POLYNOMIAL BOUNDS ON THE SOBOLEV NORMS OF THE SOLUTIONS OF THE NONLINEAR WAVE EQUATION WITH TIME DEPENDENT POTENTIAL

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Abstract. We consider the Cauchy problem for the nonlinear wave equation $u_{tt} - \Delta_x u + q(t,x)u + u^3 = 0$ with smooth and periodic in time potential $q(t,x) \geq 0$ having compact support with respect to x. The linear equation without the nonlinear term u^3 may have solutions with exponentially increasing as $t \to \infty$ norm $H^1(\mathbb{R}^3_x)$. In [2] it was established that adding the nonlinear term u^3 the $H^1(\mathbb{R}^3_x)$ norm of the solution is polynomially bounded for every choice of q. In this paper we show that $H^k(\mathbb{R}^3_x)$ norm of this global solution is also polynomially bounded. To prove this we apply a different argument based on the analysis of a sequence $\{Y_k(n\tau_k)\}_{n=0}^{\infty}$ with suitably defined energy norm $Y_k(t)$ and $0 < \tau_k < 1$.

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1. Introduction

Consider the Cauchy problem

$$\partial_t^2 u - \Delta_x u + q(t,x)u + u^3 = 0,\ u(0,x) = f_1(x),\ \partial_t u(0,x) = f_2(x),\ t \in \mathbb{R},\ x \in \mathbb{R}^3, \tag{1.1}$$

where $0 \leq q(t,x) \in C^\infty$ is periodic in time with period $T > 0$ and $q(t,x) = 0$ for $|x| \geq \rho > 0$. Set

$$\|u(t,x)\|_{\mathcal{H}} = \|u(t,x)\|_{H^1(\mathbb{R}^3)} + \|u_t(t,x)\|_{L^2(\mathbb{R}^3)}.$$

For the Cauchy problem for the linear operator $\partial_t^2 u - \Delta_x u + q(t,x)u$ there exist potentials $q(t,x) \geq 0$ for which for suitable initial data $f = (f_1, f_2) \in \mathcal{H}(\mathbb{R}^3) = H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ we have

$$\|u(t,x)\|_{H^1(\mathbb{R}^3)} \geq Ce^{\alpha|t|}$$

with $C > 0$, $\alpha > 0$ (see [1], [2]). This phenomenon is related to the so called parametric resonance. On the other hand, adding a nonlinear term u^3 for the Cauchy problem (1.1) there are no parametric resonances and for every potential q the solution $u(t,x)$ is defined globally for $t \in \mathbb{R}$ and satisfies a polynomial bound

$$\|u(t,x)\|_{H^1(\mathbb{R}^3)} \leq B_1(1 + B_0|t|)^2$$

with constants $B_0 > 0, B_1 > 0$ depending on q and the initial data $f \in \mathcal{H}$. This result has been obtained in [2] and the proof was based on the inequality

$$X'(t) \leq CX(t)^{1/2},$$

where

$$X(t) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\partial_t u|^2 + |\nabla_x u|^2 + qu^2 + \frac{1}{2}u^4\right)dx.$$

In this paper we study the problem (1.1) with initial data $f \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$, $k \geq 2$. First in Section 2 we establish a local result and we show the existence and uniqueness of solution for $t \in [s, s+\tau_k]$ with initial data $f \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ on $t = s$ and

$$\tau_k = c_k(1 + \|(f_1, f_2)\|_{\mathcal{H}(\mathbb{R}^3)})^{-\gamma},\ \gamma > 0,$$

where c_k depends on q and k (see Proposition 1). It is important to notice that τ_k depends on the norm $\|f\|_{\mathcal{H}}$ and since we have a global bound for the $\mathcal{H}$ norm of $(u, u_t)(t, x)$, the interval of local existence depends on the $\mathcal{H}$ norm of the initial data. We prove this result without using local Strichartz estimates. Next we show that the global solution in $\mathbb{R}$ is in $H^k(\mathbb{R}^3)$ for all $t \in \mathbb{R}$ and the problem is to examine if the norm $\|u(t,x)\|_{H^k(\mathbb{R}^3)}$, $k \geq 2$, is polynomially bounded. To do this, it is not possible to define a suitable energy $Y_k(t) \geq 0$ involving

$$\int_{\mathbb{R}^3} (\|u(t,x)\|^2_{H^k(\mathbb{R}^3)} + \|u_t(t,x)\|^2_{H^{k-1}(\mathbb{R}^3)}) dx$$

for which $Y_k'(t) \leq C_k Y_k^{\gamma_k}(t)$, $0 < \gamma_k < 1$. To overcome this difficulty, we follow another argument based on Lemma 1 (see Section 4) which has an independent interest and apply local Strichartz estimates for the nonlinear equation. We study first the case $k = 2$ in Section 5 and by induction we cover the case $k \geq 3$ in Section 6. Our principal result is the following

Theorem 1. *For every potential q and every $k \geq 2$ the problem* (1.1) *with initial data $f \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ has a global solution $u(t,x)$ and there exist $A_k > 0$ and $m_k \geq 2$ depending on q, k and $\|f\|_{\mathcal{H}}$ such that*

$$\|u(t,x)\|_{H^k(\mathbb{R}^3)} + \|\partial_t u(t,x)\|_{H^k(\mathbb{R}^3)} \leq A_k(1 + |t|)^{m_k},\ t \in \mathbb{R}. \tag{1.2}$$

We refer to [3] and the references therein for other results about polynomial bounds for the solutions of Hamiltonian partial differential equations. The method of the proof of Theorem 1 basically follows the approach in [3]. The main difficulty compared to [3] is that in our situation, we do not have uniform bound on the $H^1(\mathbb{R}^3)$ norm and for that purpose we need to apply the estimate of Lemma 1 below.

2. Existence and uniqueness of local solutions in $H^k(\mathbb{R}^3)$, $k \geq 3$

In this section we study the existence and uniqueness of local solutions of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta_x u + q(t,x)u + u^3 = 0, t \in [s, s+\tau],\ x \in \mathbb{R}^3, \\ u(s,x) = f_1(x), u_t(s,x) = f_2(x), \end{cases} \tag{2.1}$$

where $f = (f_1, f_2) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$, $k \geq 1, 0 < \tau < 1$. We assume that $[s, s+\tau] \subset [0, a]$, where $a > 1$ is fixed. The cases $k = 1, 2$ has been investigated in Section 3, [2] by using the norms

$$\|u\|_{S_{k-1}} := \|(u, u_t)\|_{C([s,s+\tau], H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3))}.$$

For $k = 1$ the space S_0 has been denoted as S. The number τ is given by

$$\tau = c_1(1 + \|(f_1, f_2)\|_{\mathcal{H}})^{-\gamma} < 1 \tag{2.2}$$

with some positive constants $c_1 > 0, \gamma > 0$ depending on q. The case $k \geq 3$ can be handled by a similar argument. We will show that with τ defined by (2.2) with the constant c_1 replaced by $0 < c_k \leq c_1$ depending on k and q one has a local existence and uniqueness in the interval $[s, s+\tau]$. Consider the linear problem

$$\partial_t^2 u_{n+1} - \Delta u_{n+1} + q(t,x)u_{n+1} + u_n^3 = 0,\ u_{n+1}(s,x) = f_1(x),\ \partial_t u_{n+1}(s,x) = f_2(x) \tag{2.3}$$

for $t \in [s, s+\tau]$ with $u_0 = 0$. For the solution of the above problem with right hand part $-u_n^3$ and $f = (f_1, f_2)$ we have a representation

$$(u_{n+1}, (u_{n+1})_t) = U_0(t-s)f - \int_s^t \Big[U_0(t-\tau)Q(\tau)u_{n+1}(\tau,x) + U_0(t-\tau)Q_0 u_n^3(\tau,x)\Big] d\tau. \tag{2.4}$$

Here $U_0(t,s) : \mathcal{H} \to \mathcal{H}$ is the propagator related to the free wave equation in $\mathbb{R}^3$ (see Section 2, [2]) and

$$Q(\tau) = \begin{pmatrix} 0 & 0 \\ q(\tau,x) & 0 \end{pmatrix},$$

$$Q_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

To estimate $\|u_{n+1}\|_{S_k}$, we apply the operator

$$L_k = \begin{pmatrix} (1-\Delta)^{k/2} & 0 \\ 0 & (1-\Delta)^{(k-1)/2} \end{pmatrix}.$$

Notice that this operator commute with $U_0(t-\tau)$and

$$\|U_0(t-s)\|_{\mathcal{H}\to\mathcal{H}} \leq A$$

for $|t-s| \leq 1$ with $A > 0$ independent on k. Therefore

$$\|U_0(t-s)L_k f\|_{\mathcal{H}} \leq C\|f\|_{H^{k+1}\times H^k}.$$

and

$$\Big\|\int_s^\tau U_0(t-\tau)L_k Q(\tau)u_{n+1}(\tau,x)d\tau\Big\|_{\mathcal{H}} \leq \int_s^\tau \|U_0(t-\tau)L_k Q(\tau)u_{n+1}\|_{\mathcal{H}} d\tau \leq A_k \tau \|u_{n+1}\|_{S_k}.$$

For $\tau A_k \leq 1/2$ with $A_k > 0$, depending on k and q, the term involving u_{n+1} can be absorbed by $\|u_{n+1}\|_{S_k}$ and we deduce

$$\|u_{n+1}\|_{S_k} \leq C\|(f_1,f_2)\|_{H^{k+1}(\mathbb{R}^3)\times H^k(\mathbb{R}^3)} + C\|u_n^3\|_{L^1([s,s+\tau],H^k(\mathbb{R}^3))}.$$

Here and below the constants C depend on k and q and they may change from line to line but we will omit this in the notations. Next we define the norm

$$\|f\|_{H^{s,p}(\mathbb{R}^3)} := \|(1-\Delta_x)^{s/2} f\|_{L^p(\mathbb{R}^3)},\ 1 < p \leq \infty.$$

We will use the following *product estimate*

$$\|fg\|_{H^{s,p}} \leq A_{s,p}\|f\|_{L^{q_1}}\|g\|_{H^{s,q_2}} + A_{s,p}\|g\|_{L^{r_1}}\|f\|_{H^{s,r_2}}, \tag{2.5}$$

provided

$$\frac{1}{p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2},\ q_1, r_1 \in (1,\infty],\ q_2, r_2 \in (1,\infty].$$

For the proof of the classical estimate (2.5) we refer to [4]. We apply (2.5) with $p = 2, q_1 = 3, q_2 = 6, r_1 = 6, r_2 = 3$ and get

$$\|u_n^3\|_{H^k(\mathbb{R}^3)} \leq C\|u_n\|_{H^{k,6}(\mathbb{R}^3)}\|u_n\|^2_{L^6(\mathbb{R}^3)} + C\|u_n^2\|_{H^{k,3}(\mathbb{R}^3)}\|u_n\|_{L^6(\mathbb{R}^3)}.$$

For the term involving u_n^2 we apply the same estimate with $p = 3, q_1 = q_2 = r_1 = r_2 = 6$ and deduce

$$\|u_n^2\|_{H^{k,3}(\mathbb{R}^3)} \leq 2C\|u_n\|_{H^{k,6}(\mathbb{R}^3)}\|u_n\|_{L^6(\mathbb{R}^3).}$$

Consequently, by Sobolev embedding theorem

$$\|u_n^3\|_{H^k(\mathbb{R}^3)} \leq C_1\|u_n\|_{H^{k+1}(\mathbb{R}^3)}\|\nabla_x u_n\|^2_{L^2(\mathbb{R}^3)}.$$

This implies

$$\int_s^{s+\tau} \|u_n^3\|_{H^k(\mathbb{R}^3)} dt \leq \tau \|u_n\|^2_{L^\infty([s,s+\tau]),H^1(\mathbb{R}^3))}\|u_n\|_{S_k}.$$

On the other hand, for the solution u_n we have the estimate

$$\|u_n\|_{C([s,s+\tau],H^1(\mathbb{R}^3))} \leq 2C_0\|(f_1,f_2)\|_{\mathcal{H}},\ \forall n \geq 1$$

with some constant $C_0 > 0$ depending on q (see Section 3, [2]) and we deduce the bound

$$C\|u_n^3\|_{L^1([s,s+\tau],H^k(\mathbb{R}^3))} \leq CC_1\tau(2C_0)^3\|(f_1,f_2)\|^2_{\mathcal{H}}\|u_n\|_{S_{k+1}}.$$

Thus choosing

$$2CC_1\tau(2C_0)^2\|(f_1,f_2)\|^2_{\mathcal{H}} \leq 1,$$

we may prove by induction the estimate

$$\|u_n\|_{S_k} \leq 2C\|(f_1, f_2)\|_{H^{k+1}(\mathbb{R}^3)\times H^k(\mathbb{R}^3)},\ \forall n \geq 1. \tag{2.6}$$

Repeating the argument of [2] we obtain local existence and uniqueness. Thus we get the following

Proposition 1. *For every $k \geq 1$ there exist $C_k > 0$, $c_k > 0$ and $\gamma > 0$ depending on q and k such that for every $(f_1, f_2) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ there is a unique solution $(u, u_t) \in C([s, s + \tau_k], H^k(\mathbb{R}^3)\times H^{k-1}(\mathbb{R}^3)$ of the problem (2.1) on $[s, s+\tau_k]$ with $\tau_k = c_k(1+\|(f_1, f_2)\|_{\mathcal{H}})^{-\gamma}$. Moreover, the solution satisfies*

$$\|u\|_{S_k} \leq C_k\|(f_1, f_2)\|_{H^k(\mathbb{R}^3)\times H^{k_1}(\mathbb{R}^3)}. \tag{2.7}$$

It is important to note that for every k, τ_k depends on the $\mathcal{H}$ norm of the initial data.

In [2] it was proved that one has a global solution $(u, u_t) \in C(\mathbb{R}, \mathcal{H}(\mathbb{R}^3))$ with initial data $(f_1, f_2) \in \mathcal{H}(\mathbb{R}^3)$. It is natural to expect that for $(f_1, f_2) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$ we have a global solution $(u, u_t) \in C(\mathbb{R}, H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3))$.

Let $a > 1$ be fixed and let $k \geq 1$. We wish to prove that the global solution with initial data $f \in H^{k+1}(\mathbb{R}^3) \times H^k(\mathbb{R}^3)$ is such that

$$(u, u_t)(t, x) \in H^{k+1}(\mathbb{R}^3) \times H^k(\mathbb{R}^3),\ 0 \leq t \leq a. \tag{2.8}$$

According to the result in [2], for $0 \leq t \leq a$ we have an estimate

$$\|(u, u_t)(t, x)\|_{\mathcal{H}} \leq B_a = \|f\|_{\mathcal{H}} + a(B_1 + B_2a),$$

where $B_1 > 0$ and $B_2 > 0$ depend only on $\|f\|_{\mathcal{H}}$. Consider

$$\tau_k(a) = c_k(1 + B_a)^{-\gamma}. \tag{2.9}$$

First for $0 \leq t \leq \tau_k(a)$ we apply Proposition 1. Next we apply Proposition 1 for the problem with initial data on $t = \frac{2}{3}\tau_k(a)$ which is bounded by (2.7). Thus we obtain a solution in $[0, \frac{5}{3}\tau_k(a)]$ and we continue this procedure by step $\frac{2}{3}\tau_k(a)$. On every step the norm $H^{k+1}(\mathbb{R}^3) \times H^k(\mathbb{R}^3)$ of (u, u_t) will increase with a constant C_k. Finally, if

$$\frac{3}{2}a \leq m\tau_k(a) \leq \frac{3}{2}(a + 1),$$

we deduce

$$\|(u, u_t)(a, x)\|_{H^{k+1}\times H^k} \leq C_k^m\|(f_1, f_2)\|_{H^{k+1}\times H^k} \leq e^{\frac{3}{2\tau_k(a)}\log C_k(a+1)}\|(f_1, f_2)\|_{H^{k+1}\times H^k}. \tag{2.10}$$

Hence, we established (2.8) and one has a bound of $H^{k+1} \times H^k$ norm. Since a is arbitrary, we obtain the result for all $t \in \mathbb{R}$. In Section 6 we will improve (2.10) to polynomial bounds of the Sobolev norms.

3. Local Strichartz estimate for the nonlinear wave equation

Our purpose is to establish a local Strichartz estimate for the solution of the Cauchy problem

$$\begin{cases} u_{tt} - \Delta_x u + q(t, x)u + u^3 = 0, t \in]s, s + \tau], x \in \mathbb{R}^3, \\ u(s, x) = f_1(x), u_t(s, x) = f_2(x), \end{cases} \tag{3.1}$$

where $f = (f_1, f_2) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, $0 < \tau \leq 1$. It well known (see [2]) that for the solution of the Cauchy problem

$$\begin{cases} v_{tt} - \Delta_x v = F,\ (t, x) \in]s, s + \tau] \times \mathbb{R}^3, \\ v(s, x) = h_1(x), v_t(s, x) = h_2(x), \end{cases} \tag{3.2}$$

we have an estimate

$$\|v(t, x)\|_{L^p([s,s+\tau],L^r_x(\mathbb{R}^3))} \leq C\Big(\|(h_1, h_2)\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} + \|F\|_{L^1([s,s+\tau],L^2(\mathbb{R}^3))}\Big),$$

where $\frac{1}{p} + \frac{3}{r} = \frac{1}{2}$, $2 < p \leq \infty$. We will choice later $r = \frac{4+2\epsilon}{\epsilon}$ with $0 < \epsilon \ll 1$ and this determines the choice of $p > 2$. For the solution of (3.1) we get

$$\|u(t,x)\|_{L^p([s,s+\tau],L^r_x(\mathbb{R}^3))} \leq C(p,r)\Big(\|u(s,x),u_t(s,x)\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} + \tau\|u(t,x)\|^3_{L^\infty([s,s+\tau],H^1(\mathbb{R}^3))}\Big), \tag{3.3}$$

where we have used the estimate

$$\|u^3(t,x)\|_{L^1([s,s+\tau],L^2(\mathbb{R}^3))} \leq \tau\|u(t,x)\|^3_{L^\infty([s,s+\tau],H^1(\mathbb{R}^3))}.$$

Next, for the solution $u(t,x) \in H^1(\mathbb{R}^3)$ of (3.1) in $]0,s+\tau]$ with initial data $f = (u,u_t)(0,x) \in \mathcal{H}(\mathbb{R}^3)$ we have a polynomial bound (see Section 3, [2])

$$\sup_{t\in[0,s+\tau]} \|u(t,x)\|_{H^1(\mathbb{R}^3)} \leq \|f\|_{\mathcal{H}(\mathbb{R}^3)} + s(B_1 + B_2 s)^2,$$

where $B_1 > 0, B_2 > 0$ depend only on $\|f\|_{\mathcal{H}}$, and this implies

$$\|u(t,x)\|_{L^p([s,s+\tau],L^r_x(\mathbb{R}^3))} \leq C_1(p,r,\|f\|_{\mathcal{H}})(1+s)^6. \tag{3.4}$$

Now we will examine the continuous dependence on the initial data of the local solution to (2.1) given in Section 2. Let $g_n = ((g_n)_1,(g_n)_2) \in H^{k+1}(\mathbb{R}^3)\times H^k(\mathbb{R}^3)$ be a sequence converging in $H^k(\mathbb{R}^3)\times H^{k-1}(\mathbb{R}^3)$ to $f = (f_1,f_2) \in H^k(\mathbb{R}^3)\times H^{k-1}(\mathbb{R}^3)$. Let

$$w_n(t,x) \in C([s,s+\tau],H^{k+1}(\mathbb{R}^3)) \cap C^1([s,s+\tau],H^k(\mathbb{R}^3))$$

be the local solution of (3.1) with initial data g_n. Setting $v_n = w_n - u$, we obtain for v_n the equation

$$\partial_t^2 v_n - \Delta_x v_n + q(t,x)v_n = u^3 - w_n^3.$$

By the local Strichartz estimates for the linear equation with respect to v_n, we get

$$\begin{aligned}&\|(v_n,(v_n)_t)\|_{C([s,s+\tau],H^k(\mathbb{R}^3)\times H^{k-1}(\mathbb{R}^3))} + \|v_n\|_{L^\infty_t([s,s+\tau],H^{k-1,6}_x(\mathbb{R}^3))}\\ &\leq C_k(a)\|g_n - f\|_{H^k(\mathbb{R}^3)\times H^{k-1}(\mathbb{R}^3)} + C_k(a)\|u^3 - w_n^3\|_{L^1_t([s,s+\tau],H^{k-1}_x(\mathbb{R}^3))}.\end{aligned} \tag{3.5}$$

This estimate for $k = 1,2$ has been proved in Proposition 1, [2]. The proof for $k \geq 3$ follows the same argument. The constant $C_k(a) > 0$ depends on k and on the interval $[0,a]$, where $[s,s+\tau] \subset [0,a]$. We will omit in the notations below the dependence of the constants on k and a. Applying (2.5), we have

$$\begin{aligned}\|u^3 - w_n^3\|_{H^{k-1}} &\leq C\|v_n\|_{H^{k-1,6}}\|u^2 + uw_n + w_n^2\|_{L^3} + C\|v_n\|_{L^6}\|u^2 + uw_n + w_n^2\|_{H^{k-1,3}}\\ &\leq 2C\|v_n\|_{H^{k-1,6}}\Big(\|u\|^2_{L^6} + \|w_n\|^2_{L^6}\Big)\end{aligned}$$

$$+C\|v_n\|_{L^6}\Big(2\|u\|_{H^{k-1,6}}\|u\|_{L^6} + 2\|w_n\|_{H^{k-1,6}}\|w_n\|_{L^6} + \|u\|_{H^{k-1,6}}\|w_n\|_{L^6} + \|w_n\|_{H^{k-1,6}}\|u\|_{L^6}\Big) = P_n + Q_n.$$

To handle P_n, notice that $L^\infty([s,s+\tau],L^6(\mathbb{R}^3))$ norms of u and w_n by local Strichartz estimates can be estimated by $\|f\|_{\mathcal{H}}$ and $\|g_n\|_{\mathcal{H}}$. Therefore, for $n \geq n_0$ we have

$$\Big|\int_s^{s+\tau} P_n dt\Big| \leq A_k\tau\|v_n\|_{L^\infty([s,s+\tau],H^{k-1,6}(\mathbb{R}^3))}$$

with a constant A_k depending on $C_k(a)$ and $\|f\|_{\mathcal{H}}$. Hence, we may absorb P_n by the left hand side of (3.5) choosing $0 < \tau \leq \frac{1}{2A_k}$ small. The analysis of Q_n is easy since we proved in [2] that for all $t \in [s,s+\tau]$ we have $\|\nabla_x v_n(t,x)\|_{L^2(\mathbb{R}^3)} \to 0$ as $n \to \infty$ and the term in the braked $\Big(...\Big)$ for $t \in [0,a]$ is uniformly bounded with respect to n according to the analysis in Section 2 and estimate (2.10). Finally, we conclude that

$$\|(v_n,(v_n)_t)\|_{C([s,s+\tau],H^k(\mathbb{R}^3)\times H^{k-1}(\mathbb{R}^3))} \to_{n\to\infty} 0. \tag{3.6}$$

4. Lemma

The aim in this section is to prove the following

Lemma 1. *Let $\{\alpha_n\}_{n=0}^{\infty}$ be a sequence of non-negative numbers such that with some constants $0<\gamma<1$, $C>0$ and $y\geq 0$ we have*

$$\alpha_n \leq \alpha_{n-1} + C((\alpha_{n-1})^{1-\gamma}+1)(1+n)^y,\ \forall n\geq 1.$$

Then there exists a constant $\tilde{C}>0$ such that

$$\alpha_n \leq \tilde{C}(1+n)^{\frac{1+y}{\gamma}},\ \forall n\geq 1. \tag{4.1}$$

Remark 1. *A similar estimate has been established in* [3] *for sequences $\{\alpha_n\}$ satisfying the inequality*

$$\alpha_n \leq \alpha_{n-1} + C\alpha_{n-1}^{1-\gamma}.$$

Proof. We can choose a large constant $C_1>0$ such that

$$(\alpha_{n-1})^{1-\gamma}+1 \leq C_1(\alpha_{n-1}+1)^{1-\gamma},\ \forall n\geq 1.$$

This implies with a new constant $C_2>0$ the inequality

$$\alpha_n+1 \leq \alpha_{n-1}+1+C_2(\alpha_{n-1}+1)^{1-\gamma}(1+n)^y,\ \forall n\geq 1.$$

Setting $\beta_n=\alpha_n+1$, we reduce the proof to a sequence α_n satisfying the inequality

$$\alpha_n \leq \alpha_{n-1}+C_2(\alpha_{n-1})^{1-\gamma}(1+n)^y,\ n\geq 1.$$

We will prove (4.1) by recurrence. Assume that (4.1) holds for $n-1$. Therefore

$$\begin{aligned}\alpha_n &\leq \tilde{C}n^{\frac{1+y}{\gamma}} + C_2\Big(\tilde{C}n^{\frac{1+y}{\gamma}}\Big)^{1-\gamma}(1+n)^y\\ &= \tilde{C}n^{\frac{1+y}{\gamma}}\Big[1+C_2\tilde{C}^{-\gamma}n^{-1-y}(1+n)^y\Big]\\ &= \tilde{C}(1+n)^{\frac{1+y}{\gamma}}\Big(1-\frac{1}{n+1}\Big)^{\frac{1+y}{\gamma}}\Big[1+C_2\tilde{C}^{-\gamma}n^{-1}\Big(\frac{n}{n+1}\Big)^{-y}\Big].\end{aligned}$$

To establish (4.1) for n, it is sufficient to show that for large $\tilde{C}$ one has

$$f(n) := \Big(1-\frac{1}{n+1}\Big)^{\frac{1+y}{\gamma}}\Big[1+C_2\tilde{C}^{-\gamma}n^{-1}\Big(\frac{n}{n+1}\Big)^{-y}\Big] \leq 1,\ n\geq 1. \tag{4.2}$$

Setting $C_2\tilde{C}^{-\gamma}=\epsilon$, a simple calculus yields

$$\begin{aligned}f'(n) &= \frac{1+y}{\gamma}\Big(1-\frac{1}{n+1}\Big)^{\frac{1+y}{\gamma}-1}\frac{1}{(n+1)^2}\Big[1+\frac{\epsilon}{n}\Big(\frac{n}{n+1}\Big)^{-y}\Big]\\ &+\epsilon\Big(1-\frac{1}{n+1}\Big)^{\frac{1+y}{\gamma}}\Big[-\frac{1}{n^2}\Big(\frac{n}{n+1}\Big)^{-y} - yn^{-1}\frac{1}{(n+1)^2}\Big(1-\frac{1}{n+1}\Big)^{-y-1}\Big]\\ &= \Big(1-\frac{1}{n+1}\Big)^{\frac{1+y}{\gamma}-1}\frac{1}{(n+1)^2}\Big[\frac{1+y}{\gamma}+\frac{\epsilon}{n}\frac{1+y}{\gamma}\Big(1-\frac{1}{n+1}\Big)^{-\gamma} - [\epsilon\frac{n+1}{n}+\frac{\epsilon y}{n}]\Big(1-\frac{1}{n+1}\Big)^{-y}\Big].\end{aligned}$$

Notice that since $\frac{1}{2}\leq 1-\frac{1}{n+1}$, we have

$$\Big(1-\frac{1}{n+1}\Big)^{-\gamma} \leq \Big(\frac{1}{2}\Big)^{-\gamma}$$

which implies

$$\frac{1+y}{\gamma} - \epsilon[\frac{n+1+y}{n}]\Big(1-\frac{1}{n+1}\Big)^{-y} \geq \frac{1+y}{\gamma} - \epsilon[\frac{n+1+y}{n}]\Big(\frac{1}{2}\Big)^{-y}.$$

For small $\epsilon > 0$ the right hand side of the above inequality is positive. Consequently, for the derivative we have $f'(n) > 0$ and one deduces

$$f(n) < \lim_{n \to +\infty} f(n) = 1$$

This completes the proof of (4.2). $\square$

5. Polynomial bound of the $H^2(\mathbb{R}^3)$ norm of the solution

Let

$$(u(t,x), u_t(t,x)) \in C([s, s+\tau], H^2(\mathbb{R}^3)) \times C([s, s+\tau], H^1(\mathbb{R}^3)),$$

where $u(t,x)$ is the solution for $t \in [s, s+\tau]$ of the Cauchy problem (2.1).

Taking the derivative $\partial_{x_j} = \partial_j$, $j = 1, 2, 3$, and noting $u_j = \partial_j u$, $u_{jt} = \partial_j \partial_t u$, one gets in the sense of distributions

$$(u_{jt})_t - \Delta_x u_j + (\partial_j q) u + q u_j + 3u^2 u_j = 0. \tag{5.1}$$

It is easy to see that

$$(\partial_j q) u + q u_j + 3u^2 u_j \in C([s, s+\tau], L^2(\mathbb{R}^3)).$$

In fact, our assumption implies that $u(t,x) \in C([s, s+\tau], L_x^\infty(\mathbb{R}^3))$ and this yields $u^2 u_j \in C([s, s+\tau], L^2(\mathbb{R}^3))$. Therefore

$$(u_{jt})_t - \Delta_x u_j \in C([s, s+\tau], L^2(\mathbb{R}^3)).$$

Multiplying the equality (5.1) by u_{jt}, we have

$$\begin{aligned}\int \Big((u_{jt})_t - \Delta_x u_j \Big) u_{jt} dx = - \int (\partial_j q) u u_{jt} dx - \int q u_j u_{jt} dx - 3 \int u^2 u_j u_{jt} dx \\ = I_1(t) + I_2(t) + I_3(t).\end{aligned} \tag{5.2}$$

Assuming $(u(t,x), u_t(t,x)) \in C([s, s+\tau], H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3))$, we can write

$$I_2(t) = -\frac{1}{2} \int q \partial_t (u_j^2) dx = -\frac{1}{2} \partial_t \Big(\int q u_j^2 dx \Big) + \frac{1}{2} \int q_t u_j^2 dx,$$

$$I_3(t) = -\frac{3}{2} \int u^2 \partial_t (u_j^2) dx = -\frac{3}{2} \partial_t \Big(\int u^2 u_j^2 dx \Big) + 3 \int u u_t u_j^2 dx.$$

After an integration by parts in the integral

$$\int \Delta_x (u_j) u_{jt} dx$$

for solutions $(u(t,x), u_t(t,x)) \in C([s, s+\tau], H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3))$ the equality (5.2) can be written as

$$\begin{aligned}\frac{1}{2} \partial_t \sum_{j=1}^3 \Big[\int \Big((u_{jt})^2 + |\nabla_x (u_j)|^2 + 3u^2 u_j^2 + q u_j^2 \Big)(t,x) dx \Big] = - \sum_{j=1}^3 \int (\partial_j q) u u_{jt} dx \\ + 3 \sum_{j=1}^3 \int u u_t u_j^2 dx + \frac{1}{2} \sum_{j=1}^3 \int q_t u_j^2 dx = I_1(t) + J_1(t) + J_2(t),\end{aligned} \tag{5.3}$$

where the derivative with respect to t of the left hand side is taken in sense of distributions.

5.1. **Justification of (5.3) for** $(u(t,x), u_t(t,x)) \in C([s, s+\tau], H^2 \times H^1)$**.** Introduce

$$X(t) := \frac{1}{2}\sum_{j=1}^{3}\int\Big((u_{jt})^2 + |\nabla_x(u_j)|^2 + 3u^2u_j^2 + qu_j^2\Big)(t,x)dx.$$

Notice that the function $X(t)$ is well defined. For the integral of $u^2u_j^2$ we have

$$\int u^2u_j^2 dx \leq \|u\|^2_{L^4(\mathbb{R}^3)}\|u_j\|^2_{L^4(\mathbb{R}^3)} \leq \|u\|^{1/2}_{L^2}\|\nabla_x u\|^{3/2}_{L^2}\|u_j\|^{1/2}_{L^2}\|\nabla_x u_j\|^{3/2}_{L^2}. \tag{5.4}$$

Also a similar argument shows that the right hand side of (5.3) is well defined and it is a continuous function of t. For example,

$$\Big|\int uu_tu_j^2(t,x)dx\Big| \leq \|u_j(t,x)\|^2_{L^6(\mathbb{R}^3)}\|u(t,x)\|_{L^6(\mathbb{R}^3)}\|u_t(t,x)\|_{L^2(\mathbb{R}^3)}. \tag{5.5}$$

This implies that the derivative with respect to t is taken in classical sense. Now let $(g_n, h_n) \in H^3(\mathbb{R}^3) \times H^2(\mathbb{R}^3)$ converges to $(u(s,x), u_t(s,x))$ in $H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ as $n \to \infty$. Denote as in Section 3 by $w_n(t,x)$ the local solution of (3.1) with initial data (g_n, h_n). Therefore for $t \in [s, s+\tau]$ we have

$$\int w_n^2((w_n)_j)^2(t,x)dx \to_{n\to\infty} \int u^2u_j^2(t,x)dx,$$

$$\int w_n(w_n)_t((w_n)_j)^2(t,x)dx \to_{n\to\infty} \int uu_tu_j^2(t,x)dx.$$

To justify these limits, we apply the estimates (5.4) and (5.5). For example,

$$\Big|\int w_n(w_n)_t((w_n)_j)^2(t,x)dx\Big| \leq \Big|\int (w_n - u)(w_n)_t((w_n)_j)^2dx\Big| + \Big|\int u((w_n)_t - u_t)((w_n)_j)^2dx\Big|$$
$$+\Big|\int uu_t(((w_n)_j)^2 - u_j^2)dx\Big|$$

and we use (3.6) for $k = 2$. Passing in limit in the equality (5.3) for w_n, we obtain it for u.

Consequently, after an integration with respect to t in (5.3), one deduces

$$X(s+\tau) = X(s) + 2\int_s^{s+\tau}\Big(J_1(t) + J_2(t) + I_1(t)\Big)dt.$$

5.2. **Estimation of** $\int_s^{s+\tau} J_1(t)dt$**.** Let $0 < \epsilon \ll 1$ be a small number. First by the generalized Hölder inequality one estimates

$$|J_1(t)| \leq 3\sum_{j=1}^{3}\|u(t,x)\|_{L^r(\mathbb{R}^3)}\|u_t(t,x)\|_{L^{2+\epsilon}(\mathbb{R}^3)}\|u_j(t,x)\|^2_{L^4(\mathbb{R}^3)}$$

$$\leq 3\sum_{j=1}^{3}\|u(t,x)\|_{L^r(\mathbb{R}^3)}\|u_t(t,x)\|_{L^{2+\epsilon}(\mathbb{R}^3)}\|u_j(t,x)\|^{1/2}_{L^2(\mathbb{R}^3)}\|u_j(t,x)\|^{3/2}_{L^6(\mathbb{R}^3)},$$

where

$$\frac{1}{r} = \frac{\epsilon}{4+2\epsilon}.$$

According to the estimate (2.7), for $s \leq t \leq s+\tau$ by the local existence of a solution of (3.1) with initial data $(u(s,x), u_t(s,x)) \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ on $t = s$, we obtain

$$\|u_j(t,x)\|^{3/2}_{L^6(\mathbb{R}^3)} \leq \|\nabla_x u_j(t,x)\|^{3/2}_{L^2(\mathbb{R}^3)} \leq C_2\Big(\|u(s,x)\|_{H^2(\mathbb{R}^3)} + \|u_t(s,x)\|_{H^1(\mathbb{R}^3)}\Big)^{3/2}$$

with constant $C_2 > 0$ depending on q. Next

$$\|u(s,x)\|^2_{H^2(\mathbb{R}^3)} \leq C\Big(\sum_{i,j=1}^{3} \|\partial_{x_i}\partial_{x_j} u(s,x)\|^2_{L^2(\mathbb{R}^3)} + \|u(s,x)\|^2_{H^1(\mathbb{R}^3)}\Big),$$

$$\|u_t(s,x)\|^2_{H^1(\mathbb{R}^3)} \leq C\Big(\sum_{j=1}^{3} \|u_{jt}(s,x)\|^2_{L^2(\mathbb{R}^3)} + \|u_t(s,x)\|^2_{L^2(\mathbb{R}^3)}\Big).$$

Notice that we have a polynomial bound with respect to s for the norms $\|u(s,x)\|_{H^1(\mathbb{R}^3)}$ and $\|u_t(s,x)\|_{L^2(\mathbb{R}^3)}$ of the solution $u(s,x)$ (see Theorem 2, [2]). Consequently, we obtain

$$\sup_{t\in[s,s+\tau]} \|u_j(t,x)\|^{3/2}_{L^6(\mathbb{R}^3)} \leq C_1\Big(X(s)^{3/4} + (1+s)^3\Big), \quad \sup_{t\in[s,s+\tau]} \|u_j(t,x)\|_{L^2(\mathbb{R}^3)} \leq C_0(1+s),$$

where $C_0 > 0, C_1 > 0$ depend on $\|u(0,x)\|_{H^1(\mathbb{R}^3)}$.

Now we pass to the estimate of $\|u_t(t,x)\|_{L^{2+\epsilon}(\mathbb{R}^3)}$. By Hölder inequality we obtain

$$\Big|\int u_t^{2+\epsilon} dx\Big| = \Big|\int u_t^{2(1-\frac{\epsilon}{4})} u_t^{\frac{3\epsilon}{2}} dx\Big| \leq \|u_t\|^{2(1-\epsilon/4)}_{L^2(\mathbb{R}^3)} \|u_t\|^{\frac{3\epsilon}{2}}_{L^6(\mathbb{R}^3)}$$

$$\leq C_3(1+t)^2 \|\nabla_x u_t\|^{\frac{3\epsilon}{2}}_{L^2(\mathbb{R}^3)} \leq C_4(1+s)^2\Big(X(s)^{\frac{3\epsilon}{4}} + (1+s)^{3\epsilon}\Big).$$

Hence, one deduces

$$\sup_{t\in[s,s+\tau]} \Big|\int u_t^{2+\epsilon} dx\Big|^{\frac{1}{2+\epsilon}} \leq C_5(1+s)^{3/2}\Big(X(s)^{\frac{3\epsilon}{8+4\epsilon}} + 1\Big).$$

Taking into account the above estimates, for the integral with respect to t one applies the Hölder inequality and for small ϵ we have

$$\Big|\int_s^{s+\tau} J_1(t)dt\Big| \leq C_6\tau^{1/p'}(1+s)^6 \|u(t,x)\|_{L^p([s,s+\tau];L^r_x(\mathbb{R}^3))}\Big(X(s)^{\frac{3}{4}+\frac{3\epsilon}{8}} + 1\Big),$$

where

$$\frac{1}{p} + \frac{3\epsilon}{4+2\epsilon} = \frac{1}{2}, \ \frac{1}{p'} + \frac{1}{p} = 1.$$

To complete the analysis, we apply the Strichartz estimate (3.4) and deduce

$$\|u(t,x)\|_{L^p([s,s+\tau];L^r_x(\mathbb{R}^3))} \leq C(\epsilon)(1+s)^6.$$

Finally for $0 < \tau \leq 1$ with $y = 12$ we have

$$\Big|\int_s^{s+\tau} J_1(t)dt\Big| \leq C'(\epsilon)\Big(X(s)^{\frac{3}{4}+\frac{3\epsilon}{8}} + 1\Big)(1+s)^y. \tag{5.6}$$

5.3. **Estimation of $\int_s^{s+\tau} I_1(t)dt$.** We apply a similar argument.

$$|I_1(t)| \leq C\sum_{j=1}^{3} \|u(t,x)\|_{L^2(\mathbb{R}^3)}\|u_{jt}(t,x)\|_{L^2(\mathbb{R}^3)} \leq C_7(1+|t|)^2 \sum_{j=1}^{3} \|u_{jt}(t,x)\|_{L^2(\mathbb{R}^3)}.$$

By the local existence result for $t \in [s, s+\tau]$ one has

$$\|u_{jt}(t,x)\|_{L^2(\mathbb{R}^3)} \leq C(\|u(s,x)\|_{H^2(\mathbb{R}^3)} + \|u_t(s,x)\|_{H^1(\mathbb{R}^3)})$$

and repeating the above argument, we deduce

$$\Big|\int_s^{s+\tau} I_1(t)dt\Big| \leq C_8(X(s)^{1/2} + 1)(1+s)^2. \tag{5.7}$$

5.4. **Estimation of $\int_s^{s+\tau} J_2(t)dt$.** This term is easy to be bounded since we have a polynomial estimate

$$\int u_j^2(t,x)dx \leq C_0(1+|t|)^2$$

and this yields

$$\Big|\int_s^{s+\tau} J_2(t)dt\Big| \leq C_9(1+s)^2. \tag{5.8}$$

Combining (5.6), (5.7), (5.8), finally we get

$$X(s+\tau) \leq X(s) + C_{10}\Big(X(s)^{\frac{3}{4}+\frac{3\epsilon}{8}}+1\Big)(1+s)^y. \tag{5.9}$$

5.5. **Growth of $H^2(\mathbb{R}^3)$ norm.** Let $a > 1$ be a fixed number. According to [2] and Proposition 1, there exists a solution in $[s, s+\tau(a)] \subset [0,a]$ with initial data $g \in H^2(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ on $t = s$. Here

$$\tau(a) = c\Big((1+\|f\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} + a(B_1+B_2a)\Big)^{-\gamma} < 1,$$

where $c > 0, \gamma > 0, B_1 > 0, B_2 > 0$ are independent on a and f. We choose $N(a) \in \mathbb{N}$ so that $a - \tau(a) < N(a)\tau(a) \leq a$. Setting $X(n\tau(a)) = \alpha_n$, $n \leq N(a)$, and exploiting (5.9), one deduces

$$\alpha_n \leq \alpha_{n-1} + C_{10}(\alpha_{n-1}^{7/8}+1)(1+n)^{12}.$$

We are in position to apply Lemma 1 and to obtain

$$X(N(a)\tau(a)) \leq \tilde{C}(N(a))^{104}$$
$$\leq \tilde{C}\Big(\frac{a}{c}\Big)^{104}\Big(1+\|f\|_{H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)} + a(B_1+B_2a)\Big)^{104\gamma}.$$

This estimate and the bound of the $H^1(\mathbb{R}^3)$ norm of the solution $u(a,x)$ established in [2] imply a polynomial with respect to a bound of $\|u(a,x)\|_{H^2(\mathbb{R}^3)} + \|\partial_t u(a,x)\|_{H^1(\mathbb{R}^3)}$. This implies the statement of Theorem 1 for $k = 2$.

6. Polynomial growth of the $H^k(\mathbb{R}^3)$ norm of the solution.

To examine the growth of the $H^k(\mathbb{R}^3)$ norm of the solution, we will proceed by induction. Assume that for $1 \leq k \leq s-1, s \geq 3$, we have polynomial bounds

$$\|u(t,x)\|_{H_x^k(\mathbb{R}^3)} + \|u_t(t,x)\|_{H_x^{k-1}(\mathbb{R}^3)} \leq A_k(1+|t|)^{m_k},\ t \in \mathbb{R}$$

for the global solution of the Cauchy problem of $u_{tt} - \Delta_x u + qu + u^3 = 0$ with initial data $(f_1, f_2) \in H^k(\mathbb{R}^3) \times H^{k-1}(\mathbb{R}^3)$. Consider the equality

$$\partial_t^2\partial_x^\alpha u - \Delta_x(\partial_x^\alpha u) + \partial_x^\alpha(qu) + \partial_x^\alpha(u^3) = 0$$

with $|\alpha| = s-1$. After an integration by parts which we can justify as in Section 5, we write

$$\frac{1}{2}\frac{d}{dt}\int\Big(|\nabla_x\partial_x^\alpha u|^2 + |\partial_t\partial_x^\alpha u|^2\Big)dx$$
$$= -\int \partial_x^\alpha(qu)\partial_x^\alpha\partial_t u dx - \int \partial_x^\alpha(u^3)\partial_x^\alpha\partial_t u dx = K_1(t) + K_2(t). \tag{6.1}$$

Clearly,

$$\Big|\int\Big(\partial_x^\alpha(u^3)\partial_x^\alpha\partial_t u\Big)dx\Big| \leq \|\partial_x^\alpha(u^3)\|_{L^2(\mathbb{R}^3)}\|\partial_x^\alpha\partial_t u\|_{L^2(\mathbb{R}^3)}.$$

Applying two times (2.5), one gets

$$\|\partial_x^\alpha(u^3)\|_{L^2(\mathbb{R}^3)} \leq C\|\partial_x^\alpha u\|_{L^2(\mathbb{R}^3)}\|u\|^2_{L^\infty(\mathbb{R}^3)}$$

and by Sobolev theorem $\|u\|_{L^\infty(\mathbb{R}^3)} \leq C\|u\|_{H^2(\mathbb{R}^3)}$. Thus by our assumption

$$\|\partial_x^\alpha(u^3(t,x))\|_{L^2(\mathbb{R}^3)} \leq CA_{k-1}A_2^2(1+|t|)^{m_{k-1}+2m_2}.$$

Therefore, using the notation of subsection 5.5 for $n\tau(a) \leq t \leq (n+1)\tau(a)$, one deduces

$$\|\partial_x^\alpha(u^3(t,x))\|_{L^2(\mathbb{R}^3)} \leq CA_{k-1}A_2^2(1+n)^{m_{k-1}+2m_2}.$$

On the other hand, applying (2.7), one obtains

$$\|\partial_x^\alpha \partial_t u(t,x)\|_{L^2(\mathbb{R}^3)} \leq C_k\Big((\|u(n\tau(a),x)\|_{H^s(\mathbb{R}^3)} + \|u_t(n\tau(a),x)\|_{H^{s-1}(\mathbb{R}^3)}\Big).$$

The analysis of $K_1(t)$ is easy and

$$\begin{aligned} |K_1(t)| &\leq C\|u(t,x)\|_{H^{s-1}(\mathbb{R}^3)}\|\partial_x^\alpha \partial_t u(t,x)\|_{L^2(\mathbb{R}^3)} \\ &\leq C_k A_{k-1}(1+n)^{m_{k-1}}(\|u(n\tau(a),x)\|_{H^s(\mathbb{R}^3)} + \|u_t(n\tau(a),x)\|_{H^{s-1}(\mathbb{R}^3)}). \end{aligned}$$

Now define $Y_k(t) := \|u(t,x)\|^2_{H^k(\mathbb{R}^3)} + \|\partial_t u(t,x)\|^2_{H^{k-1}(\mathbb{R}^3)}$ and integrate the equality (6.1) from $n\tau_k(a)$ to $(n+1)\tau_k(a)$ with respect to t, where $0 < \tau_k(a) < 1$ is defined by (2.9). Taking into account the above estimates, we have

$$Y_k((n+1)\tau_k(a)) \leq Y_k(n\tau_k(a)) + C_q A_{k-1}(1+n)^{m_{k-1}} + CA_{k-1}A_2^2(1+n)^{m_{k-1}+2m_2}Y_k^{1/2}(n\tau_k(a)).$$

Applying Lemma 1 and repeating the argument of subsection 5.5, we obtain a polynomial bound for $Y_k(t)$ and this completes the proof of Theorem 1.

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