DIRICHLET DYNAMICAL ZETA FUNCTION FOR BILLIARD FLOW

VESSELIN PETKOV

Abstract. We study the Dirichlet dynamical zeta function $\eta_D(s)$ for billiard flow corresponding to several strictly convex disjoint obstacles. For large $\operatorname{Re} s$ we have $\eta_D(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$, $a_n \in \mathbb{R}$ and η_D admits a meromorphic continuation to $\mathbb{C}$. We obtain some conditions of the frequencies λ_n and some sums of coefficients a_n which imply that η_D cannot be prolonged as entire function.

1. Introduction

Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with C^∞ smooth boundary and let $D = \bigcup_{j=1}^r D_j$. We assume that every D_j has non-empty interior and throughout this paper we suppose the following non-eclipse condition

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{1.1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition all periodic trajectories for the billiard flow in $\Omega = \mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangential intersections to the boundary of D. We consider the (non-grazing) billiard flow φ_t (see [Pet25, Section 2] for the definition). Next the periodic trajectories will be called periodic rays. For any periodic ray γ, denote by $\tau(\gamma) > 0$ its period, by $\tau^\sharp(\gamma) > 0$ its primitive period, and by $m(\gamma)$ the number of reflections of γ at the obstacles. Denote by P_γ the associated linearized Poincaré map (see [PS17, Section 2.3] for the definition).

Let $\mathcal{P}$ be the set of all oriented periodic rays. The counting function of the lengths of primitive periodic rays Π satisfies

$$\sharp\{\gamma \in \Pi : \tau^\sharp(\gamma) \leqslant x\} \sim \frac{\mathrm{e}^{hx}}{hx}, \quad x \to +\infty, \tag{1.2}$$

for some $h > 0$ (see for instance, [PP90, Theorem 6.5] for weak-mixing suspension symbolic flows). Thus there exists an infinite number of primitive periodic trajectories and for every small $\epsilon > 0$ we have the estimate

$$e^{(h-\epsilon)x} \leq \sharp\{\gamma \in \mathcal{P} : \tau(\gamma) \leqslant x\} \leqslant \mathrm{e}^{(h+\epsilon)x}, \; x > C_\epsilon. \tag{1.3}$$

Moreover, for some positive constants C_1, d_1, d_2 we have (see for instance [Pet99, Appendix])

$$C_1 \mathrm{e}^{d_1 \tau(\gamma)} \leqslant |\det(\mathrm{Id} - P_\gamma)| \leqslant \mathrm{e}^{d_2 \tau(\gamma)}, \quad \gamma \in \mathcal{P}. \tag{1.4}$$

By using these estimates, we define for $\mathrm{Re}(s) \gg 1$ the Dirichlet dynamical zeta function $\eta_D(s)$ by

$$\eta_{\mathrm{D}}(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma) \mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},$$

where the sums run over all oriented periodic rays. This zeta function is important for the analysis of the distribution of the scattering resonances related to the Laplacian in $\mathbb{R}^d \setminus \bar{D}$ with Dirichlet boundary conditions on ∂D (see [CP22, §1] for more details). Denote by $\sigma_a \in \mathbb{R}, \sigma_c \in \mathbb{R}$ the abscissa of absolute convergence and the abscissa of convergence of η_D, respectively.

It was proved in [CP22, Theorem 1 and Theorem 4] that η_D admits a meromorphic continuation to $\mathbb{C}$ with simple poles and integer residues. On the other hand, for $d = 2$ [Sto01] and for $d \geqslant 3$ under some conditions [Sto12] Stoyanov proved that there exists $\varepsilon > 0$ such that $\eta_{\mathrm{D}}(s)$ is analytic for $\mathrm{Re}\, s \geqslant \sigma_a - \varepsilon$.

There is a conjecture that η_D cannot be prolonged as *entire function*. This conjecture was established for obstacles with real analytic boundary (see [CP22, Theorem 3]) and for obstacles with sufficiently small diameters [Ika90b], [Sto09] and C^∞ smooth boundary. If $\eta_D(s)$ is not an entire function, then we obtain two important corollaries:

(i) η_D has infinite number of poles in some strip $\{z \in \mathbb{C} : \mathrm{Re}\, z \geq \beta\}$ (see [Pet25, Section 3] for a lower bound of the counting function of poles),

(ii) The modified Lax-Phillips conjecture (MLPC) for scattering resonances introduced by Ikawa [Ika90a] holds. (MLPC) says that there exists a strip $\{z \in \mathbb{C} : 0 < \mathrm{Im}\, z \leq \alpha\}$ containing an infinite number of scattering resonances for Dirichlet Laplacian in $\mathbb{R}^d \setminus \bar{D}$ (see [CP22, Section 1] for definitions and more precise results).

Let $\rho \in C_0^\infty(\mathbb{R}; \mathbb{R}_+)$ be an even function with $\mathrm{supp}\, \rho \subset [-1, 1]$ such that $\rho(t) > 1 \quad \text{if} \quad |t| \leqslant 1/2$. Let $(\ell_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ be sequences of positive numbers such that $\ell_j \geqslant d_0 = \min_{k \neq m} \mathrm{dist}\,(D_k, D_m) > 0$, $m_j \geqslant \max\{1, \frac{1}{d_0}\}$ and let $\ell_j \to \infty$, $m_j \to \infty$ as $j \to \infty$. Set $\rho_j(t) = \rho(m_j(t - \ell_j)), \quad t \in \mathbb{R}$, and introduce the distribution $\mathcal{F}_{\mathrm{D}}(t) \in \mathcal{S}'(\mathbb{R}^+)$ by

$$\mathcal{F}_{\mathrm{D}}(t) = \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{m(\gamma)} \tau^\sharp(\gamma) \delta(t - \tau(\gamma))}{|\det(I - P_\gamma)|^{1/2}}.$$

We have the following

Proposition 1.1. *The function $\eta_{\rm D}(s)$ cannot be prolonged as an entire function of s if and only if there exists $\alpha_0 > 0$ such that for any $\beta > \alpha_0$ we can find sequences $(\ell_j), (m_j)$ with $\ell_j \nearrow \infty$ as $j \to \infty$ such that for all $j \geqslant 0$ one has $e^{\beta \ell_j} \leqslant m_j \leqslant e^{2\beta\ell_j}$ and*

$$|\langle \mathcal{F}_{\rm D}, \rho_j\rangle| \geqslant e^{-\alpha_0 \ell_j}. \tag{1.5}$$

More precisely, if η_D cannot be prolonged as entire function, the existence of sequences (ℓ_j), (m_j) with the above properties has been proved by Ikawa [Ika90a, Prop.2.3], while in the proof of Theorem 1.1 in [Pet25] it was established that if such sequences exist, the function η_D has an infinite number of poles.

The conditions of Proposition 1.1 are difficult to verify. The purpose of this Note is to find other conditions which imply that η_D cannot be prolonged as entire function. For this purpose we exploit the local trace formula (see [Pet25, Theorem 2.1]) and the summability by typical means of Dirichlet series introduced by Hardy and Riesz [HR64] (see also [DS22, Section 2]). It is convenient to write $\eta_D(s)$ as a Dirichlet series

$$\eta_D(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \ {\rm Re}\, s \gg 1, \tag{1.6}$$

where the frequencies are arranged as follows

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_n < ...$$

and

$$a_n = \sum_{\gamma \in \mathcal{P}, \tau(\gamma) = \lambda_n} \frac{(-1)^{m(\gamma)} \tau^\sharp(\gamma)}{|\det({\rm Id} - P_\gamma)|^{1/2}}. \tag{1.7}$$

It is well known that

$$\sigma_c \geq \sigma_a - \limsup_{n \to \infty} \frac{\log n}{\lambda_n} = \sigma_a - h$$

(see for instance, [HR64, Theorem 9]). Since $\sigma_a > -\infty$, one deduces $\sigma_c > -\infty$.

Our main result is the following

Theorem 1.1. *Suppose $\sigma_c < 0$. Assume that there exist constants $C > 0$, $\delta > h + 1$, $-\gamma < \sigma_c$ and an increasing sequence $m_j \nearrow \infty$ such that*

$$\lambda_{m_j} - \lambda_{m_j - 1} \geq C e^{-\delta \lambda_{m_j}}, \tag{1.8}$$

$$|\sum_{n \geq m_j} a_n| \geq e^{-\gamma \lambda_{m_j}}. \tag{1.9}$$

Then $\eta_D(s)$ cannot be prolonged as entire function.

The condition $\sigma_c < 0$ is not a restriction since if $\sigma_c \geq 0$, the Dirichlet series

$$\eta_D(s + \sigma_c + 1) = \sum_n (a_n e^{-\lambda_n(\sigma_c+1)})e^{-\lambda_n s} = \sum_n b_n e^{-\lambda_n s}$$

is convergent for $\operatorname{Re} s > -1$, hence it has a negative abscissa of convergence σ_b and $\eta_D(s + \sigma_c + 1)$ is entire if and only if $\eta_D(s)$ is entire. Moreover, in the proof of Theorem 1.1 (see section 3), assuming η_D entire, one has the property

$$\forall A < \sigma_c,\ \exists C_A > 0,\ \ |\eta_D(s)| \leq C_A(1 + |\operatorname{Im} s|),\ \operatorname{Re} s \geq A$$

which is satisfied also for $\eta_D(s + \sigma_c + 1)$ with another constants B_A. Thus we may apply Theorem 1.1 if instead of (1.9) one has the estimate

$$|\sum_{n \geq m_j} b_n| \geq e^{-\gamma_1 \lambda_{m_j}},\ -\gamma_1 < \sigma_b. \tag{1.10}$$

The assumptions on λ_n and a_n in Theorem 1.1 are satisfied if *Bohr condition* (see for instance, [DS22, §3.13])

$$(BC) \qquad \exists C_1 > 0,\ \exists \ell > 0,\ \ \forall n > 0,\ \lambda_{n+1} - \lambda_n \geq C_1 e^{-\ell \lambda_n}$$

holds. Indeed, it is well known that in the case $\sigma_c < 0$, one has the representation

$$\sigma_c = \limsup_{n \to \infty} \frac{\log |\sum_{n \geq m} a_n|}{\lambda_m}.$$

For small $\epsilon > 0$ this implies the existence of a sequence $m_j \nearrow \infty$ such that

$$|\sum_{n \geq m_j} a_n| \geq e^{(\sigma_c - \epsilon)\lambda_{m_j}}$$

and we obtain (1.9) with $-\gamma = \sigma_c - \epsilon$.

The condition (BC) is very restrictive. The advantage of Theorem 1.1 is that (1.8) is always satisfied (see Section 3) for infinite number of frequencies $\lambda_{m_j - 1}$, λ_{m_j} and the separation by $e^{-\delta \lambda_j}$ of some frequencies λ_{m_j} only on the left is less restrictive than a separation of all frequencies on both sides. Applying Theorem 1.1, we obtain the following

Corollary 1.1. *Suppose $\sigma_c < 0$. Then if*

$$\liminf_{m \to \infty} \frac{\log |\sum_{n \geq m} a_n|}{\lambda_m} > -\infty, \tag{1.11}$$

the function $\eta_D(s)$ cannot be prolonged as entire function.

In Section 4 for $\delta > h+2$ we construct intervals $I(\lambda_k, \delta) \subset [b, b+1]$, $b \geq b_0$ with *clustering frequencies* and we obtain Corollary 4.1. We have infinite number of such intervals. Moreover, under some geometrical assumptions described in [PS12, Section 8] the number of such intervals is exponentially increasing when $b \to \infty$. Finally, assuming that the coefficients a_n have a lower bound (4.4), we prove Corollary 4.2 and we show that for every interval $I(\lambda_k, \delta)$ we have 4 possibilities concerning the behaviour of the corresponding sums. For 3 of these 4 possibilities it is possible to find frequencies satisfying (1.8), (1.9) (see Proposition 4.1).

The paper is organised as follows. In Section 2 we recall the local trace formula for η_D. Assuming η_D entire, we deduce the estimates (2.3). This makes possible to prove that that the abscissa of $k-$ summability σ_k of η_D is $-\infty$. In Section 3 we prove Theorem 1.1. Section 4 is devoted to intervals $I(\lambda_k, \delta)$ with clustering frequencies and the constructions of frequencies satisfying (1.8) and (1.9).

Acknowledgements

We would like to thank the referee for his/her valuable remarks and suggestions.

2. Summation by typical means of η_D

In this section we apply the results of [DG16], [JT25, §6.1] and [Pet25] for vector bundles. For our exposition we need only the local trace formula containing the poles of the meromorphic continuation of cut off resolvents $\mathbf{1}_{\tilde{V}_u}(-i\mathbf{P}_{k,\ell} - s)^{-1}\mathbf{1}_{\tilde{V}_u}$ of some operators

$$-i\mathbf{P}_{k,\ell,q},\ 0 \leq k \leq d,\ 0 \leq \ell \leq d^2 - d,\ q = 1, 2.$$

Here $\tilde{V}_u$ us a neighborhood of the trapping set $\tilde{K}_u$. The precise definitions of $\mathbf{P}_{k,\ell,q}$, $\tilde{K}_u$ and the corresponding setting are complicated and they are not necessary for the analysis below and we prefer to refer to [Pet25, Section 2] for the corresponding definitions and details. Denote by Res $(-i\mathbf{P}_{k,\ell,q})$ the set of the poles of the meromorphic continuation of the corresponding cut off resolvents.

For every $A > 0$ and any $0 < \epsilon \ll 1$ we have the following local trace formula (see [Pet25, Theorem 2.1])

$$\sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}\sum_{\mu\in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,2}), \mathrm{Im}\,\mu>-A}(-1)^{k+\ell}e^{-i\mu t}$$
$$-\sum_{k=0}^{d}\sum_{\ell=0}^{d^2-d}\sum_{\mu\in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,1}), \mathrm{Im}\,\mu>-A}(-1)^{k+\ell}e^{-i\mu t} \tag{2.1}$$
$$+F_A(t)=\mathcal{F}_D(t),\ t>0.$$

Here $F_A(t)\in\mathcal{S}'(\mathbb{R})$ is supported in $[0,\infty)$, the Laplace-Fourier transform $\hat{F}_A(\lambda)$ of $F_A(t)$ is holomorphic for $\mathrm{Im}\,\lambda<A-\epsilon$ and satisfies the estimate

$$|\hat{F}_A(\lambda)|=\mathcal{O}_{A,\epsilon}(1+|\lambda|)^{2d^2+2d-1+\epsilon},\ \mathrm{Im}\,\lambda<A-\epsilon. \tag{2.2}$$

Notice that the poles in $\mathrm{Res}(-i\mathbf{P}_{k,\ell,q})$ are simple with positive integer residues [CP22, Theorem 1]. For the sums with fixed q the cancellations in (2.1) could appear only between the terms with $k+\ell$ odd and $k+\ell$ even. On the other hand, taking the difference of sums with $q=2$ and $q=1$ we obtain more cancelations.

If the following we assume that η_D can be a prolonged as entire function. In particular,

$$\eta_D(-i\lambda)=\langle\mathcal{F}_D,e^{it\lambda}\rangle=\sum_{\gamma\in\mathcal{P}}\frac{(-1)^{m(\gamma)}\tau^\sharp(\gamma)e^{i\lambda\tau(\gamma)}}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}},\ \mathrm{Im}\,\lambda\geq C\gg 1$$

admits an analytic continuation for $\mathrm{Im}\,\lambda<C$. For fixed $A>0$ the function $\eta_D(-i\lambda)$ has no poles μ with $\mathrm{Im}\,\mu>-A$ and in (2.1) all terms involving poles will be canceled. Consequently, from (2.1) we obtain

$$\eta_D(-i\lambda)=\hat{F}_A(-\lambda),\ \mathrm{Im}\,\lambda>-A+\epsilon.$$

Setting $-i\lambda=s=\sigma+it$, $\sigma\in\mathbb{R}$, $t\in\mathbb{R}$, this implies

$$|\eta_D(s)|\leq C_A(1+|s|)^{2d^2+2d-1}\leq B_A(1+|t|)^{2d^2+2d-1},\ \sigma\geq -A+\epsilon. \tag{2.3}$$

Here we used the fact that $|\eta_D(s)|$ is bounded for $\sigma\geq C_0>0$ with sufficiently large $C_0>0$ and $|s|\leq\max\{A,C_0\}+|t|$ for $-A\leq\sigma\leq C_0$. We may apply the above argument for every $A>0$, so the bound (2.3) holds for every $A>0$ with constants B_A depending of A. The crucial point is that the power $2d^2+2d-1$ is *independent* of A.

Applying the Phragmént- Lindelöf principle for entire function $\eta_D(s)$ in the strip

$$\{z\in\mathbb{C}:-A+\epsilon\leq\mathrm{Re}\,z\leq C_0\},$$

one deduces

$$|\eta_D(\sigma + it)| \leq D_{\sigma,A}(1 + |t|)^{\kappa(\sigma)},\ -A + \epsilon \leq \sigma \leq C_0$$

with

$$\kappa(\sigma) = \frac{C_0 - \sigma}{C_0 + A - \epsilon}(2d^2 + 2d - 1),\ -A + \epsilon \leq \sigma \leq C_0.$$

For fixed σ, taking A sufficiently large we obtain for every small $0 < \nu \ll 1$ the estimate

$$|\eta_D(\sigma + it)| \leq B_{\sigma,\nu}(1 + |t|)^{\nu},\ \sigma \leq C_0. \tag{2.4}$$

Next we recall the *summation by typical means* of Dirichlet series (see [HR64, Section IV, §2], [DS22, Section 2] for more details). For $k > 0$ consider

$$C_\lambda^k(u) = \sum_{\lambda_n < u} (u - \lambda_n)^k a_n e^{-\lambda_n s}.$$

We say that the series $\sum_{n=1}^\infty a_n e^{-\lambda_n s}$ is (λ, k) summable if

$$\lim_{u \to \infty} \frac{C_\lambda^k(u)}{u^k} = C.$$

There exists a number σ_k such that the series is (λ, k) summable for $\sigma > \sigma_k$ and not (λ, k) summable for $\sigma < \sigma_k$ (see [HR64, Theorem 26]).The number σ_k is called abscissa of k- summability of the series. We will apply the following

Theorem 2.1 (Theorem 41, [HR64])**.** *Suppose that the series $f(s) = \sum_{n=1}^\infty a_n e^{-\lambda_n s}$ admits an analytic continuation for $\sigma > \eta$. Suppose further that k and k' are positive numbers such that $k' < k$ and for all small δ we have*

$$|f(s)| \leq \mathcal{C}_\delta(1 + |s|)^{k'}$$

uniformly for $\sigma \geq \eta + \delta > \eta$. Then $f(s)$ is (λ, k) summable for $\sigma > \eta$.

In fact, the above theorem in [HR64] is given without proof. The reader may consult Corollary 3.8 and Corollary 3.9 in [DS22] for a proof and other results related to Theorem 2.1 and (λ, k) summability. The estimates (2.4) combined with Theorem 2.1 imply the following

Proposition 2.1. *If $\eta_D(s)$ can be prolonged as entire function, for every $k > 0$ the Dirichlet series (1.6) has abscissa of $k-$summability $\sigma_k = -\infty$.*

3. Proof of Theorem 1.1

Throughout this section we assume that $\eta_D(s)$ can be prolonged as entire function. Choose $\delta > h+2$. First, it is easy to see that in every interval $[b, b+1], b \geq b_0 \gg 1$ we have subintervals $[\alpha, \beta] \subset [b+b+1]$ of length greater than $e^{-\delta b}$ which does not contain frequencies. It is sufficient to write $[b, b+1]$ as an union of $e^{\delta b}$ intervals of length $e^{-\delta b}$ and to use the bounds (1.3).

We have the following simple

Lemma 3.1. *Fix $0 < \epsilon < 1/2$ and $0 < \eta < \frac{\epsilon}{12(1+\epsilon)}$. There exists $b_0 \geq \max\{3/h, 1\}$ depending of η so that for $\alpha \geq b_0$ we have*

$$\sharp\{\gamma \in \Pi : \alpha \leq \tau^\sharp(\gamma) \leq \alpha + \epsilon\} > \frac{\epsilon(1-\eta)e^{\alpha h}}{3(\alpha+\epsilon)}. \tag{3.1}$$

Proof. For $x \geq b_0(\eta) \gg 1$ the asymptotics (1.2), imply the estimates

$$\frac{e^{hx}}{hx}(1-\eta) \leq \sharp\{\gamma \in \Pi : \tau^\sharp(\gamma) \leq x\} \leq \frac{e^{hx}}{hx}(1+\eta).$$

Therefore for $\alpha \geq b_0(\eta)$ we obtain

$$\sharp\{\gamma \in \Pi : \alpha \leq \tau^\sharp(\gamma) \leq \alpha+\epsilon\} \geq \frac{e^{h(\alpha+\epsilon)}}{h(\alpha+\epsilon)}(1-\eta) - \frac{e^{h\alpha}}{h\alpha}(1+\eta)$$

$$> \frac{(1-\eta)e^{\alpha h}}{h(\alpha+\epsilon)}\Big[1 + \epsilon h - \frac{(\alpha+\epsilon)(1+\eta)}{\alpha(1-\eta)}\Big].$$

On the other hand, we have $\frac{1}{\alpha} \leq \frac{h}{3}$ and

$$4\eta \leq \frac{\epsilon h}{3(1+\epsilon)} \leq \frac{\epsilon h}{3(1+\frac{\epsilon}{\alpha})}.$$

Then

$$\frac{(\alpha+\epsilon)(1+\eta)}{\alpha(1-\eta)} = \Big(1+\frac{\epsilon}{\alpha}\Big)\Big(1+\frac{2\eta}{1-\eta}\Big) \leq (1+\frac{\epsilon}{\alpha})(1+4\eta) \leq 1 + \frac{2\epsilon h}{3}$$

and one deduces (3.1). ☐

Proof of Theorem 1.1. We start with the formula for the abscissa of $k-$summability $\sigma_k < 0$ in the case when $k \in \mathbb{N}$ is an integer established by Kuniyeda [Kun16, Theorem E]. More precisely, we have

$$\sigma_k = \limsup_{u \to \infty} \frac{\log |R^k(u)|}{u^k}, \tag{3.2}$$

where $R^k(u) = \sum_{\lambda_n > u} a_n(\lambda_n - u)^k$. We are interesting in the case $k = 1$. Let $\delta, -\gamma$ be the constants given in (1.8), (1.9), respectively. By Proposition 2.1, for $\eta_D(s)$ we have $\sigma_1 = -\infty$. We fix $\gamma_1 > 0$ so that

$-\gamma_1 < -\delta - \gamma - 1$. Then (3.2) implies that there exists $M = M(\gamma_1) > 1$ such that

$$|R(u)| = |\sum_{\lambda_n > u} a_n(\lambda_n - u)| \leq e^{-\gamma_1 u},\ \forall u \geq M.$$

Let

$$\lambda_{m_j} - \lambda_{m_j - 1} \geq Ce^{-\delta\lambda_{m_j}},\ \lambda_{m_j-2} \geq M,\ |\sum_{n \geq m_j} a_n| \geq e^{-\gamma\lambda_{m_j}}.$$

Obviously, for M large by using (3.1), we get $\lambda_{m_j} - \lambda_{m_j-1} < 1$.

Choose u_{m_j-1}, u_{m_j} so that $\lambda_{m_j-2} < u_{m_j-1} < \lambda_{m_j-1} < u_{m_j} < \lambda_{m_j}$ and write

$$R(u_{m_j-1}) - R(u_{m_j}) = a_{m_j-1}(\lambda_{m_j-1} - u_{m_j-1}) + (u_{m_j} - u_{m_j-1}) \sum_{\lambda_n > u_{m_j}} a_n.$$

We choose $\lambda_{m_j-1} - u_{m_j-1} = \epsilon_j \ll 1$ sufficiently small to arrange

$$|a_{m_j-1}|(\lambda_{m_j-1} - u_{m_j-1}) \leq e^{-\gamma_1\lambda_{m_j}}.$$

(Exploiting (1.4), we obtain an upper bound $|a_n| \leq e^{c\lambda_n}$, $\forall n$ with $c > 0$ independent of λ_n, but this is not necessary for the estimation above.) Next

$$u_{m_j} - u_{m_j-1} = (u_{m_j} - \lambda_{m_j}) + (\lambda_{m_j} - \lambda_{m_j-1}) + (\lambda_{m_j-1} - u_{m_j-1}).$$

Taking ϵ_j very close to 0, if it is necessary, we choose $u_{m_j} = \lambda_{m_j} - \epsilon_j$, and deduce

$$u_{m_j} - u_{m_j-1} = \lambda_{m_j} - \lambda_{m_j-1} \geq Ce^{-\delta\lambda_{m_j}}.$$

Then

$$Ce^{(-\delta-\gamma)\lambda_{m_j}} \leq (u_{m_j} - u_{m_j-1})|\sum_{n \geq m_j} a_n|$$
$$= |R(u_{m_j-1}) - R(u_{m_j}) - a_{m_j-1}(\lambda_{m_j-1} - u_{m_j-1})|$$
$$\leq e^{-\gamma_1 u_{m_j-1}} + e^{-\gamma_1 u_{m_j}} + e^{-\gamma_1\lambda_{m_j}}$$
$$\leq \Big(2e^{\gamma_1(\lambda_{m_j} - u_{m_j-1})} + 1\Big)e^{-\gamma_1\lambda_{m_j}}.$$

Since

$$\lambda_{m_j} - u_{m_j-1} = \lambda_{m_j} - \lambda_{m_j-1} + \epsilon_j < 3/2,$$

the above inequality yields

$$1 \leq \frac{1}{C}\Big(2e^{\frac{3}{2}\gamma_1} + 1\Big)e^{(-\gamma_1+\delta+\gamma)\lambda_{m_j}}$$

and we obtain a contradiction for $\lambda_{m_j} \to \infty$. This completes the proof.

Since (1.8) is always satisfied for suitable frequencies $\lambda_{m_j-1}, \lambda_{m_j}$ (see Section 4), exploiting (1.11), we may arrange the condition (1.9) for λ_{m_j} large enough. An application of Theorem 1.1 yields Corollary 1.1.

4. Intervals with clustering frequencies

We fix $\delta > h+2$ and $e^{-b} < \epsilon \ll 1/2$ and consider an interval $[b, b+1]$, $b \geq b_0(\epsilon)$. Let $\lambda_k \in [b+e^{-b}, b+1-e^{-b}]$. To examine the clustering around λ_k, we construct some sets. Introduce

$$J_\delta(\mu) = (\mu, \mu + \mathrm{e}^{-\delta b}).$$

If $\lambda_{k+1} \notin J_\delta(\lambda_k)$, we stop the construction on the right. If $\lambda_{k+1} \in J_\delta(\lambda_k)$, one considers $J_\delta(\lambda_{k+1})$. In the case $\lambda_{k+2} \notin J_\delta(\lambda_{k+1})$, we stop the construction. Otherwise, we continue with $J_\delta(\lambda_{k+2})$ up to the situation when $\lambda_{k+q+1} \notin J_\delta(\lambda_{k+q})$. It is clear that such q exists.We repeat the same construction moving on the left introducing

$$G_\delta(\mu) = (\mu - \mathrm{e}^{-\delta b}, \mu).$$

We stop when $\lambda_{k-p-1} \notin G_\delta(\lambda_{k-p})$. Set $I(\lambda_k, \delta) = [\lambda_{k-p}, \lambda_{k+q}]$. The integers p, q depend on λ_k, but we omit this in the notations below. Clearly, if we take another frequency $\lambda_{k'} \in I(\lambda_k, \delta)$, we obtain by the above construction the same interval. It is not excluded that $I(\lambda_k, \delta) = \{\lambda_k\}$. In the particular case, one has $q = p = 0$. The number of the frequencies in $I(\lambda_k, \delta)$ is bounded by $e^{(h+\epsilon)(b+1)}$ and

$$\lambda_{k+q} - \lambda_{k-p} \leq e^{(h-\delta+\epsilon)b+(h+\epsilon)} < e^{-b},\ b \geq b_0(\epsilon). \tag{4.1}$$

This estimate implies $\lambda_{k+q} < b+1,\ \lambda_{k-p} > b$, so $I(\lambda_k, \delta) \subset (b, b+1)$. By Lemma 3.1, the intervals without frequencies have lengths less than ϵ. Let $M(\epsilon, \delta, b)$ be the number of the sets

$$I(\lambda_k, \delta) \cup (\lambda_{k+q}, \lambda_{k+q+1}), e^{-\delta b} \leq \lambda_{k+q+1} - \lambda_{k+q} < \epsilon, \lambda_k \in [b+e^{-b}, b+1-e^{-b}].$$

Taking the union of such sets, we obtain

$$M(\epsilon, \delta, b)(\epsilon + e^{-b}) \geq 1 - 2\epsilon - 2e^{-b}.$$

For large b thus implies

$$M(\epsilon, \delta, b) > \frac{1 - 2\epsilon - 2e^{-b}}{\epsilon + e^{-b}} = \frac{1}{\epsilon} - 2 + \mathcal{O}_\epsilon(e^{-b}). \tag{4.2}$$

Hence we have at least $\left[\frac{1}{\epsilon}\right] - 2$ frequencies $\lambda_{m_j} \in [b+e^{-b}, b+1-e^{-b}]$ with

$$\lambda_{m_j-p_j} - \lambda_{m_j-p_j-1} > e^{-\delta \lambda_{m_j-p_j-1}},\ \lambda_{m_j+q_j+1} - \lambda_{m_j+q_j} > e^{-\delta \lambda_{m_j+q_j}}, \tag{4.3}$$

where $[a]$ denotes the entire part of a.

Now let $\gamma \gg 1$ be fixed. Given an interval $I(\lambda_k, \delta) \subset (b, b+1)$, we have 2 possibilities:

$$(i) \quad |\sum_{n \geq k-p} a_n| \geq e^{-\gamma \lambda_{k-p}}, \;\; (ii) \quad |\sum_{n \geq k-p} a_n| < e^{-\gamma \lambda_{k-p}}.$$

In the case (i) the conditions (1.8), (1.9) are satisfied for λ_{k-p-1} and λ_{k-p}. If one has (ii), and $|\sum_{n \geq k+q+1} a_n| < e^{-\gamma \lambda_{k+q+1}}$, by triangle inequality one deduces

$$|\sum_{n=k-p}^{k+q} a_n| \leq e^{-\gamma \lambda_{k-p}} + e^{-\gamma \lambda_{k+q+1}} < 2e^{-\gamma \lambda_{k-p}}.$$

Thus if (ii) holds, and $|\sum_{n=k-p}^{k+q} a_n| \geq 2e^{-\gamma \lambda_{k-p}}$ the conditions (1.8), (1.9) are satisfied for λ_{k+q} and λ_{k+q+1}. Taking into account (4.3) and applying Theorem 1.1, we obtain the following

Corollary 4.1. *Suppose $\sigma_c < 0$. Suppose that there exist constants $\delta > h+2, \gamma \gg 1$ and a sequence of intervals*

$$I(\lambda_{m_j}, \delta) = [\lambda_{m_j - p_j}, \lambda_{m_j + q_j}], \; \lambda_{m_j} \nearrow \infty$$

satisfying (4.3) such that

$$|\sum_{n=m_j-p_j}^{m_j+q_j} a_n| \geq 2e^{-\gamma \lambda_{m_j - p_j}}.$$

Then η_D cannot be prolonged as entire function.

It is important to increase the number of intervals included in $[b, b+1]$ satisfying (4.3). By using Lemma 3.1, we see that for $\epsilon \searrow 0$ we have $b_0(\epsilon) \nearrow \infty$ so a more precise asymptotics for the counting functions of the number of frequencies with remainder is necessary. Under some geometrical assumptions, it was proved (see [PS12, Theorem 4]) that we may replace ϵ by $e^{-\mu b}$ with small $0 < \mu < h$ and obtain a lower bound of

$$\sharp\{\gamma \in \Pi : \alpha \leq \tau^{\sharp}(\gamma) \leq \alpha + e^{-\mu b}\}.$$

These assumptions are satisfied for $d = 2$, while for $d \geq 3$ one make some restrictions. We refer to [PS12, Section 8] for precise results and more details. Under these assumptions we obtain $M(e^{-\mu b}, \delta, b) \sim e^{\mu b}$ as $b \to \infty$ so the number of intervals satisfying (4.3) increase exponentially as $b \to +\infty$. The issue is that the possibilities to satisfy the conditions of Theorem 1.1 increase exponentially, too.

To obtain a lower bound for $|a_n|$, $\forall n \geq n_0$, introduce the condition (L) There exist constants $c_1 > 0, c_2 > 0$, independent of n such that

$$|a_n| \geq c_1 e^{-c_2 \lambda_n},\ \forall n \geq n_0. \tag{4.4}$$

The condition (4.4) holds in the case when the lengths of primitive periodic rays $\gamma \in \Pi$ are rationally independent, because (1.7) will contain only one term and from (1.4) one deduces (4.4) with $c_1 = \min_{i \neq j} \text{dist}\,(D_i, D_j)$ and $c_2 = d_2/2$. This rational independence has been proved for generic domains (see [PS17, Theorem 6.2.3]). Then if (L) holds and

$$|\sum_{k \geq m} a_k| < e^{-\gamma \lambda_m},\ |\sum_{k \geq m+1} a_k| < e^{-\gamma \lambda_{m+1}} \tag{4.5}$$

with $\gamma > c_2 + 1$, one has $c_1 e^{-c_2 \lambda_m} \leq |a_m| < 2e^{-\gamma \lambda_m}$ which is impossible for large λ_m. Hence at least one of the estimates (4.5) does not hold. We may study also the existence of 3 consecutive frequencies which are exponentially separated from each other.

Corollary 4.2. *Assume (L) satisfied. Suppose that there exist constants $\delta > 0$, $C > 0$ and a sequence $\lambda_{m_j} \nearrow +\infty$ such that*

$$\lambda_{m_j} - \lambda_{m_j - 1} > Ce^{-\delta \lambda_{m_j}},\ \lambda_{m_{j+1}} - \lambda_{m_j} > Ce^{-\delta \lambda_{m_{j+1}}}. \tag{4.6}$$

Then η_D cannot be prolonged as entire function.

For the proof we exploit (4.6) and we arrange easily (1.9). Then we apply Theorem 1. We conjecture that for generic domains there exists a sequence $\{\lambda_{m_j}\}$ satisfying (4.6).

Going back to intervals $I(\lambda_k, \delta)$, notice that for λ_{k+q} one has also 2 possibilities:

$$(iii)\ \ |\sum_{n \geq k+q} a_n| \geq e^{-\gamma \lambda_{k+q}},\ \ (iv)\ \ |\sum_{n \geq k+q} a_n| < e^{-\gamma \lambda_{k+q}}.$$

Assuming (L) and $\gamma > c_2 + 1$, in the case (iv) the conditions (1.8), (1.9) are satisfied for λ_{k+q} and λ_{k+q+1}. Consequently, we obtain the following

Proposition 4.1. *Assume (L) satisfied and $\gamma > c_2 + 1$. Then for every interval $I(\lambda_k, \delta)$ we have 4 possibilities: $(i)-(iii), (i)-(iv), (ii)-(iii), (ii)-(iv)$. If (i) holds, or if we have $(ii)-(vi)$, we may find an interval $[\lambda_{k-p-1}, \lambda_{k-p}]$ or $[\lambda_{k+q}, \lambda_{k+q+1}]$ satisfying (1.8) and (1.9).*

A more fine analysis of the estimates of the sums $|\sum_{n=k_j-p_j}^{k_j+q_j} a_n|$ should imply more precise results.

References

[CP22] Yann Chaubet and Vesselin Petkov. Dynamical zeta functions for billiards. *https://doi.org/10.48550/arXiv.2201.00683*, 2022. Ann. Inst. Fourier, (Grenoble), to appear.

[DG16] Semyon Dyatlov and Colin Guillarmou. Pollicott–Ruelle resonances for open systems. *Annales Henri Poincaré*, 17(11):3089–3146, 2016.

[DS22] Andreas Defant and Ingo Schoolmann. Holomorphic functions of finite order generated by Dirichlet series. *Banach J. Math. Anal.*, 16(2):Paper No. 33, 65, 2022.

[HR64] G. H. Hardy and M. Riesz. *The general theory of Dirichlet's series*, volume No. 18 of *Cambridge Tracts in Mathematics and Mathematical Physics*. Stechert-Hafner, Inc., New York, 1964.

[Ika90a] Mitsuru Ikawa. On the distribution of poles of the scattering matrix for several convex bodies. In *Functional-analytic methods for partial differential equations (Tokyo, 1989)*, volume 1450 of *Lecture Notes in Math.*, pages 210–225. Springer, Berlin, 1990.

[Ika90b] Mitsuru Ikawa. Singular perturbation of symbolic flows and poles of the zeta functions. *Osaka J. Math.*, 27(2):281–300, 1990.

[JT25] Long Jin and Zhongkai Tao. Counting Pollicott-Ruelle resonances for Axiom A flows. *Comm. Math. Physics*, 406, (2), 2025. article 26.

[Kun16] M. Kuniyeda. Note on Perron's integral and summability-abscissae of Dirichlet's series. *Quart. J.*, 47:193–219, 1916.

[Pet99] Vesselin Petkov. Analytic singularities of the dynamical zeta function. *Nonlinearity*, 12(6):1663–1681, 1999.

[Pet25] Vesselin Petkov. On the number of poles of the dynamical zeta functions for billiard flows. *Discrete and Continuous Dynamical Systems*, 95(9):3174–3194, 2025.

[PP90] William Parry and Mark Pollicott. Zeta functions and the periodic orbit structure of hyperbolic dynamics. *Astérisque*, (187-188), 1990. 268 pp.

[PS12] Vesselin Petkov and Luchezar Stoyanov. Distribution of periods of closed trajectories in exponentially shrinking intervals. *Comm. Math. Phys.*, 310(3):675–704, 2012.

[PS17] Vesselin M. Petkov and Luchezar N. Stoyanov. *Geometry of the generalized geodesic flow and inverse spectral problems*. John Wiley & Sons, Ltd., Chichester, second edition, 2017.

[Sto01] Luchezar Stoyanov. Spectrum of the Ruelle operator and exponential decay of correlations for open billiard flows. *American Journal of Mathematics*, 123(4):715–759, 2001.

[Sto09] Luchezar Stoyanov. Scattering resonances for several small convex bodies and the Lax-Phillips conjecture. *Memoirs Amer. Math. Soc.*, 199(933), 2009. vi+76 pp.

[Sto12] Luchezar Stoyanov. Non-integrability of open billiard flows and Dolgopyat-type estimates. *Ergodic Theory and Dynamical Systems*, 32(1):295–313, 2012.

Université de Bordeaux, Institut de Mathématiques de Bordeaux, 351, Cours de la Libération, 33405 Talence, France

Email address: `petkov@math.u-bordeaux.fr`