WEYL FORMULA FOR THE NEGATIVE DISSIPATIVE EIGENVALUES OF MAXWELL'S EQUATIONS

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ABSTRACT. Let $V(t) = e^{tG_b}$, $t \geq 0$, be the semigroup generated by Maxwell's equations in an exterior domain $\Omega \subset \mathbb{R}^3$ with dissipative boundary condition $E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0, \gamma(x) > 0, \forall x \in \Gamma = \partial\Omega$. We study the case when $\Omega = \{x \in \mathbb{R}^3 : |x| > 1\}$ and $\gamma \neq 1$ is a constant. We establish a Weyl formula for the counting function of the negative real eigenvalues of G_b.

1. INTRODUCTION

Let $K \subset \{x \in \mathbb{R}^3 : |x| \leq a\}$ be an open connected domain and let $\Omega = \mathbb{R}^3 \setminus \bar{K}$ be connected domain with C^∞ smooth boundary Γ. Consider the boundary problem

$$\partial_t E = \operatorname{curl} B, \qquad \partial_t B = -\operatorname{curl} E \quad \text{in} \quad \mathbb{R}_t^+ \times \Omega,$$

$$E_{tan} - \gamma(x)(\nu \wedge B_{tan}) = 0 \quad \text{on} \quad \mathbb{R}_t^+ \times \Gamma, \qquad (1.1)$$

$$E(0, x) = E_0(x), \qquad B(0, x) = B_0(x).$$

with initial data $f = (E_0, B_0) \in L^2(\Omega; \mathbb{C}^6) = \mathcal{H}$. Here $\nu(x)$ is the unit outward normal to $\partial\Omega$ at $x \in \Gamma$ pointing into Ω, $\langle\,,\rangle$ denotes the scalar product in $\mathbb{C}^3$, $u_{tan} := u - \langle u, \nu\rangle\nu$, and $\gamma(x) \in C^\infty(\Gamma)$ satisfies $\gamma(x) > 0$ for all $x \in \Gamma$. Let

$$G = \begin{pmatrix} 0 & \operatorname{curl} \\ -\operatorname{curl} & 0 \end{pmatrix}$$

and let G_b be the operator G with domain $D(G_b)$ which is the closure in the graph norm

$$|||u||| = (\|u\|_{\mathcal{H}}^2 + \|Gu\|_{\mathcal{H}}^2)^{1/2}$$

of functions $u = (v, w) \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^6)$ satisfying the boundary condition $v_{tan} - \gamma(\nu \wedge w_{tan}) = 0$ on Γ. The operator G_b generates a contraction semigroup $V(t)$ in $\mathcal{H}$ (see for instance Theorem 3.1.8 and Section 3.8 in [6]) and the solution of the problem (1.1) is described by

$$(E, B) = V(t)f = e^{tG_b}f, \ t \geq 0.$$

In [1] it was proved that the spectrum of G_b in the open half plan $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is formed by isolated eigenvalues with finite multiplicities. Note that if $G_b f = \lambda f$ with $\operatorname{Re}\lambda < 0$, the solution $u(t, x) = V(t)f = e^{\lambda t}f(x)$ of (1.1) has exponentially decreasing global energy. Such solutions are called **asymptotically disappearing** and they are very important for the inverse scattering problems (see [1]). In particular, the eigenvalues λ with $\operatorname{Re}\lambda \to -\infty$ imply a very fast decay of the corresponding solutions. In [2] the existence of eigenvalues of G_b has been studied for the ball $B_3 = \{x \in \mathbb{R}^3 : |x| < 1\}$ assuming γ constant. It was proved that for

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$\gamma = 1$ there are no eigenvalues in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$, while for $\gamma = \text{const}, \gamma \neq 1$ there is always an infinite number of real eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ and with exception of λ_1 they satisfy the estimate

$$\lambda_n \leq -\frac{1}{\max\{(\gamma_0 - 1), \sqrt{\gamma_0 - 1}\}} = -c_0, \qquad (1.2)$$

where $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$.

In this Note we study the distribution of the negative eigenvalues and our purpose is to obtain a Weyl formula for the counting function

$$N(r) = \#\{\lambda \in \sigma_p(G_b) \cap \mathbb{R}^- : |\lambda| \leq r\}, \ r > r_0(\gamma),$$

where every eigenvalues λ_n is counted with its algebraic multiplicity given by

$$\text{mult}(\lambda_n) = \text{rank} \ \frac{1}{2\pi \mathbf{i}} \int_{|\lambda_n - z| = \epsilon} (z - G_b)^{-1} dz,$$

where $0 < \epsilon \ll 1$. Our main result is the following

Theorem 1.1. *Let $\gamma \neq 1$ be a constant and let $\gamma_0 = \max\{\gamma, \frac{1}{\gamma}\}$. Then the counting function $N(r)$ for the ball B_3 has the asymptotic*

$$N(r) = (\gamma_0^2 - 1)r^2 + \mathcal{O}_\gamma(r), \ r \geq r_0(\gamma) > c_0. \qquad (1.3)$$

The proof of Theorem 1.1 is based on a precise analysis of the roots of the equation (3.1) involving spherical Hankel functions $h_n^{(1)}(\lambda)$ of first kind. We show in Section 3 that for $\gamma > 1$ this equation has only one real root $\lambda_n < 0$. Moreover, we have $\lambda_{n+1} < \lambda_n, \ \forall n \in \mathbb{N}$, so we have a decreasing sequence of eigenvalues. The geometric multiplicity of λ_n is $2n + 1$. Since C_b is not a self-adjoint operator the geometric multiplicity could be less than the algebraic one. In our case these multiplicities coincide and the proof is based on a representation of $(G_b - z)^{-1}$. To estimate λ_n as $n \to \infty$, we apply an approximation of the exterior semiclassical Dirichlet to Neumann map for the operator $(h^2\Delta + z)$ established in [7] (see also [9]) combined with an application of Rouché theorem.

We conjecture that in the general case of strictly convex obstacles and $\min_{y \in \Gamma} \gamma(y) = \gamma_1 > 1$ we have the asymptotic

$$N(r) = \frac{1}{4\pi} \Big(\int_\Gamma (\gamma^2(y) - 1) dS_y \Big) r^2 + \mathcal{O}_\gamma(r), \ r \geq r_0(\gamma).$$

For the ball B_3 this agrees with (1.3).

2. Boundary problem for Maxwell system

Our purpose is to study the eigenvalues of G_b in case the obstacle is equal to the ball $B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$. Setting $\lambda = \mathbf{i}\mu, \ \operatorname{Im} \mu > 0$, an eigenfunction $(E, B) \neq 0$ of G_b satisfies

$$\text{curl} \ E = -\mathbf{i}\mu B, \qquad \text{curl} \ B = \mathbf{i}\mu E. \qquad (2.1)$$

Replacing B by $H = -B$ yields for $(E, H) \in H^2(\{|x| \geq 1\}; \mathbb{C}^6)$ the problem,

$$\begin{cases} \text{curl} \ E = \mathbf{i}\mu H, \qquad \text{curl} \ H = -\mathbf{i}\mu E, \quad \text{for} \quad x \in B_3, \\ E_{tan} + \gamma(\nu \wedge H_{tan}) = 0, \quad \text{for} \quad x \in \mathbb{S}^2. \end{cases} \qquad (2.2)$$

The functions $E(x), H(x)$ are solutions in $\{|x| > 1\}$ of the Helmholtz equation

$$\Delta v + \mu^2 v = 0$$

and since $(E, H) \in \mathcal{H}$ these solutions are outgoing. By using spherical coordinates ω on $\mathbb{S}^2$, we can expand $E(x), H(x)$ by the spherical functions $Y_n^m(\omega)$, $n = 0, 1, 2, ..., |m| \leq n$, $\omega \in \mathbb{S}^2$ and the spherical Hankel functions of first kind

$$h_n^{(1)}(z) := \frac{H_{n+1/2}^{(1)}(z)}{\sqrt{z}}, n \geq 1.$$

An application of Theorem 2.50 in [3] (in the notation of [3] it is necessary to replace k by $\mu \in \mathbb{C} \setminus \{0\}$) says that the outgoing solution of the system

$$\operatorname{curl} E = \mathbf{i}\mu H, \qquad \operatorname{curl} H = -\mathbf{i}\mu E, \quad \text{for} \quad x \in B_3$$

for $x = |x|\omega, r = |x| > 0, \omega = \frac{x}{r}$ has the form

$$E(x) = \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[\alpha_n^m \sqrt{n(n+1)} \frac{h_n^{(1)}(\mu r)}{r} Y_n^m(\omega)\omega \right.$$
$$\left. + \frac{\alpha_n^m}{r}(rh_n^{(1)}(\mu r))' U_n^m(\omega) + \beta_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right], \tag{2.3}$$

$$H(x) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[\beta_n^m \sqrt{n(n+1)} \frac{h_n^{(1)}(\mu r)}{r} Y_n^m(\omega)\omega \right.$$
$$\left. + \frac{\beta_n^m}{r}(rh_n^{(1)}(\mu r))' U_n^m(\omega) + \mu^2 \alpha_n^m h_n^{(1)}(\mu) V_n^m(\omega) \right]. \tag{2.4}$$

Here $U_n^m(\omega) = \frac{1}{\sqrt{n(n+1)}} \operatorname{grad}_{\mathbb{S}^2} Y_n^m(\omega)$ and $V_n^m(\omega) = \nu \wedge U_n^m(\omega)$ for $n \in \mathbb{N}, -n \leq m \leq n$ form a complete orthonormal basis in

$$L_t^2(\mathbb{S}^2) = \{u(\omega) \in (L^2(\mathbb{S}^2; \mathbb{C}^3) : \langle \omega, u(\omega) \rangle = 0 \text{ on } \mathbb{S}^2\}.$$

To find a representation of $\nu \wedge H_{tan}$, observe that $\nu \wedge (\nu \wedge U_n^m) = -U_n^m$, so for $r = 1$ one has

$$(\nu \wedge H_{tan})(\omega) = -\frac{1}{\mathbf{i}\mu} \sum_{n=1}^{\infty} \sum_{|m| \leq n} \left[\beta_n^m \left(h_n^{(1)}(\mu) + \frac{d}{dr} h_n^{(1)}(\mu r)|_{r=1} \right) V_n^m(\omega) \right.$$

$$\left. -\mu^2 \alpha_n^m h_n^{(1)}(\mu) U_n^m(\omega) \right]$$

and the boundary condition in (2.2) is satisfied if

$$\alpha_n^m \left[h_n^{(1)}(\mu) + \frac{d}{dr}(h_n^{(1)}(\mu r))|_{r=1} - \gamma \mathbf{i}\mu h_n^{(1)}(\mu) \right] = 0, \ \forall n \in \mathbb{N}, \ |m| \leq n, \tag{2.5}$$

$$-\frac{\beta_n^m \gamma}{\mathbf{i}\mu} \left[h_n^{(1)}(\mu) + \frac{d}{dr}(h_n^{(1)}(\mu r))|_{r=1} - \frac{\mathbf{i}\mu}{\gamma} h_n^{(1)}(\mu) \right] = 0, \ \forall n \in \mathbb{N}, \ |m| \leq n. \tag{2.6}$$

3. Roots of the equation $g_n(\lambda) = 0$

To examine the eigenvalues of G_b, it is necessary to find the roots of the equations (2.5) and (2.6). Since $h_n^{(1)}(\mu) \neq 0$ for $\operatorname{Im} \mu > 0$, the problem is reduced to study the roots $\lambda \in \mathbb{R}^-$ of the equation

$$1 + \frac{d}{dr} h_n^{(1)}(-\mathbf{i}\lambda r)\Big|_{r=1} (h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma = 0 \tag{3.1}$$

and the same equation with γ replaced by $\frac{1}{\gamma}$. Clearly, if $\mu = -\mathbf{i}\lambda$ is such that the expressions in the brackets [...] in (2.5) and (2.6) are non-vanishing for every $n \geq 1$, we must have $\alpha_n^m = \beta_n^m = 0$ which implies $E_{tan} = B_{tan} = 0$. Hence $(E, B) = 0$ because the boundary problem with $\gamma = 0$ has no eigenvalues in $\{z \in \mathbb{C} : \operatorname{Re} z < 0\}$. In this section we suppose that $\gamma \neq 1$ and examine the equation

$$g_n(\lambda) := \frac{1}{\lambda} + \frac{d}{d\lambda}\left(h_n^{(1)}(-\mathbf{i}\lambda)\right)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma = 0. \tag{3.2}$$

It is well known that (see [5])

$$h_n^{(1)}(-\mathbf{i}\lambda) = (-\mathbf{i})^{n+1} \frac{e^\lambda}{-\mathbf{i}\lambda} R_n\left(\frac{\mathbf{i}}{-2\mathbf{i}\lambda}\right) = (-\mathbf{i})^n \frac{e^\lambda}{\lambda} R_n\left(-\frac{1}{2\lambda}\right)$$

with

$$R_n(z) := \sum_{m=0}^{n} a_{m,n} z^m, \ a_{m,n} = \frac{(n+m)!}{m!(n-m)!} > 0.$$

We will prove the following

Proposition 3.1. *For $\lambda < 0$ we have*

$$G_{n,n+1}(\lambda) = \frac{\frac{d}{d\lambda} h_{n+1}^{(1)}(-\mathbf{i}\lambda)}{h_{n+1}^{(1)}(-\mathbf{i}\lambda)} - \frac{\frac{d}{d\lambda} h_n^{(1)}(-\mathbf{i}\lambda)}{h_n^{(1)}(-\mathbf{i}\lambda)} > 0. \tag{3.3}$$

Proof. The purpose is to show that

$$\left(h_n^{(1)}(-\mathbf{i}\lambda)\frac{d}{d\lambda}h_{n+1}^{(1)}(-\mathbf{i}\lambda) - h_{n+1}^{(1)}(-\mathbf{i}\lambda)\frac{d}{d\lambda}h_n^{(1)}(-\mathbf{i}\lambda)\right)\left(h_{n+1}^{(1)}(-\mathbf{i}\lambda)h_n^{(1)}(-\mathbf{i}\lambda)\right)^{-1} > 0.$$

Introduce the functions

$$\xi_n(\lambda) := \frac{e^\lambda}{\lambda} R_n\left(-\frac{1}{2\lambda}\right), \ \eta_n(\lambda) := \lambda\xi_n(\lambda).$$

Then $h_n^{(1)}(-\mathbf{i}\lambda) = (-\mathbf{i})^n \xi_n(\lambda)$ and the above inequality is equivalent to

$$\left(\xi_n(\lambda)\frac{d}{d\lambda}\xi_{n+1}(\lambda) - \xi_{n+1}(\lambda)\frac{d}{d\lambda}\xi_n(\lambda)\right)\left(\xi_{n+1}(\lambda)\xi_n(\lambda)\right)^{-1}$$

$$= \left(\eta_n(\lambda)\frac{d}{d\lambda}\eta_{n+1}(\lambda) - \eta_{n+1}(\lambda)\frac{d}{d\lambda}\eta_n(\lambda)\right)\left(\eta_{n+1}(\lambda)\eta_n(\lambda)\right)^{-1} > 0.$$

Since $\eta_n(\lambda)\eta_{n+1}(\lambda) > 0$ for $\lambda < 0$, it suffices to show that the function

$$F(\lambda) = \eta_n(\lambda)\frac{d}{d\lambda}\eta_{n+1}(\lambda) - \eta_{n+1}(\lambda)\frac{d}{d\lambda}\eta_n(\lambda)$$

has positive values for $\lambda \in (-\infty, 0)$. Consider the derivative

$$F'(\lambda) = \eta_n(\lambda)\frac{d^2}{d\lambda^2}\eta_{n+1}(\lambda) - \eta_{n+1}(\lambda)\frac{d^2}{d\lambda^2}\eta_n(\lambda).$$

We have

$$\eta_n(\lambda) = \mathbf{i}^{n+1} h_n^{(1)}(-\mathbf{i}\lambda)(-\mathbf{i}\lambda) = \mathbf{i}^{n+1}\Xi_n(-\mathbf{i}\lambda) = -\mathbf{i}^{n-1}\Xi_n(-\mathbf{i}\lambda).$$

The function $\Xi_n(z) = z h_n^{(1)}(z)$ satisfies the equation

$$\Xi_n''(z) + \left(1 - \frac{n^2 + n}{z^2}\right)\Xi_n(z) = 0$$

and

$$\frac{d^2}{d\lambda^2}\eta_n(\lambda) = \mathbf{i}^{n-1}\Xi_n''(-\mathbf{i}\lambda) = -\mathbf{i}^{n-1}\left(1 + \frac{n^2 + n}{\lambda^2}\right)\Xi_n(-\mathbf{i}\lambda)$$

$$= \left(1 + \frac{n^2 + n}{\lambda^2}\right)\eta_n(\lambda).$$

Consequently,

$$F'(\lambda) = \left[\frac{(n+1)^2 + n + 1}{\lambda^2} - \frac{n^2 + n}{\lambda^2}\right]\eta_n(\lambda)\eta_{n+1}(\lambda)$$

$$= 2(n+2)\frac{\eta_n(\lambda)\eta_{n+1}(\lambda)}{\lambda^2} > 0.$$

On the other hand,

$$F(\lambda) = e^\lambda R_n\left(-\frac{1}{2\lambda}\right)\frac{d}{d\lambda}\left(e^\lambda R_{n+1}\left(-\frac{1}{2\lambda}\right)\right) - e^\lambda R_{n+1}\left(-\frac{1}{2\lambda}\right)\frac{d}{d\lambda}\left(e^\lambda R_n\left(-\frac{1}{2\lambda}\right)\right)$$

$$= \frac{e^{2\lambda}}{2\lambda^2}\left[R_n\left(-\frac{1}{2\lambda}\right)R_{n+1}'\left(-\frac{1}{2\lambda}\right) - R_{n+1}\left(-\frac{1}{2\lambda}\right)R_n'\left(-\frac{1}{2\lambda}\right)\right]$$

and

$$\lim_{\lambda \to -\infty} F(\lambda) = 0, \ \lim_{\lambda \nearrow 0} F(\lambda) = +\infty$$

since

$$\lim_{w \to +\infty}\left[R_n(w)R_{n+1}'(w) - R_{n+1}(w)R_n'(w)\right] = +\infty.$$

Finally, the function $F(\lambda)$ in the interval $(-\infty, 0]$ is increasing from 0 to $+\infty$ and this completes the proof. □

Now if $\lambda_n < 0$ is a solution the equation (3.2) one has

$$g_{n+1}(\lambda_n) = \frac{1}{\lambda_n} + \left(\frac{d}{d\lambda}h_{n+1}^{(1)}(-i\lambda_n)\right)(h_{n+1}^{(1)}(-i\lambda_n))^{-1} - \gamma = G_{n,n+1}(\lambda_n) > 0,$$

so λ_n is not a root of the equation

$$g_{n+1}(\lambda) = \frac{1}{\lambda} + \left(\frac{d}{d\lambda}h_{n+1}^{(1)}(-\mathbf{i}\lambda)\right)(h_{n+1}^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma = 0.$$

In the following we assume that $\gamma > 1$. Then for $\lambda \to -\infty$ we have $g_{n+1}(\lambda) \to 1 - \gamma < 0$, and since $g_{n+1}(\lambda_n) > 0$ the equation $g_{n+1}(\lambda) = 0$ has at least one root $-\infty < \lambda_{n+1} < \lambda_n$.

Lemma 3.1. *Let $\gamma > 1$. For every $n \geq 1$ the equation $g_n(\lambda) = 0$ in the interval $(-\infty, 0)$ has exactly one root $\lambda_n < 0$.*

Proof. Setting $w = -\frac{1}{2\lambda}$, we write the equation (3.2) as $\mathcal{R}_n(w) := w^2 R_n'(w) + \alpha R_n(w) = 0$, where $\alpha = \frac{1-\gamma}{2} < 0$. We will show that this equation has exactly one positive root. Since

$$w^2 R_n'(w) = \sum_{k=1}^n k a_{k,n} w^{k+1}, \ R_n(w) = \sum_{k=0}^n a_{k,n} w^k,$$

the polynomial $\mathcal{R}_n(w)$ has the representation

$$\mathcal{R}_n(w) = \sum_{k=0}^{n+1} b_{k,n} w^k$$

with

$$\begin{cases} b_{k,n} = (k-1)a_{k-1,n} + \alpha a_{k,n}, \ 0 \le k \le n, \ a_{-1,n} = 0, \\ b_{n+1,n} = \frac{(2n)!}{(n-1)!}. \end{cases}$$

Taking into account the form of $a_{k,n}$, we deduce

$$b_{k,n} = \frac{(n+k-1)!}{(n-k+1)!k!}\Big(k(k-1) + \alpha(n+k)(n-k+1)\Big), \ 0 \le k \le n+1. \quad (3.4)$$

Thus the sign of $b_{k,n}$ depends on the sign of the function

$$B(k) := (1-\alpha)k^2 + (\alpha-1)k + \alpha(n^2+n)$$

which for $k \ge 1$ is increasing since

$$B'(k) = 2(1-\alpha)k + \alpha - 1 \ge 1 - \alpha > 0.$$

Clearly, $b_{0,n} = \alpha < 0$ and $b_{n+1,n} > 0$. There are two cases:

(i) $b_{1,n} \le 0$. Then there is only one change of sing in the Descartes' sequence $\{b_{n+1,n}, b_{n,n}, ..., b_{1,n}, b_{0,n}\}$.

(ii) $b_{1,n} > 0$. Then $b_{k,n} > 0$ for $1 \le k \le n+1$ and in the Descartes' sequence $\{b_{n+1,n}, b_{n,n}, ..., b_{1,n}, b_{0,n}\}$ one has again only one change of sign.

Applying the Descartes' rule of signs, we conclude that the number of the positive roots of $\mathcal{R}_n(w) = 0$ is exactly one.

$\square$

Combining Proposition 3.1 and Lemma 3.1, one obtain the following

Corollary 3.1. *Let $\gamma > 1$. Then the generator G_b has an infinite sequence of real eigenvalues*

$$-\infty < ... < \lambda_n < ... < \lambda_2 < \lambda_1 < 0$$

and λ_n has geometric multiplicity $2n+1$.

The geometric multiplicity is $2n+1$ since the functions $\{Y_{m,n}(\omega)\}_{m=-n}^m$ are linearly independent. The algebraic multiplicity of λ_m will be discussed in Section 5.

4. ESTIMATION OF THE ROOTS

Throughout this section we assume $\gamma > 1$. Set $\lambda = \frac{i\sqrt{z}}{h}$, $0 < h \ll 1$ with $z = -1 + i\eta$, $0 \le |\eta| \le h^{1/2}, \eta \in \mathbb{R}$. Consider the Dirichlet problem

$$\begin{cases} (h^2\Delta + z)w = 0, \ |x| > 1, w \in H^2(|x| > 1), \\ w = f, \ |x| = 1 \end{cases} \quad (4.1)$$

and note that $\Delta + \frac{z}{h^2} = \Delta - \lambda^2$. The solution of (4.1) has the form

$$w(r\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} h_n^{(1)}(-i\lambda r)(h_n^{(1)}(-i\lambda))^{-1}\alpha_{n,m}Y_{n,m}(\omega),$$

where

$$f(\omega) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{n,m} Y_{n,m}(\omega).$$

The semiclassical Dirichlet-to-Neumann operator $\mathcal{N}_{ext}(h,z) = \frac{h}{\mathbf{i}} \frac{d}{dr} w|_{r=1}$ related to (4.1) becomes

$$\mathcal{N}_{ext}(h,z) = -\mathbf{i}\sqrt{z} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (h_n^{(1)})'(-\mathbf{i}\lambda)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} \alpha_{n,m} Y_{n,m}$$

$$= \sqrt{z} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{d}{d\lambda}\left(h_n^{(1)}(-\mathbf{i}\lambda)\right)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} \alpha_{n,m} Y_{n,m}.$$

By using the approximation of $\mathcal{N}_{ext}(h,z)$ established in [9],[7] for $z = -1 + \mathbf{i}\eta$, one deduces

$$\|\mathcal{N}_{ext}(h,z)f - Op_h(\rho)f\|_{L^2(\mathbb{S}^2)} \leq C \frac{|\sqrt{z}|}{|\lambda|} \|f\|_{L^2(\mathbb{S}^2)}, \ 0 < h \leq h_0$$

with $\rho = \sqrt{z - r_0(x', \xi')}$ and a constant $C > 0$ independent of z, λ and f. Here $r_0(x', \xi')$ is the principal symbol of the semiclasssical Laplace-Beltrami operator $-h^2 \Delta_{\mathbb{S}^2} = \frac{z}{\lambda^2} \Delta_{\mathbb{S}^2}$ and $Op_h(\rho)$ is a $h-$pseudodifferential operator with symbol ρ. Moreover, $\sqrt{z} = \mathbf{i}\sqrt{1 - \mathbf{i}\eta} = \mathbf{i}(1 - \frac{\mathbf{i}\eta}{2} + \mathcal{O}(\eta^2))$ and

$$\operatorname{Re}\lambda = -\frac{1}{h} + \mathcal{O}(1), \ \operatorname{Im}\lambda = \mathcal{O}(h^{-1/2}).$$

Hence, for $0 < h \leq h_0$ we get

$$\lambda \in \Lambda_0 = \{z \in \mathbb{C} : \ |\operatorname{Im} z| \leq c h_0^{1/2} |\operatorname{Re} z|, \ \operatorname{Re}\lambda < -\epsilon < 0, \ |\lambda| \geq \lambda_0\}.$$

On the other hand, it easy to see that

$$\left\|Op_h(\rho) - \sqrt{z}\left(\sqrt{1 - \frac{\Delta_{\mathbb{S}^2}}{\lambda^2}}\right)\right\|_{L^2(\mathbb{S}^2) \to L^2(\mathbb{S}^2)} \leq C_1 |\lambda|^{-1}, \ \lambda \in \Lambda_0.$$

Applying the spectral calculus for the operator $\Delta_{\mathbb{S}^2}$, one deduces

$$\left(\sqrt{1 - \frac{\Delta_{S^2}}{\lambda^2}}\right)f = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\sqrt{1 + \frac{n(n+1)}{\lambda^2}}\right) \alpha_{n,m} Y_{n.m}$$

and

$$\left\|\left(\mathcal{N}_{ext}(h,-z) - \sqrt{z}\left(\sqrt{1 - \frac{\Delta_{S^2}}{\lambda^2}}\right)\right)f\right\|_{L^2(S^2)}^2 = |z| \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left|\frac{d}{d\lambda}\left(h_n(-i\lambda)\right)(h_n(-i\lambda))^{-1}\right.$$

$$\left. - \sqrt{1 + \frac{n(n+1)}{\lambda^2}}\right|^2 |a_{n,m}|^2.$$

This implies

$$\left|\frac{d}{d\lambda}\left(h_n^{(1)}(-\mathbf{i}\lambda)\right)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \sqrt{1 + \frac{n(n+1)}{\lambda^2}}\right| \leq C_2 |\lambda|^{-1}, \ \forall n \in \mathbb{N}, \ \lambda \in \Lambda_0 \ (4.2)$$

which we write as

$$\left|\left[\frac{1}{\lambda} + \frac{d}{d\lambda}\left(h_n^{(1)}(-\mathbf{i}\lambda)\right)(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \gamma\right] - \left[\sqrt{1 + \frac{n(n+1)}{\lambda^2}} - \gamma\right]\right| \leq C_0 |\lambda|^{-1}. \ (4.3)$$

Remark 4.1. *For bounded $1 \leq n \leq N_0$ and sufficiently large $|\lambda|$ the estimate (4.2) follows easily from the fact that $\frac{R'_n(w)}{R_n(w)} = n(n+1) + \mathcal{O}(|w|)$ as $|w| \to 0$.*

Remark 4.2. *The estimate (4.2) is similar to that in Proposition 2.1 in [8], where the function $\frac{J'_\nu(\lambda)}{J_\nu(\lambda)}$ for $\nu \geq 0$ and $0 < C \leq |\operatorname{Im}\lambda| \leq \delta |\operatorname{Re}\lambda|$, $\operatorname{Re}\lambda > C_1$ has been approximated. Here $J_\nu(z)$ is the Bessel function, while the boundary problem examined in [8] is in the bounded domain $\{x \in \mathbb{R}^3 : |x| < 1\}$.*

Put $z = \lambda$ and for $z \in \Lambda_0$ consider the function

$$f_n(z) := \sqrt{1 + \frac{n(n+1)}{z^2}} - \gamma$$

with zeros

$$z_n^\pm = \pm\sqrt{\frac{n^2 + n}{\gamma^2 - 1}}.$$

In the following we set $z_n = -\sqrt{\frac{n(n+1)}{\gamma^2-1}}$. Clearly,

$$f'_n(z) = -\frac{1}{z} \frac{\frac{n(n+1)}{z^2}}{\sqrt{1 + \frac{n(n+1)}{z^2}}}$$

and $\frac{n(n+1)}{z_n^2} = \gamma^2 - 1$, $f'_n(z_n) = -\frac{\gamma^2-1}{\gamma z_n}$. A calculus yields the second derivative

$$f''_n(z) = \frac{1}{z^2}\left[\frac{3n(n+1)}{z^2}\left(\sqrt{1 + \frac{n(n+1)}{z^2}}\right)\right.$$

$$\left. - \frac{n^2(n+1)^2}{z^4}\left(\sqrt{1 + \frac{n(n+1)}{z^2}}\right)^{-1/2}\right]\left(1 + \frac{n(n+1)}{z^2}\right)^{-1}.$$

For n large enough and $a > 0$ to be fixed below introduce the contour

$$C_n(a) := \{z = z_n + ae^{\mathbf{i}\varphi}, \; 0 \leq \varphi < 2\pi\} \subset \Lambda_0.$$

Our purpose is to choose a so that

$$|f_n(z)| \geq \frac{C_0}{|z|}, \quad \forall z \in C_n(a). \tag{4.4}$$

We have

$$z^2 = z_n^2 + 2z_n ae^{\mathbf{i}\varphi} + a^2 e^{2\mathbf{i}\varphi}$$

and

$$\frac{n(n+1)}{z^2} = (\gamma^2 - 1)\left(1 + \mathcal{O}\left(\frac{1}{n}\right)a + \mathcal{O}\left(\frac{1}{n^2}\right)a^2\right)^{-1}, \; z \in C_n(a). \tag{4.5}$$

On the other hand,

$$\sqrt{\frac{n(n+1)}{z^2} + 1} = \left[\frac{\gamma^2 + \mathcal{O}\left(\frac{1}{n}\right)a + \mathcal{O}(\frac{1}{n^2})a^2}{1 + \mathcal{O}\left(\frac{1}{n}\right)a + \mathcal{O}(\frac{1}{n^2})a^2}\right]^{1/2}.$$

Clearly, one has the estimate

$$|f_n(z)| \geq \frac{\gamma^2 - 1}{\gamma |z_n|}a - \frac{a^2}{2}\sup_{z \in C_n(a)}|f''_n(z)|, \; z \in C_n(a). \tag{4.6}$$

Set $C_\gamma = \frac{\gamma^2 - 1}{\gamma} > 0$ and choose $a > 0$ so that $C_\gamma a > 4C_0$. We fix a and obtain

$$\frac{C_\gamma a}{2|z_n|} > \frac{2C_0}{|z_n|} > \frac{C_0}{|z_n||1 + \frac{ae^{i\varphi}}{z_n}|}, \quad 0 \leq \varphi < 2\pi,$$

taking n large enough to satisfy the inequality

$$\frac{1}{\left|1 + \frac{ae^{i\varphi}}{z_n}\right|} < 2.$$

Next we arrange the inequality

$$\frac{C_\gamma a}{2|z_n|} - \frac{a^2}{2} \sup_{z \in C_n(a)} |f_n''(z)| > 0. \tag{4.7}$$

It is clear that

$$f_n''(z) = \frac{1}{z^2} G\Big(\frac{n(n+1)}{z^2}\Big),$$

where

$$G(\zeta) = \Big[3\zeta\sqrt{\zeta + 1} - \zeta^2(\zeta + 1)^{-1/2}\Big](\zeta + 1)^{-1}.$$

Note that for $z \in C_n(a)$ and n large enough according to (4.4), the function $|G(\frac{n(n+1)}{z^2})|$ is bounded by a constant $B_{\gamma, a}$ depending on γ and a . Thus for large n we get

$$\sup_{z \in C_n(a)} |f_n''(z)| \leq B_{\gamma, a} \sup_{z \in C_n(a)} \frac{1}{|z|^2} = B_{\gamma, a} \frac{1}{|z_n|^2} \sup_{z \in C_n(a)} \frac{1}{|1 + \frac{ae^{i\varphi}}{z_n}|^2} \leq 4B_{\gamma, a} \frac{1}{|z_n|^2}$$

and the proof of (4.7) is reduced to

$$C_\gamma > 4B_{\gamma, a} \frac{a}{|z_n|}$$

which is satisfied taking again n large. Finally, we proved the estimate (4.3) and we can apply Rouché theorem for the functions $g_n(z)$ and $f_n(z)$ and conclude that the function $g_n(z)$ has exactly one simple zero λ_n in $C_n(a)$. Since $g_n(z)$ has only real zeros (see Appendix in [2]), this implies the following

Lemma 4.1. *There exist $n_0(\gamma)$ and $a(\gamma) > 0$ depending on γ such that for $n \geq n_0(\gamma)$ the negative root λ_n of the equation (3.2) satisfies the estimate*

$$\left|\lambda_n + \sqrt{\frac{n(n+1)}{\gamma^2 - 1}}\right| \leq a(\gamma). \tag{4.8}$$

Remark 4.3. *According to Proposition 2.1, $n_0(\gamma)$ must satisfy the inequality*

$$n_0(\gamma) \geq \frac{\sqrt{\gamma^2 - 1}}{\max\{\gamma - 1, \sqrt{\gamma - 1}\}}.$$

5. Weyl asymptotics

We start with the analysis of the multiplicity of λ_n.

Lemma 5.1. *For $n \geq n_0(\gamma)$ we have $\mathrm{mult}(\lambda_n) = 2n + 1$.*

Proof. Since the geometric multiplicity of λ_n is $2n+1$, it is sufficient to show that

$$\text{mult}(\lambda_n) \leq 2n+1. \tag{5.1}$$

Let $\lambda \in \Lambda_0$, where Λ_0 is the set introduced in the previous section and let $\lambda \notin \sigma(G_b)$. If $0 \neq (f,g) \in \text{Image } G_b \cap L^2(\Omega)$, one has $\text{div } f = \text{div } g = 0$ and for $(u,v) = (G_b - \lambda)^{-1}(f,g)$ we get $\text{div } u = \text{div } v = 0$. Consider the skew self-adjoint operator

$$A = \begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix}$$

with boundary condition $\nu \wedge u = 0$ on $\mathbb{S}^2$. Then $\sigma(A) \subset \mathbf{i}\mathbb{R}$ and let $(u_0(x;\lambda), v_0(x;\lambda)) = (A - \lambda)^{-1}(f,g)$, that is

$$\begin{cases} (A - \lambda) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} & \text{for } |x| > 1, \\ \nu \wedge u_0 = 0 \text{ on } \mathbb{S}^2. \end{cases} \tag{5.2}$$

Since $\text{div } u_0 = \text{div } v_0 = 0$, the well known coercive estimates yield $(u_0, v_0) \in H^1(\Omega)$. Moreover the resolvent $(A - \lambda)^{-1}$ is analytic in $\{z \in \mathbb{C} : \text{Re } z < 0\}$ and $u_0(x;\lambda)$, $v_0(x;\lambda)$ depend analytically on λ. We write $(u,v) = (u_0, v_0) + (u_1, v_1)$, where $(u_1(x;\lambda), v_1(x;\lambda))$ is the solution of the problem

$$\begin{cases} (G - \lambda) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{for } |x| > 1, \\ (u_1)_{tan} - \gamma(\nu \wedge (v_1)_{tan}) = -\gamma(\nu \wedge (v_0)_{tan}(x;z)) \text{ on } \mathbb{S}^2. \end{cases} \tag{5.3}$$

To solve (5.3), note that $-\gamma(\nu \wedge (v_0)_{tan}(\omega;z)) = F(\omega;\lambda) \in L^2(\mathbb{S}^2)$ with $F(\omega;\lambda)$ analytical in λ for $\lambda \in \Lambda_0$. Thus we may write

$$F(\omega;\lambda) = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \tilde{\alpha}_n^m(\lambda) U_n^m(\omega) + \tilde{\beta}_n^m(\lambda) V_n^m(\omega)$$

with analytical coefficients $\tilde{\alpha}_n^m(\lambda), \tilde{\beta}_n^m(\lambda)$. Now we can solve (2.5), (2.6) with right hand part $(\tilde{\alpha}_n^m(\lambda), \tilde{\beta}_n^m(\lambda))$. Finally, we obtain a representation of the solution of (5.3) with meromorphic coefficients

$$\alpha_n^m(\lambda) = \frac{\tilde{\alpha}_n^m(\lambda)}{h_n^{(1)}(-\mathbf{i}\lambda)\left[1 + \frac{d}{dr}(h_n^{(1)}(-\mathbf{i}\lambda r))|_{r=1}(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma\right]},$$

$$\beta_n^m(\lambda) = -\frac{\lambda\tilde{\beta}_n^m(\lambda)}{\gamma h_n^{(1)}(-\mathbf{i}\lambda)\left[1 + \frac{d}{dr}(h_n^{(1)}(-\mathbf{i}\lambda r))|_{r=1}(h_n^{(1)}(-\mathbf{i}\lambda))^{-1} - \lambda\gamma^{-1}\right]}.$$

If $\gamma > 1$ the analysis in the previous section shows that for $\lambda \in \Lambda_0$ the meromorphic function $\alpha_n^m(\lambda)$ has a simple pole at $\lambda_n < 0$, while $\beta_n^m(\lambda)$ is analytic in Λ_0. For $0 < \gamma < 1$ the function $\alpha_n^m(\lambda)$ is analytic in Λ_0 and $\beta_n^m(\lambda)$ is meromorphic. Next we integrate $(u(x;\lambda), v(x;\lambda))$ over the circle $|\lambda_n - \lambda| = \epsilon$, where ϵ is sufficiently small. The integral of $(u_0(x;\lambda), v_0(x;\lambda))$ vanish, while for the integral of $(u_1(x;\lambda), v_1(x;\lambda))$, taking into account the representation of the solution of (5.3), we will obtain a sum

$$S_n = \begin{cases} c_n \sum_{m=-n}^{m} \tilde{\alpha}_n^m(\lambda_n) U_n^m(\omega), \ c_n \neq 0, \ \gamma > 1, \\ d_n \sum_{m=-n}^{m} \lambda_n \tilde{\beta}_n^m(\lambda_n) \gamma^{-1} V_n^m(\omega), \ d_n \neq 0, \ 0 < \gamma < 1. \end{cases}$$

This completes the proof of (5.1).   □

Passing to the analysis of $N(r)$, consider first the case $\gamma > 1$. The root λ_n has algebraic multiplicity $2n+1$ and to find a lower bound of $N(r)$ we apply the estimate

$$|\lambda_n| \leq \sqrt{\frac{n(n+1)}{\gamma^2 - 1}} + a(\gamma) < \frac{n+1}{\sqrt{\gamma^2 - 1}} + a(\gamma) \leq r$$

for $r \geq a(\gamma) + \frac{n_0(\gamma)+1}{\sqrt{\gamma^2-1}}$. Then

$$N(r) \geq \sum_{j=n_0(\gamma)}^{[(r-a(\gamma))\sqrt{\gamma^2-1}-1]} (2j+1) = (\gamma^2 - 1)r^2 + \mathcal{O}_\gamma(r) + A_\gamma.$$

To get a upper bound for $N(r)$, we use the estimate

$$|\lambda_n| \geq \sqrt{\frac{n(n+1)}{\gamma^2 - 1}} - a(\gamma) > \frac{n}{\sqrt{\gamma^2 - 1}} - a(\gamma) \geq r$$

for

$$n \geq (r + a(\gamma))\sqrt{\gamma^2 - 1} \geq 2a(\gamma)\sqrt{\gamma^2 - 1} + n_0(\gamma) + 1,$$

hence

$$N(r) \leq \sum_{j=n_0(\gamma)}^{[(r+a(\gamma))\sqrt{\gamma^2-1}]+1} (2j+1) + D_\gamma = (\gamma^2 - 1)r^2 + \mathcal{O}_\gamma(r) + A'_\gamma.$$

If $0 < \gamma < 1$, we have $\frac{1}{\gamma} > 1$ and one applies our argument to the the equation (2.6). This completes the proof of theorem 1.1

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