})$. Then by [25, Theorem 8.2.4], the pull-back

$$\iota_\Delta^* K_{\mathbf{T}} \in \mathcal{D}'(V,\ \mathrm{End}(\mathcal{E}))$$

is well defined, where we used the identification

$$\iota_\Delta^*(\mathcal{E} \boxtimes \mathcal{E}^*) \simeq \mathcal{E} \otimes \mathcal{E}^* \simeq \mathrm{End}(\mathcal{E}).$$

If $K_{\mathbf{T}}$ is compactly supported, we define the *flat trace* of $\mathbf{T}$ by

$$\mathrm{tr}^\flat\, \mathbf{T} = \langle \mathrm{tr}_{\mathrm{End}(\mathcal{E})}(\iota_\Delta^* K_{\mathbf{T}}), 1\rangle,$$

where again the pairing is taken with respect to μ. It is not hard to see that the flat trace does not depend on the choice of the density μ.

3.2. Flat trace of the cut-off resolvent

We introduce a cut-off function $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ such that $\widetilde{\chi} \equiv 1$ on $\widetilde{K}_u$. For $\varrho \in C_c^\infty(\mathbb{R}^+ \setminus \{0\})$ define

$$\mathbf{T}_\varrho^{k,\ell}\mathbf{u} = \left(\int_0^\infty \varrho(t)\widetilde{\chi}(\mathrm{e}^{-t\mathbf{Q}_{k,\ell}}\mathbf{u})\widetilde{\chi}\mathrm{d}t\right), \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

As in [49], we need to introduce some notations. For simplicity we denote $\mathcal{H} = \mathcal{E} \otimes \mathcal{F}$, where $\mathcal{E} \to G$, $\mathcal{F} \to G$ are bundles over G. Let $e^{-t\mathbf{X}}$ be a transport operator. For $\omega \in G$ and $t > 0$ introduce the *parallel transport* map

$$\alpha_{\omega,t} = \alpha_{1,\omega,t} \otimes \alpha_{2,\omega,t} : \mathcal{E}_\omega \otimes \mathcal{F}_\omega \longrightarrow \mathcal{E}_{\tilde{\varphi}_t(\omega)} \otimes \mathcal{F}_{\tilde{\varphi}_t(\omega)}$$

given by

$$\mathbf{u} \otimes \mathbf{v} \longmapsto (e^{-t\mathbf{X}}(\mathbf{u} \otimes \mathbf{v}))(\widetilde{\varphi}(t)),$$

where $\mathbf{u}, \mathbf{v}$ are some sections of $\mathcal{E}_\omega$ and $\mathcal{F}_\omega$ over ω, respectively. The definition does not depend on the choice of $\mathbf{u}$ and $\mathbf{v}$ (see [13, (0.8)]). Now if $\widetilde{\gamma}(t) = \{\widetilde{\varphi}_t(\omega_0) : 0 \leqslant t \leqslant \tau(\widetilde{\gamma})\}$ is a periodic orbit, we have

$$\alpha_{j,\omega_0,\tau(\tilde{\gamma})} = \alpha^{-1}_{j,\omega_0,t} \circ \alpha_{j,\tilde{\gamma}(t),\tau(\tilde{\gamma})} \circ \alpha_{j,\omega_0,t}, \quad j = 1,2,$$

and therefore the trace

$$\operatorname{tr}(\alpha_{\tilde{\gamma}}) = \operatorname{tr}(\alpha_{1,\tilde{\gamma}(t),\tau(\tilde{\gamma})}) \operatorname{tr}(\alpha_{2,\tilde{\gamma}(t),\tau(\tilde{\gamma})}) \tag{3.2}$$

is independent of t. (Here we use the flow $\widetilde{\varphi}_t$ instead of $\widetilde{\psi}_t$ since for periodic orbits the action of both flows is the same.) Returning to the bundle $\mathcal{E}_{k,\ell}$, the flow $\Phi_t^{k,\ell}$ and the operator $e^{-t\mathbf{Q}_{k,\ell}}$, introduced in Sections 2.6 and 2.8, the corresponding parallel transport map will be denoted by $\alpha^{k,\ell}_{\omega,t}$ and the trace $\operatorname{tr}(\alpha^{k,\ell}_{\tilde{\gamma}})$ is well defined.

In the same way we define the linearized Poincaré map for $\omega \in \widetilde{\gamma}$ and $\tau(\widetilde{\gamma})$ by

$$P_{\omega,\tau(\tilde{\gamma})} = \mathrm{d}\widetilde{\varphi}_{-\tau(\tilde{\gamma})}(\omega)\big|_{\tilde{E}_s(\omega)\oplus\tilde{E}_u(\omega)}.$$

As above, given a periodic orbit $\widetilde{\gamma}$, the map $P_{\omega,\tau(\tilde{\gamma})}$ is conjugated to $P_{\omega',\tau(\tilde{\gamma})}$ if ω and ω' lie on $\widetilde{\gamma}$ and we define $\widetilde{P}_\gamma$ as $P_{\omega,\tau(\tilde{\gamma})}$. To define the flat trace,we must check the condition (3.1) concerning the intersection of the wave front $\mathrm{WF}'(K_{\mathbf{T}^{k,\ell}_\varrho})$ of the kernel $K_{\mathbf{T}^{k,l}_\varrho}$ of $\mathbf{T}^{k,\ell}_\varrho$ and the conormal bundle $N^*(\Delta_{\tilde{V}_u})$, $\Delta_{\tilde{V}_u}$ being the diagonal in $(\widetilde{V}_u \times \widetilde{V}_u)$. This is down in Section 3.1 of [49]. We omit the repetition and refer to this paper for a detailed exposition.

We may apply the Guillemin trace formula [21, Section 2 of Lecture 2] (we refer to [49, Lemma 3.1] for a detailed presentation based on the argument of [14, Appendix B]), which implies that the flat trace of $\mathbf{T}^{k,\ell}_\varrho$ is well defined, and

$$\operatorname{tr}^\flat(\mathbf{T}^{k,\ell}_\varrho) = \sum_{\tilde{\gamma}} \frac{\varrho(\tau(\gamma))\tau^\sharp(\gamma)\operatorname{tr}(\alpha^{k,\ell}_{\tilde{\gamma}})}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|}, \tag{3.3}$$

where the sum runs over all periodic orbits $\widetilde{\gamma}$ of $\widetilde{\varphi}_t$. Here,

$$\widetilde{P}_\gamma = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})\big|_{\tilde{E}_u(\omega_{\tilde{\gamma}})\oplus\tilde{E}_s(\omega_{\tilde{\gamma}})}$$

is the linearized Poincaré map of the closed orbit

$$t \longmapsto \widetilde{\gamma}(t) = (\gamma(t), E_u(\gamma(t)))$$

of the flow $\widetilde{\varphi}_t$ and $\omega_{\tilde{\gamma}} \in \operatorname{Im}(\widetilde{\gamma})$ is any reference point taken in the image of $\widetilde{\gamma}$. Note that if we take another point $\omega'_{\tilde{\gamma}} \in \operatorname{Im}(\widetilde{\gamma})$, then the map $\mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega'_{\tilde{\gamma}})$ is conjugated to $\mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})$ by $\mathrm{d}\widetilde{\varphi}_{t_1}(\omega_{\tilde{\gamma}})$, where $t_1 \in \mathbb{R}$ is chosen so that $\widetilde{\varphi}_{t_1}(\omega'_{\tilde{\gamma}}) = \omega_{\tilde{\gamma}}$. Hence the determinant $\det(\mathrm{Id} - \widetilde{P}_\gamma)$ does not depend on the reference point $\omega_{\tilde{\gamma}}$ and is well defined. The number $\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell})$ is the trace of the linear map

$$\alpha_{\omega_{\tilde{\gamma}},\tau(\gamma)}^{k,\ell} : \mathcal{E}_{k,\ell}|_{\omega_{\tilde{\gamma}}} \longrightarrow \mathcal{E}_{k,\ell}|_{\omega_{\tilde{\gamma}}},$$

where for $t \in \mathbb{R}$ and $\omega \in G$, we denote by

$$\alpha_{\omega,t}^{k,\ell} : \mathcal{E}_{k,\ell}|_{\omega} \longrightarrow \mathcal{E}_{k,\ell}|_{\tilde{\varphi}_t(\omega)}$$

the restriction of the map $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ to the fiber $\mathcal{E}_{k,\ell}|_{\omega}$. Again, if we take another reference point $\omega'_{\tilde{\gamma}}$, the map $\alpha_{\omega'_{\tilde{\gamma}},\tau(\gamma)}^{k,\ell}$ is conjugated to $\alpha_{\omega_{\tilde{\gamma}},\tau(\gamma)}^{k,\ell}$, hence its trace depends only on $\widetilde{\gamma}$, and this justifies the notation $\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell})$.

Next, we follow the strategy of [49, Section 3.1] which is based on that used in [14, Section 4] for Anosov flows on closed manifolds to compute the flat trace of the (shifted) resolvent defined below. We may apply formula (3.3) with the functions $\varrho_{s,T}(t) = \mathrm{e}^{-st}\varrho_T(t)$, where the support of $\varrho_T \in C_c^\infty(\mathbb{R}^+)$ satisfies $\mathrm{supp}\,\varrho_T \subset [\varepsilon/2, T+1]$ for $0 < \varepsilon < d_0 = \min_{\gamma\in\mathcal{P}} \tau(\gamma)$ small and $\varrho_T \equiv 1$ on $[\varepsilon, T]$. Then taking the limit $T \to \infty$, we obtain, with (2.14) in mind,

$$\mathrm{tr}^\flat\, \mathbf{R}_\varepsilon^{k,\ell}(s) = \sum_{\tilde{\gamma}} \frac{\mathrm{e}^{-s\tau(\gamma)}\tau^\sharp(\gamma)\,\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell})}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|}, \quad \mathrm{Re}(s) \gg 1. \tag{3.4}$$

Here for $\mathrm{Re}(s)$ large enough and $\varepsilon > 0$ small, we set

$$\mathbf{R}_\varepsilon^{k,\ell}(s) = \widetilde{\chi}\mathrm{e}^{-\varepsilon(s+\mathbf{Q}_{k,\ell})}(\mathbf{Q}_{k,\ell} + s)^{-1}\widetilde{\chi},$$

and ε is chosen so that $\mathrm{e}^{-\varepsilon\mathbf{Q}_{k,\ell}}\,\mathrm{supp}(\widetilde{\chi}) \subset \widetilde{V}_u$, so that $\mathbf{R}_\varepsilon^{k,\ell}(s)$ is well defined. The equality (3.4) is exactly the equation concerning $\lim_{T\to\infty} \mathrm{tr}^\flat(B_T)$ with $f \equiv 1$ on page 668 in [49], and we refer to this work for a detailed proof. Note that the flat trace $\mathrm{tr}^\flat\, \mathbf{R}_\varepsilon^{k,\ell}(s)$ is well defined thanks to the information on the wavefront set $\mathrm{WF}'(K_{\mathbf{R}_\varepsilon^{k,\ell}(s)})$ obtained from (2.16), together with the multiplication properties satisfied by wavefront sets (see [25, Theorem 8.2.14]).

Next, one states the following result, similar to that in [16, Section 2]. This crucial lemma explains the reason to introduce the bundles $\mathcal{E}_{k,\ell}$. For the sake of completeness, we present a detailed proof.

LEMMA 3.1. — *For any periodic orbit $\widetilde{\gamma}$ related to a periodic orbit γ, we have*

$$\frac{1}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d^2-d} (-1)^{k+\ell}\,\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) = |\det(\mathrm{Id} - P_\gamma)|^{-1/2}.$$

Proof. — Let $\gamma(t)$ be a periodic orbit and let $\widetilde{\gamma}(t) = (\gamma(t), E_u(\gamma(t))$ with $\omega_{\tilde{\gamma}} \in \widetilde{\gamma}$ and $z \in \gamma$. Set

$$P_{\gamma,u} = \mathrm{d}\varphi_{-\tau(\gamma)}(z)|_{E_u(z)}, \quad P_{\gamma,s} = \mathrm{d}\varphi_{-\tau(\gamma)}(z)|_{E_s(z)},$$

$$P_{\gamma,\perp} = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})|_{\ker \mathrm{d}\pi_G(\omega)}, \quad P_{\gamma,\perp}^{-1} = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})^{-1}|_{\ker \mathrm{d}\pi_G(\omega)}.$$

The linearized Poincaré map $\widetilde{P}_\gamma$ of the closed orbit $\widetilde{\gamma}$ satisfies

$$\begin{aligned} \det(\mathrm{Id} - \widetilde{P}_\gamma) &= \det\left(\mathrm{Id} - \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}|_{\tilde{E}_s(\omega)\oplus\tilde{E}_u(\omega)}\right) \\ &= \det\left(\mathrm{Id} - P_\gamma\right)\det\left(\mathrm{Id} - P_{\gamma,\perp}\right) \end{aligned} \tag{3.5}$$

since $\widetilde{E}_s(\omega) \simeq E_s(z) \oplus \ker \mathrm{d}\pi_G(\omega)$ and $\widetilde{E}_u(\omega) \simeq E_u(z)$ by Lemma 2.1. Recall the well known formula

$$\det(\mathrm{Id} - A) = \sum_{j=0}^{k} (-1)^j \operatorname{tr} \wedge^j A$$

for any endomorphism A of a k-dimensional vector space. Moreover, notice that

$$\operatorname{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) = b_{\tau(\gamma)}(\omega_{\tilde{\gamma}}) \operatorname{tr} \wedge^k P_{\gamma,u} \operatorname{tr} \wedge^\ell P_{\gamma,\perp}^{-1},$$

since by (2.8), $\alpha_{\tilde{\gamma}}^{k,\ell}$ coincides with the map

$$\begin{aligned} b_{\tau(\gamma)}(\omega_{\tilde{\gamma}}) \wedge^k \left[\mathrm{d}\varphi_{\tau(\gamma)}(\pi_G(\omega_{\tilde{\gamma}}))^{-\top}\right] \otimes \wedge^\ell \left[\mathrm{d}\widetilde{\varphi}_{\tau(\gamma)}(\omega_{\tilde{\gamma}})\right] : \\ \wedge^k \mathcal{E}^*|_{\omega_{\tilde{\gamma}}} \otimes \wedge^\ell \mathcal{F}|_{\omega_{\tilde{\gamma}}} \longrightarrow \wedge^k \mathcal{E}^*|_{\omega_{\tilde{\gamma}}} \otimes \wedge^\ell \mathcal{F}|_{\omega_{\tilde{\gamma}}}. \end{aligned}$$

Therefore, one gets

$$\begin{aligned} &\sum_{\ell=0}^{d^2-d} \sum_{k=0}^{d-1} (-1)^{k+\ell} \operatorname{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) \\ &\qquad = b_{\tau(\gamma)}(\omega_{\tilde{\gamma}}) \left(\sum_{k=0}^{d-1} (-1)^k \operatorname{tr} \wedge^k P_{\gamma,u}\right) \left(\sum_{\ell=0}^{d^2-d} (-1)^\ell \operatorname{tr} \wedge^\ell P_{\gamma,\perp}^{-1}\right) \\ &\qquad = |\det(P_{\gamma,u})|^{-1/2} |\det(P_{\gamma,\perp})| \det(\mathrm{Id} - P_{\gamma,u}) \det(\mathrm{Id} - P_{\gamma,\perp}^{-1}). \end{aligned} \tag{3.6}$$

Here we have used the equality

$$\begin{aligned} b_{\tau(\gamma)}(\omega_{\tilde{\gamma}}) &= |\det \mathrm{d}\varphi_{\tau(\gamma)}(\pi_G(\omega_{\tilde{\gamma}}))|_{[\omega_{\tilde{\gamma}}]}|^{1/2} \cdot |\det\left(\mathrm{d}\widetilde{\varphi}_{\tau(\gamma)}(\omega_{\tilde{\gamma}})|_{\ker \mathrm{d}\pi_G}\right)|^{-1} \\ &= |\det(P_{\gamma,u})|^{-1/2} |\det(P_{\gamma,\perp})| \end{aligned}$$

which holds because $P_{\gamma,u}$ and $P_{\gamma,\perp}$ are defined with $\mathrm{d}\varphi_{-t}$ and $\mathrm{d}\widetilde{\varphi}_{-t}$, respectively. Therefore (3.5) yields

$$\sum_{k,\ell}(-1)^{k+\ell}\frac{\operatorname{tr}(\alpha^{k,\ell}_{\tilde{\gamma}})}{|\det(\mathrm{Id}-\widetilde{P}_\gamma)|} = \frac{\det(\mathrm{Id}-P_{\gamma,u})\det(\mathrm{Id}-P^{-1}_{\gamma,\perp})|\det(P_{\gamma,u})|^{-1/2}}{|\det(\mathrm{Id}-P_\gamma)||\det(\mathrm{Id}-P_{\gamma,\perp})||\det(P_{\gamma,\perp})|^{-1}}. \tag{3.7}$$

Since P_γ is a linear symplectic map, we have

$$\det(\mathrm{Id}-P^{-1}_{\gamma,s}) = \det(\mathrm{Id}-P_{\gamma,u}),\ \det(P_{\gamma,s}) = \det(P^{-1}_{\gamma,u}),$$

and one deduces

$$\begin{aligned}|\det(\mathrm{Id}-P_\gamma)| &= |\det(\mathrm{Id}-P_{\gamma,u})||\det(\mathrm{Id}-P_{\gamma,s})| \\ &= |\det(P_{\gamma,s})||\det(\mathrm{Id}-P_{\gamma,u})||\det(\mathrm{Id}-P^{-1}_{\gamma,s})| \\ &= |\det(P_{\gamma,u})|^{-1}|\det(\mathrm{Id}-P_{\gamma,u})|^2.\end{aligned}$$

For $t>0$ the map $\mathrm{d}\widetilde{\varphi}_t = (\mathrm{d}\widetilde{\varphi}_{-t})^{-1}$ is contracting on $\ker \mathrm{d}\pi_G \subset \widetilde{E}_s(\omega_{\tilde{\gamma}})$ (resp. $\mathrm{d}\varphi_{-t}$ is contracting on $E_u(z)$) and this yields $\det(\mathrm{Id}-P^{-1}_{\gamma,\perp})>0$ (resp. $\det(\mathrm{Id}-P_{\gamma,u})>0$). Thus the terms involving $P_{\gamma,\perp}$ in (3.7) cancel and since

$$|\det(\mathrm{Id}-P_\gamma)|^{-1/2} = |\det(P_{\gamma,u})|^{1/2}\det(\mathrm{Id}-P_{\gamma,u})^{-1},$$

the right hand side of (3.7) is equal to $|\det(\mathrm{Id}-P_\gamma)|^{-1/2}$. □

3.3. Meromorphic continuation of η_{N}

From Lemma 3.1 and (3.4), we deduce that for $\mathrm{Re}(s)\gg 1$, we have

$$\eta_{\mathrm{N}}(s) = \sum_{k=0}^{d-1}\sum_{\ell=0}^{d^2-d}(-1)^{k+\ell}\operatorname{tr}^\flat \mathbf{R}^{k,\ell}_\varepsilon(s),$$

where $\eta_{\mathrm{N}}(s)$ is defined by

$$\eta_{\mathrm{N}}(s) = \sum_\gamma \frac{\tau^\sharp(\gamma)\mathrm{e}^{-\tau(\gamma)s}}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}}.$$

Since for every k,ℓ the family $s\mapsto \mathbf{R}^{k,\ell}_\varepsilon(s)$ extends to a meromorphic family on the whole complex plane, so does $s\mapsto\eta_{\mathrm{N}}(s)$. Indeed, it follows from the proof of [13, Lemma 3.2] that $s \mapsto K_{\mathbf{R}_\varepsilon^{k,\ell}(s)}$ is continuous as a map[(1)]

$$\mathbb{C} \setminus \operatorname{Res}(\mathbf{R}_\varepsilon^{k,\ell}) \longrightarrow \mathcal{D}'_\Gamma(G \times G, \mathcal{E}_{k,\ell} \boxtimes \mathcal{E}^*_{k,\ell}).$$

Here for $s \notin \operatorname{Res}(\mathbf{R}_\varepsilon^{k,\ell})$ the distribution $K_{\mathbf{R}_\varepsilon^{k,\ell}(s)}$ is the Schwartz kernel of $\mathbf{R}_\varepsilon^{k,\ell}(s)$ and (see (2.16) for the notation)

$$\Gamma = \Delta_\varepsilon \cup \Upsilon_{+,\varepsilon} \cup \widetilde{E}^*_+ \times \widetilde{E}^*_-,$$

where $\Delta_\varepsilon = \{(\Psi_\varepsilon(\omega,\Omega),\omega,\Omega) : (\omega,\Omega) \in T^*(\widetilde{V}_u) \setminus \{0\}\}$ and

$$\Upsilon_{+,\varepsilon} = \{(\Psi_t(\omega,\Omega),\omega,\Omega) : (\omega,\Omega) \in T^*(\widetilde{V}_u) \setminus \{0\},\ t \geqslant \varepsilon,\ \langle \widetilde{Y}(\omega),\Omega\rangle = 0\},$$

while $\mathcal{D}'_\Gamma(G\times G, \mathcal{E}_{k,\ell} \boxtimes \mathcal{E}^*_{k,\ell})$ is the space of distributions valued in $\mathcal{E}_{k,\ell} \boxtimes \mathcal{E}^*_{k,\ell}$ whose wavefront set is contained in Γ. This space is endowed with its usual topology (see [25, Section 8.2]). Thus, outside the set of poles $\operatorname{Res}(\mathbf{R}_\varepsilon^{k,\ell})$, we apply the procedure with a flat trace. In particular, $s \mapsto \operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell}(s)$ is continuous on $\mathbb{C} \setminus \operatorname{Res}(\mathbf{R}_\varepsilon^{k,\ell})$ by [25, Theorem 8.2.4]. Finally, Cauchy's formula implies that this map is meromorphic on $\mathbb{C}$ and this completes the proof that the Dirichlet series $\eta_{\mathrm{N}}(s)$ admits a meromorphic continuation in $\mathbb{C}$.

Next, we establish that $\eta_{\mathrm{N}}(s)$ has simple poles with integer residues. To do this, we may proceed as in [13, Section 4]. For the sake of completeness we reproduce the argument. Let $s_0 \in \operatorname{Res}(\mathbf{R}_{k,\ell})$ for some k,ℓ. Recalling the development (2.15), it is enough to show that

$$\operatorname{tr}^\flat \left(\widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})} \left[(\mathbf{Q}_{k,\ell}+s_0)^{j-1}\Pi_{s_0}^{k,\ell}\right] \widetilde{\chi}\right) = 0, \quad j \geqslant 2, \tag{3.8}$$

and

$$\operatorname{tr}^\flat \left(\widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})}\Pi_{s_0}^{k,\ell}\widetilde{\chi}\right) = \operatorname{rank}\Pi_{s_0}^{k,\ell}. \tag{3.9}$$

In the following we fix k and ℓ. We may write

$$\Pi_{s_0}^{k,\ell} = \sum_{i=1}^m \mathbf{u}_i \otimes \mathbf{v}_i,$$

where $\otimes$ denotes the tensor product and by (2.17) for $i = 1,\dots,m$ we have

$$\begin{aligned} &\mathbf{u}_i \in \mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell}), \quad \operatorname{supp}(\mathbf{u}_i) \subset \Gamma_+, \quad \mathrm{WF}'(\mathbf{u}_i) \subset \widetilde{E}^*_+,\\ &\mathbf{v}_i \in \mathcal{D}'(\widetilde{V}_u, \mathcal{E}^*_{k,\ell}), \quad \operatorname{supp}(\mathbf{v}_i) \subset \Gamma_-, \quad \mathrm{WF}'(\mathbf{v}_i) \subset \widetilde{E}^*_-.\end{aligned} \tag{3.10}$$

The relations

$$\widetilde{E}^*_+ \cap \widetilde{E}^*_- \cap (T^*(\widetilde{V}) \setminus \{0\}) = \emptyset,$$

(1) This follows from the fact that the estimates on the wavefront set of $\mathbf{R}_\varepsilon^{k,\ell}(s)$ given in [13, Lemma 3.5] are locally uniform with respect to $s \in \mathbb{C}$.

make possible to define the pairing $\langle \mathbf{u}_i, \mathbf{v}_p \rangle$ on $\mathcal{E}_{k,\ell} \times \mathcal{E}^*_{k,\ell}$ for $i, p = 1, \dots, m$ which yields a distribution on $\widetilde{V}_u$. This distribution is compactly supported since

$$\operatorname{supp} \mathbf{u}_i \cap \operatorname{supp} \mathbf{v}_p \subset \Gamma_+ \cap \Gamma_- = \widetilde{K}_u.$$

The family $(\mathbf{u}_i)$ is a basis of the range of $\Pi^{k,\ell}_{s_0}$. By definition of the flat trace using the information on the wavefront sets and the supports of $\mathbf{u}_i$ and $\mathbf{v}_j$, we can write

$$\begin{aligned}(3.11)\quad \operatorname{tr}^\flat\left(\widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})}\left[(\mathbf{Q}_{k,\ell}+s_0)^{j-1}\Pi^{k,\ell}_{s_0}\right]\widetilde{\chi}\right) \\ = \sum_{i=1}^m \int_{\tilde{V}_u} \langle \widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q})}(\mathbf{Q}_{k,\ell}+s_0)^{j-1}\mathbf{u}_i, \widetilde{\chi}\mathbf{v}_i\rangle.\end{aligned}$$

Here the integrals make sense taking into account the estimates of the supports and the wavefront sets of $\mathbf{u}_i$ and $\mathbf{v}_p$ mentioned above. Since $\Pi^{k,\ell}_{s_0}$ is a projector it holds $\Pi^{k,\ell}_{s_0} \circ \Pi^{k,\ell}_{s_0} = \Pi^{k,\ell}_{s_0}$, therefore the family $(\mathbf{v}_p)$ is dual to the basis $(\mathbf{u}_i)$ in the sense that

$$(3.12)\qquad \int_{\tilde{V}_u} \langle \mathbf{u}_i, \mathbf{v}_p \rangle = \delta_{ip}, \quad 1 \leqslant i, p \leqslant m,$$

where δ_{ip} are the Kronecker symbols. Introduce

$$C^{(j)}_{s_0,k,\ell} = \left\{\mathbf{u} \in \mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell}) : \operatorname{supp} u \subset \Gamma_+, \mathrm{WF}(u) \subset \widetilde{E}^*_+, (\mathbf{Q}_{k,\ell}+s_0)^j \mathbf{u} = 0\right\}.$$

Then, since $\widetilde{\chi} = 1$ near $\widetilde{K}_u$, by applying (3.11), one deduces that the operator $\widetilde{\chi}\mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})}(\mathbf{Q}_{k,\ell}+s_0)\Pi^{k,\ell}_{s_0}\widetilde{\chi}$ maps

$$\widetilde{\chi} C^{(j+1)}_{s_0,k,\ell} \longrightarrow \widetilde{\chi} C^{(j)}_{s_0,k,\ell}, \quad \text{and} \quad \widetilde{\chi} C^{(1)}_{s_0,k,\ell} \longrightarrow \{0\}, \qquad j \geqslant 1.$$

This fact and (3.12) show that (3.8) holds. To prove (3.9), we write

$$\begin{aligned}&\sum_i \int_{\tilde{V}_u} \langle \widetilde{\chi} \mathrm{e}^{-\varepsilon(s_0+\mathbf{Q}_{k,\ell})}\mathbf{u}_i, \widetilde{\chi}\mathbf{v}_i\rangle \\ &= \sum_i \int_{\tilde{V}_u} \langle \mathbf{u}_i, \mathbf{v}_i \rangle - \sum_i \int_0^\varepsilon \mathrm{d}t \int_{\tilde{V}_u} \left\langle \widetilde{\chi} \mathrm{e}^{-t(s_0+\mathbf{Q}_{k,\ell})}(\mathbf{Q}_{k,\ell}+s_0)\mathbf{u}_i, \widetilde{\chi}\mathbf{v}_i \right\rangle.\end{aligned}$$

Now, we replace ε by t in (3.8) for any $t \in [0, \varepsilon]$, and we obtain that the last sum in the right hand side of the above equation vanishes. Finally, applying (3.12), we obtain (3.9).

4. Dynamical zeta function for particular rays

In this section we adapt the above construction to prove the following result.

Theorem 4.1. — *Let $q \in \mathbb{N}_{\geqslant 1}$. The function $\eta_q(s)$ defined by*

$$\eta_q(s) = \sum_{\gamma \in \mathcal{P},\, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma) \mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \quad \mathrm{Re}(s) \gg 1,$$

where the sum runs over all periodic rays γ with $m(\gamma) \in q\mathbb{N}$, admits a meromorphic continuation to the whole complex plane with simple poles and residues valued in $\mathbb{Z}/q$.

Note that for large $\mathrm{Re}(s)$ we have the formula

$$\eta_{\mathrm{D}}(s) = 2\eta_2(s) - \eta_{\mathrm{N}}(s). \tag{4.1}$$

In particular, Theorem 4.1 implies that $\eta_{\mathrm{D}}(s)$ also extends meromorphically to the whole complex plane, since $\eta_{\mathrm{N}}(s)$ does by the preceding section. In particular, we obtain Theorem 1.1 since $2\eta_2(s)$ has simple poles with residues in $\mathbb{Z}$.

4.1. The q-reflection bundle

For $q \geqslant 2$ define the *q-reflection bundle* $\mathcal{R}_q \to M$ by

$$\mathcal{R}_q = \left(\left[S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right)\right] \times \mathbb{R}^q\right) \Big/ \approx, \tag{4.2}$$

where the equivalence classes of the relation $\approx$ are defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right)$ and $\xi \in \mathbb{R}^q$, we set

$$[(x, v, \xi)] = \{(x, v, \xi), (x, v', A(q) \cdot \xi)\} \quad \text{if } (x, v) \in \mathcal{D}_{\mathrm{in}},\ (x, v') \in \mathcal{D}_{\mathrm{out}},$$

where $A(q)$ is the $q \times q$ matrix with entries in $\{0, 1\}$ given by

$$A(q) = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Clearly, the matrix $A(q)$ yields a shift permutation

$$A(q)(\xi_1, \xi_2, \dots, \xi_q) = (\xi_q, \xi_1, \dots, \xi_{q-1}).$$

This indeed defines an equivalence relation since $(x, v') \in \mathcal{D}_{\text{out}}$ whenever $(x, v) \in \mathcal{D}_{\text{in}}$. Note that

$$A(q)^q = \text{Id}, \quad \text{tr}\, A(q)^j = 0, \quad j = 1, \ldots, q-1. \tag{4.3}$$

Let us describe the smooth structure of $\mathcal{R}_q$, using the charts of M and the notations of Section 2.2. For $z_\star \in \mathcal{D}_{\text{in}}$, let $U_{z_\star} = B(0,\delta) = \{x \in \mathbb{R}^{d-1} : |x| < \delta\}$ be a neighborhood of 0 used for the definition of $F_{z_\star}$ (see Section 2.2) and let

$$\Psi_{z_\star}^{-1} : \mathcal{O}_{z_\star} \longrightarrow]-\epsilon, \epsilon[\times B(0,\delta) \times B(0,\delta) = W_{z_\star}$$

be a chart. Then the bundle $\mathcal{R}_q \to M$ can be defined by defining its transition maps, as follows. Let $W = \Psi^{-1}(B \backslash \pi^{-1}(\partial D))$ be a chart. In the smooth coordinates introduced in Section 2.2, we have $W_{z_\star} \cap W = W_+ \sqcup W_-$, where

$$W_+ =]0, \varepsilon[\times B(0,\delta) \times B(0,\delta) \quad \text{and} \quad W_- =]-\varepsilon, 0[\times B(0,\delta) \times B(0,\delta).$$

Then we define the transition map $\alpha_{z_\star} : W_{z_\star} \cap W \to \text{GL}(\mathbb{R}^q)$ of the bundle $\mathcal{R}_q$ with respect to the pair of charts $(\Psi_{z_\star}, \Psi)$ to be the locally constant map defined by

$$\alpha_{z_\star}(z) = \begin{cases} \text{Id} & \text{if } z \in W_-, \\ A(q) & \text{if } z \in W_+. \end{cases}$$

For $z_\star, z'_\star \in \mathcal{D}_{\text{in}}$, the transition map of $\mathcal{R}_q$ for the pair of charts $(\Psi_{z_\star}, \Psi_{z'_\star})$ is declared to be constant and equal to Id on $W_{z_\star} \cap W_{z'_\star}$. In this way we obtain a smooth bundle $\mathcal{R}_q$ over M, which is clearly homeomorphic to the quotient space (4.2). Since the transition maps of $\mathcal{R}_q$ are locally constant, there is a natural flat connection d^q on $\mathcal{R}_q$ which is given in the charts by the trivial connection on $\mathbb{R}^q$.

Consider a small smooth neighborhood V of K. As in Section 2.4, we embed V into a smooth compact manifold without boundary N, and we fix an extension of $\mathcal{R}_q$ to N (this is always possible if we choose N to be the double manifold of V). Consider any connection ∇^q on the extension of $\mathcal{R}_q$ which coincides with d^q near K, and denote by

$$P_{q,t}(z) : \mathcal{R}_q(z) \longrightarrow \mathcal{R}_q(\varphi_t(z))$$

the parallel transport of ∇^q along the curve $\{\varphi_\tau(z) : 0 \leqslant \tau \leqslant t\}$. We have a smooth action of φ_t^q on $\mathcal{R}_q$ which is given by the horizontal lift of φ_t

$$\varphi_t^q(z, \xi) = (\varphi_t(z), P_{q,t}(z) \cdot \xi), \quad (z, \xi) \in \mathcal{R}_q.$$

As in (3.2), we see that for a periodic orbit γ we define $P_{q,\gamma}$ as an endomorphism on $\mathbb{R}^q$. From (4.3), and the fact that ∇^q coincides with d^q near

K, we easily deduce that for any periodic orbit $\gamma = (\varphi_\tau(z))_{\tau\in[0,\tau(\gamma)]}$, we have

$$\operatorname{tr}(P_{q,\gamma}) = \begin{cases} q & \text{if } m(\gamma) = 0 \mod q, \\ 0 & \text{if } m(\gamma) \neq 0 \mod q. \end{cases} \tag{4.4}$$

4.2. Transfer operators acting on G

Now, consider the bundle

$$\mathcal{E}^q_{k,\ell} = \mathcal{E}_{k,\ell} \otimes \pi_G^* \mathcal{R}_q,$$

where $\pi_G^*\mathcal{R}_q$ is the pullback of $\mathcal{R}_q$ by π_G and $\mathcal{E}_{k,\ell}$ is defined in Section 2.6, so that $\pi_G^*\mathcal{R}_q \to G$ is a vector bundle over G. We may lift the flow φ_t^q to a flow $\Phi_t^{k,\ell,q}$ on $\mathcal{E}^q_{k,\ell}$ which is defined locally near $\widetilde{K}_u$ by

$$\begin{aligned} &\Phi_t^{k,\ell,q}(\omega, u\otimes v\otimes \xi) \\ &= \left(\widetilde{\varphi}_t(\omega),\ b_t(\omega)\cdot \left[\left(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\right)^{\wedge k}(u) \otimes (\mathrm{d}\widetilde{\varphi}_t(\omega))^{\wedge \ell}(v) \otimes P_{q,t}(z)\cdot \xi\right]\right) \end{aligned}$$

for any $\omega = (z, E) \in G$, $u \otimes v \otimes \xi \in \mathcal{E}^q_{k,\ell}(\omega)$ and $t \in \mathbb{R}$. Here $b_t(\omega)$ is defined in Section 2.6. As in Section 2.8, we consider a smooth connection $\nabla^{k,\ell,q} = \nabla^{k,\ell} \otimes \pi_G^*\nabla^q$ on $\mathcal{E}^q_{k,\ell}$. Define the transfer operator

$$\Phi_{-t}^{k,\ell,q,*} : C^\infty(G, \mathcal{E}^q_{k,\ell}) \longrightarrow C^\infty(G, \mathcal{E}^q_{k,\ell})$$

by

$$\Phi_{-t}^{k,\ell,q,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell,q}[\mathbf{u}(\widetilde{\varphi}_{-t}(\omega)], \ \mathbf{u} \in C^\infty(G, \mathcal{E}^q_{k,\ell}).$$

Then the operator

$$\mathbf{P}_{k,\ell,q} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_{-t}^{k,\ell,q,*}\right)\Big|_{t=0}, \ \mathbf{u} \in C^\infty(G, \mathcal{E}^q_{k,\ell})$$

which is defined near $\widetilde{K}_u$, can be written locally as $\nabla^{k,\ell,q}_{\widetilde{X}} + \mathbf{A}_{k,\ell,q}$ for some $\mathbf{A}_{k,\ell,q} \in C^\infty(\widetilde{U}_u, \operatorname{End} \mathcal{E}^q_{k,\ell})$ defined in some small neighborhood $\widetilde{U}_u$ of $\widetilde{K}_u$. Next, we choose some $\mathbf{B}_{k,\ell,q} \in C^\infty(G, \operatorname{End} \mathcal{E}^q_{k,\ell})$ which coincides $\mathbf{A}_{k,\ell,q}$ near $\widetilde{K}_u$. We consider $\widetilde{V}_u$ and $\widetilde{Y}$ as in Section 2.7, and set

$$\mathbf{Q}_{k,\ell,q} = \nabla^{k,\ell,q}_{\widetilde{Y}} + \mathbf{B}_{k,\ell,q} : C^\infty(G, \mathcal{E}^q_{k,\ell}) \longrightarrow C^\infty(G, \mathcal{E}^q_{k,\ell}).$$

4.3. Meromorphic continuation of $\eta_q(s)$

For $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ such that $\widetilde{\chi} \equiv 1$ near $\widetilde{K}_u$, define

$$\mathbf{R}_\varepsilon^{k,\ell,q}(s) = \widetilde{\chi} \mathrm{e}^{-\varepsilon(\mathbf{Q}_{k,\ell,q}+s)}(\mathbf{Q}_{k,\ell,q}+s)^{-1}\widetilde{\chi}.$$

Repeating the argument of the preceding section, one can obtain an analog of (3.4), where the factor $\operatorname{tr}(\alpha_{\tilde{\gamma}}^{k,\ell})$ must be replaced by $\operatorname{tr}(\alpha_{\tilde{\gamma}}^{k,\ell})\operatorname{tr}(P_{q,\gamma})$. This leads to a meromorphic continuation of $\mathbf{R}_\varepsilon^{k,\ell,q}(s)$.

On the other hand, by (4.4) one gets $\operatorname{tr}(P_{q,\gamma}) = \mathbf{1}_{q\mathbb{N}}(m(\gamma))$. In particular, proceeding exactly as in the preceding section, we obtain that for $\operatorname{Re}(s)$ large enough we have

$$\sum_{k=0}^{d-1}\sum_{\ell=0}^{d^2-d}(-1)^{k+\ell}\operatorname{tr}^\flat \mathbf{R}_\varepsilon^{k,\ell,q}(s) = q \sum_{\substack{\gamma\in\mathcal{P} \\ m(\gamma)\in q\mathbb{N}}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}}. \tag{4.5}$$

Therefore, repeating the argument of Section 3, we establish a meromorphic continuation of the function $s \mapsto \eta_q(s)$. Finally, by using (4.5), we may proceed exactly as in Section 3.3 to show that $q\eta_q(s)$ has integer residues. This completes the proof of Theorem 4.1.

5. Modified Lax–Phillips conjecture for real analytic obstacles

In this section, we assume that the obstacles $D_1, \dots, D_r$ have real analytic boundary. Then the smooth structure on M defined in Section 2.2 induces an analytic structure on M. Indeed, with notations of Section 2.2, the local parametrizations $F_{z_\star}$ of $\mathcal{D}_{\mathrm{in}}$ can be chosen to be real analytic, as $\mathcal{D}_{\mathrm{in}}$ is a real analytic submanifold of $S\mathbb{R}^{d-1}$. This makes the transition maps (2.3) real analytic, and thus we obtain a real analytic structure on M. In the charts defined by $\Psi_{z_\star}$ and Ψ (see Section 4.1), the billiard flow φ_t is a translation and it defines a real analytic flow. Of course, the Grassmannian bundle $G \to M$ also becomes real analytic. Consequently, the lifted flow $\widetilde{\varphi}_t$ on G, which is defined by (2.6), is real analytic as well.

Consider the bundles $\mathcal{E}_{k,\ell}^q \to G$ defined in Section 4.2 for $q \geqslant 2$, $1 \leqslant k \leqslant d-1$ and $1 \leqslant \ell \leqslant d^2-d$. In the case $q=1$ the bundles $\mathcal{E}_{k,\ell}^1 \to G$ are isomorphic to $\mathcal{E}_{k,\ell}$, $\mathcal{E}_{k,\ell}$ being the bundles defined in Section 2.6. As before, we naturally extend the flow $\widetilde{\varphi}_t$ to a flow $\Phi_t^{k,\ell,q}$ (which is non-complete) on $\mathcal{E}_{k,\ell}^q$. We set

$$\mathcal{E}_q^+ = \bigoplus_{k+\ell \text{ even}} \mathcal{E}_{k,\ell}^q \quad \text{and} \quad \mathcal{E}_q^- = \bigoplus_{k+\ell \text{ odd}} \mathcal{E}_{k,\ell}^q.$$

Define the flows $\Phi^+_{t,q}$ and $\Phi^-_{t,q}$, acting respectively on the bundles $\mathcal{E}^+_q$ and $\mathcal{E}^-_q$, by

$$\Phi^+_{t,q} = \bigoplus_{k+\ell \text{ even}} \Phi_t^{k,\ell,q} \quad \text{and} \quad \Phi^-_{t,q} = \bigoplus_{k+\ell \text{ odd}} \Phi_t^{k,\ell,q}.$$

Then $\Phi^\pm_{t,q}$ is a virtual lift of $\widetilde{\varphi}_t$ to the virtual bundle $\mathcal{E}^{\text{virtual}}_q = \mathcal{E}^+_q - \mathcal{E}^-_q$, in the sense of [17, p. 176]. Next, given a periodic ray γ, a point $\omega = (z, E) \in G$, $z \in \gamma$, and a bundle $\xi \to G$ over G, one considers the transformation $\Phi_{\tau(\gamma)} : \xi_\omega \to \xi_\omega$, where ξ_ω is the fibre over ω and Φ_t is the lift of the flow $\widetilde{\varphi}_t$ to ξ. Then we set $\chi_\gamma(\xi) = \operatorname{tr} \Phi_{\tau(\gamma)}$. Following [17, p. 176], one defines $\chi_\gamma(\mathcal{E}^+_q - \mathcal{E}^-_q) = \chi_\gamma(\mathcal{E}^+_q) - \chi_\gamma(\mathcal{E}^-_q)$. For a periodic ray γ related to a primitive periodic ray $\gamma^\sharp$ one defines $\mu(\gamma) \in \mathbb{N}$ determined by the equality $\tau(\gamma) = \mu(\gamma)\tau(\gamma^\sharp)$.

After this preparation introduce the zeta function

$$\zeta_q(s) = \exp\Big(-\frac{1}{q} \sum_{\tilde{\gamma}} \frac{\chi_\gamma(\mathcal{E}^+_q - \mathcal{E}^-_q)}{\mu(\gamma)|\det(\mathrm{Id} - \widetilde{P}_\gamma)|} \mathrm{e}^{-s\tau(\gamma)} \Big), \quad \mathrm{Re}(s) \gg 1.$$

This function corresponds exactly to the *flat-trace function* $s \mapsto T^\flat(s)$ introduced by Fried [17, p. 177]. On the other hand, one has

$$\chi_\gamma(\mathcal{E}^+_q - \mathcal{E}^-_q) = \operatorname{tr} \Phi^+_{\tau(\gamma),q} - \operatorname{tr} \Phi^-_{\tau(\gamma),q} = \sum_{k,\ell} (-1)^{k+\ell} \operatorname{tr} \Phi^{k,\ell,q}_{\tau(\gamma)}(\omega_{\tilde{\gamma}}).$$

According to the analysis of Section 3 for the function $\zeta_{\mathrm{N}}(s)$, one deduces

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \zeta_1(s) = \sum_{\gamma \in \mathcal{P}} \frac{\tau(\gamma^\sharp)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}} = \eta_{\mathrm{N}}(s), \quad \mathrm{Re}\, s \gg 1.$$

Similarly, the argument of Section 4 implies

$$\frac{\mathrm{d}}{\mathrm{d}s} \log(\zeta_2(s)^2) = 2 \sum_{\substack{\gamma \in \mathcal{P} \\ m(\gamma) \in 2\mathbb{N}}} \frac{\tau(\gamma^\sharp)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}} = 2\eta_2(s), \quad \mathrm{Re}\, s \gg 1.$$

Consequently, the representation (4.1) yields

$$\eta_{\mathrm{D}}(s) = \frac{\mathrm{d}}{\mathrm{d}s} \log\Big(\frac{\zeta_2(s)^2}{\zeta_1(s)} \Big), \quad \mathrm{Re}\, s \gg 1. \tag{5.1}$$

For obstacles with real analytic boundary the flow $\widetilde{\varphi}_t$ is real analytic and the bundles $\mathcal{E}^\pm_q$ are real analytic, too.

For convenience of the reader, we recall the definition of the order of a function f meromorphic on the complex plane (see for instance [24]). For $r \geqslant 0$, denote by $n(r, f)$ the number of poles of f in the disk $\{|z| \leqslant r\}$

counted with their multiplicity. Introduce the (Nevalinna) counting function

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} \mathrm{d}t + n(0, f) \log r.$$

Let $\log^+ : \mathbb{R} \to \mathbb{R}^+$ be the function defined by

$$\log^+ x = \begin{cases} \log x & \text{if } x \geqslant 1, \\ 0 & \text{if } x \leqslant 1. \end{cases}$$

The proximity function $m(r, f)$ is defined by

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r\mathrm{e}^{i\theta})| \mathrm{d}\theta,$$

assuming that $f(z)$ has no poles for $|z| = r$. Then $T(r, f) = N(r, f) + m(r, f)$ is called the (Nevalinna) characteristic of f. Finally, the order $\rho(f)$ of f is defined by

$$\rho(f) = \limsup_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}.$$

We are now in position to apply the principal result of Fried [17, Theorem on p. 180, see also pp. 177–178] saying that the zeta functions $s \mapsto \zeta_k(s)$, $k = 1, 2$, are entire functions with finite orders $\rho(\zeta_k)$. Thus ζ_2^2/ζ_1 is a meromorphic function with order $\max\{\rho(\zeta_1), \rho(\zeta_2)\}$.

Proof that $\eta_D(s)$ is not an entire function. — We will show that the Dirichlet series $\eta_{\mathrm{D}}(s)$ cannot be continued as an entire function to $\mathbb{C}$, that is, $\eta_{\mathrm{D}}(s)$ has at least one pole. We proceed by contradiction and assume that $\eta_{\mathrm{D}}(s)$ is an entire function. Applying the representation (5.1), this means that $\zeta_2(s)^2/\zeta_1(s)$ has neither poles nor zeros. As we have mentioned above, this function has finite order, so by the Hadamard factorisation theorem we deduce that $\zeta_2(s)^2/\zeta_1(s) = \exp(Q(s))$ for some polynomial $Q(s)$. This implies that $\eta_{\mathrm{D}}(s) = Q'(s)$ is a polynomial, which is impossible. Indeed, since $\eta_{\mathrm{D}}(s) \to 0$ as $\mathrm{Re}(s) \to +\infty$, this implies that $Q'(s)$ must be the zero polynomial. By uniqueness of the development of Dirichlet series of the form $\sum_n a_n \mathrm{e}^{-\lambda_n s}$ [43] absolutely convergent for $\mathrm{Re}\, s \geqslant s_0$, this leads to a contradiction. □

Appendix A. Hyperbolicity of the billiard flow

In this appendix we show that the non-grazing flow ϕ_t defined in Section 2.1 in Euclidean metric is uniformly hyperbolic on the trapped set K_e. Throughout this section we work with the Euclidean metric. As it was mentioned in Section 2.4, we can obtain the uniform hyperbolicity of the flow φ_t on K in the smooth model from that for ϕ_t on $\mathring{B}\cap K_e$. The flow ϕ_t is hyperbolic on $\mathring{B}\cap K_e$ if for every $z=(x,v)\in\mathring{B}\cap K_e$ we have a splitting

$$T_z\mathbb{R}^d=\mathbb{R}X(z)\oplus E_s(z)\oplus E_u(z),$$

where $X(z)=v$ and $E_s(z)/E_u(z)$ are stable/unstable spaces such that $\mathrm{d}\phi_t(z)$ maps $E_{s/u}(z)$ onto $E_{s/u}(\phi_t(z))$ whenever $\phi_t(z)\in\mathring{B}\cap K_e$, and if for some constants $C>0,\nu>0$ independent of $z\in K_e$, we have

$$\|\mathrm{d}\phi_t(z)\cdot v\|\leqslant\begin{cases}C\mathrm{e}^{-\nu t}\|v\|, & v\in E_s(z), \quad t\geqslant 0,\\ C\mathrm{e}^{-\nu|t|}\|v\|, & v\in E_u(z), \quad t\leqslant 0.\end{cases}\tag{A.1}$$

First, we consider the case of periodic points. Our purpose is to define the unstable and stable manifolds $E_u(z)$ and $E_s(z)$ at a periodic point $z\in\mathring{B}\cap K_e$, and to estimate the norm of $\mathrm{d}\phi_t(z)|_{E_b(z)}$ for $b=u,s$. Consider a periodic ray γ with reflection points $z_i=(q_i,\omega_i)$, $q_i\in\partial D$, $\omega_i\in\mathbb{S}^{d-1}$, $i=0,\dots,m(\gamma)=m$ with period $T>0$. Let π_x be the projection $\pi_x:(t,x,\tau,\xi)\ni T^*(\mathbb{R}\times\Omega)\to x\in\Omega$ and let $\widetilde{\gamma}\subset T^*(\mathbb{R}\times\Omega)$ be the generalized bicharacteristic of the wave operator $\partial_t^2-\Delta_x$ for which $\pi_x(\widetilde{\gamma})=\gamma$ (see Section 1.2 in [47] for the definition of generalized bicharacteristics). Let $\rho=(x,\xi)\in\widetilde{\gamma}\cap\big(T^*(\mathring{\Omega})\setminus\{0\}\big)$ be such that $\pi_x(\rho)\neq q_i$, $i=0,\dots,m$. Then the flow ϕ_T maps a small conic neighborhood $\mathcal{V}\subset T^*(\mathring{\Omega})\setminus\{0\}$ of ρ to a conic neighborhood $\mathcal{W}\subset T^*(\mathring{\Omega})\setminus\{0\}$ of ρ and

$$\mathrm{d}\phi_T(\rho):T_\rho(T^*(\mathring{\Omega}))\longrightarrow T_\rho(T^*(\mathring{\Omega})).$$

The tangent vector ζ to $\widetilde{\gamma}$ at ρ and the cone axis $\eta=\{t\xi,t>0\}$ are invariant with respect to $\mathrm{d}\phi_T(\rho)$ and we define the quotient $\Sigma_\rho=T_\rho(T^*(\mathring{\Omega}))/E_\rho$, E_ρ being the two dimensional space spanned by ζ and η. Then

$$P_\gamma(\rho)=\mathrm{d}\phi_T(\rho)|_{\Sigma_\rho}$$

is the linear Poincaré map corresponding to γ at ρ. It is easy to see that if $\mu\in\widetilde{\gamma}\cap\big(T^*(\mathring{\Omega})\setminus\{0\}\big)$ is another periodic point, the maps $P_\gamma(\rho)$ and $P_\gamma(\mu)$ are conjugated and the eigenvalues of $P_\gamma(\rho)$ are independent of the choice of ρ.

Recall the billiard ball map $\mathbf{B}$ introduced in Section 2.4. The advantage is that $\mathbf{B}$ is smooth (see [36]). We will apply the representation of the Poincaré map for billiard ball map $\mathbf{B}$ established in Theorem 2.3.1 and Proposition 2.3.2 in [47]. To do this, we recall some notations given in Section 2 of [47]. Let $\Pi_i\subset\mathbb{R}^d$ be the plane passing through q_i and orthogonal to ω_i and let Π_i' be the plane passing through q_i and orthogonal to ω_{i-1}. For $j=i\pmod m$ we set $\Pi_j=\Pi_i$, $q_j=q_i$. Set $\lambda_i=\|q_{i-1}-q_i\|$ and let

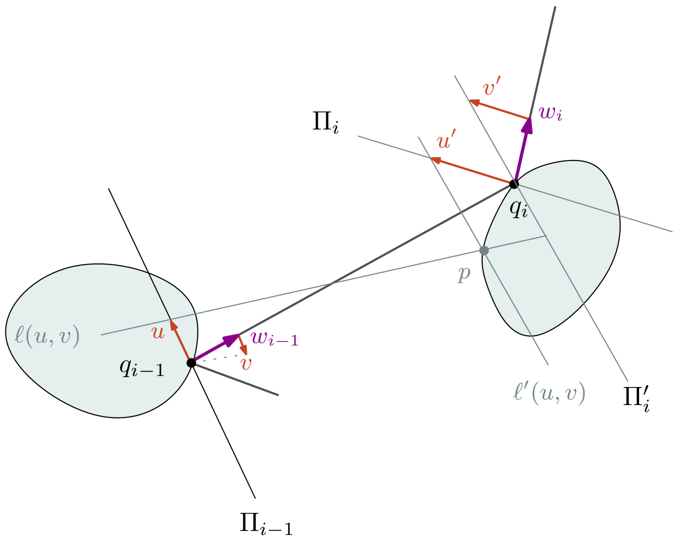

Figure A.1. The map $\Psi_i : (u, v) \mapsto (u', v')$

σ_i be the symmetry with respect to the tangent plane $\alpha_i = T_{q_i}(\partial D)$. (If $u = u_t + u_n \in \mathbb{S}^{d-1}$ with $u_t \in \alpha_i, u_n \perp \alpha_i$, then $\sigma_i(u) = u_t - u_n$.) Clearly,

$$\sigma_i(\omega_i) = \omega_{i+1}, \quad \sigma_i(\Pi'_i) = \Pi_i, \quad \Pi_0 = \Pi_m.$$

We identify Π_{i-1} and Π'_i by using a translation along the line determined by the segment $[q_{i-1}, q_i]$ and we will write $\sigma_i(\Pi_{i-1}) = \Pi_i$.

Given $(u, v) \in \Pi_{i-1} \times \Pi_{i-1}$ sufficiently close to $(0, 0)$, consider the line $\ell(u, v)$ passing through $q_{i-1} + u$ and having direction $\omega_{i-1} + v$ (the point v is identified with the vector v). Then $\ell(u, v)$ intersects ∂D at a point $p = p(u, v)$ close to q_i. Let $\ell'(u, v)$ be the line symmetric to $\ell(u, v)$ with respect to the tangent plane to ∂D at p and let $q_i + u' \in \Pi_i$ be the intersection point of $\ell'(u, v)$ with Π_i. There exists an unique $v' \in \Pi_i$ for which $\omega_i + v'$ has the direction of $\ell'(u, v)$. Thus we get a map

$$\Psi_i : \Pi_{i-1} \times \Pi_{i-1} \ni (u, v) \longmapsto (u', v') \in \Pi_i \times \Pi_i$$

defined for (u, v) in a small neighborhood of $(0, 0)$ (see Figure A.1). The smoothness of the billiard ball map $\mathbf{B}$ implies the smoothness of Ψ_i. Next consider the second fundamental form $S(\xi, \eta) = \langle G_i(\xi), \eta \rangle$ for D at q_i, where

$$G_i = \mathrm{d}n_j(q_i) : \alpha_i \longrightarrow \alpha_i$$

is the Gauss map. Here recall that $n_j(q)$ is the inward unit normal vector to ∂D_j at q pointing into D_j. Introduce a symmetric linear map $\widetilde{\psi}_i$ on Π_i defined for $\xi, \eta \in \Pi_i'$ by

$$\langle \widetilde{\psi}_i \sigma_i(\xi), \sigma_i(\eta) \rangle = -2\langle \omega_{i-1}, n_j(q_i) \rangle \langle G_i(\pi_i(\xi)), \pi_i(\eta) \rangle,$$

where $\langle \cdot , \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$, $\pi_i : \Pi_i' \to \alpha_i$ is the projection on α_i along $\mathbb{R}\omega_{i-1}$.

Notice that the non-eclipse condition (1.1) implies that there exists $\beta_0 \in]0, \pi/2[$ depending only on D such that for all incoming directions ω_{i-1} and all reflection points $q_i \in \partial D_j$, one has

$$-\langle \omega_{i-1}, n_j(q_i) \rangle = \langle \omega_i, n_j(q_i) \rangle \geqslant \cos \beta_0 > 0.$$

Consequently, the symmetric map $\widetilde{\psi}_i$ has spectrum included in $[\mu_1, \mu_2]$ with $0 < \mu_1 < \mu_2$ depending only on $\kappa = \cos \beta_0$ and the sectional curvatures of ∂D. Finally, define the symmetric map

$$\psi_i = s_i^{-1} \widetilde{\psi}_i s_i : \Pi_m \longrightarrow \Pi_m$$

with $s_i = \sigma_i \circ \sigma_{i-1} \circ \cdots \circ \sigma_1$. By Theorem 2.3.1 in [47], the map $\mathrm{d}\Psi_i(0,0)$ has the form

$$\mathrm{d}\Psi_i(0,0) = \begin{pmatrix} I & \lambda_i I \\ \widetilde{\psi}_i & I + \lambda_i \widetilde{\psi}_i \end{pmatrix} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},$$

and the linearized Poincaré map P_γ related to γ is given by

$$P_\gamma = \mathrm{d}(\Psi_m \circ \cdots \circ \Psi_1)(0,0) : \Pi_0 \times \Pi_0 \longrightarrow \Pi_0 \times \Pi_0,$$

which implies

$$P_\gamma = \begin{pmatrix} s_m & 0 \\ 0 & s_m \end{pmatrix} \begin{pmatrix} I & \lambda_m I \\ \psi_m & I + \lambda_m \psi_m \end{pmatrix} \cdots \begin{pmatrix} I & \lambda_1 I \\ \psi_1 & I + \lambda_1 \psi_1 \end{pmatrix}.$$

Here the space $\Pi_0 \times \Pi_0$ is identified with the space $T_\rho(T^*(\Omega))/E_\rho$, where $\pi_x(\rho) = (q_0, \omega_0)$.

Now we repeat without changes the argument of Proposition 2.3.2 in [47]. For $k = 0, 1, \ldots, m$, consider the space $\mathcal{M}_k$ of linear symmetric non-negative definite maps $M : \Pi_k \to \Pi_k$. Next, let $\mathcal{M}_k(\varepsilon) \subset \mathcal{M}_k$ be the space of maps such that $M \geqslant \varepsilon I$ with $\varepsilon > 0$. To study the spectrum of P_γ, consider the subspace

$$L_0 = \{(u, M_0 u) : u \in \Pi_0\}, \quad M_0 \in \mathcal{M}_0,$$

which is Lagrangian with respect to the symplectic structure on $\Pi_0 \times \Pi_0$ induced from the symplectic structure on $T_\rho(T^*(\Omega))$ by the factorisation with E_ρ. The action of the map $\mathrm{d}\Psi_1(0,0)$, transforms the Lagrangian space L_0 into

$$L_1 = \{\sigma_1(I + \lambda_1 M_0)u, \sigma_1((I + \lambda_1\psi_1)M_0 + \psi_1)u :\ u \in \Pi_0\} \subset \Pi_1 \times \Pi_1.$$

Introducing the operator

$$\mathcal{A}_i : \mathcal{M}_{i-1} \longrightarrow \mathcal{M}_i$$

defined by

$$\mathcal{A}_i(M) = \sigma_i M(I + \lambda_i M)^{-1}\sigma_i^{-1} + \widetilde{\psi}_i,$$

we write $L_1 = \{(u, M_1 u) :\ u \in \Pi_1\}$ with $M_1 = \mathcal{A}_1(M_0)$. By recurrence, define

$$L_k = \{(u, M_k u) :\ u \in \Pi_k\}, \quad M_k = \mathcal{A}_k(M_{k-1}), \quad k = 1, 2, \dots, m.$$

The maps $\mathcal{A}_k$ are contractions from $\mathcal{M}_{k-1}(\varepsilon)$ to $\mathcal{M}_k(\varepsilon)$, hence

$$\mathcal{A} = \mathcal{A}_m \circ \dots \circ \mathcal{A}_1$$

becomes also a contraction from $\mathcal{M}_0(\varepsilon)$ to $\mathcal{M}_0(\varepsilon)$. We choose $M_0 \in \mathcal{M}_0(\varepsilon)$ as a *fixed point* of $\mathcal{A}$ and thus we fix L_0. Notice that $\varepsilon > 0$ can be chosen uniformly for all periodic rays. Thus we deduce

$$P_\gamma \begin{pmatrix} u \\ M_0 u \end{pmatrix} = \begin{pmatrix} Su \\ M_0 Su \end{pmatrix}$$

with a map $S : \Pi_0 \to \Pi_0$ having the form

$$S = \sigma_m(I + \lambda_m \mathcal{A}'_{m-1}(M_0)) \circ \sigma_{m-1}(I + \lambda_{m-1}\mathcal{A}'_{m-2}(M_0)) \circ \dots \circ \sigma_1(I + \lambda_1 M_0),$$

where $\mathcal{A}'_k = \mathcal{A}_k \circ \mathcal{A}_{k-1} \circ \dots \circ \mathcal{A}_1$. Setting

$$d_0 = \min_{i \neq j} \operatorname{dist}(D_i, D_j) > 0, \quad d_1 = \max_{i \neq j} \operatorname{dist}(D_i, D_j),$$

and $\beta = \log(1 + \varepsilon d_0)$, one obtains

$$\|Su\| \geqslant (1 + d_0\varepsilon)^m \|u\| = \mathrm{e}^{\beta m}\|u\|.$$

Obviously, the eigenvalues of S are eigenvalues of P_γ and we conclude that P_γ has $(d-1)$ eigenvalues $\nu_1, \dots, \nu_{d-1}$ satisfying

$$|\nu_j| \geqslant \mathrm{e}^{\beta m}, \quad j = 1, \dots, d-1.$$

For $0 < \tau < \lambda_1$, consider a point $\rho = \phi_\tau(z) \in \mathring{B} \cap \gamma$, where $z = (x, v) \in \mathcal{D}_{\mathrm{in}}$. The map $\phi_\tau : \mathcal{D}_{\mathrm{in}} \to \mathring{B}$ is smooth near z and moreover $\mathrm{d}\phi_\tau(z) : \Sigma_z \to \Sigma_\rho$. We identify $\Pi_0 \times \Pi_0$ with Σ_z and $\Sigma_{\phi_\tau(z)}$ with the image

$$\mathrm{d}\phi_\tau(z)\Sigma_z = \begin{pmatrix} I & \tau I \\ 0 & I \end{pmatrix} (\Pi_0 \times \Pi_0).$$

Next we define the unstable subspace of Σ_ρ as

$$E_u(\rho) = \mathrm{d}\phi_\tau(z)(L_0) = \begin{pmatrix} I & \tau I \\ 0 & I \end{pmatrix} (L_0).$$

Let $0 < \sigma < \lambda_{p+1}$ with $p \geqslant 1$ and set $t = -\tau + \sum_{j=1}^{p} \lambda_j + \sigma$.

Then $\phi_{-\tau}$ is smooth near ρ, the map $\mathbf{B}^p$ is smooth and

$$\mathrm{d}\phi_t(\rho)|_{\Sigma_\rho} = \mathrm{d}\phi_\sigma(\mathbf{B}^p(z)) \circ \mathrm{d}\mathbf{B}^p(z) \circ \mathrm{d}\phi_{-\tau}(\rho) : \Sigma_\rho \longrightarrow \Sigma_{\phi_t(\rho)}.$$

This is illustrated by the diagram

$$\begin{array}{ccccccc} & & E_u(\rho) & \xrightarrow{\mathrm{d}\phi_t(\rho)} & E_u(\phi_t(\rho)) & & \\ & & \Big\downarrow \mathrm{d}\phi_{-\tau}(\rho) & & \Big\uparrow \mathrm{d}\phi_\sigma(\mathbf{B}^p(z)) & & \\ \Pi_0 & \xrightarrow{\chi_0} & L_0 & \xrightarrow{\mathrm{d}\mathbf{B}^p(z)} & L_p & \xleftarrow{\chi_p} & \Pi_p, \end{array}$$

where $\chi_0 : \Pi_0 \ni u \mapsto (u, M_0 u) \in L_0 \subset \Pi_0 \times \Pi_0$ and $\chi_p : \Pi_p \ni u \mapsto (u, M_p u) \in L_p \subset \Pi_p \times \Pi_p$. It is easy to obtain an estimate of the action of $\mathrm{d}\phi_t(\rho)|_{E_u(\rho)}$ for $\rho = \phi_\tau(z)$, $v = \mathrm{d}\phi_\tau(z)(u, M_0 u) \in E_u(\rho)$. Clearly,

$$\mathrm{d}\phi_t(\rho) \cdot v = (\mathrm{d}\phi_\sigma(\mathbf{B}^p(z)) \circ \mathrm{d}\mathbf{B}^p(z))(u, M_0 u).$$

By the above argument we deduce

$$\mathrm{d}\mathbf{B}^p(z)(u, M_0 u) = (S_p u, M_p S_p u) \in L_p$$

with

$$S_p = \sigma_p(I + \lambda_p \mathcal{A}'_{p-1}(M_0)) \circ \sigma_{p-1}(I + \lambda_{p-1}\mathcal{A}'_{p-2}(M_0)) \circ \cdots \circ \sigma_1(I + \lambda_1 M_0).$$

Setting $\beta_0 = \beta/d_1$ and $w = (u, M_0 u) = \mathrm{d}\phi_{-\tau}(\rho) \cdot v$, we have

$$\begin{aligned} \text{(A.2)} \quad & \|\mathrm{d}\mathbf{B}^p(z) \cdot w\| \\ & = \|(S_p u, M_p S_p u)\| \geqslant \|S_p u\| \geqslant \mathrm{e}^{\frac{\beta}{d_1} p d_1} \|u\| \geqslant \mathrm{e}^{\beta_0(t+\tau-\sigma)}\|u\|, \end{aligned}$$

and

$$\text{(A.3)} \qquad \|\mathrm{d}\mathbf{B}^p(z) \cdot w\| \leqslant C_0 \mathrm{e}^{-\beta_0 d_1} e^{\beta_0 t}\|w\| = C_0 \mathrm{e}^{-\beta_0 d_1} e^{\beta_0 t} \|\mathrm{d}\phi_{-\tau}(\rho) v\|.$$

Here we used the estimate

$$\|w\| = \Big(\|u\|^2 + \|M_0 u\|^2\Big)^{1/2} \leqslant (1 + B_0^2)^{1/2}\|u\|$$

with $\|M_0\|_{\Pi_0 \to \Pi_0} \leqslant B_0$ and we set $C_0 = (1 + B_0^2)^{-1/2}$. The constant B_0 can be chosen uniformly for all M_k and all periodic points since for every non-negative symmetric map M one has

$$\|M(I + \lambda_k M)^{-1}\| \leqslant \frac{1}{\lambda_k} \leqslant \frac{1}{d_0},$$

while the norms $\|\widetilde{\psi}_k\|$ are uniformly bounded by a constant depending of the sectional curvatures of D and $\kappa > 0$. Consequently,

$$\|\mathcal{A}_k(M)\| \leqslant B_0, \tag{A.4}$$

the same is true for the norm of the fixed point $M_0 = \mathcal{A}_m(M_{m-1})$ and the estimate (A.4) is uniform for all periodic points. Finally, estimating the norm of $\mathrm{d}\phi_{-\sigma}(\mathbf{B}^p(z)) = \left(\begin{smallmatrix} I & -\sigma I \\ 0 & I \end{smallmatrix}\right)$, we obtain $\|\mathrm{d}\phi_{-\sigma}(\mathbf{B}^p(z))\zeta\| \geqslant (1+d_1)^{-1}\|\zeta\|$ and

$$\begin{aligned} \|\mathrm{d}\phi_t(\rho)v\| &\geqslant (1+d_1)^{-1}C_0 e^{-\beta_0 d_1}\mathrm{e}^{\beta_0 t}\|\mathrm{d}\phi_{-\tau}(\rho)v\| \\ &\geqslant (1+d_1)^{-2}C_0 e^{-\beta_0 d_1}\mathrm{e}^{\beta_0 t}\|v\|. \end{aligned}$$

It remains to treat the case $\rho = \phi_\tau(z)$, $z \in \mathcal{D}_{in}$, $0 < t = \tau + \sigma < \lambda_1$. Then $\phi_t(\rho) = \phi_{\tau+\sigma}(z) \in \mathring{B} \cap \gamma$ and we obtain easily an estimate for $\|(\mathrm{d}\phi_{\tau+\sigma}(z)) \cdot v\|$.

Our case is a partial one of a more general setting (see [39]) concerning Lagrangian spaces $\{(u, Mu)\}$ with positive definite linear maps M. Such spaces are called positive Lagrangian. A linear symplectic map L is called monotone if it maps positive Lagrangian onto positive Lagrangian. In [39] it is proved that any monotone symplectic map is a contraction on the manifold of positive Lagrangian spaces. After a suitable conjugation the map L has the representation (see Proposition 3 in [39])

$$L = \begin{pmatrix} A^{-1} & 0 \\ 0 & A^* \end{pmatrix} \begin{pmatrix} I & R \\ P & I + PR \end{pmatrix}$$

with positive definite matrices P and R. In our situation, we have $A = I$, $R = \lambda_i I$ and $P = \psi_i$.

To determine the stable space $E_s(z)$ at z, we will study the flow ϕ_t for $t < 0$ and repeat the above argument leading to a fixed point. For completeness we present some details. The linear map P_γ^{-1} for a periodic ray γ with m reflections has the representation

$$P_\gamma^{-1} = (\mathrm{d}\Psi_1)^{-1} \circ \cdots \circ (\mathrm{d}\Psi_m)^{-1} : \Pi_0 \times \Pi_0 \longrightarrow \Pi_0 \times \Pi_0,$$

where

$$(\mathrm{d}\Psi_k)^{-1} = \begin{pmatrix} \sigma_k^{-1} & 0 \\ 0 & \sigma_k^{-1} \end{pmatrix} \begin{pmatrix} I + \lambda_k \psi_k & -\lambda_k I \\ -\psi_k & I \end{pmatrix}.$$

Recall that $\Pi_0 = \Pi_m$. Consider a Lagrangian

$$Q_0 = Q_m = \{(u, -N_m u) : u \in \Pi_0\}$$

with a symmetric non-negative definite map $N_m \in \mathcal{M}_0$. Then

$$(\mathrm{d}\Psi_m)^{-1}Q_m = \big\{\big(\sigma_m^{-1}(I+\lambda_m(\psi_m+N_m))u, -\sigma_m^{-1}(\psi_m+N_m)u\big) \,:\, u\in\Pi_0\big\}$$
$$= \{(u, -N_{m-1}u) :\ u \in \Pi_{m-1}\},$$

where

$$N_{m-1} = \sigma_m^{-1}(\psi_m+N_m)\Big(I+\lambda_m(\psi_m+N_m)\Big)^{-1}\sigma_m :\ \Pi_{m-1} \longrightarrow \Pi_{m-1}.$$

By recurrence, introduce the Lagrangian spaces

$$Q_k = \{(u, -N_k u) : u\in\Pi_k\}, \quad N_k = \mathcal{B}_k(N_{k+1}), \quad k = 0,\dots,m-1,$$

where

$$\mathcal{B}_k(M) = \sigma_{k+1}^{-1}(\psi_{k+1}+M)\Big(I+\lambda_{k+1}(\psi_{k+1}+M)\Big)^{-1}\sigma_{k+1} :\ \Pi_k \longrightarrow \Pi_k.$$

It is easy to see that $\mathcal{B}_k$ are contractions from $\mathcal{M}_{k+1}(\varepsilon)$ to $\mathcal{M}_k(\varepsilon)$ since

$$\sigma_{k+1}\Big(\mathcal{B}_k(M_1)-\mathcal{B}_k(M_2)\Big)\sigma_{k+1}^{-1}$$
$$= (I+\lambda_{k+1}(\psi_{k+1}+M_1))^{-1}(M_1-M_2)(I+\lambda_{k+1}(\psi_{k+1}+M_2))^{-1}.$$

Therefore, $\mathcal{B} = \mathcal{B}_0\circ\cdots\circ\mathcal{B}_{m-1}$ will be contraction from $\mathcal{M}_0(\varepsilon)$ to $\mathcal{M}_0(\varepsilon)$ and there exists a *fixed point* $N_m \in \mathcal{M}_0(\varepsilon)$ of $\mathcal{B}$. Moreover,

$$P_\gamma^{-1}\begin{pmatrix} u \\ -N_m u\end{pmatrix} = \begin{pmatrix} \widetilde{S}u \\ -N_m\widetilde{S}u\end{pmatrix}, \quad u\in\Pi_0,$$

where

$$\widetilde{S} = \sigma_1^{-1}(I+\lambda_1(\psi_1+\mathcal{B}_1'(N_m)))\circ\sigma_2^{-1}(I+\lambda_2(\psi_2+\mathcal{B}_2'(N_m)))$$
$$\circ\cdots\circ\sigma_m^{-1}(I+\lambda_m(\psi_m+N_m))$$

and $\mathcal{B}_k' = \mathcal{B}_k\circ\cdots\circ\mathcal{B}_{m-1}$, $k = 1,\dots,m-1$. Clearly,

$$\|\widetilde{S}u\| \geqslant (1+d_0\varepsilon)^m\|u\|,\ u\in\Pi_0,$$

where $\varepsilon>0$ depends of the sectional curvatures of D. Thus the stable manifold at $\phi_\sigma(z), -\lambda_{m-1}<\sigma<0$ can be defined as $E_s(\phi_\sigma(z)) = \mathrm{d}\phi_\sigma(z)(Q_m)$ and we may repeat the above argument for the estimate of $\mathrm{d}\phi_t(\phi_\sigma(z))$ acting on $E_s(\phi_\sigma(z))$ for $t<0$.

The intersection of the unstable and stable manifolds at $y = \phi_t(z)$, $0 < t < \lambda_p$ is $(0,0)$. Indeed, we have

$$E_u(y) = \mathrm{d}\phi_t(z)(L_{p-1}), \quad E_s(y) = \mathrm{d}\phi_{t-\lambda_p}(\phi_{\lambda_p}(z))(Q_p),$$

where $L_{p-1} = \{(u, M_{p-1}u) :\ u\in\Pi_{p-1}\times\Pi_{p-1}\}$ and $Q_p = \{(-u,-N_p u) :\ u\in\Pi_p\times\Pi_p\}$. Assume that $E_u(y)\cap E_s(y)\neq(0,0)$. Then there exists $0\neq v\in L_{p-1}\cap\mathrm{d}\phi_{-\lambda_p}(\phi_{\lambda_p}(z))(Q_p)$. By the above argument $\mathrm{d}\phi_{-\lambda_p}(\phi_{\lambda_p}(z))(Q_p) =$

$\{(u, -N_{p-1}u) : u \in \Pi_{p-1} \times \Pi_{p-1}\}$. This implies the existence of $u \neq 0$ for which $(M_{p-1} + N_{p-1})u = 0$ which is impossible since $M_{p-1} + N_{p-1}$ is a definite positive map. Consequently, $E_u(y)$ and $E_s(y)$ are transversal subspaces of dimension $d-1$ of Σ_y and we have a direct sum $\Sigma_y = E_u(y) \oplus E_s(y)$.

Now we pass to the estimates of $\mathrm{d}\phi_t(z)|_{E_u(z)}$, where $z \in \mathring{B} \cap K_e$ is not a periodic point. Since $z \in K_e$, the trajectory $\gamma = \{\phi_t(z) : t \in \mathbb{R}\}$ has infinite number successive reflection points $q_k \in \partial D_{i_k}$, $k \in \mathbb{Z}$, with an infinite sequence

$$J_0 = (i_j)_{j \in \mathbb{Z}}, \quad i_j \neq i_{j+1}.$$

For every $p \geqslant p_0 \gg 1$ define the configuration

$$\alpha_p = \begin{cases} (i_{-p}, \dots, i_0, \dots, i_p) & \text{if } i_p \neq i_{-p}, \\ (i_{-p}, \dots, i_0, \dots, i_{p+1}) & \text{if } i_p = i_{-p}. \end{cases}$$

Repeating α_p infinite times, one obtains an infinite configuration. Following the arguments of the proof of Proposition 10.3.2 in [47], there exists a periodic ray γ_p following this configuration and we obtain a sequence of periodic rays $(\gamma_{p_0+k})_{k \geqslant 0}$. Let $\{q_{p,k} \in \partial D_{i_k}\}$ be the reflexion points of γ_p. For the periodic ray γ_p passing through $q_{p,0} \in \partial D_{i_0}$ consider the linear space

$$L_{p,0} = \{(u, M_{p,0}u) : \ u \in \Pi_{p,0}\} \subset \Pi_{p,0} \times \Pi_{p,0}.$$

Our purpose is to show that the symmetric linear maps $M_{p,0} \in \mathcal{M}_{p,0}(\varepsilon)$ composed by some unitary maps converge as $p \to \infty$ to a symmetric linear map $\widetilde{M}_0 \in \mathcal{M}_0(\varepsilon)$ on Π_0. This composition is necessary since the maps $M_{p,0}$, $p \geqslant p_0$, are defined on different spaces. To do this, we will use Lemmas 10.2.1, 10.4.1 and 10.4.2 in [47]. Consider the rays γ_{p_0+q}, $q \geqslant 1$, and γ. These rays have reflection points passing successively through the obstacles

$$L' = D_{i_{-p_0-1}}, D_{i_{-p_0}}, \dots, D_{i_0}, \dots, D_{i_{p_0}}, D_{i_{p_0+1}} = L''.$$

According to Lemma 10.2.1 in [47], there exist uniform constants $C > 0$ and $\delta \in (0,1)$ such that for any $|k| \leqslant p_0$ and $j = 1, \dots, q$, one has

$$\|q_{p_0+1,k} - q_{p_0+j,k}\| \leqslant C(\delta^{p_0+k} + \delta^{p_0-k}), \quad \|q_{p_0+j,k} - q_k\| \leqslant C(\delta^{p_0+k} + \delta^{p_0-k}).$$

We need to introduce some notations from [47, Section 10.4]. Let $x \in \partial D_i$ and $y \in \partial D_j$ with $i \neq j$, and assume that the segment $[x,y]$ is transversal to both ∂D_i and ∂D_j. Let Π be the plane orthogonal to $[x,y]$, passing through x. Let $e = (x-y)/\|x-y\|$, and introduce the projection $\pi : \Pi \to T_x(\partial D)$ along the vector e. As above, we define the symmetric linear map $\widetilde{\psi} : \Pi \to \Pi$ by

$$\langle \widetilde{\psi}(u), u \rangle = 2\langle e, n(x) \rangle \langle G_x(\pi(u)), \pi(u) \rangle, \quad u \in \Pi,$$

and notice that

$$\operatorname{spec}\widetilde{\psi} \subset [\mu_1, \mu_2], \quad 0 < \mu_1 < \mu_2.$$

Setting $D_0 = 2C$, we have the estimates

$$\|q_{p_0+j,k} - q_k\| \leqslant D_0 \delta^{p_0+k}, \quad k = -p_0+1, \dots, 0, \quad j = 1, \dots, q.$$

Fix $1 \leqslant j \leqslant q$ and introduce the vectors

$$e_k = \frac{q_{k+1} - q_k}{\|q_{k+1} - q_k\|}, \quad e'_k = \frac{q_{p_0+j,k+1} - q_{p_0+j,k}}{\|q_{p_0+j,k+1} - q_{p_0+j,k}\|}.$$

Consider the maps $\widetilde{\psi}_k : \Pi_k \to \Pi_k$ and $\widetilde{\psi}'_k : \Pi'_k \to \Pi'_k$ related to the segments $[q_{k-1}, q_k]$ and $[q_{p_0+j,k-1}, q_{p_0+j,k}]$, respectively. Let $M_{-p_0+1} : \Pi_{-p_0+1} \to \Pi_{-p_0+1}$ and $M'_{-p_0+j} : \Pi'_{-p_0+j} \to \Pi'_{-p_0+j}$ be symmetric non-negative definite linear operators. By recurrence, define

$$M_k = \sigma_k M_{k-1}(I + \lambda_k M_{k-1})^{-1}\sigma_k + \widetilde{\psi}_k, \quad k = -p_0+2, \dots, 0,$$

where $\lambda_k = \|q_{k-1} - q_k\|$ and σ_k is the symmetry with respect to $T_{q_k}(\partial D)$. Similarly, we define M'_k, $k = -p_0+2, \dots, 0$, replacing $\widetilde{\psi}_k, \lambda_k$ and σ_k by $\widetilde{\psi}'_k$, $\lambda_{p_0+j,k} = \|q_{p_0+j,k-1} - q_{p_0+j,k}\|$ and σ'_k, respectively. Next, introduce the constants

$$b = (1 + 2\mu_1 \kappa d_0)^{-1} < 1, \quad a_1 = \max\{\delta, b\} < 1,$$

where $d_0 > 0$ and $\kappa > 0$ were defined above. We choose M_{-p_0+1} so that $\|M_{-p_0+1}\| \leqslant B_0$ and by induction one deduces $\|M_k\| \leqslant B_0$. Here $B_0 > 0$ is the constant in (A.4). We have uniform estimates

$$\|M_k\| \leqslant B_0, \quad \|M'_k\| \leqslant B_0, \quad k = -p_0+1, \dots, 0. \tag{A.5}$$

Applying [47, Lemma 10.4.1], there exists a linear isometry $A_k : \mathbb{R}^d \to \mathbb{R}^d$ such that $A_k(\Pi'_k) = \Pi_k$, and A_k satisfies the estimates

$$\|A_k - I\| \leqslant C_1 D_0 (1+\delta)\delta^k, \quad \|\widetilde{\psi}_k - A_k \widetilde{\psi}'_k A_k^{-1}\| \leqslant C_2 D_0 (1+\delta)\delta^k, \tag{A.6}$$

for any $k = -p_0+1, \dots, 0$. Now we are in position to apply [47, Lemma 10.4.2] saying that with some constant $E > 0$, depending only of D, κ, δ and b, for $k = -p_0+1, \dots, 0$ we have

$$\begin{aligned} &\|M_k - A_k M'_k A_k^{-1}\| \\ &\qquad \leqslant D_0 E a_1^{p_0+k} + b^{2(k+p_0-1)} \|M_{-p_0+1} - A_{-p_0+1} M'_{-p_0+1} A^{-1}_{-p_0+1}\|. \end{aligned} \tag{A.7}$$

The norm of the second term on the right hand side is bounded by the quantity $2B_0 b^{2(k+p_0-1)}$ and for $k = 0$ we obtain

$$\|M_0 - A_0 M'_0 A_0^{-1}\| \leqslant D_0 E a_1^{p_0} + 2B_0 b^{2(p_0-1)}.$$

Applying the above estimate for the rays γ_{p_0+q}, the maps M'_0, A_0 will depend of the ray γ_{p_0+q} and for this reason we denote them by $M'_{q,0}, A_{q,0}$. Now we use these estimates for the maps $M'_{q,0}$, $M_{q',0}$ related to the rays γ_{p_0+q} and $\gamma_{p_0+q'}$ and by the triangle inequality one deduces

$$\|A_{q,0}M'_{q,0}A^{-1}_{q,0} - A_{q',0}M'_{q',0}A^{-1}_{q',0}\| \leqslant 2D_0Ea_1^{p_0} + 4B_0b^{2(p_0-1)}. \tag{A.8}$$

Here $A_{q,0}(\Pi'_{q,0}) = \Pi_0$ and $A_{q',0}(\Pi'_{q',0}) = \Pi_0$ are some isometries satisfying the estimates (A.6). Clearly, one obtain a Cauchy sequence $(A_{q,0}M'_{q,0}A^{-1}_{q,0})_{q\geqslant 1}$ which converges to a symmetric non-negative linear map $\widetilde{M}_0$ in Π_0. Moreover, if for every q we have $M'_{q,0} \geqslant \varepsilon I$, then $\widetilde{M}_0 \geqslant \varepsilon I$.

After this preparation, for any $0 < \tau < \|q_1 - q_0\|$ we define the unstable manifold at $\rho = \phi_\tau(z_0)$ with $z_0 = (q_0, v_0)$ as the subspace

$$E_u(\rho) = \mathrm{d}\phi_\tau(z_0)\{(u, \widetilde{M}_0u) \in \Pi_0 \times \Pi_0 : \ u \in \Pi_0\} \subset \Sigma_\rho.$$

It is important to note that the procedure leading to the estimate (A.7) can be repeated starting with $\widetilde{M}_0$ instead of M_{-p_0+1}. Then if $\widetilde{M}_k$ are the maps obtained from $\widetilde{M}_0$ after successive reflections, we obtain an estimate

$$\|\widetilde{M}_k - A_kM'_kA_k^{-1}\| \leqslant D_0Ea_1^{p_0+k} + b^{2(k+p_0-1)}\|\widetilde{M}_0 - A_0\widetilde{M}'_0A_0^{-1}\|$$

for $k = 1, \dots, p_0/2$.

We can repeat the above argument for $v \in E_u(\rho)$ and

$$t = -\tau + \sum_{j=1}^{p} \lambda_j + \sigma,$$

where $0 < \tau < \lambda_1$ and $0 < \sigma < \lambda_{p+1}$, to estimate

$$\|\mathrm{d}\phi_t(\rho) \cdot v\|.$$

We apply (A.2) and (A.3) with the expansion map $\widetilde{S}_p$ defined as the composition of the maps $(I + \lambda_k\mathcal{A}'_{k-1}(\widetilde{M}_0))$ and we get an estimate for $\|\mathrm{d}\phi_t(\rho) \cdot v\|$. Finally, the construction of the stable space $E_s(\phi_\sigma(z_0))$ for $\|q_{-1} - q_0\| < \sigma < 0$ can be obtained by a similar argument and we omit the details.

Appendix B. Ikawa's criterion and proof of Theorem 1.3

In this appendix we prove Theorem 1.3 for all dimensions $d \geqslant 2$. The result of Ikawa [31, Theorem 2.1] was established for d odd and it yields only an infinite number of resonances in a suitable band. To obtain a stronger result we apply the argument of [35]. The proof is based on Lemma 2.2, Proposition 2.3 and Theorem 2.4 in [31]. Recall the notation

$$\Lambda_\omega = \Big\{\mu_j \in \mathbb{C} \setminus e^{i\frac{\pi}{2}\overline{\mathbb{R}^+}} :\ 0 < \operatorname{Im}\mu_j \leqslant \omega|\operatorname{Re}\mu_j|,\ 0 < \arg\mu_j < \pi\Big\}.$$

For the modification covering all dimensions $d \geqslant 2$, it is necessary only to modify Lemma 2.2 in [31] since the other results are independent of the dimension d. Below we consider only the resonances μ_j for which $0 < \arg\mu_j < \pi$ and we omit this in the notation.

Let $\rho \in C_c^\infty(\mathbb{R};\mathbb{R}_+)$ be an even function with $\operatorname{supp}\rho \subset [-1,1]$ such that

$$\rho(t) > 1 \quad \text{if} \quad |t| \leqslant 1/2,$$

and the property that its Fourier transform is non-negative,

$$\widehat{\rho}(k) = \int \mathrm{e}^{itk}\rho(t)\mathrm{d}t \geqslant 0, \quad k \in \mathbb{R}.$$

(As in [31], we use the above Fourier transform, since we deal with $\langle e^{i\mu_j t}, \rho\rangle$). It is easy to construct ρ with the above properties. Let $\phi \in C_c^\infty(\mathbb{R};[0,1])$ be an even function with support in $[-1/2,1/2]$ such that $\phi(x) \equiv 1$ for $|x| \leqslant 3/8$. Define

$$\Phi(t) := (\phi \star \phi)(t) = \int_{-\infty}^{\infty} \phi(x)\phi(t-x)\mathrm{d}x \geqslant 0.$$

Clearly, $\Phi(t)$ is even, has support in $[-1,1]$ and $\widehat{\Phi}(k) = (\widehat{\phi}(k))^2$. For $k \in \mathbb{R}$ the function $\widehat{\phi}(k)$ is real valued and $\widehat{\Phi}(k) \geqslant 0$ for $k \in \mathbb{R}$. On the other hand, for $|t| \leqslant 1/2$ we have

$$\Phi(t) \geqslant \int_{-1/4}^{1/4} \phi(t-x)\mathrm{d}x = \int_{t-1/4}^{t+1/4} \phi(s)\mathrm{d}s \\ \geqslant \operatorname{mes}([t-1/4, t+1/4] \cap [-3/8, 3/8]) \geqslant \frac{1}{8}$$

and we may take $\rho(t) = 9\Phi(t)$.

Let $(\ell_q)_{q\in\mathbb{N}}$ and $(m_q)_{q\in\mathbb{N}}$ be sequences of positive numbers such that $\ell_q \geqslant d_0 = \min_{k\neq j}\operatorname{dist}(D_k, D_j) > 0$, $m_q \geqslant \max\{1, \frac{1}{d_0}\}$ and let $\ell_q, m_q \to \infty$ as $q \to \infty$. Finally, set

$$\rho_q(t) = \rho(m_q(t-\ell_q)), \quad t \in \mathbb{R},$$

and $c_0 = \int \rho(t)\mathrm{d}t \geqslant 1$. The result [31, Lemma 2.2] must be modified as follows.

Lemma B.1. — *Let $0 < \delta < 1$ be fixed. Assume that for $\alpha \geqslant 1$we have*

$$N(\alpha) = \#\{\mu_j \in \Lambda_\omega :\ 0 < \operatorname{Im}\mu_j \leqslant \alpha,\ |\mu_j| \leqslant r\} \leqslant P(\alpha,\delta)r^\delta$$

with $P(\alpha,\delta) < \infty$. Then we have

$$\sum_{\mu_j \in \Lambda_\omega} |\widehat{\rho}_q(\mu_j)| \leqslant C_0 e^{\alpha} m_q^{d+1} \mathrm{e}^{-\alpha \ell_q} + C_1 P(\alpha,\delta) m_q^{-(1-\delta)} \tag{B.1}$$

for some constants $C_0 > 0, C_1 > 0$ independent of α and q.

Proof. — We write

$$\sum_{\mu_j \in \Lambda_\omega} = \sum_{\substack{\mu_j \in \Lambda_\omega \\ \operatorname{Im} \mu_j > \alpha}} + \sum_{\substack{\mu_j \in \Lambda_\omega \\ \operatorname{Im} \mu_j \leqslant \alpha}} = (\mathrm{I}) + (\mathrm{II}).$$

For (I) one integrates by parts,

$$\begin{aligned} \int_{\ell_q - \frac{1}{m_q}}^{\ell_q + \frac{1}{m_q}} \rho(m_q(t-\ell_q)) \mathrm{e}^{it\mu_j} \mathrm{d}t \\ = \frac{(-1)^{d+2} m_q^{d+2}}{(i\mu_j)^{d+2}} \int_{\ell_q - \frac{1}{m_q}}^{\ell_q + \frac{1}{m_q}} \rho^{(d+2)}(m_q(t-\ell_q)) \mathrm{e}^{it\mu_j} \mathrm{d}t. \end{aligned} \tag{B.2}$$

We have $|e^{it\mu_j}| \leqslant \mathrm{e}^{-t \operatorname{Im} \mu_j}$ and since $\operatorname{supp} \rho_q \subset [\ell_q - m_q^{-1}, \ell_q + m_q^{-1}]$ and $\operatorname{Im} \mu_j > \alpha$, we get

$$|e^{it\mu_j}| \leqslant \mathrm{e}^{-\alpha(\ell_q - m_q^{-1})} \leqslant \mathrm{e}^{-\alpha(\ell_q - 1)}.$$

In particular, the right hand side of (B.2) is estimated by

$$C e^{\alpha} \frac{e^{-\alpha \ell_q} m_q^{d+1}}{|\mu_j|^{d+2}} \|\rho\|_{C^{d+2}(\mathbb{R})}$$

with a constant $C > 0$ independent of j and q. On the other hand, for d even by the results of Vodev [54, 55] we have the estimate

$$\#\{\mu_j : 0 \leqslant \arg \mu_j \leqslant \pi,\ |\mu_j| \leqslant k\} \leqslant C_2 k^d.$$

and for d odd we have the same bound (see Section 4.3 in [15]). Consequently, the series

$$\sum_{|\mu_j| \geqslant 1} \frac{1}{|\mu_j|^{d+2}} = \sum_{k=1}^{\infty} \sum_{k \leqslant |\mu_j| < k+1} \frac{1}{|\mu_j|^{d+2}} \leqslant C_2 \sum_{k=1}^{\infty} \frac{(k+1)^d}{k^{d+2}} \leqslant C_3$$

is convergent. This yields the first term on the right hand side of (B.1). Passing to the estimate of (II), we apply the argument of the proof of Theorem 2 in [35]. First,

$$\int e^{i\zeta t} \rho_q(t) \mathrm{d}t = m_q^{-1} e^{i\zeta \ell_q} \widehat{\rho}(\zeta m_q^{-1}).$$

Applying the Paley–Winner theorem for $\operatorname{Im}\zeta \geqslant 0$ and $N \geqslant 2$ one deduces

$$\begin{aligned}\left|\int e^{i\zeta t}\rho_q(t)\mathrm{d}t\right| &\leqslant C_N m_q^{-1} e^{-\operatorname{Im}\zeta(\ell_q - m_q^{-1})}(1+|\zeta m_q^{-1}|)^{-N}\\ &\leqslant C_N m_q^{-1}(1+|\zeta m_q^{-1})^{-N}.\end{aligned}$$

Therefore

$$\begin{aligned}\left|\sum_{\operatorname{Im}\mu_j>\alpha}\langle e^{i\mu_j t},\rho_q(t)\rangle\right| &\leqslant C_N m_q^{-1}\int_0^\infty (1+m_q^{-1}r)^{-N}\mathrm{d}N_\alpha(r)\\ &\leqslant -C_N m_q^{-1}\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}r}\Big((1+m_q^{-1}r)^{-N}\Big)N_\alpha(r)\mathrm{d}r\\ &\leqslant B_N P(\alpha,\delta)m_q^{-1+\delta}\int_0^\infty (1+y)^{-N-1}y^\delta\mathrm{d}y\\ &= A_N P(\alpha,\delta)m_q^{-(1-\delta)}.\end{aligned}$$

Notice that the other terms in the trace formula of Zworski (1.3) are easily estimated. In fact, since $\lambda \mapsto \psi(\lambda)$ has compact support, one gets

$$\begin{aligned}\left|\int\Big(\int \psi(\lambda)\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}(\lambda)\cos(\lambda t)\mathrm{d}\lambda\Big)\rho(m_q(t-\ell_q))\mathrm{d}t\right| &\leqslant C_\psi\int\rho(m_q(t-\ell_q))\mathrm{d}t\\ &\leqslant C_\psi c_0 m_q^{-1}.\end{aligned}$$

Here we integrate by parts in the integral with respect to λ and exploit the fact that $\sigma(\lambda)$ is bounded on the support of $\psi(\lambda)$ (see Section 3.10 in [15] for the estimates of $\sigma(\lambda)$). Similarly,

$$\left|\int v_{\omega,\psi}(t)\rho(m_q(t-\ell_q))\mathrm{d}t\right| \leqslant C_{\omega,\psi}\int\rho(m_q(t-\ell_q))\mathrm{d}t \leqslant c_0 C_{\omega,\psi}m_q^{-1}.$$

We can put the estimates of these terms in $C_1 P(\alpha,\delta)m_q^{-(1-\delta)}$ increasing the constant C_1. This completes the proof. □

Define the distribution $\widehat{F}_{\mathrm{D}} \in \mathcal{S}'(\mathbb{R}^+)$ by

$$\widehat{F}_{\mathrm{D}}(t) = \sum_{\gamma\in\mathcal{P}}\frac{(-1)^{m(\gamma)}\tau^\sharp(\gamma)\delta(t-\tau(\gamma))}{|\det(I-P_\gamma)|^{1/2}}. \tag{B.3}$$

As we mentioned above, the following results are proved in [31] and their proofs are independent of the dimension d. For convenience of the reader we present the statements.

Proposition B.2 ([31, Propsition 2.3]). — *Suppose that $\eta_{\mathrm{D}}(s)$ cannot be continued as an entire function of s. Then there exists $\alpha_0 > 0$ such that* *for any $\beta > \alpha_0$ we can find sequences $(\ell_q), (m_q)$ with $\ell_q \to \infty$ as $q \to \infty$ and such that for all $q \geqslant 0$ one has*

$$e^{\beta \ell_q} \leqslant m_q \leqslant \mathrm{e}^{2\beta \ell_q} \quad \text{and} \quad |\langle \widehat{F}_{\mathrm{D}}, \rho_q \rangle| \geqslant \mathrm{e}^{-\alpha_0 \ell_q}.$$

THEOREM B.3 (Theorem 2.4, [31]). — *There are constants $C > 0$ and $\alpha_1 > 0$ such that for any sequences (ℓ_q) and (m_q) with $\ell_q \to \infty$ as $q \to \infty$, it holds*

$$|\langle u, \rho_q \rangle| \geqslant |\langle \widehat{F}_{\mathrm{D}}, \rho_q \rangle| - C\mathrm{e}^{\alpha_1 \ell_q} m_q^{-1}. \tag{B.4}$$

Remark B.4. — In [31, Theorem 2.4], on the right hand side of (B.4), one has the term $m_q^{-\epsilon}$ for some $\epsilon > 0$ instead of m_q^{-1}. In particular the above estimate holds, increasing $\beta > \alpha_0$.

The above theorem is given in [31] without proof. However its proof repeats that of Proposition 2.2 in [30] following the procedure described in [27, Section 3] and exploiting the construction of asymptotic solutions in [28]. The first term on the right hand side of (B.4) is obtained by the leading term in (1.4) applying the stationary phase argument to a trace of a global parametrix (see Chapter 4 in [47]) or to the trace of the asymptotic solutions given below. For the second one we must estimate a sum

$$\sum_{\substack{\gamma \in \mathcal{P} \\ \tau(\gamma) \leqslant \ell_q + m_q^{-1}}} \int_{\ell_q - m_q^{-1}}^{\ell_q + m_q^{-1}} \rho_q(t) r_\gamma(t) \mathrm{d}t,$$

where r_γ is a function in $L^1_{\mathrm{loc}}(\mathbb{R})$, which is obtained from the lower order terms in the application of the stationary phase argument. Since $r_\gamma(t)$ could increase as $t \to \infty$, we need a precise analysis of the behavior of $r_\gamma(t)$.

We discuss briefly the approach of Ikawa and refer to [27, 28] for more details. First one expresses the distribution $u(t)$ defined in Introduction by the kernels $E(t, x, y)$, $E_0(t, x, y)$ of the operators $\cos(t\sqrt{-\Delta}) \oplus 0$ and $\cos(t\sqrt{-\Delta_0})$, respectively (recall that $-\Delta$ is the Laplacian in $Q = \mathbb{R}^d \setminus D$ with Dirichlet boundary conditions on ∂D). Consider

$$\widehat{E}(t, x, y) = \begin{cases} E(t, x, y) & \text{if } (x, y) \in Q \times Q, \\ 0 & \text{if } (x, y) \notin Q \times Q. \end{cases}$$

If $D \subset \{x : \ |x| \leqslant a_0\}$, then

$$\operatorname{supp}_{x,y}\Big(\widehat{E}(t, x, y) - E_0(t, x, y)\Big) \subset \big\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \ |x| \leqslant a_0 + t, \ |y| \leqslant a_0 + t\big\}.$$

For $t \in \operatorname{supp} \rho_q$ we must study the trace

$$\int_{\Omega_q} \langle \widehat{E}(t,x,x) - E_0(t,x,x), \rho_q \rangle \mathrm{d}x$$

with $Q_q = \{x \in Q : |x| \leqslant a_0 + \ell_q + 1\}$. For odd dimensions the kernel $E_0(t,x,x)$ vanishes for $t > 0$. For even dimensions, $x \mapsto E_0(t,x,x)$ is smooth for any $t > 0$ and we can easily estimate

$$\left| \int_{\Omega_q} \langle E_0(t,x,x), \rho_q \rangle \mathrm{d}x \right| \leqslant A_0 m_q^{-1}$$

with $A_0 > 0$ independent of q by using the representation of the kernel $E_0(t,x,y)$ by oscillatory integrals with phases $\langle x-y, \eta\rangle \pm t$ (see for example, [47, Section 3.1]).

Now, choose $g \in C_c^\infty(Q_q)$ and write the kernel $E(t,x,y)$ of $\cos(t\sqrt{-\Delta})$ as

$$E(t,x,y)g(y) = (2\pi)^{-d} \int_{\mathbb{S}^{d-1}} \mathrm{d}\eta \int_0^\infty k^{d-1} u(t,x;k,\eta) e^{-ik\langle y,\eta\rangle} g(y) \mathrm{d}k,$$

where $u(t,x;k,\eta)$ is the solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \quad \text{in} \quad \mathbb{R}_t \times Q, \\ u = 0 \quad \text{on} \quad \mathbb{R}_t \times \partial Q, \\ u(0,x) = \widetilde{g}(x)\mathrm{e}^{ik\langle x,\eta\rangle}, \quad \partial_t u(0,x) = 0, \end{cases}$$

with a function $\widetilde{g} \in C_c^\infty(Q)$ equal to 1 on $\operatorname{supp} g$. In the works [8, 27, 28, 30]) of Ikawa and Burq, asymptotic solutions $w^{(N)} = w_{q,+}^{(N)} + w_{q,-}^{(N)}$ of the above problem have been constructed. They have the form

$$w_{q,\pm}^{(N)}(t,x;k,\eta) = \sum_{|\mathbf{j}|d_0 \leqslant a_0+\ell_q+1} \mathrm{e}^{ik(\varphi_{\mathbf{j}}^{\pm}(x,\eta) \mp t)} \sum_{h=0}^{N} v_{\mathbf{j},h}^{\pm}(t,x,\eta)(ik)^{-h}.$$

Here $\mathbf{j} = \{j_1, j_2, \ldots, j_n\}$, $j_k \in (1,\ldots,r)$, $j_k \neq j_{k+1}$, $k = 1,2,\ldots,n-1$, $|\mathbf{j}| = n$ is a configuration related to the rays reflecting successively on $\partial D_{j_1}, \partial D_{j_2}, \ldots, \partial D_{j_n}$ (see Section 2.3). The phases $\varphi_{\mathbf{j}}^{\pm}$ are constructed successively starting from $\langle x,\eta\rangle$ and following the reflections on obstacles determined by the configuration $\mathbf{j}$. The amplitudes $v_{\mathbf{j},h}^{\pm}$ are determined by transport equations. The reader may consult [27, Section 3], [30, Equations (3.2) and (3.3)], [28, Section 4] and [8] for the construction of $v_{\mathbf{j},h}^{\pm}$. The function $u - w^{(N)}$ is solution of the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)(u - w^{(N)}) = k^{-N} F_N(t,x;k,\eta) \quad \text{in} \quad \mathbb{R}_t \times Q, \\ u - w^{(N)} = k^{-N} b_N(t,x;k,\eta) \quad \text{on} \quad \mathbb{R}_t \times \partial Q, \\ (u - w^{(N)})(0,x;k,\eta) = \partial_t(u - w^{(N)})(0,x;k,\eta) = 0. \end{cases}$$

Here F_N is obtained as the action of $(\partial_t^2 - \Delta_x)$ to the amplitudes $v^{\pm}_{\mathbf{j},N}$, while b_N is obtained by the traces on ∂Q of the amplitudes $v^{\pm}_{\mathbf{j},N}$. It is important to note that the asymptotic solutions $w^{(N)}$ are independent of the sequence (m_q). The integral involving $u - w^{(N)}$ is easily estimated and it yields a term $\mathcal{O}(m_q^{-1})$ (see [27]). For the integral involving $w^{(N)}_{q,\pm}$ one applies the stationary phase argument as $k \to \infty$ for the integration with respect to $x \in Q_q$, $\eta \in \mathbb{S}^{d-1}$, considering t as a parameter. Next, in [28], estimates of the derivatives of order p of $v^{\pm}_{\mathbf{j},h}(x,t,\eta)$ with respect to $x \in Q_q$, $\eta \in \mathbb{S}^{d-1}$ with bound $C_p e^{-\alpha_2 \ell_q}(t+1)^h$, $\alpha_2 > 0$ have been established. Here $C_p > 0$ and $\alpha_2 > 0$ are independent of ℓ_q. By using a partition on unity $\sum_j \psi_j(x) = 1$ on Q_q, for large fixed N one deduces the estimate

$$\left|\langle u - \widehat{F}_{\mathrm{D}}, \rho_q\rangle\right| \leqslant A\mathrm{e}^{-\alpha_2\ell_q} \# \left\{\mathbf{j} : |\mathbf{j}| \leqslant \frac{2\ell_q}{d_0}\right\} \ell_q^{2N+2} m_q^{-1}$$

with constant $A > 0$ independent of q. Finally, since

$$\# \left\{\mathbf{j} : |\mathbf{j}| \leqslant \frac{2\ell_q}{d_0}\right\} \leqslant \mathrm{e}^{\alpha_3 \ell_q}, \ \forall\, q \geqslant 1$$

with constant $\alpha_3 > 0$ independent of q, we obtain (B.4).

Combining Proposition B.2 and the estimates (B.1) and (B.4), it is easy to obtain a contradiction with the assumption that $P(\alpha, \delta) < \infty$ for all $\alpha \geqslant 1$. Indeed, let

$$\alpha = \frac{(2d+3)}{1-\delta}(\alpha_0 + \alpha_1 + 1), \quad \beta = \frac{\alpha}{2d+3}.$$

Then

$$m_q^{d+1}\mathrm{e}^{-\alpha\ell_q} \leqslant \mathrm{e}^{(d+1)2\beta\ell_q}\mathrm{e}^{-\alpha\ell_q} = \mathrm{e}^{-\beta\ell_q} \leqslant e^{-\beta(1-\delta)\ell_q}$$

and

$$\alpha_1 + \alpha_0 - \beta = \alpha_0 + \alpha_1 - \frac{\alpha_0 + \alpha_1 + 1}{1-\delta} = -1 - \frac{\delta(\alpha_0 + \alpha_1 + 1)}{1-\delta}.$$

From (B.1), (B.4) and Proposition B.2 one deduces

$$\begin{aligned}\Big(C_0\mathrm{e}^{\alpha} + C_1 P(\alpha,\delta)\Big)\mathrm{e}^{-\beta(1-\delta)\ell_q} &\geqslant C_0 e^{\alpha} m_q^{d+1} e^{-\alpha\ell_q} + C_1 P(\alpha,\delta) e^{-\beta(1-\delta)\ell_q} \\ &\geqslant |\langle u, \rho_q\rangle| \geqslant \mathrm{e}^{-\alpha_0\ell_q} - C\mathrm{e}^{\alpha_1\ell_q}\mathrm{e}^{-\beta\ell_q} \\ &= \mathrm{e}^{-\alpha_0\ell_q}\Big(1 - Ce^{-\ell_q - \frac{\delta(\alpha_0+\alpha_1+1)}{1-\delta}\ell_q}\Big).\end{aligned}$$

Since $\beta(1-\delta) > \alpha_0$, letting $q \to \infty$ we obtain a contradiction. This completes the proof of Theorem 1.3.

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Manuscrit reçu le 30 août 2022,
révisé le 19 octobre 2023,
accepté le 12 avril 2024.

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