ASYMPTOTIC OF THE DISSIPATIVE EIGENVALUES OF MAXWELL'S EQUATIONS

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Abstract. Let $\Omega = \mathbb{R}^3 \setminus \bar{K}$, where K is an open bounded domain with smooth boundary Γ. Let $V(t) = e^{tG_b}$, $t \geq 0$, be the semigroup related to Maxwell's equations in Ω with dissipative boundary condition $\nu \wedge (\nu \wedge E) + \gamma(x)(\nu \wedge H) = 0, \gamma(x) > 0, \forall x \in \Gamma$. We study the case when $\gamma(x) \neq 1$, $\forall x \in \Gamma$, and we establish a Weyl formula for the counting function of the eigenvalues of G_b in a polynomial neighbourhood of the negative real axis.

Keywords: Dissipative boundary conditions, Dissipative eigenvalues, Weyl formula

1. Introduction

Let $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$ be an open connected domain and let $\Omega = \mathbb{R}^3 \setminus \bar{K}$ be connected domain with C^∞ smooth boundary Γ. Consider the boundary problem

$$\begin{cases} \partial_t E = \operatorname{curl} H, \qquad \partial_t H = -\operatorname{curl} E \quad \text{in} \quad \mathbb{R}_t^+ \times \Omega, \\ \nu \wedge (\nu \wedge E) + \gamma(x)(\nu \wedge H) = 0 \quad \text{on} \quad \mathbb{R}_t^+ \times \Gamma, \\ E(0,x) = E_0(x), \qquad H(0,x) = H_0(x) \end{cases} \tag{1.1}$$

with initial data $F_0 = (E_0, H_0) \in \mathcal{H} = L^2(\Omega; \mathbb{C}^3) \times L^2(\Omega; \mathbb{C}^3)$. Here $\nu(x)$ is the unit outward normal at $x \in \Gamma$ pointing into Ω, and $\gamma(x) \in C^\infty(\Gamma)$ satisfies $\gamma(x) > 0$ for all $x \in \Gamma$. The solution of the problem (1.1) is described by a contraction semigroup

$$(E, H)(t) = V(t)F_0 = e^{tG_b}F_0, \ t \geq 0,$$

where the generator G_b is the operator

$$G = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}$$

with domain $D(G_b) \subset \mathcal{H}$ which is the closure in the graph norm $|||u||| = (\|u\|^2 + \|Gu\|^2)^{1/2}$ of functions $u = (v, w) \in C^\infty_{(0)}(\mathbb{R}^3; \mathbb{C}^3) \times C^\infty_{(0)}(\mathbb{R}^3; \mathbb{C}^3)$ satisfying the boundary condition $\nu \wedge (\nu \wedge v) + \gamma(\nu \wedge w) = 0$ on Γ.

In [1] it was proved that the spectrum of G_b in the open half plan $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is formed by isolated eigenvalues with finite multiplicities. Notice that if $G_b f = \lambda f$ with $\operatorname{Re} \lambda < 0$, the solution $u(t,x) = V(t)f = e^{\lambda t} f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are important for the scattering problems (see [1], [2], [8], [9]). In particular, the eigenvalues λ with $\operatorname{Re} \lambda \to -\infty$ imply a very fast decay of the corresponding solutions. Let $\sigma_p(G_b)$ be the point spectrum of G_b. Concerning the scattering problems, we mention three properties related to the existence of eigenvalues of G_b. First, let $W_\pm$ be the wave operators

$$W_- f = \lim_{t\to+\infty} V(t)JU_0(-t)f,\ \ W_+ f = \lim_{t\to+\infty} V^*(t)JU_0(t)f,$$

where $U_0(t)$ is the unitary group in $\mathcal{H}_0 = L^2(\mathbb{R}^3;\mathbb{C}^3)\times L^2(\mathbb{R}^3;\mathbb{C}^3)$ related to the Cauchy problem for Maxwell system, $J : \mathcal{H}_0 \to \mathcal{H}$ is a projection and $V^*(t) = e^{tG^*}$ is the adjoint semigroup (see [6], [7]). If $\sigma_p(G_p)\cap\{z : \text{Re } z < 0\} \neq \emptyset$, the wave operators $W_\pm$ are not complete (see [1]), that is Ran $W_+ \neq$ Ran W_- and we cannot define the scattering operator by $S = W_+^{-1}\circ W_-$. We may define the scattering operator by using another evolution operator (see [6], [7]). Second, in a suitable representation the scattering operator becomes an operator valued function $S(z) : L^2(\mathbb{S}^2;\mathbb{C}^3) \to L^2(\mathbb{S}^2;\mathbb{C}^3)$, $z \in \mathbb{C}$, and Lax and Phillips (see [6]) proved that the existence of z_0, Im $z_0 > 0$, for which the kernel of $S(z_0)$ is not trivial implies $\mathbf{i}z_0 \in \sigma_p(G_b)$. The existence of such z_0 leads to problems in inverse scattering. Third, for dissipative systems Lax and Phillips developed a scattering theory in [6] and they introduced the representation of the energy space $\mathcal{H}$ as a direct sum $\mathcal{H} = D_a^- \oplus K_a \oplus D_a^+$. A function f is called *outgoing* (resp. *incoming*) if its component in D_a^- (resp. D_a^+) is vanishing. If f is an eigenfunction with eigenvalue $\lambda \in \sigma_p(G_b)$, Re $\lambda < 0$, it is easy to see that f is incoming and, moreover, $V(t)f$ remains incoming for all $t \geq 0$. On the other hand, $V^*(t)f$ is not converging to 0 as $t \to +\infty$. In fact, assuming $V^*(t)f \to 0$ for $t \to +\infty$, by the result in [1] one deduces that f must be disappearing, that is there exists $T > 0$ such that $V(t)f = 0$ for $t \geq T$ which is impossible for an eigenfunction.

The existence of infinite number eigenvalues of G_b presents an interest for applications. However to our best knowledge this problem has been studied only for the ball $B_3 = \{x \in \mathbb{R}^3,\ |x| < 1\}$ assuming γ constant (see [2]). It was proved in [2] that for $\gamma = 1$ there are no eigenvalues in $\{z \in \mathbb{C} : \text{Re } z < 0\}$, while for $\gamma \neq 1$ there is always an infinite number of negative real eigenvalues λ_j and with exception of one they satisfy the estimate

$$\lambda_j \ \leq\ -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}} = -c_0\,, \tag{1.2}$$

where $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$. On the other hand, a Weyl formula for the counting function of the negative eigenvalues of G_b for $K = B_3$ and $\gamma \neq 1$ constant has been established in [3].

The distribution of the eigenvalues of G_b in the complex plane has been studied in [2] and it was established that if $\gamma(x) \neq 1, \forall x \in \Gamma$, then for every $\epsilon > 0$ and every $M \in \mathbb{N}$, the eigenvalues lie in $\Lambda_\epsilon \cup \mathcal{R}_M$, where

$$\Lambda_\epsilon = \{z \in \mathbb{C} :\ |\text{Re } z| \leq C_\epsilon(1 + |\text{Im } z|^{1/2+\epsilon}),\ \text{Re } z < 0\},$$

$$\mathcal{R}_M = \{z \in \mathbb{C} :\ |\text{Im } z| \leq C_M(1 + |\text{Re } z|)^{-M},\ \text{Re } z < 0\}.$$

An eigenvalue $\lambda_j \in \sigma_p(G_b)\cap\{z :\ \text{Re}\, z < 0\}$ has (algebraic) multiplicity given by

$$\text{mult}(\lambda_j) = \text{tr}\, \frac{1}{2\pi\mathbf{i}} \int_{|\lambda_j - z| = \epsilon} (z - G_b)^{-1} dz,$$

where $0 < \epsilon \ll 1$. Introduce the set

$$\Lambda := \{z \in \mathbb{C} :\ |\text{Im } z| \leq C_2(1 + |\text{Re } z|)^{-2},\ \text{Re } z \leq -C_0 \leq -1\}.$$

We choose $C_0 \geq 2C_2$ and $\mathcal{R}_M \subset \Lambda$, $M \geq 2$ modulo a compact set containing a finite number eigenvalues.

Throughout this paper we assume that either $0 < \gamma(x) < 1, \forall x \in \Gamma$ or $1 < \gamma(x), \forall x \in \Gamma$. Our purpose is to prove the following

Theorem 1.1. *Let $\gamma(x) \neq 1$, $\forall x \in \Gamma$, and let $\gamma_0(x) = \max\{\gamma(x), \frac{1}{\gamma(x)}\}$. Then the counting function of the eigenvalues in Λ counted with their multiplicities for $r \to \infty$ has the asymptotic*

$$\sharp\{\lambda_j \in \sigma_p(G_b) \cap \Lambda : |\lambda_j| \leq r, r \geq C_{\gamma_0}\} = \frac{1}{4\pi}\Big(\int_\Gamma (\gamma_0^2(x) - 1) dS_x\Big) r^2 + \mathcal{O}_{\gamma_0}(r). \quad (1.3)$$

The proof of the above theorem follows the approach in [10] and [9]. In comparison with [9], we will discuss briefly some difficulties and new points. For the analysis of $\sigma_p(G_b)$ we prove in Section 2 a trace formula involving the operator $C(\lambda)f = \mathcal{N}(\lambda)f + \frac{1}{\gamma(x)}(\nu \wedge f)$, where $\mathcal{N}(\lambda)f = \nu \wedge H|_\Gamma$ and (E, H) is the solution of the problem (2.2) with $U_1 = U_2 = 0$. Setting $\lambda = -\frac{1}{\tilde{h}}$, $\tilde{h} = h(1 + \mathbf{i}t)$ with $0 < h \leq h_0$, $t \in \mathbb{R}$, $|t| \leq h^2$, we are going to study the semiclassical problem (2.8) with $z = -\mathbf{i}(1 + \mathbf{i}t)^{-1}$. In a recent work Vodev [12] constructed a semiclassical parametrix for this problem assuming $\theta = |\mathrm{Im}\, z| \geq h^{2/5 - \epsilon}$, $0 < \epsilon \ll 1$. Moreover, in [12] an approximation for $\mathcal{N}(-\frac{1}{\tilde{h}})$ has been obtained by a semiclassical pseudo-differential matrix valued operator.

We deal with the elliptic case, where $\theta \geq 1 - h^2$. In this case according to the results in [12], an approximation $T_N(h, z)(\nu \wedge f)$ of $\mathcal{N}(-\frac{1}{\tilde{h}})f$ can be constructed with a remainder having norm $\mathcal{O}(h^N)\|f\|^2$ choosing $N \in \mathbb{N}$ very large. The principal symbol of $T_N(h, z)$ has matrix symbol $m = \frac{1}{z}\Big(\rho I + \frac{\mathcal{B}}{\rho}\Big)$ (see Section 3), where $\mathcal{B}$ is a symmetric matrix, $\rho = \sqrt{z^2 - r_0}$ and $r_0(x', \xi')$ is the principal symbol of Laplace Beltrami operator $-h^2\Delta|_\Gamma$. To approximate $C(-\frac{1}{\tilde{h}})$, we use the self-adjoint operator $\mathcal{P}(h) = -T_N(h, -\mathbf{i}) - \frac{1}{\gamma(x)}$. In the case $\gamma_0(x) = \frac{1}{\gamma(x)} > 1$, $\forall x \in \Gamma$, there exist values of h for which $\mathcal{P}(h)$ is not invertible. The semiclassical analysis of $\mathcal{P}(h)$ is related to the eigenvalues of the principal symbol $-m - \gamma_0$ which has a double eigenvalue $\sqrt{1 + r_0} - \gamma_0$ and an eigenvalue $r(h) = (1 + r_0)^{-1/2} - \gamma_0$. The symbol $r(h)$ is elliptic but $\lim_{|\xi'| \to \infty} r(h) = -\gamma_0$ and this leads to problems in the semiclassical analysis of the spectrum of $\mathcal{P}(h)$ (see Section 12 in [4] and hypothesis (H2)). To overcome this difficulty, we introduce a global diagonalisation of m with a unitary matrix U and write $(Op_h(U))^{-1}\mathcal{P}(h)Op_h(U)$ in a block matrix form (see Section 4). We study the eigenvalues $\mu_k(h)$ of a self-adjoint operator $Q(h)$ and show that the invertibility of $Q(h)$ implies that of $\mathcal{P}(h)$. This approach is more convenient that the investigation of $\det \mathcal{P}(h)$. If h_k, $0 < h_k \leq h_0$, is such that $\mu_k(h_k) = 0$, then $Q(h_k)$ is not invertible and in this direction our analysis is very similar to that in [10] and [9]. The next step is to express the trace formula involving $\mathcal{P}(h)^{-1}$ with a trace one involving $Q(h)^{-1}$ (see Proposition 5.2). Finally, the problem is reduced to the count of the negative eigenvalues of $Q(1/r)$ for $1/r < h_0$. This strategy is not working if $\gamma_0(x) = \gamma(x) > 1$. To cover this case, we consider the problem (3.10) and the operator $\mathcal{N}_1(-\frac{1}{\tilde{h}})$ related to the solution of it. Then we introduce the operators $C_1(h), \mathcal{P}_1(h), Q_1(h)$. We study the eigenvalues of the self-adjoint operator $Q_1(h)$ and repeat the analysis in the case $0 < \gamma(x) < 1$. The eigenvalues of the semiclassical principal symbols of both operators $Q(h)$ and $Q_1(h)$ are $\sqrt{1+r_0} - \gamma_0$.

The argument of our paper with technical complications can be applied to study the non homogenous Maxwell equations (see (2.1) for the notation)

$$\begin{cases} \operatorname{curl} E = -\lambda\mu(x)H, \ x \in \Omega, \\ \operatorname{curl} H = \lambda\epsilon(x)E, \ \ x \in \Omega, \\ \frac{1}{\gamma(x)}(\nu \wedge (\nu \wedge E)) + (\nu \wedge H) = 0 \text{ for } x \in \Gamma, \\ (E, H) : \mathbf{i}\lambda - \text{outgoing}. \end{cases} \tag{1.4}$$

Here $\epsilon(x) > 0$, $\mu(x) > 0$ are scalar valued functions in $C^\infty(\bar{\Omega})$ which are equal to constants ϵ_0, μ_0 for $|x| \geq c_0 > a$. For this purpose it is necessary to generalise the results for eigenvalues free regions in [2] and to apply the construction in [12] concerning the non homogeneous case.

The paper is organised as follows. In Section 2 in the case $0 < \gamma(x) < 1, \forall x \in \Gamma$ we introduce the operators $\mathcal{N}(\lambda), C(\lambda), P(\lambda)$ and prove a trace formula (see Proposition 2.1). Similarly, in the case $\gamma(x) > 1, \forall x \in \Gamma$, the operators $\mathcal{N}_1(\lambda), C_1(\lambda), \mathcal{P}_1(\lambda)$ are introduced. In Section 3 we collect some facts concerning the construction of a semiclassical parametrix for the problems (2.8), (2.10) build in [12]. Setting $\lambda = -\frac{1}{\tilde{h}}$, $\tilde{h} = h(1 + \mathbf{i}t)$, $0 < h \leq h_0$, we treat the case $z = \mathbf{i}h\lambda = -\mathbf{i}(1+\mathbf{i}t)^{-1}$ with $|t| \leq h^2$ and this implies some simplifications. The self-adjoint operators $Q(h), Q_1(h)$ and their eigenvalues $\mu_k(h)$ are examined in Section 4. Finally, in Section 5 we compare the trace formulas involving $\mathcal{P}(h)$ and $\mathcal{C}(h)$ and show that they differ by negligible terms. The proof of Theorem 1.1 is completed by the asymptotic of the negative eigenvalues of $Q(1/r), Q_1(1/r)$.

2. Trace formula for Maxwell's equations

An eigenfunction $D(G_b) \ni (E, H) \neq 0$ of G_b with eigenvalue $\lambda \in \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ satisfies

$$\begin{cases} \operatorname{curl} E = -\lambda H, \ x \in \Omega, \\ \operatorname{curl} H = \lambda E, \ \ x \in \Omega, \\ \frac{1}{\gamma(x)}(\nu \wedge (\nu \wedge E)) + (\nu \wedge H) = 0 \text{ for } x \in \Gamma, \\ (E, H) : \mathbf{i}\lambda - \text{outgoing}. \end{cases} \tag{2.1}$$

The $\mathbf{i}\lambda$- outgoing condition means that every component of $E = (E_1, E_2, E_3)$ and $H = (H_1, H_2, H_3)$ satisfies the $\mathbf{i}\lambda$-outgoing condition for the equation $(\Delta - \lambda^2)u = 0$, that is

$$\frac{d}{dr}\Big(E_j(r\omega)\Big) - \lambda E_j(r\omega) = \mathcal{O}\Big(\frac{1}{r^2}\Big), \ j = 1, 2, 3, \ r \to \infty$$

uniformly with respect to $\omega \in S^2$ and the same condition holds for H_j, $j = 1, 2, 3$. This condition can be written in several equivalent forms and for Maxwell's equation it is known also as Silver-Müller radiation condition (see Remark 3.31 in [5]). Notice that we can present E and H by integrals involving the outgoing resolvent $(\Delta_0 - \lambda^2)^{-1}$ of the free Laplacian in $\mathbb{R}^3$ with kernel $R_0(x, y; \lambda) = \frac{e^{\lambda|x-y|}}{4\pi|x-y|}$, $x \neq y$, and if (E, H) satisfy the $\mathbf{i}\lambda$- outgoing condition, we can apply the Green formula

$$\int_\Omega \operatorname{div}(A \wedge B) dx = \int_\Omega (\langle B, \operatorname{curl} A\rangle - \langle A, \operatorname{curl} B\rangle) dx = \int_\Gamma \langle (\nu \wedge B), A\rangle dS_x,$$

where $\langle ., . \rangle$ is the scalar product in $\mathbb{C}^3$.

First we treat the case $0 < \gamma(x) < 1, \forall x \in \Gamma$. The case $\gamma(x) > 1, \forall x \in \Gamma$ will be discussed at the end of this section. Introduce the spaces

$$\mathcal{H}_s^t(\Gamma) = \{f \in H^s(\Gamma; \mathbb{C}^3) : \langle \nu(x), f(x) \rangle = 0\}$$

and consider the boundary problem

$$\begin{cases} \operatorname{curl} E = -\lambda H + U_1, \ x \in \Omega, \\ \operatorname{curl} H = \lambda E + U_2, \ x \in \Omega, \\ \nu \wedge E = f \text{ for } x \in \Gamma, \\ (E, H) : \mathbf{i}\lambda - \text{outgoing} \end{cases} \tag{2.2}$$

with $U_1, U_2 \in L^2(\Omega; \mathbb{C}^3)$, $\operatorname{div} U_1, \operatorname{div} U_2 \in L^2(\Omega)$, $f \in \mathcal{H}_1^t(\Gamma)$. Consider the operator

$$\mathcal{N}(\lambda) : \mathcal{H}_1^t(\Gamma) \ni f \longrightarrow \nu \wedge H|_\Gamma \in \mathcal{H}_0^t(\Gamma),$$

(E, H) being the solution of (2.2) with $U_1 = U_2 = 0$. According to Theorem 3.1 in [12], this operator is well defined and it plays the role of the Dirichlet-to-Neumann operator for the Helmoltz equation $(\Delta - \lambda^2)u = 0$. By $\mathcal{N}(\lambda)$, we write the boundary condition in (2.1) as follows

$$C(\lambda)f := \mathcal{N}(\lambda)f + \frac{1}{\gamma(x)}(\nu \wedge f) = 0, \ \nu \wedge E|_\Gamma = f \in \mathcal{H}_1^t(\Gamma). \tag{2.3}$$

Introduce the operator $P(\lambda)(f) := \mathcal{N}(\lambda)(\nu \wedge f)$, that is $\mathcal{N}(\lambda)f = -P(\lambda)(\nu \wedge f)$. Therefore, since $\gamma_0(x) = \frac{1}{\gamma(x)}$ the condition (2.3) becomes

$$\tilde{C}(\lambda)g := P(\lambda)g - \gamma_0(x)g = 0, \ g = \nu \wedge f = -E_{tan}|_\Gamma \tag{2.4}$$

and $P(\lambda) : \mathcal{H}_1^t(\Gamma) \longrightarrow \mathcal{H}_0^t(\Gamma)$. For $\lambda \in \mathbb{R}^-$, it is easy to see that the operator $P(\lambda)$ is self-adjoint in $\mathcal{H}_0^t$. To do this, we must prove that for $u, v \in \mathcal{H}_1^t(\Gamma)$ we have

$$\begin{aligned} -(P(\lambda)(\nu \wedge u), \nu \wedge v) &= (\mathcal{N}(\lambda)u, \nu \wedge v) \\ = (\nu \wedge u, \mathcal{N}(\lambda)v) &= -(\nu \wedge u, P(\lambda)(\nu \wedge v)), \end{aligned} \tag{2.5}$$

where $(., .)$ is the scalar product in $\mathcal{H}_0^t(\Gamma)$. Let (E, H) (resp. (X, Y)) be the solution of the problem (2.2) with $U_1 = U_2 = 0$ and f replaced by u (resp. v). By applying the Green formula, we get

$$\lambda \int_\Omega (\langle E, X \rangle + \langle Y, H \rangle) dx = \int_\Omega \langle E, \operatorname{curl} Y \rangle dx - \int_\Omega \langle Y, \operatorname{curl} E \rangle dx$$

$$= -\int_\Gamma \langle \nu \wedge Y, E \rangle = \int_\Gamma \overline{\langle Y, u \rangle} = \int_\Gamma \overline{\langle \nu \wedge Y, \nu \wedge u \rangle} = (\nu \wedge u, \mathcal{N}(\lambda)v).$$

Similarly,

$$\lambda \int_\Omega (\langle X, E \rangle + \langle H, Y \rangle) dx = \int_\Omega \langle X, \operatorname{curl} H \rangle dx - \int_\Omega \langle H, \operatorname{curl} X \rangle dx$$

$$= -\int_\Gamma \langle \nu \wedge H, X \rangle = \int_\Gamma \overline{\langle H, v \rangle} = \int_\Gamma \overline{\langle \nu \wedge H, \nu \wedge v \rangle} = \overline{(\mathcal{N}(\lambda)u, \nu \wedge v)}$$

and for real λ we obtain (2.5).

Let $(E, H) = R_C(\lambda)(U_1, U_2)$ be the solution of (2.2) with $f = 0$. Then $R_C(\lambda) = (G_C - \lambda)^{-1}$, where G_C is the operator G with boundary condition $\nu \wedge E|_\Gamma = 0$ and domain $D(G_C) \subset \mathcal{H}$. The operator $\mathbf{i}G_C$ is self-adjoint and $(G_C - \lambda)^{-1}$ is analytic operator valued function for $\text{Re}\,\lambda < 0$. On the other hand, it is easy to express $\mathcal{N}(\lambda)$ by $R_C(\lambda)$. Given $f \in \mathcal{H}_1^t(\Gamma)$, let $e_0(f) \in H^{3/2}(\Omega; \mathbb{C}^3)$ be an extension of $-(\nu \wedge f)$ with compact support. Consider

$$\begin{pmatrix} u \\ v \end{pmatrix} = -R_C(\lambda)\Big((G - \lambda)\begin{pmatrix} e_0(f) \\ 0 \end{pmatrix}\Big) + \begin{pmatrix} e_0(f) \\ 0 \end{pmatrix}.$$

Then (u, v) satisfies (2.2) with $U_1 = U_2 = 0$ and $\mathcal{N}(\lambda)f = \nu \wedge v|_\Gamma$ implies that $\mathcal{N}(\lambda)$ is analytic for $\text{Re}\,\lambda < 0$. Consequently, $C(\lambda) : H_1^t(\Gamma) \to H_0^t(\Gamma)$ is also analytic for $\text{Re}\,\lambda < 0$. On the other hand, for $\text{Re}\,\lambda < 0$ the operator $\mathcal{N}(\lambda)$ is invertible. Indeed, if $\mathcal{N}(\lambda)f = 0$, let (E, H) be a solution of the problem

$$\begin{cases} \text{curl}\, E = -\lambda H, \ x \in \Omega, \\ \text{curl}\, H = \lambda E, \ x \in \Omega, \\ \nu \wedge H = 0 \text{ for } x \in \Gamma, \\ (E, H) : \mathbf{i}\lambda - \text{outgoing}. \end{cases} \tag{2.6}$$

By Green formula one gets

$$\bar{\lambda}\int_\Omega (|E|^2 + |H|^2)dx = \int_\Omega \langle E, \text{curl}\, H\rangle dx - \int_\Omega \langle H, \text{curl}\, E\rangle dx = -\int_\Gamma \langle \nu \wedge H, E\rangle = 0.$$

This implies $E = H = 0$, hence $f = 0$. Thus we conclude that for $\text{Re}\,\lambda < 0$ the operator $\mathcal{N}(\lambda)^{-1}$ is analytic and

$$C(\lambda)f = \mathcal{N}(\lambda)\Big(I + \mathcal{N}(\lambda)^{-1}\gamma_0(x)i_\nu\Big)f.$$

Here $i_\nu(x)$ is a (3×3) matrix such that $i_\nu(x)f = \nu(x) \wedge f$. The operator $\mathcal{N}(\lambda)^{-1} : H_0^t(\Gamma) \to H_1^t(\Gamma)$ is compact and by the analytic Fredholm theorem one deduces that

$$C(\lambda)^{-1} = \Big(I + \mathcal{N}(\lambda)^{-1}\gamma_0(x)i_\nu\Big)^{-1}\mathcal{N}(\lambda)^{-1}$$

is a meromorphic operator valued function.

To establish a trace formula involving $(G_b - \lambda)^{-1}$, consider $(G_b - \lambda)(u, v) = (F_1, F_2) = X,\ (u, v) \in D(G_b)$. Then

$$\begin{cases} \text{curl}\, u = -\lambda v + F_2, \ x \in \Omega, \\ \text{curl}\, v = \lambda u + F_1, \ x \in \Omega \end{cases}$$

and $(u, v) = R_C(\lambda)X + K(\lambda)f = (G_b - \lambda)^{-1}X$, where $K(\lambda)f$ is solution of (2.2) with $U_1 = U_2 = 0$. Let $R_C(\lambda)X = ((R_C(\lambda)X)_1, (R_C(\lambda)X)_2)$. Notice that for $\text{Re}\,\lambda < 0$, $(R_C(\lambda)X)_j,\ j = 1, 2$, are analytic vector valued functions. To satisfy the boundary condition, we must have

$$\gamma_0(x)(\nu \wedge f) + \Big(\mathcal{N}(\lambda)(f) + \nu \wedge (R_C(\lambda)X)_2|_\Gamma\Big) = 0,$$

hence

$$f = -C(\lambda)^{-1}\Big(\nu \wedge (R_C(\lambda)X)_2|_\Gamma\Big),$$

provided that $C(\lambda)^{-1}$ exists.

Assuming that $C(\lambda)^{-1}$ has no poles on a closed positively oriented curve $\delta \subset \{z \in \mathbb{C} :\ \text{Re}\, z < 0\}$, we apply Lemma 2.2 in [10] and exploit the cyclicity of the trace to conclude that the operators

$$\mathcal{H} \ni X \longrightarrow -\int_\delta K(\mu)\Big(C(\lambda)^{-1}\Big(\nu \wedge (R_C(\lambda)X)_2|_\Gamma\Big)\Big)d\lambda \in \mathcal{H}$$

and

$$H_1^1(\Gamma) \ni w \longrightarrow -\int_\delta C(\lambda)^{-1}\Big(\nu \wedge \Big(R_C(\lambda)(K(\lambda)(w))\Big)_2\big|_\Gamma\Big)d\lambda \in H_1^t(\Gamma)$$

have the same traces. On the other hand,

$$\begin{cases}(G-\lambda)\frac{\partial K(\lambda)(w)}{\partial\lambda} = K(\lambda)(w),\\ \nu \wedge \Big(\frac{\partial K(\lambda)(w)}{\partial \lambda}\Big)_1\big|_\Gamma = 0.\end{cases}.$$

This implies $\Big(R_C(\lambda)(K(\lambda)(w))\Big)_2\big|_\Gamma = \Big(\frac{\partial K(\lambda)(w)}{\partial\lambda}\Big)_2\big|_\Gamma$ and

$$\nu \wedge \Big(\frac{\partial K(\lambda)(w)}{\partial\lambda}\Big)_2\big|_\Gamma = \frac{\partial \mathcal{N}(\lambda)w}{\partial\lambda} = \frac{\partial C(\lambda)(w)}{\partial\lambda}.$$

The integrals involving the analytic terms $(R_C(\lambda)X)_j,\ j = 1,2$, vanish and we obtain the following

Proposition 2.1. *Let $0 < \gamma(x) < 1,\ \forall x \in \Gamma$ and let $\delta \subset \{z \in \mathbb{C} :\ \operatorname{Re} z < 0\}$ be a closed positively oriented curve without self intersections such that $C(\lambda)^{-1}$ has no poles on δ. Then*

$$\operatorname{tr}_{\mathcal{H}} \frac{1}{2\pi\mathbf{i}}\int_\delta (\lambda - G_b)^{-1}d\lambda = \operatorname{tr}_{\mathcal{H}_1^t(\Gamma)}\frac{1}{2\pi\mathbf{i}}\int_\delta C(\lambda)^{-1}\frac{dC(\lambda)}{d\lambda}d\lambda. \tag{2.7}$$

The left hand side of (2.7) is equal to the number of the eigenvalues of G_b in the domain bounded by δ counted with their multiplicities. Set $\lambda = -\frac{1}{\tilde h}$ with $0 < \operatorname{Re}\tilde h \ll 1$. For $\lambda \in \Lambda$ we have $\tilde h \in L$, where

$$L := \{\tilde h \in \mathbb{C} :\ |\operatorname{Im}\tilde h| \le C_1|\tilde h|^4,\ |\tilde h| \le C_0^{-1},\ \operatorname{Re}\tilde h > 0\}.$$

Write $\tilde h = h(1 + \mathbf{i}t),\ 0 < h \le h_0 \le C_0^{-1},\ t \in \mathbb{R}$. Then for $\tilde h \in L$, it is easy to see that $|t| \le h^2$ for $\tilde h \in L$ and the problem (2.2) with $U_1 = U_2 = 0$ becomes

$$\begin{cases}-\mathbf{i}h \operatorname{curl} E = zH,\ x \in \Omega,\\ -\mathbf{i}h \operatorname{curl} H = -zE,\ x \in \Omega,\\ \nu \wedge E = f,\ x \in \Gamma,\\ (E,H) - \text{outgoing}\end{cases} \tag{2.8}$$

with $-\mathbf{i}z = h\lambda,\ z = -\mathbf{i}(1+\mathbf{i}t)^{-1}$. We introduce the operator $\mathcal{C}(\tilde h) = C(-\tilde h^{-1})$ and the trace formula is transformed in

$$\operatorname{tr}_{\mathcal{H}} \frac{1}{2\pi\mathbf{i}}\int_\delta (\lambda - G_b)^{-1}d\lambda = \operatorname{tr}_{\mathcal{H}_1^t(\gamma)}\frac{1}{2\pi\mathbf{i}}\int_{\tilde\delta} \mathcal{C}(\tilde h)^{-1}\frac{d\mathcal{C}(\tilde h)}{d\tilde h}d\tilde h, \tag{2.9}$$

where $\tilde\delta = \{z \in \mathbb{C} :\ z = -\frac{1}{w},\ w \in \delta\}$.

To deal with the case $\gamma(x) > 1,\ \forall x \in \Gamma$, we write the boundary condition in (1.1) in the form

$$-(\nu \wedge E) + \gamma_0(x)(\nu \wedge (\nu \wedge H)) = 0,\ x \in \Gamma.$$

Consider the boundary problem

$$\begin{cases}\operatorname{curl} E = -\lambda H,\ x \in \Omega,\\ \operatorname{curl} H = \lambda E,\ x \in \Omega,\\ \nu \wedge H = f \text{ for } x \in \Gamma,\\ (E,H) : \mathbf{i}\lambda - \text{outgoing}\end{cases} \tag{2.10}$$

and introduce the operator

$$\mathcal{N}_1(\lambda) : \mathcal{H}_1^t(\Gamma) \ni f \longrightarrow \nu \wedge E|_\Gamma \in \mathcal{H}_0^t(\Gamma),$$

where (E, H) is the solution of (2.10). The above boundary condition becomes

$$C_1(\lambda) := \mathcal{N}_1(\lambda) f - \gamma_0(x)(\nu \wedge f) = 0,\ f = \nu \wedge H|_\Gamma.$$

Now we introduce the operator $P_1(\lambda) f = -\mathcal{N}_1(\lambda)(\nu \wedge f)$ and write the boundary condition as

$$\tilde{C}_1(\lambda) f := P_1(\lambda) g - \gamma_0(x) g = 0,\ x \in \Gamma,\ g = -H_{tan}|_\Gamma. \tag{2.11}$$

Comparing (2.11) with (2.4), we see that both boundary conditions are writen by $\gamma_0(x)$. Clearly, we may repeat the above argument and obtain a trace formula involving $C_1(\lambda)^{-1}$ and $\frac{d}{d\lambda} C_1(\lambda)$.

3. Semiclassical parametrix in the elliptic region

In this section we will collect some results in [12] concerning the construction of a semi-classical parametrix of the problem (2.8) and we refer to this work for more details. Let $\theta = |\text{Im } z| = \frac{1}{1+t^2} \leq 1$. Then the condition $\theta > h^{2/5-\epsilon}$, $0 < \epsilon \ll 1$ in [12] is trivially satisfied for small h_0. Moreover, $\theta \geq 1 - t^2 \geq 1 - h_0^2$ so θ has lower bound independent of h. This simplifies the construction in [12]. In the exposition we will use h-pseudo-differential operators and we refer to [4] for more details. Let (x_1, x') be local geodesic coordinates in a small neighbourhood $\mathcal{U} \subset \mathbb{R}^3$ of $y_0 \in \Gamma$. We set $x_1 = \text{dist}(y, \Gamma)$, $x' = s^{-1}(y)$, where $x' = (x_2, x_3)$ are local coordinates in a neighborhood $\mathcal{U}_0 \subset \mathbb{R}^2$ of $(0,\ 0)$ and $s : \mathcal{U}_0 \to \mathcal{U} \cap \Gamma$ is a diffeomorphism. Set $\nu(x') = \nu(s(x')) = (\nu_1(x'), \nu_2(x'), \nu_3(x'))$. Then $y = s(x') + x_1 \nu(x')$ and (see Section 2 in [2] and Section 2 in [12])

$$\frac{\partial}{\partial y_j} = \nu_j(x') \frac{\partial}{\partial x_1} + \sum_{k=2}^{3} \alpha_{j,k}(x) \frac{\partial}{\partial x_k},\ j = 1, 2, 3.$$

The functions $\alpha_{j,k}(x)$ are determined as follows. Let $\zeta_1 = (1, 0, 0), \zeta_2 = (0, 1, 0), \zeta_3 = (0, 0, 1)$ be the standard orthonormal basis in $\mathbb{R}^3$ and let $d(x)$ be a smooth matrix valued function such that

$$d(x)\zeta_1 = \nu(x'), d(x)\zeta_k = (\alpha_{1,k}(x), \alpha_{2,k}(x), \alpha_{3,k}(x)),\ k = 2, 3$$

and $\nabla_y = d(x)\nabla_x$. Denote by $\xi = (\xi_1, \xi')$ the dual variables of (x_1, x'). Then the symbol of the operator $-\mathbf{i}\,\nabla|_{x_1=0}$ in the coordinates (x, ξ) has the form $\xi_1 \nu(x') + \beta(x', \xi')$, where $\beta(x', \xi')$ is vector valued symbol given by

$$\beta(x', \xi') = \sum_{k=2}^{3} \xi_k d(0, x') \zeta_k = \Big(\sum_{k=2}^{3} \xi_k \alpha_{j,k}(x) \Big)_{j=1,2,3}$$

and $\langle \nu(x'), \beta(x', \xi') \rangle = 0$. The principal symbol of the operator $-\Delta|_{x_1=0}$ becomes

$$\xi_1^2 + \langle \beta(x', \xi'), \beta(x', \xi') \rangle,$$

while the principal symbol of the Laplace-Beltrami operator $-\Delta_\Gamma$ has the form

$$r_0(x', \xi') = \langle \beta(x', \xi'), \beta(x', \xi') \rangle.$$

It is important to note that $\beta(x', \xi')$ is defined globally and it is invariant when we change the coordinates x'. In fact if $\tilde{x}'$ are new coordinates, $x_1 = \tilde{x}_1$ and

$$y = \tilde{s}(\tilde{x}') + x_1 \nu(\tilde{x}) = s(x') + x_1 \nu(x'),$$

in the intersection of the domains $s(\mathcal{U}_0) \cap \tilde{s}(\tilde{\mathcal{U}_0})$, where the coordinates x' and $\tilde{x}'$ are defined, then $\nu(x') = \nu(\tilde{x}')$. From the equality $\nabla|_{x_1=0} = \nabla|_{\tilde{x}_1=0}$, we deduce $\beta(x', \xi') = \beta(\tilde{x}', \tilde{\xi}')$.

Let $a(x, \xi'; h) \in C^\infty(T^*(\Gamma) \times (0, h_0])$. Given $k \in \mathbb{R}, 0 < \delta < 1/2$, denote by $\mathcal{S}^k_\delta$ the set of symbols so that

$$\left|\partial^\alpha_{x'} \partial^\beta_{\xi'} a(x, \xi'; h)\right| \leq C_{\alpha,\beta} h^{-\delta(|\alpha|+|\beta|} \langle \xi' \rangle^{k-|\beta|},\ \forall \alpha, \forall \beta, \quad (x', \xi) \in T^*(\Gamma)$$

with $\langle \xi' \rangle = (1 + |\xi'|^2)^{1/2}$ and constants $C_{\alpha,\beta}$ independent of h. A matrix symbol m belongs to $\mathcal{S}^k_\delta$ if all entries of m are in the class $\mathcal{S}^k_\delta$. The $h-$pseudo-differential operator with symbol $a(x, \xi; h)$ acts by

$$(Op_h(a)f)(x) := \ (2\pi h)^{-2} \iint_{T^*(\Gamma)} e^{\mathbf{i}\langle y' - x', \xi \rangle / h} a(x, \xi'; h) f(y) d\xi' dy'.$$

By using the change $\xi' = h\eta'$, the operator can be written also as a classical pseudo-differential operator

$$(Op_h(a)f)(x) := \ (2\pi)^{-2} \iint_{T^*(\Gamma)} e^{\mathbf{i}\langle y' - x', \xi \rangle} a(x, h\xi'; h) f(y) d\xi' dy'.$$

Next for a positive function $\omega(x', \xi') > 0$ we define the space of symbols $a(x', \xi'; h) \in S^k_{\delta_1,\delta_2}(\omega)$ for which

$$\left|\partial^\alpha_{x'} \partial^\beta_{\xi'} a(x, \xi'; h)\right| \leq C_{\alpha,\beta} \omega^{k - \delta_1|\alpha| - \delta_2|\beta|},\ \forall \alpha, \forall \beta, \quad (x', \xi') \in T^*(\Gamma).$$

We denote $S^k_{\delta_1,\delta_2} = S^k_{\delta_1,\delta_2}(\langle \xi' \rangle)$ and introduce the norm

$$\|u\|_{H^k_h(\Gamma)} := \|Op_h(\langle \xi' \rangle^k) u\|_{L^2(\Gamma)}.$$

Let $\rho(x', \xi', z) = \sqrt{z^2 - r_0(x', \xi')}$, $\operatorname{Im} \rho > 0$, be the root of the equation $\xi_1^2 + r_0(x', \xi') - z^2 = 0$ with respect to ξ_1. Set $z = -\mathbf{i} + t(h)$, $t(h) = \mathcal{O}(h^2)$. We have $\rho \in \mathcal{S}^1_0$,

$$\sqrt{z^2 - r_0} = \mathbf{i}\sqrt{1 + r_0} - \frac{\mathcal{O}(h^2)}{\mathbf{i}\sqrt{1 + 2\mathbf{i}t(h) - t^2(h) + r_0} + \mathbf{i}\sqrt{1 + r_0}}$$

and $\rho - \mathbf{i}\sqrt{1 + r_0} \in \mathcal{S}^{-1}_0$.

The local parametrix of (2.8) constructed in [12] in local coordinates (x_1, x') has the form

$$\tilde{E} = (2\pi h)^{-2} \iint e^{\frac{\mathbf{i}}{h}(\langle y', \xi' \rangle + \varphi(x, \xi', z))} \phi_0^2(x_1/\delta) a(x, y', \xi', z, h) d\xi' dy',$$

$$\tilde{H} = (2\pi h)^{-2} \iint e^{\frac{\mathbf{i}}{h}(\langle y', \xi' \rangle + \varphi(x, \xi', z))} \phi_0^2(x_1/\delta) b(x, y', \xi', z, h) d\xi' dy',$$

where $\phi_0(s) \in C_0^\infty(\mathbb{R})$ is equal to 1 for $|s| \leq 1$ and to 0 for $|s| \geq 2$ and $0 < \delta \ll 1$. Set $\chi(x_1) = \phi_0^2(x_1/\delta)$. The phase function φ satisfies for N large the equation

$$\langle d\nabla_x \varphi, d\nabla_x \varphi \rangle - z^2 \varphi = x_1^N \Phi$$

and has the form

$$\varphi = \sum_{k=0}^{N-1} x_1^k \varphi_k(x', \xi', z),\ \varphi_0 = -\langle x', \xi' \rangle,\ \varphi_1 = \rho.$$

Moreover, for δ small enough we have $\text{Im}\,\varphi \geq x_1 \text{Im}\,\rho/2$ for $0 \leq x_1 \leq 2\delta$. The construction of φ is given in [11], [12]. For $z = -\mathbf{i}$ we have $\varphi = -\langle x'.\xi'\rangle + \mathbf{i}\tilde{\varphi}$ with real valued phase $\tilde{\varphi}$ (see [11] and Section 3 in [9]). Introduce a function $\eta \in C^\infty(T^*(\Gamma))$ such that $\eta = 1$ for $r_0 \leq C_0$, $\eta = 0$ for $r_0 \geq 2C_0$, where $C_0 > 0$ is independent on h. Choosing C_0 big enough, one arranges the estimates

$$\begin{aligned} C_1 \leq |\rho| \leq C_2,\ \text{Im}\,\rho \geq C_3,\ (x', \xi') \in \text{supp}\,\eta, \\ |\rho| \geq \text{Im}\,\rho \geq C_4|\xi'|,\ (x', \xi') \in \text{supp}\,(1-\eta) \end{aligned} \tag{3.1}$$

with positive constants $C_j > 0$. Following [12], we say that a symbol $\omega \in C^\infty(T^*(\Gamma))$ is in the class $S^{k_1}_{\delta_1,\delta_2}(\omega_1) + S^{k_2}_{\delta_3,\delta_4}(\omega_2)$ if $\eta\omega \in S^{k_1}_{\delta_1,\delta_2}(\omega_1)$ and $(1-\eta)\omega \in S^{k_2}_{\delta_3,\delta_4}(\omega_2)$. The amplitudes a and b have the form $a = \sum_{j=0}^{N-1} h^j a_j$, $b = \sum_{j=0}^{N-1} h^j b_j$ and a_j, b_j for $0 \leq j \leq N-1$ satisfy the system

$$\begin{cases} (d\nabla_x\varphi) \wedge a_j - z b_j = \mathbf{i}(d\nabla_x) \wedge a_{j-1} + x_1^N \Psi_j, \\ (d\nabla_x\varphi) \wedge b_j + z a_j = \mathbf{i}(d\nabla_x) \wedge b_{j-1} + x_1^N \tilde{\Psi}_j, \\ \nu \wedge a_j = \begin{cases} g,\ j = 0, \\ 0,\ j \geq 1 \end{cases} \quad \text{on } x_1 = 0, \end{cases} \tag{3.2}$$

where $a_{-1} = b_{-1} = 0$ and

$$g = -\nu(x') \wedge (\nu(y') \wedge f(y')) = f(y') - (v(x') - v(y')) \wedge (\nu(y') \wedge f(y')).$$

On the other hand, the function a_j, b_j have the presentation

$$a_j = \sum_{k=0}^{N-1} x_1^k a_{j,k},\ b_j = \sum_{k=0}^{N-1} x_1^k b_{j,k}.$$

The symbols $a_{j,k}, b_{j,k}$ are expressed by terms involving g. Moreover,

$$a_{j,k} = A_{j,k}(x', \xi')\tilde{f}(y'),\ b_{j,k} = B_{j,k}(x', \xi')\tilde{f}(y'),$$

where $\tilde{f}(y') = \nu(y') \wedge f(y') = i_\nu(y') f(y')$ with a (3×3) matrix $i_\nu = \sum_{j=1}^3 \nu_j I_j$, I_j being (3×3) constant matrices. Here $A_{j,k}, B_{j,k}$ are smooth matrix valued functions. The important point proved in Lemma 4.3 in [12] is that we have the properties

$$\begin{aligned} A_{j,k} \in S^{-1-3k-5j}_{2,2}(|\rho|) + S^{-j}_{0,1}(|\rho|),\ j \geq 0,\ k \geq 0, \\ B_{j,k} \in S^{-1-3k-5j}_{2,2}(|\rho|) + S^{1-j}_{0,1}(|\rho|),\ j \geq 0,\ k \geq 0. \end{aligned} \tag{3.3}$$

Since by (3.1), the function $|\rho|$ is bounded from below for $(x', \xi') \in \text{supp}\,\eta$, in the above properties we may replace absorb $S^{-1-3k-5j}_{2,2}(|\rho|)$ and obtain the class $S^{-j}_{0,1}(|\rho|)$ (resp. $S^{1-j}_{0,1}(|\rho|)$) for all (x', ξ'). For the principal symbols $a_{0,0}$, $b_{0,0}$ we have form (3.2) the system

$$\begin{cases} \psi_0 \wedge a_{0,0} - z b_{0,0} = 0, \\ \psi_0 \wedge b_{0,0} + z a_{0,0} = 0, \\ \nu \wedge a_{0,0} = g, \end{cases} \tag{3.4}$$

with $\psi_0 = d(0, x')\nabla_x\varphi|_{x_1=0} = \rho\nu - \beta$. The solution of (3.4) is given by (4.4) in [12] and one has

$$\begin{aligned} a_{0,0} = -\nu \wedge g + \rho^{-1}\langle \nu, \beta \wedge g\rangle \nu, \\ \nu \wedge b_{0,0} = \frac{1}{z}\Big(\rho(\nu \wedge g) + \rho^{-1}\langle \beta, \nu \wedge g\rangle \beta\Big). \end{aligned} \tag{3.5}$$

Thus we obtain $\nu \wedge \tilde{E}|_{x_1=0} = f$ and

$$\nu \wedge \tilde{H}|_{x_1=0} = i_\nu(x')\tilde{H}|_{x_1=0} = \sum_{j=0}^{N-1} h^j Op_h(i_\nu B_{j,0})\tilde{f}.$$

Following [12] and using (3.5), for the principal symbol of $\nu \wedge \tilde{H}|_{x_1=0}$ one deduces

$$i_\nu B_{0,0}\tilde{f} = \nu \wedge b_{0,0} = m(\nu \wedge g) = m\tilde{f} + m i_\nu \sum_{j=1}^{3}(\nu_j(y') - \nu_j(x'))I_j\tilde{f}$$

with a matrix symbol $m := \frac{1}{z}(\rho I + \rho^{-1}\mathcal{B})$ and matrix valued symbol $\mathcal{B}$ defined by $\mathcal{B}v = \langle \beta, v\rangle\beta,\ v \in \mathbb{R}^3$. Then we obtain

$$Op_h(i_\nu B_{0,0})\tilde{f} = Op_h(m)\tilde{f} + hOp_h(\tilde{m})\tilde{f}$$

with $\tilde{m} \in \mathcal{S}_0^0$. Choosing $\tilde{f} = \psi(x')(\nu \wedge f)$, we obtain a local parametrix $T_{N,\psi}(h,z)$ and in Theorem 1.1 in [12] the estimate

$$\|\mathcal{N}(\lambda)(\psi f) - Op_h(m + h\tilde{m})(\nu \wedge \psi f)\|_{\mathcal{H}_0^t} \leq Ch\theta^{-5/2}\|f\|_{\mathcal{H}_{-1}^t} \tag{3.6}$$

has been established in a more general setting assuming a lower bound $\theta > h^{2/5}$. With the last condition one can study the case $z = 1 + \mathbf{i}\theta = h\lambda,\ h = |\mathrm{Re}\,\lambda|^{-1}$, provided $|\mathrm{Re}\,\lambda| \geq |\mathrm{Im}\,\lambda|$.

In this paper we need a parametrix in the elliptic case $z = -\mathbf{i} + t(h)$ and in (3.6) we can obtain an approximation modulo $\mathcal{O}(h^{-\ell_2+N})$ adding lower order terms of $T_{N,\psi}(h,z)$ and exploiting the bound $\theta \geq 1 - h^2$ as well as the estimates (3.1), (3.3). According to Lemma 4.2 in [12], one has the estimates

$$\left|\partial_{x'}^\alpha \partial_{\xi'}^\beta \left(e^{\frac{\mathbf{i}\varphi}{h}}\right)\right| \leq C_{\alpha,\beta}|\xi'|^{-|\beta|}e^{-C|\xi'|x_1/h}$$

for $0 \leq x_1 \leq 2\delta$ with constants $C > 0, C_{\alpha,\beta} > 0$ independent of x_1, z and h. In fact, the above estimates are proved for $(x', \xi') \in \mathrm{supp}\,(1-\eta)$, while for $(x', \xi') \in \mathrm{supp}\,\eta$ the factor $|\xi'|$ is bounded. Then (see (4.31) in [12])

$$h^{-N}x_1^N e^{\mathbf{i}\varphi/h} \in S_{0,1}^{-N}$$

uniformly in x_1 and h. Now let

$$-\mathbf{i}h\nabla \wedge \tilde{E} - z\phi\tilde{H} = (2\pi h)^{-2}\iint e^{\frac{\mathbf{i}}{h}(\langle y',\xi'\rangle + \varphi)}V_1(x,y',\xi',h,z)d\xi' dy' = U_1,$$

$$-\mathbf{i}h\nabla \wedge \tilde{H} + z\phi\tilde{E} = (2\pi h)^{-2}\iint e^{\frac{\mathbf{i}}{h}(\langle y',\xi'\rangle + \varphi)}V_2(x,y',\xi',h,z)d\xi' dy' = U_2$$

with

$$V_1 = h\tilde{\chi}a + h^N\chi(d\nabla_x) \wedge a_{N-1} + x_1^N\sum_{j=0}^{N-1}h^j\chi\Psi_j,$$

$$V_2 = h\tilde{\chi}b + h^N\chi(d\nabla_x) \wedge b_{N-1} + x_1^N\sum_{j=0}^{N-1}h^j\chi\tilde{\Psi}_j,$$

where $\tilde{\chi}$ has support in $\delta \leq x_1 \leq 2\delta$. Clearly,

$$(h\partial_x)^\alpha U_j(x_1,.) = Op_h\Big(h^{|\alpha|}\partial_x^\alpha(e^{\mathbf{i}\varphi/h}V_j)\Big)\tilde{f},\ j = 1,2.$$

Combing this with the properties (3.3) and the proof of Lemma 4.3 in [12], we obtain the estimate

$$\|(h\partial_x)^\alpha U_j\|_{L^2(\Gamma)} \leq C_{\alpha,N} h^{-\ell_\alpha + N} \|f\|_{H_h^{-1}(\Gamma)} \tag{3.7}$$

with ℓ_α independent of h, N and f. Thus by the argument in Section 4 in [12] we construct a local parametrix in the elliptic region and

$$\|\mathcal{N}\Big(-\frac{1}{\tilde{h}}\Big)(\psi f) - T_{N,\psi}(h,z)(\nu \wedge \psi f)\|_{H_h^s(\Gamma)} \leq C_N h^{-\ell_s + N} \|f\|_{L^2(\Gamma)},\ s \geq 0,\ N \geq N_s. \tag{3.8}$$

Choosing a partition of unity $\sum_{j=1}^M \psi_j(x') \equiv 1$ on Γ, we construct a parametrix $T_N(h,z) = \sum_{j=1}^M T_{N,\psi_j}(h,z)$ and obtain

$$\|\mathcal{N}\Big(-\frac{1}{\tilde{h}}\Big) f - T_N(h,z)(\nu \wedge f)\|_{H_h^s(\Gamma)} \leq C_N h^{-\ell_s + N} \|f\|_{L^2(\Gamma)},\ s \geq 0,\ N \geq N_s. \tag{3.9}$$

For the operator $P(-\frac{1}{\tilde{h}})f = \mathcal{N}(-\frac{1}{\tilde{h}})(\nu \wedge f) = \mathcal{N}(\lambda)(\nu \wedge f)$ one has an approximation by $-T_N(h,z)f$. Moreover, for $z = -\mathbf{i}$ the principal symbol of $-T_N(h,-\mathbf{i})$ becomes

$$-m = \frac{1}{\mathbf{i}}\Big(\mathbf{i}\sqrt{1+r_0}I + \frac{\mathcal{B}}{\mathbf{i}\sqrt{1+r_0}}\Big) = \sqrt{1+r_0}I - \frac{1}{\sqrt{1+r_0}}\mathcal{B}.$$

Now we discuss briefly the existence of the parametrix for the problem

$$\begin{cases} -\mathbf{i}h \operatorname{curl} E = zH,\ x \in \Omega, \\ -\mathbf{i}h \operatorname{curl} H = -zE,\ x \in \Omega, \\ \nu \wedge H = f,\ x \in \Gamma, \\ (E,H) - \text{outgoing} \end{cases} \tag{3.10}$$

with $-\mathbf{i}z = h\lambda,\ z = -\mathbf{i}(1+\mathbf{i}t)^{-1}$. We follow the construction above with the same phase function. The transport equations for a_j, b_j have the form

$$\begin{cases} (d\nabla_x\varphi) \wedge b_j + z a_j = \mathbf{i}(d\nabla_x) \wedge b_{j-1} + x_1^N \Psi_j, \\ (d\nabla_x\varphi) \wedge a_j - z b_j = \mathbf{i}(d\nabla_x) \wedge a_{j-1} + x_1^N \tilde{\Psi}_j, \\ \nu \wedge b_j = \begin{cases} g,\ j = 0, \\ 0,\ j \geq 1 \end{cases} \quad \text{on } x_1 = 0, \end{cases} \tag{3.11}$$

where $a_{-1} = b_{-1} = 0$. This system is the same as (3.2) if we replace z by $-z$ and a_j, b_j by b_j, a_j, respectively. Therefore, by using (3.5), we obtain

$$\begin{aligned} b_{0,0} &= -\nu \wedge g + \rho^{-1}\langle \nu, \beta \wedge g\rangle \nu, \\ \nu \wedge a_{0,0} &= -\frac{1}{z}\Big(\rho(\nu \wedge g) + \rho^{-1}\langle \beta, \nu \wedge g\rangle \beta\Big). \end{aligned} \tag{3.12}$$

We obtain an analog of (3.9) with $\mathcal{N}(h), T_N(h,z)$ replaced by $\mathcal{N}_1(h), T_{1,N}(h,z)$. For the operator $P_1(-\frac{1}{\tilde{h}})f = -\mathcal{N}_1(-\frac{1}{\tilde{h}})(\nu \wedge f)$ we have an approximation with $T_{1,N}(h,z)f$ and by (3.12) the principal symbol of $T_{1,N}(h,-\mathbf{i})$ becomes

$$m_1 = \frac{1}{\mathbf{i}}\Big(\mathbf{i}\sqrt{1+r_0}I + \frac{\mathcal{B}}{\mathbf{i}\sqrt{1+r_0}}\Big) = \sqrt{1+r_0}I - \frac{1}{\sqrt{1+r_0}}\mathcal{B}.$$

4. Properties of the operator $\mathcal{P}(h)$

In this section we study the case $\tilde{h}$ real. Recall that the operator $\mathcal{B}(hD_{x'})$ has matrix symbol $\mathcal{B}(x', h\xi')$ such that

$$\mathcal{B}f = \langle \beta, f\rangle \beta,\ f \in \mathbb{R}^3,$$

where $\beta = \beta(x', \xi') \in \mathbb{R}^3$ is vector valued homogeneous polynomial of order 1 in ξ' introduced in the previous section. The equality $\langle \beta, \beta\rangle = r_0$ implies $\mathcal{B}(x', \xi')\beta(x', \xi') = r_0(x', \xi')\beta(x', \xi')$ and $\mathcal{B}(x', \xi')(\nu(x') \wedge \beta(x', \xi')) = 0$. Thus the matrix $\mathcal{B}(x', \xi')$ has three eigenvectors $\nu(x'), \nu(x') \wedge \beta(x', \xi'),\ \beta(x', \xi')$ with corresponding eigenvalues $0, 0, r_0(x', \xi')$. These eigenvalues are defined globally on Γ. Let $\|\xi'\|_g$ be the induced Riemann metric on $T^*(\Gamma)$ and let $b(x', \xi') = \beta(x', \frac{\xi'}{\sqrt{r_0(x', \xi')}})$. For $\|\xi'\|_g = 1$ introduce the unitary (3×3) matrix

$$U(x', \xi') = \begin{pmatrix} | & | & | \\ \nu(x') & \nu(x') \wedge b(x', \xi') & b(x', \xi') \\ | & | & | \end{pmatrix}.$$

Then for $\|\xi'\|_g = 1$ one obtains a global diagonalisation

$$U^T(x', \xi')\mathcal{B}(x', \xi')U(x', \xi') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & r_0 \end{pmatrix},$$

where A^T denotes the transpose matrix of A. Writing $\xi' = \omega\|\xi'\|_g$ with $\|\omega\|_g = 1$ and using the fact that $B(x', \xi')$ and $r_0(x', \xi')$ are homogeneous of order 2 in ξ', one concludes that the above diagonalisation is true for all ξ'.

First we study the case $0 < \gamma(x) < 1$, $\forall x \in \Gamma$, which yields $\gamma_0(x) = \frac{1}{\gamma(x)}$. Introduce the self-adjoint operator $\mathcal{P}(h) = -T_N(h, -\mathbf{i}) - \gamma_0(x)I$ with principal symbol

$$p_1 = \sqrt{1 + h^2 r_0}\Big(I - \frac{\mathcal{B}(h\xi)}{1 + h^2 r_0}\Big) - \gamma_0(x)I = \tilde{p}_1 - \gamma_0(x)I,$$

where I is the (3×3) identity matrix. We assume that N is fixed sufficiently large and we omit this in the notation $\mathcal{P}(h)$. Moreover, as it was mentioned in Section 2, we can write the pseudo-differential operator as a classical one and

$$U^T p_1 U = \begin{pmatrix} \sqrt{1 + h^2 r_0} - \gamma_0 & 0 & 0 \\ 0 & \sqrt{1 + h^2 r_0} - \gamma_0 & 0 \\ 0 & 0 & (1 + h^2 r_0)^{-1/2} - \gamma_0, \end{pmatrix}. \tag{4.1}$$

Moreover, $U^{-1} = U^T$ and U^T is the principal symbol of $(Op_h(U))^{-1}$. To examine the invertibility of $\mathcal{P}(h)$, observe that the symbol

$$-\gamma_0(x) \leq (1 + h^2 r_0)^{-1/2} - \gamma_0(x) \leq -(\gamma_0(x) - 1)$$

is elliptic. Write $(Op_h(U))^{-1}\mathcal{P}(h)Op_h(U)$ in a block matrix form

$$(Op_h(U))^{-1}\mathcal{P}(h)Op_h(U) = \begin{pmatrix} R(h) & S(h) \\ S^*(h) & r(h) \end{pmatrix},$$

where $R(h)$ is a (2×2) matrix valued operator, $S(h)$ is (2×1) matrix valued operator with symbol in $h\mathcal{S}_0^0(\Gamma)$, the adjoint operators $S^*(h)$ is (1×2) matrix valued operator, while

$$\begin{pmatrix} R(h) & 0 \\ 0 & r(h) \end{pmatrix}$$

has principal symbol (4.1). The equation $(Op_h(U))^{-1}\mathcal{P}(h)Op_h(U)(Y, y_3) = (F, f_3)$ with a vector $Y = (y_1, y_2)$ and $F = (f_1, f_2)$ implies

$$r(h)y_3 + S^*(h)Y = f_3.$$

Then $y_3 = -r(h)^{-1}S^*(h)Y + r(h)^{-1}f_3$ and for Y one obtains the equation

$$Q(h)Y = \Big(R(h) - S(h)r(h)^{-1}S^*(h)\Big)Y = F - S(h)r(h)^{-1}f_3.$$

The invertibility of the operator

$$Q(h) := R(h) - S(h)r(h)^{-1}S^*(h)$$

depends of that of $R(h)$ and $R(h)$ has principal symbol

$$q_1 = \begin{pmatrix} \sqrt{1+h^2r_0} - \gamma_0 & 0 \\ 0 & \sqrt{1+h^2r_0} - \gamma_0 \end{pmatrix}.$$

Let

$$c_0 = \min_{x\in\Gamma} \gamma_0(x) = (\max_{x\in\Gamma} \gamma(x))^{-1},\ c_1 = \max_{x\in\Gamma} \gamma_0(x) = (\min_{x\in\Gamma} \gamma(x))^{-1}.$$

Introduce the constants $C = \frac{1}{c_1^2}$, $\epsilon = \frac{C}{2}(c_0-1)^2 < 1/2$ and set $\langle hD\rangle = (1-h^2\Delta_\Gamma)^{1/2}$. We say that $A \geq B$ if $(Au, u) \geq (Bu, u)$, $\forall u \in L^2(\Gamma; \mathbb{C}^3)$. We need the following

Proposition 4.1. *The operator $Q(h)$ satisfies the estimate*

$$h\frac{\partial Q(h)}{\partial h} + CQ(h)\langle hD\rangle^{-1}Q(h) \geq \epsilon\langle hD\rangle. \tag{4.2}$$

Proof. The proof is a repetition of that of Prop. 4.1 in [9]. For the sake of completeness we present the details. We have

$$h\frac{\partial q_1}{\partial h} = \frac{h^2r_0}{\sqrt{1+h^2r_0}}I = \sqrt{1+h^2r_0}I - (1+h^2r_0)^{-1/2}I,$$

where I is the (2×2) identity matrix. The operator $CQ(h)\langle hD\rangle^{-1}Q(h)$ has principal symbol

$$C\sqrt{1+h^2r_0}I - 2C\gamma_0 I + C\gamma_0^2(1+h^2r_0)^{-1/2}I$$

and the principal symbol of the left hand side of (4.2) becomes

$$(1+C-\epsilon)\sqrt{1+h^2r_0}I + \epsilon\sqrt{1+h^2r_0}I - 2C\gamma_0 I + (C\gamma_0^2 - 1)(1+h^2r_0)^{-1/2}I.$$

We write the last term in the form

$$(1-C\gamma_0^2)\Big(1-(1+h^2r_0)^{-1/2}\Big)I + (C\gamma_0^2-1)I = A_1 + A_2.$$

Since $1 - C\gamma_0^2(x) \geq 0$ and $1-(1+h^2r_0)^{-1/2} \geq 0$, the term A_1 is symmetric non-negative definite matrix and we may apply the semi-classical strict Gärding inequality to bound from below $(Op_h(A_1)u, u)$ by $-C_1h\|u\|^2$. Next

$$(1+C-\epsilon)(\langle hD\rangle u, u) \geq (1+C-\epsilon)\|u\|^2$$

and

$$\Big((C(\gamma_0-1)^2-\epsilon)u, u\Big) \geq (C(c_0-1)^2-\epsilon)\|u\|^2 = \epsilon\|u\|^2.$$

The lower order symbol hq_0 of the operator $Q(h)$ yields a term

$$h(Op(q_0)u, u) \geq -h\|Op(q_0)\|_{L^2\to L^2}\|u\|^2 = -hC_2\|u\|^2$$

and we may absorb these terms taking $0 < h \leq \epsilon(C_1+C_2)^{-1} = \frac{\epsilon}{C_3}$. □

For the analysis of the eigenvalues of $Q(h)$ we will follow the approach of [10]. Introduce the semiclassical Sobolev space $H^s(\Gamma;\mathbb{C}^2)$ with norm $\|u\|_s = \|\langle hD\rangle^s u\|_{L^2(\Gamma;\mathbb{C}^2)}$. Let

$$\mu_1(h) \leq \mu_2(h) \leq ... \leq \mu_k(h) \leq ...$$

be the eigenvalues of $Q(h)$ repeated with their multiplicities. Fix $0 < h_0 \leq \frac{\epsilon}{C_3}$, where $\epsilon > 0$ is the constant in Proposition 4.1 and let $k_0 \in \mathbb{N}$ be chosen so that $\mu_k(h_0) > 0$ for $k \geq k_0$. This follows from the fact that the number of the non-positive eigenvalues of $Q(h_0)$ is given by a Weyl formula (see for instance Theorem 12.3 in [4])

$$(2\pi h_0)^{-2} \iint_{\sqrt{1+r_0} - \gamma_0 \leq 0} dx' d\xi' + \mathcal{O}(h_0^{-1}).$$

By using Proposition 4.1 and choosing $0 < \delta \leq \frac{c_0 - 1}{2}$, one obtains

$$\frac{\epsilon}{2} \leq h \frac{d\mu_k(h)}{dh} \leq C_0,\ k \geq k_0,$$

whenever $\mu_k(h) \in [-\delta, \delta],\ 0 < h \leq h_0$ (see Section 4 in [9]). Now if $0 < \frac{1}{r} < h_0$ and $\mu_k(1/r) < 0$, then there exists unique $h_k, 1/r < h_k < h_0$ such that $\mu_k(h_k) = 0$. Clearly, the operator $Q(h_k)$ is not invertible and for the invertibility of $Q(h)$ we must avoid small intervals around h_k. The purpose is to obtain a bijection between the set of $h_k \in (0, h_0]$ and the eigenvalues in Λ. Repeating the argument in Sections 4 in [9] and [10], one obtains the following

Proposition 4.2 (Prop.4.1, [10])**.** *Let $p > 3$ be fixed. The inverse operator $Q(h)^{-1} : L^2(\Gamma;\mathbb{C}^2) \to L^2(\Gamma;\mathbb{C}^2)$ exists and has norm $\mathcal{O}(h^{-p})$ for $h \in (0, h_0] \setminus \Omega_p$, where Ω_p is a union of disjoint closed intervals $J_{1,p}, J_{2,p}, ...$ with $|J_{k,p}| = \mathcal{O}(h^{p-1})$ for $h \in J_{k,p}$. Moreover, the number of such intervals that intersect $[h/2, h]$ for $0 < h \leq h_0$ is at most $\mathcal{O}(h^{1-p})$.*

If the operator $Q(h)^{-1}$ exists, it is easy to see that $\mathcal{P}(h)$ is also invertible. First, we have

$$\begin{pmatrix} I & S(h)r^{-1}(h) \\ 0_{2,1} & 1 \end{pmatrix} \begin{pmatrix} Q(h) & 0_{1,2} \\ S^*(h) & r(h) \end{pmatrix} = \begin{pmatrix} R(h) & S(h) \\ S^*(h) & r(h) \end{pmatrix}, \tag{4.3}$$

where I is the identity (2×2) matrix and $0_{1,2}$, $0_{2,1}$ are (1×2) and (2×1) matrices, respectively, with zero entires. Second, the operator $r^{-1}(h)$ has principal symbol $\frac{\sqrt{1+h^2 r_0}}{1 - \sqrt{1+h^2 r_0}\gamma_0} \in \mathcal{S}_0^0$, so $r^{-1}(h) : H^s(\Gamma;\mathbb{C}) \to H^s(\Gamma;\mathbb{C})$ is bounded for every s. On the other hand, $S(h) : H^s(\Gamma;\mathbb{C}) \to H^s(\Gamma;\mathbb{C}^2)$ has norm $\mathcal{O}_s(h)$. Consequently, the operator

$$\begin{pmatrix} I & S(h)r^{-1}(h) \\ 0_{2,1} & 1 \end{pmatrix}^{-1}$$

is bounded in $\mathcal{L}(H^s(\Gamma;\mathbb{C}^3), H^s(\Gamma;\mathbb{C}^3))$, while

$$\begin{pmatrix} Q(h) & 0_{1,2} \\ S^*(h) & r(h) \end{pmatrix}^{-1} = \begin{pmatrix} Q(h)^{-1} & 0_{1,2} \\ -r^{-1}(h)S^*(h)Q(h)^{-1} & r^{-1}(h) \end{pmatrix}.$$

We deduce that the operator on the right hand side of (4.3) is invertible, whenever $Q(h)$ is invertible and since $Op_h(U)$ is invertible this implies the invertibility of $\mathcal{P}(h)$. Finally, the statement of Proposition 4.2 holds for the operator $\mathcal{P}(h)^{-1}$ with the same intervals $J_{k,p}$ and we have a bound $\|\mathcal{P}(h)^{-1}\|_{L^2(\Gamma;\mathbb{C}^3) \to L^2(\Gamma;\mathbb{C}^3)} = \mathcal{O}(h^{-p})$ for $h \in (0, h_0] \setminus \Omega_p$.

The analysis of the case $\gamma(x) > 1$, $\forall x \in \Gamma$, is completely similar to that of the case $0 < \gamma(x) < 1$ examined above and we have $\gamma_0(x) = \gamma(x)$. We study the operators $\mathcal{N}_1(\lambda), C_1(\lambda)$ and $P_1(h) = P_1(-\frac{1}{h})$ introduced at the end of Section 2. For the self adjoint operator $\mathcal{P}_1(h) = T_{1,N}(h, -\mathbf{i}) - \gamma_0(x)I$, the argument at the end of Section 3 shows that $\mathcal{P}_1(h)$ has principal symbol $\tilde{p}_1(x', h\xi') - \gamma_0(x)I$. Thus we obtain the statements of Proposition 4.1 and Proposition 4.2 with a self-adjoint operator $Q_1(h)$ having principal symbol

$$\begin{pmatrix} \sqrt{1+h^2 r_0} - \gamma & 0 \\ 0 & \sqrt{1+h^2 r_0} - \gamma_0 \end{pmatrix}.$$

Notice that both operators $Q(h), Q_1(h)$ have the same principal symbol. Next for the operator $\mathcal{P}_1(h)$ we obtain the same statements as those for $\mathcal{P}(h)$.

5. Relation between the trace integrals for $\mathcal{P}(\tilde{h})$ and $\mathcal{C}(\tilde{h})$

The purpose in this section is to study the operators $\mathcal{P}(\tilde{h})$ and $\mathcal{C}(\tilde{h})$ for complex $\tilde{h} = h(1 + \mathbf{i}t) \in L$, $|t| \leq h^2$. We change the notations and we will use the notation h for the points in $L \subset \mathbb{C}$ with $|\mathrm{Im}\, h| \leq (\mathrm{Re}\, h)^2$, $0 < \mathrm{Re}\, h \leq h_0 \ll 1$. First we study the case $0 < \gamma(x) < 1$, $\forall x \in \Gamma$. The operator $T_N(h, z)$ can be extended for $h \in L$ as a holomorphic function of h. The same is true for $\mathcal{P}(h) = -T_N(h, z) - \gamma_0(x)I$. To study $\mathcal{P}(h)^{-1}$, we must examine the inverse of the operator on the left hand side of (4.3) for $h \in L$. Clearly, $S(h)$, $S^*(h)$ and $r(h)$ can be extended for $h \in L$ and $h^{-1}S(h)$, $r(h)^{-1}$ are bounded as operators from $H^s(\Gamma; \mathbb{C})$ to $H^s(\Gamma; \mathbb{C}^2)$ and from $H^s(\Gamma, \mathbb{C})$ to $H^s(\Gamma, \mathbb{C})$, respectively. Since $b(x', h\xi') = b(x', \xi')$, the symbol of $U(x', \xi')$ may be trivially extended for $h \in L$. It remains to study $Q(h)^{-1}$. Repeating the proof of Lemma 5.1 in [10] and using Proposition 4.1, we get

$$\|Q(h)^{-1}\|_{\mathcal{L}(H^{-1/2}(\Gamma;\mathbb{C}^2), H^{1/2}(\Gamma;\mathbb{C}^2))} \leq C \frac{\mathrm{Re}\, h}{|\mathrm{Im}\, h|},\ \mathrm{Im}\, h \neq 0,\ h \in L. \tag{5.1}$$

Here we have used the estimate

$$\|r(h)^{-1}\|_{H^s(\Gamma;\mathbb{C}) \to H^s(\Gamma;\mathbb{C})} \leq C_s' \leq C_s' \frac{\mathrm{Re}\, h}{|\mathrm{Im}\, h|},\ \mathrm{Im}\, h \neq 0,\ h \in L$$

since $|\mathrm{Im}\ h| \leq (\mathrm{Re}\, h)^2 \leq h_0 \mathrm{Re}\, h$. To obtain an estimate of

$$\|Q(h)^{-1}\|_{\mathcal{L}(H^s(\Gamma;\mathbb{C}^2), H^{s+1}(\Gamma;\mathbb{C}^2))},$$

as in Section 5 in [9], we introduce a C^∞ symbol

$$\chi(x', \xi') = \begin{cases} 2,\ x' \in \Gamma,\ \|\xi'\|_g \leq B_0, \\ 0,\ x' \in \Gamma,\ \|\xi'\|_g \geq B_0 + 1. \end{cases}$$

Here $B_0 > 0$ is a constant such that $\sqrt{C_3} B_0 \geq 2c_1$, $r_0(x', \xi') \geq C_3 \|\xi'\|_g^2$. Then we extend homorphically $\chi(x', \mathrm{Re}\, h D_{x'})$ to $\zeta(x', h D_{x'})$ for $h \in L$ and consider the operator $M(h) = Q(h) + \gamma_0(x')\zeta(x', hD_{x'})$. This modification implies the property $Q(h) - M(h) : \mathcal{O}_s(1) : H^{-s}(\Gamma; \mathbb{C}^2) \to H^s(\Gamma; \mathbb{C}^2)$ for every s and the operator $M(h)$ with principal symbol $M(x', \xi') \in \mathcal{S}_0^1$ becomes elliptic. Then $M(h)^{-1} : H^s(\Gamma; \mathbb{C}^2) \to H^{s+1}(\Gamma; \mathbb{C}^2)$ is bounded by $\mathcal{O}_s(1)$ and repeating the argument in Section 5, [9] and using (5.1), one deduces

$$\|Q(h)^{-1}\|_{\mathcal{L}(H^s(\Gamma;\mathbb{C}^2), H^{s+1}(\Gamma;\mathbb{C}^2))} \leq C_s \frac{\mathrm{Re}\, h}{|\mathrm{Im}\, h|},\ \mathrm{Im}\, h \neq 0. \tag{5.2}$$

Taking the inverse operators in (4.3), one obtains with another constant C_s the estimate

$$\|\mathcal{P}(h)^{-1}\|_{\mathcal{L}(H^s(\Gamma;\mathbb{C}^3),H^{s+1}(\Gamma;\mathbb{C}^3))} \leq C_s \frac{\mathrm{Re}\, h}{|\mathrm{Im}\, h|},\ \mathrm{Im}\, h \neq 0. \tag{5.3}$$

Following [10], we introduce piecewise smooth positively oriented curve $\gamma_{k,p} \subset \mathbb{C}$ which is a union of four segments: $\mathrm{Re}\, h \in J_{k,p}$, $\mathrm{Im}\, h = \pm(\mathrm{Re}\, h)^{p+1}$ and $\mathrm{Re}\, h \in \partial J_{k,p}$, $|\mathrm{Im}\, h| \leq (\mathrm{Re}\, h)^{p+1}$, $J_{k,p}$ being the interval in Ω_p introduced in Proposition 4.3.

Proposition 5.1. *For every $h \in \gamma_{k,p}$ the inverse operator $\mathcal{P}(h)^{-1}$ exists and*

$$\|\mathcal{P}(h)^{-1}\|_{\mathcal{L}(H^s(\Gamma;\mathbb{C}^3),H^{s+1}(\Gamma;\mathbb{C}^3))} \leq C_{k,s}(\mathrm{Re}\, h)^{-p}. \tag{5.4}$$

The proof is the same as in Proposition 5.2 in [10]. It is based on the estimate of $\|\mathcal{P}(h)^{-1}\|_{L^2(\Gamma;\mathbb{C}^3)\to L^2(\Gamma;\mathbb{C}^3)}$ for $h \in (0, h_0] \setminus \Omega_p$, the Taylor expansion of $\mathcal{P}(h)$ for $0 \leq |\mathrm{Im} h| \leq (\mathrm{Re} h)^{p+1}$ and the application of (5.3). We omit the details. Of course, by the same argument an analog to (5.4) holds for the norm of the operator $Q(h)^{-1}$ and $h \in \gamma_{k,p}$.

To obtain an estimate for $\mathcal{C}(h)^{-1}$, with N large enough write

$$\mathcal{C}(h)f = \mathcal{N}(-\frac{1}{h})f + \gamma_0(x)(\nu \wedge f) = T_N(h,z)(\nu \wedge f) + \gamma_0(x)(\nu \wedge f) + \mathcal{R}_q(h,z)(\nu \wedge f)$$
$$= -\mathcal{P}(h)i_\nu f + \mathcal{R}_q(h,z)i_\nu f,\ q \gg 2p$$

with $\mathcal{R}_q(h,z) : \mathcal{O}_s((\mathrm{Re}\, h)^q)) : H^s \to H^{s+q-1}$. This yields

$$\mathcal{P}(h)^{-1}\mathcal{C}(h)f = -\Big(Id - \mathcal{P}(h)^{-1}\mathcal{R}_q(h,z)\Big)i_\nu f$$

and by (5.4) one deduces

$$\big\|\mathcal{P}(h)^{-1}\mathcal{R}_q(h,z)\big\|_{\mathcal{L}(H^s,H^{s+q})} \leq C_s(\mathrm{Re}\, h)^{-p+q}.$$

For small $\mathrm{Re}\, h$ this implies

$$i_\nu\Big(Id - \mathcal{P}(h)^{-1}\mathcal{R}_q(h,z)\Big)^{-1}\mathcal{P}(h)^{-1}\mathcal{C}(h) = Id.$$

Repeating the argument in Section 5 of [9], we obtain

$$\|\mathcal{C}(h)^{-1}\|_{\mathcal{L}(H^s,H^{s+1})} \leq C_s(\mathrm{Re}\, h)^{-p},\ h \in \gamma_{k,p}. \tag{5.5}$$

In the same way writing

$$\mathcal{C}(h)^{-1} - i_\nu\mathcal{P}(h)^{-1} = i_\nu\Big(\Big(Id - \mathcal{P}(h)^{-1}\mathcal{R}_q(h,z)\Big)^{-1} - Id\Big)\mathcal{P}(h)^{-1},$$

one gets

$$\|\mathcal{C}(h)^{-1} - i_\nu\mathcal{P}(h)^{-1}\|_{\mathcal{L}(H^s,H^{s+q-1})} \leq C_s(|h|^{q-2p}),\ h \in \gamma_{k,p}. \tag{5.6}$$

On the other hand, $i_\nu\mathcal{P}(h)^{-1} = \Big(-\mathcal{P}(h)i_\nu\Big)^{-1}$ since $i_\nu i_\nu = -Id$. By using the Cauchy formula

$$\frac{d}{dh}\Big(\mathcal{C}(h) - (-\mathcal{P}(h)i_\nu\Big) = \frac{1}{2\pi\mathbf{i}}\int_{\tilde{\gamma}_{k,p}} \frac{C(\zeta) + \mathcal{P}(\zeta)i_\nu}{\zeta - h}d\zeta$$
$$= \int_{\tilde{\gamma}_{k,p}} \mathcal{R}_q(\zeta,z)i_\nu d\zeta,$$

where $\tilde{\gamma}_{k,p}$ is the boundary of a domain containing $\gamma_{k,p}$, one deduces

$$\Big\|\frac{d}{dh}\mathcal{C}(h) - \frac{d}{dh}(-\mathcal{P}(h)i_\nu)\Big\|_{\mathcal{L}(H^s, H^{q+q-1})} \leq C_s(\mathrm{Re}\, h)^q. \tag{5.7}$$

Now we pass to a trace formula involving $\mathcal{P}(h)^{-1}$ and $Q(h)^{-1}$. Recall that $k_0 \in \mathbb{N}$ is fixed so that $\mu_k(h_0) > 0,\ k \geq k_0$. Let $\mu_k(h_k) = 0,\ 0 < h_k < h_0,\ k \geq k_0$. Since $\mu_k(h)$ is increasing when $\mu_k(h) \in [-\delta, \delta]$, the function $\mu_k(h)$ has no other zeros for $0 < h \leq h_0$. We define the multiplicity of h_k as the multiplicity of the eigenvalues $\mu_k(h)$ of $Q(h)$ and denote by $\dot{A}$ the derivative of A with respect to h.

Proposition 5.2. *Let $\beta \subset L$ be a closed positively oriented simple C^1 curve without self intersections such that there are no points h_k on β with $\mu_k(h_k) = 0, k \geq k_0$. Then*

$$\mathrm{tr}_{H^{1/2}(\Gamma;\mathbb{C}^3)}\frac{1}{2\pi\mathbf{i}}\int_\beta \mathcal{P}(h)^{-1}\dot{\mathcal{P}}(h)dh = \mathrm{tr}_{H^{1/2}(\Gamma;\mathbb{C}^2)}\frac{1}{2\pi\mathbf{i}}\int_\beta Q(h)^{-1}\dot{Q}(h)dh \tag{5.8}$$

is equal to the number of h_k counted with their multiplicities in the domain bounded by β.

Proof. Since β is related to the eigenvalues of $Q(h)$, repeating without any changes the argument of the proof of Proposition 5.3 in [10], one deduces the existence of the trace on the right hand side of (5.8) and the fact that this trace is equal to the number of h_k in the domain bounded by β. Next

$$\int_\beta \begin{pmatrix} Q(h) & 0_{1,2} \\ S^*(h) & r(h)\end{pmatrix}^{-1}\begin{pmatrix} \dot{Q}(h) & 0_{1,2} \\ \dot{S}^*(h) & \dot{r}(h)\end{pmatrix} dh = \int_\beta \begin{pmatrix} Q(h)^{-1}\dot{Q}(h) & 0 \\ Y_{1,2}(h) & r^{-1}(h)\dot{r}(h)\end{pmatrix} dh$$

and the integral of $r^{-1}(h)\dot{r}(h)$ vanishes since this operator is analytic in the domain bounded by β. Thus the trace of the right hand side of the above equality is equal to the right hand side of (5.8) multiplies by $(2\pi\mathbf{i})$. Write

$$\begin{pmatrix} Q(h) & 0_{1,2} \\ S^*(h) & r(h)\end{pmatrix}^{-1}\begin{pmatrix} I & S(h)r^{-1}(h) \\ 0_{2,1} & 1\end{pmatrix}^{-1}\frac{d}{dh}\Big[\begin{pmatrix} I & S(h)r^{-1}(h) \\ 0_{2,1} & 1\end{pmatrix}\begin{pmatrix} Q(h) & 0_{1,2} \\ S^*(h) & r(h)\end{pmatrix}\Big]$$
$$= \begin{pmatrix} Q(h) & 0_{1,2} \\ S^*(h) & r(h)\end{pmatrix}^{-1}\begin{pmatrix} \dot{Q}(h) & 0_{1,2} \\ \dot{S}^*(h) & \dot{r}(h)\end{pmatrix} + Z(h).$$

The integral of $Z(h)$ vanishes by the cyclicity of trace since the product

$$\begin{pmatrix} I & S^*(h)r^{-1}(h) \\ 0_{2,1} & 1\end{pmatrix}^{-1}\frac{d}{dh}\begin{pmatrix} I & S^*(h)r^{-1}(h) \\ 0_{2,1} & 1\end{pmatrix}.$$

is an analytic function of h. By applying the equality (4.3), we obtain that the trace of integral involving $\begin{pmatrix} R(h) & S(h) \\ S^*(h) & r(h)\end{pmatrix}$ is equal to the trace on the right hand side of (5.8). By the same manipulation as above taking the product with $(Op_h(U))^{-1}$ on the right and by $Op_h(U)$ on the left, one obtains (5.8). □

Notice that by the cyclicity of the trace we get

$$\mathrm{tr}_{H^{1/2}(\Gamma;\mathbb{C}^3)}\int_\beta \mathcal{P}(h)^{-1}\dot{\mathcal{P}}(h)dh = \mathrm{tr}_{H^{1/2}(\Gamma;\mathbb{C}^3)}\int_\beta \Big(-\mathcal{P}(h)i_\nu\Big)^{-1}\frac{d}{dh}\Big(-\mathcal{P}(h)i_\nu\Big)dh.$$

Applying the estimate (5.6) for $\mathcal{C}(h)^{-1} - i_\nu\mathcal{P}(h)^{-1}$ and (5.7) for $\frac{d}{dh}\mathcal{C}(h) - \frac{d}{dh}\Big(-\mathcal{P}(h)i_\nu\Big)$ and taking into account Proposition 5.2, we conclude as in Section 5 of [9] that in the case $0 < \gamma(x) < 1$, $\forall x \in \Gamma$, we have

$$\operatorname{tr} \frac{1}{2\pi \mathbf{i}} \int_{\gamma_{k,p}} \mathcal{C}(h)^{-1}\dot{\mathcal{C}}(h)dh = \operatorname{tr} \frac{1}{2\pi \mathbf{i}} \int_{\gamma_{k,p}} \mathcal{P}(h)^{-1}\dot{\mathcal{P}}(h)dh$$
$$= \operatorname{tr} \frac{1}{2\pi \mathbf{i}} \int_{\gamma_{k,p}} Q(h)^{-1}\dot{Q}(h)dh.$$

The analysis in the case $\gamma(x) > 1$, $\forall x \in \Gamma$, is completely similar and we have trace formula involving the operator $\mathcal{C}_1(h)$ introduced at the end of Section 2 and trace formula involving $Q_1(h)$ and $\mathcal{P}_1(h) = T_{1,N}(h, z) - \gamma_0 I$. In this case

$$\operatorname{tr} \frac{1}{2\pi \mathbf{i}} \int_{\gamma_{k,p}} \mathcal{C}_1(h)^{-1}\dot{\mathcal{C}}_1(h)dh = \operatorname{tr} \frac{1}{2\pi \mathbf{i}} \int_{\gamma_{k,p}} \mathcal{P}_1(h)^{-1}\dot{\mathcal{P}}_1(h)dh$$
$$= \operatorname{tr} \frac{1}{2\pi \mathbf{i}} \int_{\gamma_{k,p}} Q_1(h)^{-1}\dot{Q}_1(h)dh.$$

The equality of traces shows that the proof of the asymptotic (1.3) is reduced to the count of h_k with their multiplicities for which we have $\mu_k(h_k) = 0$ in the domain $\beta_{k,j}$ bounded by $\gamma_{k,p}$. Here $\mu_k(h)$ are the eigenvalues of $Q(h)$ (resp. $Q_1(h)$) if $0 < \gamma(x) < 1$ (resp. if $\gamma(x) > 1$). We obtain a bijection $\beta_{k,j} \ni h_k \to \ell(h_k) = \lambda_j \in \sigma_p(G_b) \cap \Lambda$ which preserves the multiplicities. The existence of h_k with $1/r < h_k < h_0$ is equivalent to $\mu_k(1/r) < 0$ and we are going to study the asymptotic of the counting function of the negative eigenvalues of $Q(1/r)$ (resp. $Q_1(1/r)$). The semiclassical principal symbol of both operators $Q(h)$, $Q_1(h)$ has a double eigenvalue $q(x', \xi') = \sqrt{1 + r_0(x', \xi')} - \gamma_0(x')$. Applying Theorem 12.3 in [4], we obtain

$$\sharp\{\lambda \in \sigma_p(G_b) \cap \Lambda : \ |\lambda| \leq r, \ r \geq C_{\gamma_0}\} = \frac{r^2}{(2\pi)^2} \int_{q(x',\xi') \leq 0} dx'd\xi' + \mathcal{O}_{\gamma_0}(r).$$

Finally,

$$\int_{q(x',\xi') \leq 0} dx'd\xi' = \int_{r_0(x',\xi') \leq \gamma_0^2(x') - 1} dx'd\xi' = \pi \int_\Gamma (\gamma_0^2(x') - 1)dx'$$

and this completes the proof of Theorem 1.1.

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