

ABSENCE OF EIGENVALUES OF DISSIPATIVE OPERATOR FOR STRICTLY CONVEX OBSTACLES

VESSELIN PETKOV

ABSTRACT. We study the wave equation in the exterior of a strictly convex bounded domain $K \subset \mathbb{R}^d, d \geq 3$, odd, with dissipative boundary condition $\partial_\nu u - \gamma(x)\partial_t u = 0$ on the boundary Γ and $0 < \gamma(x) < 1, \forall x \in \Gamma$. The solutions are described by a contraction semigroup $V(t) = e^{tG}, t \geq 0$. In [10] we established that for $\gamma \equiv \text{const}$ and $K = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ the operator G has no eigenvalues and we conjectured that the same result holds for every strictly convex obstacle. In this paper we prove this conjecture.

1. INTRODUCTION

Let $K \subset \mathbb{R}^d, d \geq 3, d$ odd, be a bounded non-empty domain with C^∞ strictly convex boundary Γ . Let $\Omega = \mathbb{R}^d \setminus \bar{K}$ be connected and $K \subset \{x \in \mathbb{R}^d : |x| < \rho\}$. Consider the boundary problem

$$\begin{cases} u_{tt} - \Delta_x u = 0 \text{ in } \mathbb{R}_t^+ \times \Omega, \\ \partial_\nu u - \gamma(x)\partial_t u = 0 \text{ on } \mathbb{R}_t^+ \times \Gamma, \\ u(0, x) = f_1, u_t(0, x) = f_2 \end{cases} \quad (1.1)$$

with initial data $(f_1, f_2) \in \mathcal{H} = H^1(\Omega) \times L^2(\Omega)$. Here $\nu(x)$ is the unit outward normal at $x \in \Gamma$ pointing into Ω and $\gamma(x) > 0$ is a C^∞ function on Γ . Introduce the contraction semi-group $V(t)$ in $\mathcal{H}$ with generator

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}.$$

The operator G has domain $\mathcal{D}$ given by the closure in the graph norm

$$|||f||| = (\|f\|_{\mathcal{H}}^2 + \|Gf\|_{\mathcal{H}}^2)^{1/2}$$

of functions $f = (f_1, f_2) \in C_{(0)}^\infty(\mathbb{R}^d) \times C_{(0)}^\infty(\mathbb{R}^d)$ satisfying the boundary condition $\partial_\nu f_1 - \gamma f_2 = 0$ on Γ .

The solution of the problem (1.1) has the form $V(t)f = e^{tG}f, t \geq 0$. The point spectrum $\sigma_p(G)$ of G in $\mathbb{C}_- = \{z \in \mathbb{C} : \text{Re } z < 0\}$ is formed by isolated eigenvalues with finite multiplicity, $\sigma_p(G) \cap i\mathbb{R} = \emptyset$ and G has continuous spectrum $\sigma_c(G) = i\mathbb{R}$ (see Section 8, [6]). Notice that if $Gf = \lambda f$ with $0 \neq f \in \mathcal{D}, \text{Re } \lambda < 0$, we have solution $u(t, x) = V(t)f = e^{\lambda t}f(x)$ of (1.1) with exponentially decreasing global energy. Such solutions are called asymptotically disappearing and they are important for scattering problems (see for more details [8], [10]). The location in $\mathbb{C}_-$ of the eigenvalues of G has been studied in [8] improving previous results of Majda [7]. In the special case when $\gamma \equiv 1$, and K is the unit ball $B_3 = \{x \in \mathbb{R}^3 : |x| \leq 1\}$ it was proved in [8] that G has no eigenvalues. The case $\gamma(x) > 1, \forall x \in \Gamma$ has been

examined in [8] and a Weyl formula for the counting function of the eigenvalues of G close to the negative real axis has been established in [9].

Throughout this paper we assume that $0 < \gamma(x) < 1$, $\forall x \in \Gamma$. This case has been studied in [8], [10]. In particular, for $\gamma \equiv \text{const}$ and $K = B_3$ in Appendix, [10] it was proved that G has no eigenvalues. We conjectured in [10] that the same is true for every strictly convex obstacle. Our main result is the following

Theorem 1.1. *For strictly convex obstacles the operator G has no eigenvalues λ with $\text{Re } \lambda < 0$.*

The eigenvalues of G in $\mathbb{C}_-$ are incoming functions (see Section 2 for the definition of outgoing/incoming functions). The absence eigenvalues of G for $\text{Re } \lambda > 0$ is based on the representation of the outgoing resolvent

$$(G - \lambda)^{-1} = - \int_0^\infty e^{tG} e^{-\lambda t} dt, \quad \text{Re } \lambda > 0 \quad (1.2)$$

which is analytic for $\text{Re } \lambda > 0$. For the incoming eigenvalues it should be useful to have a similar representation with integral on $(-\infty, 0]$ involving some evolution operator. However, the problem

$$\begin{cases} (\partial_t^2 - \Delta_x) f = 0 \text{ in } \mathbb{R}_t^- \times \Omega, \\ \partial_\nu f - \gamma(x) \partial_t f = 0 \text{ on } \mathbb{R}_t^- \times \Gamma, \\ f(0, x) = 0, \quad f_t(0, x) = \chi(x) \end{cases} \quad (1.3)$$

is not L^2 well posed (see Theorem 1 in [7]). On the other hand, the problem (1.3) is well posed with some loss of regularity (see [3], [4]) and for $\chi \in H_{\text{comp}}^m(\Omega)$, $m > d/2 + 3$ it is possible to obtain a solution of (1.3) with the property $e^{c_{m-3/2} t} f(t, x) \in H^{m-2}(\mathbb{R}_t^- \times \Omega)$ with some large $c_{m-3/2} > 0$ (see Proposition 3.2). The idea is to consider the incoming function

$$w(x, \lambda) = \int_{-\infty}^0 e^{-\lambda t} P(t) \chi dt, \quad \text{Re } \lambda \leq -c_{m-3/2},$$

where $P(t)\chi$ is a solution of (1.3) and to modify it to obtain an approximative resolvent $R_a^\sharp(\lambda)$ of $(G - \lambda)^{-1}$.

We expose below the strategy of the proof of Theorem 1.1. In Section 3 we examine the operator $\mathcal{C}(\lambda) = N(\lambda) - \lambda\gamma(x)$, where $N(\lambda) : H^s(\Gamma) \rightarrow H^{s-1}(\Gamma)$ is the Dirichlet-to-Neumann operator. We study the existence of $\mathcal{C}(\lambda)^{-1}$ for $\text{Re } \lambda \leq -c_0 < 0$. For this purpose we apply a weak version of Theorem 2.1 in [3] concerning only $N(\lambda)$. We expect that it is possible to obtain (3.2) for general obstacles and we will study this problem in a further work. In Proposition 3.2 we justify the existence of an operator $P(t)$ satisfying (3.13). In Section 4 we follow the approach of the proof of Theorem 4.43 in [1] by using a cut-off function $\zeta_a(t, x)$. Let $\chi_a \in H_{\text{comp}}^m(\Omega)$, $\chi_a(x) \equiv 1$ for $|x| \leq \rho + 2\varepsilon$, $\varepsilon > 0$. We study the operator $\zeta_a P(t) \chi_a$ which satisfies

$$\begin{cases} (\partial_t^2 - \Delta_x)(\zeta_a P(t) \chi_a) =: F_a(t), \\ (\partial_\nu - \gamma(x) \partial_t)(\zeta_a P(t) \chi_a)|_{\mathbb{R}_t^- \times \Gamma} =: J_a(t). \end{cases}$$

To treat the remainder operators $F_a(t)$ and $J_a(t)$, we define an approximative resolvent

$$R_a^\sharp(\lambda) = \int_{-\infty}^0 e^{-\lambda t} \left[\zeta_a P(t) \chi_a + (1 - \chi_c) W_a(t) - B_a(t) \right] dt, \quad \text{Re } \lambda \leq 0.$$

with cut-off function $\chi_c(x)$ and suitable correction operators $W_a(t)$, $B_a(t)$. The kernels of these operators have compact support with respect to t and this makes possible to extend analytically $R_a^\sharp(\lambda)$ from $\operatorname{Re} \lambda \leq -c_{m-3/2}$ to $\operatorname{Re} \lambda < 0$. The operator $R_a^\sharp(\lambda)$ satisfies $(\Delta - \lambda^2)R_a^\sharp(\lambda) = \chi_a(\operatorname{Id} + L(\lambda))$ and for sufficiently large $|\lambda| \geq A_0$ we obtain the existence of $(\operatorname{Id} + L(\lambda))^{-1}$. This leads to the absence of eigenvalues λ of G with $|\lambda| \geq A_0$. Finally, Section 5 is devoted to the absence of eigenvalues with $|\lambda| \leq A_0$. We introduce the operator $\mathcal{C}_\varepsilon(\lambda) = N(\lambda) - \lambda\varepsilon\gamma(x)$ with $\varepsilon \in [0, 1]$. The idea is to compare the number of eigenvalues of G in a bounded set $\omega = \{z \in \mathbb{C} : -c_0 \leq \operatorname{Re} \lambda \leq 0, |\operatorname{Im} z| \leq A_0\}$ with the number of eigenvalues in ω corresponding to the problem (1.1) with boundary condition $(\partial_\nu u - \varepsilon\gamma(x)\partial_t u)|_\Gamma = 0$. For this purpose we apply a trace formula (5.6) and examine the limit $\varepsilon \searrow 0$ passing to Neumann problem for which there are no eigenvalues.

2. PRELIMINARIES

In this section we collect some facts from [8], [10]. For $\lambda \in \mathbb{C}$ introduce the exterior Dirichlet-to-Neumann map

$$N(\lambda) : H^s(\Gamma) \ni f \longrightarrow \partial_\nu u|_\Gamma \in H^{s-1}(\Gamma),$$

where $u = K(\lambda)f$ is the solution of the problem

$$\begin{cases} (-\Delta + \lambda^2)K(\lambda)f = 0 \text{ in } \Omega, \\ K(\lambda)f = f \text{ on } \Gamma, \\ K(\lambda)f : \lambda - \text{incoming}. \end{cases} \quad (2.1)$$

A function $u(x)$ is called λ -outgoing (λ -incoming) if there exists $R > \rho$ and $g \in L_{comp}^2(\mathbb{R}^d)$ such that

$$u(x) = R_0^\pm(\lambda)g, \quad |x| \geq R,$$

where $R_0^\pm(\lambda) = (-\Delta_0 + \lambda^2)^{-1}$ is the outgoing (+) (incoming (-)) resolvent of the free Laplacian $-\Delta_0$ in $\mathbb{R}^d$. The resolvents $R_0^\pm(\lambda)$ are analytic in $\mathbb{C}$ and they have kernels

$$R_0^\pm(\lambda, x, y) = \frac{(-1)^{(d-1)/2}}{2(2\pi)^{(d-1)/2}} \left(\frac{1}{r} \partial_r \right)^{(d-3)/2} \left(\frac{e^{\mp \lambda r}}{r} \right) \Big|_{r=|x-y|}.$$

If for $\operatorname{Re} \lambda < 0$ we have $G(f_1, f_2) = \lambda(f_1, f_2)$, then $f_1 \in H^2(\Omega)$ is λ -incoming solution of $(-\Delta + \lambda^2)f_1 = 0$ (see [6] and Section 1 in [10]). Let $R_D(\lambda) = (-\Delta_D + \lambda^2)^{-1}$ be the incoming resolvent of the Dirichlet Laplacian Δ_D in Ω with domain $\mathbf{D} = H^2(\Omega) \cap H_0^1(\Omega)$ which is analytic for $\lambda \in \mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$. Let

$$\mathbf{D}_{loc} = \{u \in L_{loc}^2(\Omega) : \chi(x) \in C_0^\infty(\mathbb{R}^d), \chi(x) \equiv 1 \text{ in a neighborhood of } \bar{K} \Rightarrow \chi u \in \mathbf{D}\}.$$

The incoming resolvent has meromorphic extension

$$R_D(\lambda) : L_{comp}^2(\Omega) \rightarrow \mathbf{D}_{loc}$$

from $\mathbb{C}_-$ to $\mathbb{C}$. The solution of the problem (2.1) with $f \in H^{3/2}(\Gamma)$ has the form

$$u = e(f) + R_D(\lambda)((\Delta - \lambda^2)(e(f))), \quad (2.2)$$

where $e(f) : H^{3/2}(\Gamma) \ni f \rightarrow e(f) \in H_{comp}^2(\Omega)$ is an extension operator. Clearly, we may find $\partial_\nu u|_\Gamma$ by applying (2.2). By (2.2) we conclude that $N(\lambda)$ is analytic in $\bar{\mathbb{C}}_-$.

We write the boundary condition in (1.1) as

$$\mathcal{C}(\lambda)v := (N(\lambda) - \lambda\gamma)v = \left(Id - \lambda\gamma N(\lambda)^{-1} \right) N(\lambda)v = 0, \quad v = f_1|_\Gamma \in H^{3/2}(\Gamma).$$

For $\operatorname{Re} \lambda \leq 0$ the operator $\mathcal{C}(\lambda) : H^s(\Gamma) \rightarrow H^{s-1}(\Gamma)$ has the same singularities as $N(\lambda)$. On the other hand, $N(\lambda)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ is compact operator and by analytic Fredholm theorem (see [8], [10]), the operator $\mathcal{C}(\lambda)$ is a meromorphic operator valued function for $\operatorname{Re} \lambda \leq 0$. Notice that if $\mathcal{C}(\lambda)$ is analytic at λ_0 , then $\mathcal{C}(\lambda_0) : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$. Here and below a meromorphic operator valued function $B(z)$ means that $B(z)$ have Laurent expansion with finite number negative powers of z and coefficients having finite rank. In [9] it was proved that we can extend the incoming resolvent $(G - \lambda)^{-1}$, $\lambda \notin \sigma_p(G)$, $\operatorname{Re} \lambda < 0$ as meromorphic function $(G - \lambda)^{-1} : \mathcal{H}_{comp} \rightarrow \mathcal{D}_{loc}$ for $\lambda \in \mathbb{C}$, where

$$\mathcal{D}_{loc} = \{u \in \mathcal{H}_{loc}, \chi(x) \in C_0^\infty(\mathbb{R}^d), \chi(x) \equiv 1 \text{ in a neighborhood of } \bar{K} \Rightarrow \chi u \in \mathcal{D}\}.$$

Let $\begin{pmatrix} u \\ w \end{pmatrix} = (G - \lambda)^{-1} \begin{pmatrix} f \\ g \end{pmatrix}$ with $(f, g) \in \mathcal{H}_{comp}$. It was established in [8], [10] that $(G - \lambda)^{-1}$ is analytic at λ_0 if and only if $\mathcal{C}(\lambda_0)$ is invertible and $\mathcal{C}(\lambda_0)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ is analytic at λ_0 . Moreover, if $\mathcal{C}(\lambda)^{-1}$ is analytic, $w = \lambda u + f$, where

$$u(x, \lambda) = -R_D(\lambda)(g + \lambda f) + K(\lambda)\mathcal{C}(\lambda)^{-1} \left[\partial_\nu \left(R_D(\lambda)(g + \lambda f) \right) \Big|_\Gamma + \gamma f|_\Gamma \right] \quad (2.3)$$

is a solution of the problem

$$\begin{cases} (\Delta - \lambda^2)u = g + \lambda f \text{ in } \Omega, \\ (\partial_\nu - \gamma(x)\lambda)u|_\Gamma = \gamma f|_\Gamma, \\ u : \lambda - \text{incoming.} \end{cases} \quad (2.4)$$

Notice that for $(f, g) \in \mathcal{H}_{comp}$ we obtain $u(x, \lambda) \in H^2(\Omega)$. Indeed,

$$\mathcal{C}(\lambda)^{-1} \left[\partial_\nu R_D(\lambda)(g + \lambda f) \Big|_\Gamma + \gamma f|_\Gamma \right] \in H^{3/2}(\Gamma)$$

and applying $K(\lambda)$, we obtain the result.

It is easy to see that if $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ for some $s_0 \geq 0$ has a pole $\lambda \in \bar{\mathbb{C}}_-$, then this operator has a pole λ_0 for every $s \geq 0$. This follows from the fact that there exists $0 \neq v \in H^s(\Gamma)$ such that $\mathcal{C}(\lambda_0)v = (\operatorname{Id} - \lambda_0\gamma N(\lambda_0)^{-1})v = 0$. Since $N(\lambda_0)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ is bounded, this implies $v \in C^\infty(\Gamma)$ and $(\operatorname{Id} - \lambda_0\gamma N(\lambda_0)^{-1})$ is not invertible in $H^s(\Gamma)$ for all $s \geq 0$. In Section 4 we need the following

Lemma 2.1. *Assume that there exist $a_1 > \rho$ and $k \geq 2$ such that for $f = 0$ and every $g \in H^k(\Omega)$ with $\operatorname{supp} g \subset B(0, a_1)$ there exists a solution $u(x, \lambda)$ of (2.4) which is analytic in a neighbourhood $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \delta\}$ of $\lambda_0 \in \mathbb{C}_-$. Then the operator $\mathcal{C}(\lambda)^{-1} : H^{k+1}(\Gamma) \rightarrow H^{k+2}(\Gamma)$ is analytic at λ_0 .*

Proof. Suppose that $\mathcal{C}(\lambda)^{-1}$ has a pole at λ_0 . Then there exist $h \in H^{k+1}(\Gamma)$, $0 \neq h_0 \in H^{k+2}(\Gamma)$ and $m \geq 1$ such that for $0 \neq |\lambda - \lambda_0| < \varepsilon$ with sufficiently small $0 < \varepsilon \leq \delta$ we have

$$\mathcal{C}(\lambda)^{-1}h = \frac{h_0}{(\lambda - \lambda_0)^m} + \text{lower order singularities.}$$

Let $q = N(\lambda_0)^{-1}h \in H^{k+2}(\Gamma)$. It is easy to construct $\tilde{g} \in H^{k+2}(\Omega)$ with $\operatorname{supp} \tilde{g} \subset B(0, a_1)$ such that $\tilde{g}|_\Gamma = q$, $\partial_\nu \tilde{g}|_\Gamma = 0$. To do this, by using a partition of unity on Γ ,

we pass to a construction in a neighbourhood $\mathcal{U} \subset \mathbb{R}^d$ of a point $x_0 \in \Gamma$. Consider in $\mathcal{U}$ local normal coordinates $(y_1, \dots, y_d) = (y_1, y')$ such that

$$x = \alpha(y_1, y') = \beta(y') + y_1 \nu(y'),$$

where $y_1 = \text{dist}(x, \Gamma)$ and $\nu(y')$ is an extension of unit normal vector to unit vector field. Then the boundary Γ is given by $y_1 = 0$ and $\partial_\nu f(x) = \partial_{y_1} f(\alpha(y_1, y'))$. If $q(\alpha(0, y')) = s(y')$, we take $\tilde{g}(y) = s(y')$. Next for $f = 0$ and $g = (\Delta - \lambda_0^2)\tilde{g} \in H^k(\Omega)$ by our assumption there exists a solution $u(x, \lambda)$ of (2.4) which is analytic for $|\lambda - \lambda_0| < \delta$. The function $v = R_D(\lambda_0)g$ satisfies

$$(-\Delta + \lambda_0^2)(v + \tilde{g}) = 0, \quad (v + \tilde{g})|_\Gamma = q,$$

hence $\partial_\nu v|_\Gamma = \partial_\nu(v + \tilde{g})|_\Gamma = N(\lambda_0)q = h$. Then in the representation (2.3) for $u(x, \lambda)|_\Gamma$ we will have a singular term given by $\mathcal{C}(\lambda)^{-1} \partial_\nu R_D(\lambda_0)g|_\Gamma$ and we obtain a contradiction with the analyticity of $u(x, \lambda)$. $\square$

Remark 2.1. *By the same argument we may construct a modification $\tilde{g} \in H^m(\Omega)$ of $g \in H^m(\Omega)$ such that $\tilde{g}|_\Gamma = 0$, $\partial_\nu \tilde{g}|_\Gamma = g|_\Gamma$. To do this, in local normal coordinates (y_1, y') we take $\tilde{g}(y) = y_1 g(\alpha(y_1, y'))$.*

3. EXISTENCE AND ESTIMATE OF $\mathcal{C}(\lambda)^{-1}$

3.1. Analyticity of $\mathcal{C}(\lambda)^{-1}$. Throughout the paper we will use the notation $\lambda = \mu + ik$, $\mu, k \in \mathbb{R}$. In this subsection we study the analyticity of $\mathcal{C}(\lambda)^{-1}$ for sufficiently negative μ . Denote by $(\cdot, \cdot)_s$, $\|\cdot\|_s$ are the scalar product and the norm in $H^s(\Gamma)$, respectively. Consider the problem

$$\begin{cases} (-\Delta + \lambda^2)K_1(\lambda)f = 0 \text{ in } \Omega, \\ K_1(\lambda)f = f \text{ on } \Gamma, \\ K_1(\lambda)f : \lambda - \text{outgoing}. \end{cases}$$

and the Dirichlet-to-Neumann operator $N_1(\lambda)f = \partial_\nu K_1 f(\lambda)|_\Gamma$. Ikawa established the following

Theorem 3.1 (Theorem 2.1 in [3]). *Let K be a strictly convex obstacle in $\mathbb{R}^3$. Then there exist constants $r > 0, C_m > 0$ such that for every $m \in \mathbb{N}$ and $\mu \geq r$ we have*

$$-\text{Re}(N_1(\lambda)\varphi, \varphi)_m \geq (\mu - C_m)\|\varphi\|_m^2, \quad \forall \varphi \in C^\infty(\Gamma) \quad (3.1)$$

with C_m independent of λ and φ .

After the change $\eta = -\lambda$, we deduce for operator $N(\lambda)$ related to (2.1) and $\mu \leq -r$ the estimate

$$-\text{Re}(N(\lambda)\varphi, \varphi)_m \geq (|\mu| - C_m)\|\varphi\|_m^2, \quad \forall \varphi \in C^\infty(\Gamma). \quad (3.2)$$

Remark 3.2. *We apply the estimate (3.1) in [3] with $\gamma \equiv 0$. This estimate can be proved for odd dimensions $d > 3$ by similar argument. Notice that some estimates of the operator $\frac{N(\lambda)}{\lambda}$ in suitable regions have been established in Section 4 in [10].*

Let $\delta = \max_{x \in \Gamma} \gamma(x) < 1$. For $m = 0$ we have $(\gamma\varphi, \varphi)_0 \leq \delta\|\varphi\|_0^2$ and we obtain the estimate

$$-\text{Re}((N(\lambda) - \lambda\gamma(x))\varphi, \varphi)_0 \geq ((1 - \delta)|\mu| - C_0)\|\varphi\|_0^2.$$

Setting $\alpha = 1 - \delta > 0$, $c_0 = \max\{\frac{2C_0}{\alpha}, r\}$, for $\mu \leq -c_0$ we may absorb the term $C_0\|\varphi\|_0^2$ and obtain

$$\|\mathcal{C}(\lambda)\varphi\|_0 \geq \frac{\alpha|\mu|}{2}\|\varphi\|_0, \quad \forall \varphi \in C^\infty(\Gamma). \quad (3.3)$$

In the following we suppose that $\mu \leq -c_0$. Set $M(\lambda) = (Id - \lambda\gamma N^{-1}(\lambda))$ and note that $\mathcal{C}(\lambda) = M(\lambda)N(\lambda)$. If for $s \geq 0$ the operator $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ has a pole at λ_0 , then $M(\lambda_0) : H^s(\Gamma) \rightarrow H^s(\Gamma)$ is not invertible and there exists $0 \neq \varphi_0 \in L^2(\Gamma)$ such that $\mathcal{C}(\lambda_0)\varphi_0 = M(\lambda_0)\varphi_0 = 0$. Since $N(\lambda_0)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$, we conclude that $\varphi_0 \in C^\infty(\Gamma)$. This leads to a contradiction with (3.3), hence $\mathcal{C}(\lambda)^{-1}$ is analytic for $\mu \leq -c_0$ and c_0 is independent of s .

Remark 3.3. *The invertibility of $\mathcal{C}(\lambda)$ for $s \in \mathbb{N}$ and $\mu \geq -c_0$ has been proved in [3] following Lemma 3.3 in [2]. Our argument above is short and we use (3.2) only for $m = 0$.*

Now we will estimate the norm of $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \rightarrow H^s(\Gamma)$. Let $h \in L^2(\Gamma)$ and let $\mathcal{C}(\lambda)g = h$ with $g \in H^1(\Gamma)$. (We omit in the notations the dependence of g of λ). Choose a sequence $\varphi_j \rightarrow_{j \rightarrow \infty} g$ in $H^1(\Gamma)$ with $\varphi_j \in C^\infty(\Gamma)$. Therefore

$$\|\mathcal{C}(\lambda)(\varphi_j - g)\|_0 \leq B_0(\lambda)\|\varphi_j - g\|_1$$

and $\mathcal{C}(\lambda)\varphi_j \rightarrow \mathcal{C}(\lambda)g$ in $L^2(\Gamma)$. On the other hand, $\|\varphi_j\|_0 \rightarrow \|g\|_0$. We apply (3.3) with φ_j and passing to limit, we deduce

$$\|\mathcal{C}(\lambda)^{-1}h\|_0 \leq \frac{2}{\alpha|\mu|}\|h\|_0, \quad \forall h \in L^2(\Gamma). \quad (3.4)$$

Next assume that for $\mu \leq -c_m \leq -c_0$ we have the estimate

$$\|\mathcal{C}(\lambda)^{-1}h\|_m \leq \frac{B_m}{\alpha|\mu|}\|h\|_m, \quad \forall h \in H^m(\Gamma). \quad (3.5)$$

Applying (3.2), we get

$$-\operatorname{Re}((N(\lambda) - \lambda\gamma(x))\varphi, \varphi)_{m+1} \geq (|\mu| - C_{m+1})\|\varphi\|_{m+1}^2 + \mu(\gamma\varphi, \varphi)_{m+1}.$$

On the other hand,

$$|\mu(\gamma\varphi, \varphi)_{m+1}| \leq |\mu|\delta\|\varphi\|_{m+1}^2 + A_m|\mu|\|\varphi\|_m\|\varphi\|_{m+1},$$

where A_m depends of the norms of the derivatives of $\gamma(x)$ of order $j \leq m$. The above estimates imply

$$A_m|\mu|\|\varphi\|_m\|\varphi\|_{m+1} + \|\mathcal{C}(\lambda)\varphi\|_{m+1}\|\varphi\|_{m+1} \geq (\alpha|\mu| - C_{m+1})\|\varphi\|_{m+1}^2.$$

We estimate $|\mu|\|\varphi\|_m$ from (3.5) and obtain

$$(A_mB_m\alpha^{-1} + 1)\|\mathcal{C}(\lambda)\varphi\|_{m+1} \geq (\alpha|\mu| - C_{m+1})\|\varphi\|_{m+1}.$$

Choosing $c_{m+1} = \max\{\frac{2C_{m+1}}{\alpha}, c_m\}$ and repeating the above argument with an approximation $\varphi_j \rightarrow g$ in $H^{m+2}(\Gamma)$, we conclude for $\mu \leq -c_{m+1}$ that

$$\|\mathcal{C}(\lambda)^{-1}h\|_{m+1} \leq \frac{B_{m+1}}{\alpha|\mu|}\|h\|_{m+1}, \quad \forall h \in H^{m+1}(\Gamma).$$

By iteration we establish this estimate for every $m \in \mathbb{N}$. Next applying an interpolation argument for the spaces $H^m(\Gamma)$ and $H^{m+1}(\Gamma)$, we extend it to $H^s(\Gamma)$, $m < s < m+1$ and obtain the following

Proposition 3.1. *For every $s \geq 0$ there exist constants $c_s > 0$, $B_s > 0$ such that $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ is analytic for $\mu \leq -c_s$ and in this domain we have*

$$\|\mathcal{C}(\lambda)^{-1}h\|_s \leq \frac{B_s}{\alpha|\mu|} \|h\|_s, \quad \forall h \in H^s(\Gamma). \quad (3.6)$$

3.2. Existence of solution of the problem (1.3). Let $\chi \in H^m(\Omega)$ with $m \in \mathbb{N}$, $m > d/2 + 3$ and $\text{supp } \chi \subset B(0, a) = \{x : |x| \leq a\}$, $a > \rho$. Consider an extension $\tilde{\chi} \in H^m(\mathbb{R}^d)$ such that $\tilde{\chi}|_\Omega = \chi$ and $\|\tilde{\chi}\|_{H^m(\mathbb{R}^d)} \leq M\|\chi\|_{H^m(\Omega)}$. Let $g_0(t, x)$ be the solution of

$$\begin{cases} (\partial_t^2 - \Delta_x)g_0 = 0 \text{ in } \mathbb{R} \times \Omega, \\ g_0(0, x) = 0, \quad \partial_t g_0(0, x) = \tilde{\chi}(x). \end{cases} \quad (3.7)$$

The function g_0 has the representation $g_0 = \frac{\sin(t\sqrt{-\Delta_0})}{\sqrt{-\Delta_0}} \tilde{\chi}$, where $-\Delta_0$ is the free Laplacian in Ω . Set

$$w_0(\lambda, x) = \int_{-\infty}^{\infty} e^{-\lambda t} H(t) g_0(t, x) dt, \quad \text{Re } \lambda \leq 0,$$

where throughout our paper we use the Heaviside function $H(t) = 1$ for $t \leq 0$ and $H(t) = 0$ for $t > 0$. We obtain $(-\Delta + \lambda^2)w_0 = \tilde{\chi}$ and

$$q(\lambda, x) = (\partial_\nu - \lambda\gamma(x))w_0(\lambda, x)|_\Gamma = \int_{-\infty}^{\infty} e^{-\lambda t} H(t) (\partial_\nu - \gamma\partial_t)g_0(t, x)|_\Gamma dt. \quad (3.8)$$

Thus $w_0(\lambda, x) = R_0(\lambda)\tilde{\chi} \in H^{m+2}(\mathbb{R}^d)$, where $R_0(\lambda) = (-\Delta_0 + \lambda^2)^{-1}$ is the incoming resolvent of $-\Delta_0$ and $w_0(\lambda, x)$ is analytic for $\text{Re } \lambda \leq 0$. Let $\|\cdot\|_{H^m(B(0,b))} = \|\cdot\|_{m,b}$, $b > \rho$. It is well known that

$$\|R_0(\lambda)\chi\|_{k,b} \leq C_{a,b}|\lambda|^{k-1}\|\tilde{\chi}\|_{0,a}, \quad \text{Re } \lambda \leq 0, |\lambda| \geq L_0 > 1, \quad k = 0, 1. \quad (3.9)$$

In fact, for $|x| \leq b$, the sharp Huyghens principle yields $g_0(t, x) = 0$ for $|t| > a + b$ and (3.9) follows after an integration by parts. The estimate (3.9) implies

$$\|\partial_x^\alpha w_0\|_{k,b} \leq C_{a,b}|\lambda|^{k-1}\|\chi\|_{m,a}, \quad |\alpha| \leq m, \quad k = 0, 1$$

and from the equation $\Delta\partial_x^\alpha w_0 - \lambda^2\partial_x^\alpha w_0 = \partial_x^\alpha \tilde{\chi}$ we deduce

$$\|\partial_x^\alpha w_0\|_{m-1,b} \leq C_{m,a,b}|\lambda|^{k-2}\|\chi\|_{m,a}, \quad |\alpha| = k = 0, 1.$$

Since $m-3 > d/2$, the functions $w_0(\lambda, x)$ and $\partial_{x_j}w_0(\lambda, x)$, $j = 1, \dots, d$ are continuous in $|x| \leq b$ with uniform bound $\mathcal{O}(|\lambda|^{-2})$ and $\mathcal{O}(|\lambda|^{-1})$, respectively, and

$$\|q(\lambda, x)\|_{H^{m-3/2}(\Gamma)} \leq D_m|\lambda|^{-1}\|\chi\|_{m,a}.$$

Below up to end of this subsection we suppose that $\mu \leq -c_{m-3/2}$. Introduce the analytic function

$$h(\lambda, x) = \mathcal{C}(\lambda)^{-1}q(\lambda, x) \in H^{m-1/2}(\Gamma).$$

Applying the estimates (3.6), we deduce

$$\|h(\lambda, x)\|_{H^{m-3/2}(\Gamma)} \leq B_{m-3/2}D_m(\alpha|\lambda\mu|)^{-1}\|\chi\|_{m,a}.$$

Next, let $w_1(\lambda, x) = K(\lambda)h(\lambda, x) \in H^{m-2}(\Omega)$ be the solution of the Dirichlet problem (2.1) with $f = h(\lambda, x)$. Clearly,

$$(\partial_\nu - \lambda\gamma(x))w_1(\lambda, x)|_\Gamma = (N(\lambda) - \lambda\gamma(x))h(\lambda, x) = q(\lambda, x). \quad (3.10)$$

On the other hand, for $w_1(\lambda, x)$ we have the estimates

$$\sum_{k+j \leq m-2} |\lambda|^j \|w_1\|_{H^k(\Omega)} \leq K_m \|h\|_{H^{m-3/2}(\Gamma)} \leq M_m (\alpha |\lambda \mu|)^{-1} \|\chi\|_{m,a}. \quad (3.11)$$

Consequently, the inclusion $H^{m-3}(\Omega) \subset C(\Omega)$ implies

$$|w_1(\lambda, x)| \leq M_m |\lambda|^{-2} |\mu|^{-1} \|\chi\|_{m,a}$$

uniformly with respect to $\mu \leq -c_{m-3/2}$ and $x \in \Omega$. For $(t, x) \in \mathbb{R}_t \times \bar{\Omega}$ and $c \geq c_{m-3/2}$ define

$$g_1(t, x) = \frac{1}{2\pi i} \int_{\mu=-c}^{\infty} e^{\lambda t} w_1(\lambda, x) d\lambda = \frac{e^{-ct}}{2\pi} \int_{-\infty}^{\infty} e^{ikt} w_1(-c + ik, x) dk.$$

We may shift the contour of integration to the left, hence $g_1(\lambda, x)$ does not depend of the choice of $c \geq c_{m-3/2}$. It is easy to see that $g_1(t, x) \equiv 0$ for $t \geq 0$. For fixed $t_0 \geq 0, x_0 \in \Omega$ one obtains

$$|g_1(t_0, x_0)| \leq \frac{e^{-ct_0}}{2\pi} \int_{-\infty}^{\infty} |w_1(-c + ik, x_0)| dk \leq \frac{Ae^{-ct_0}}{c} \int_{-\infty}^{\infty} \frac{1}{c_{m-3/2}^2 + k^2} dk \leq \frac{Be^{-ct_0}}{c}$$

and taking $c \rightarrow \infty$, we conclude that $g_1(t_0, x_0) = 0$.

According to (3.11), we get

$$\|w_1\|_{H^{m-2-j}(\Omega)} = \mathcal{O}_m(|\mu|^{-1} |\lambda|^{-j-1}), \quad j = 0, \dots, m-2.$$

In particular, since $m \geq 5$, we have $\|\Delta w_1\|_{L^2} \leq \|w_1\|_{H^{m-3}} = \mathcal{O}_m(|\mu|^{-1} |\lambda|^{-2})$. Therefore,

$$\begin{aligned} (2\pi i) \Delta g_1 &= \int_{\mu=-c}^{\infty} e^{\lambda t} \Delta w_1(\lambda, x) d\lambda = \lim_{R \rightarrow \infty} \int_{-c-iR}^{-c+iR} e^{\lambda t} \lambda^2 w_1(\lambda, x) d\lambda \\ &= \lim_{R \rightarrow \infty} \partial_t^2 \int_{-c-iR}^{-c+iR} e^{\lambda t} w_1(\lambda, x) d\lambda = (2\pi i) \partial_t^2 g_1(t, x). \end{aligned}$$

Here in the last equality we take the derivative ∂_t^2 in the sens of distributions and we exchange ∂_t^2 and the limit $R \rightarrow \infty$. This implies

$$(\partial_t^2 - \Delta_x) g_1 = 0, \quad t < 0, x \in \Omega.$$

Finally, taking into account (3.8) and (3.10) and applying the inverse Laplace transformation, for $t < 0$ we deduce

$$\begin{aligned} (\partial_\nu - \gamma \partial_t) g_1(t, x)|_\Gamma &= \frac{1}{2\pi i} \int_{\mu=-c}^{\infty} e^{\lambda t} q(\lambda, x) d\lambda \\ &= \frac{1}{2\pi i} \int_{\mu=-c}^{\infty} e^{\lambda t} \left(\int_{-\infty}^0 e^{-\lambda s} (\partial_\nu - \gamma \partial_s) g_0(s, x)|_\Gamma ds \right) d\lambda = (\partial_\nu - \gamma \partial_t) g_0(t, x)|_\Gamma. \end{aligned}$$

Thus we obtain a solution $f(t, x) = g_0(t, x) - g_1(t, x)$ of the problem (1.3). Introduce the space $\mathcal{H}^m(\mathbb{R}_t^- \times \Omega)$ with norm

$$\|w(t, x)\|_m = \sum_{j+|\alpha| \leq m} \|\partial_t^j \partial_x^\alpha w(t, x)\|_{L^2(\mathbb{R}_t^- \times \Omega)}.$$

The estimate (3.11) implies $e^{c_{m-3/2}t} f(t, x) \in \mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega)$ and we obtain the following

Proposition 3.2. *For $m > d/2 + 3$ there exists an operator*

$$P(t) : H_{comp}^m(\Omega) \longrightarrow e^{-c_m - 3/2t} \mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega) \quad (3.12)$$

satisfying

$$\begin{cases} (\partial_t^2 - \Delta_x)P(t) = 0, & P(0) = 0, & \partial_t P(0) = \text{Id}, \\ (\partial_\nu - \gamma(x)\partial_t)P(t)|_{\mathbb{R}_t^- \times \Gamma} = 0. \end{cases} \quad (3.13)$$

4. ABSENCE OF EIGENVALUES $|\lambda| \geq A_0$

To obtain an analytic continuation of the incoming resolvent for $|\lambda| \geq A_0$, we will follow the approach in section 4.6 in [1]. Let $\chi_a \in C_0^\infty(\mathbb{R}^d)$ be such that $\text{supp } \chi_a \subset B(0, a)$ and $\chi_a \equiv 1$ for $|x| \leq \rho + 2\varepsilon$, $a \geq \rho + 3\varepsilon$, $\varepsilon > 0$. Introduce a function $\zeta_a(t, x) \in C^\infty$ such that

$$\zeta_a(t, x)|_{|x| > \rho + 2\varepsilon} = \begin{cases} 1 & \text{for } 0 \geq t \geq -(|x| + T_a), \\ 0 & \text{for } t \leq -(x| + T_a + 1), \end{cases}$$

and $\zeta_a(t, x) = \varphi_a(t)$ for $|x| < \rho + \varepsilon$ with

$$\varphi_a(t) = \begin{cases} 1 & \text{for } 0 \geq t \geq -(\rho + \varepsilon + T_a), \\ 0 & \text{for } t \leq -(\rho + \varepsilon + T_a + 1). \end{cases}$$

Here $T_a \geq a$ is a fixed constant. Moreover, we suppose that $\zeta_a(t, x) = 1$ for $0 \geq t \geq -T_a$, $x \in \Omega$. Then for $|x| \leq \rho + \varepsilon$ we have $\partial_t \zeta_a(t, x) = \psi_a(t) \in C_0^\infty(\mathbb{R}_t^-)$.

For $m > d/2 + 3$ consider the operator $P(t)$ given by Proposition 3.2 and introduce

$$\zeta_a P(t) \chi_a : H^m(\Omega) \longrightarrow e^{-c_m - 3/2t} \mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega).$$

Therefore,

$$(\partial_t^2 - \Delta_x)(\zeta_a P(t) \chi_a) = [\square, \zeta_a] P(t) \chi_a =: F_a(t)$$

and

$$(\partial_\nu - \gamma(x)\partial_t)(\zeta_a P(t) \chi_a)|_{\mathbb{R}_t^- \times \Gamma} = -\gamma(x)\psi_a(t)P(t)\chi_a|_{\mathbb{R}_t^- \times \Gamma} =: J_a(t).$$

Clearly, for $g \in H^m(\Omega)$ we have

$$\text{supp } (F_a(t)g) \subset \{(t, x) \in \mathbb{R}_t^- \times \Omega : -(\max\{|x|, \rho + \varepsilon\} + T_a + 1) \leq t \leq -T_a\},$$

$$\text{supp } (J_a(t)g) \subset \{(t, x) \in \mathbb{R}_t^- \times \Gamma : -(\rho + \varepsilon + T_a + 1) \leq t \leq -(\rho + \varepsilon + T_a)\}.$$

The operator $P(t)$ has a Schwartz kernel $p(t, x, y) \in \mathcal{D}'(\mathbb{R}_t^- \times \Omega \times \mathbb{R}_y^d)$ such that

$$(P(t)f)(x) = \int_{\mathbb{R}^d} p(t, x, y)f(y)dy, \quad f \in C_0^\infty(\mathbb{R}^d),$$

where the integral is taken in the sens of distributions. This implies that $J_a(t)$ has kernel $j_a(t, x, y) = \gamma(x)\psi_a(t)p(t, x, y)\chi_a(y)|_{x \in \Gamma}$. According to (3.13), we have

$$\gamma(x)\psi_a(t)P(t)\chi_a|_\Gamma : H^m(\Omega) \rightarrow \mathcal{H}^{m-5/2}(\mathbb{R}_t^- \times \Gamma).$$

We will construct a kernel $\tilde{b}_a(t, x, y)$ such that

$$\tilde{b}_a(t, x, y) = 0 \text{ for } -(\rho + \varepsilon + T_a + 1) \leq t \leq 0, \quad x \in \Gamma, \quad (4.1)$$

$$\partial_\nu \tilde{b}_a(t, x, y) = j_a(t, x, y) \text{ for } -(\rho + \varepsilon + T_a + 1) \leq t \leq 0, \quad x \in \Gamma. \quad (4.2)$$

To arrange the above conditions we apply Remark 2.1 considering the variables t, y as parameters. By the construction with local coordinates, we have $\tilde{b}_a(t, x, y) = \alpha(x)j_a(t, x, y)$ with some smooth function satisfying $\alpha(x)|_\Gamma = 0$, $\partial_\nu \alpha(x)|_\Gamma = 1$.

Choose $\chi_b \in C_0^\infty(B(0, a))$ such that $\chi_b = 1$ near $B(0, \rho + \varepsilon)$ and $\chi_a \equiv 1$ on the support of χ_b and define $b_a(t, x, y) = \chi_b(x)\tilde{b}_a(t, x, y)$. The conditions (4.1), (4.2) yield

$$(\partial_\nu - \gamma(x)\partial_t)b_a(t, x, y)|_{x \in \Gamma} = j_a(t, x, y).$$

Next let $B_a(t) : H^m(\Omega) \rightarrow H^{m-2}(\mathbb{R}_t^- \times \Omega)$ be the operator with kernel $b_a(t, x, y)$. The smoothness of $b_a(t, x, y)$ follows from that of $j_a(t, x, y)$. The choice of the correction $B_a(t)$ leads to the equality

$$(\partial_\nu - \gamma\partial_t)\left(\zeta_a P(t)\chi_a - B_a(t)\right)\Big|_{x \in \Gamma} = 0$$

and we have

$$\text{supp}(B_a(t)g) \subset \{(t, x) : -(\rho + \varepsilon + T_a + 1) \leq t \leq -(\rho + \varepsilon + T_a), |x| \leq a\}.$$

Denote by $W_a(t)$ the solution of the problem

$$\square_0 W_a(t) = -(1 - \chi_b)F_a(t), W_a(t) \equiv 0 \text{ for } t \geq 0. \quad (4.3)$$

Here $\square_0 = \partial_t^2 - \Delta_0$ and Δ_0 is the Laplacian in $\mathbb{R}^d$. Repeating the argument of the proof of Theorem 4.3 in [1], it is easy to see that there exists $C_a > T_a$ such that

$$\text{supp}(W_a(t)g) \subset \{(t, x) \in \mathbb{R}_t^- \times \Omega : -(|x| + C_a) \leq t \leq -(|x| + T_a)\}. \quad (4.4)$$

For completeness we present the details. Write

$$W_a(t) = -(1 - \chi_b)\zeta_a H(t)P(t)\chi_a + H(t)U_0(t)((1 - \chi_b)\chi_a) + Q_a(t),$$

where $U_0(t)$ is the solution of the Cauchy problem

$$\begin{cases} \square_0 U_0 = 0, \\ U_0(0) = 0, (\partial_t U_0)(0) = (1 - \chi_b)\zeta_a \chi_a. \end{cases}.$$

Then

$$\begin{aligned} \square_0 Q_a &= \square_0 W_a + (1 - \chi_b)\square_0(\zeta_a H(t)P(t)\chi_a) + [\Delta, \chi_b]\zeta_a H(t)P(t)\chi_a \\ &\quad - \square_0(H(t)U_0(t)(1 - \chi_b)\chi_a) = (1 - \chi_b)\chi_a + [\Delta, \chi_b]\zeta_a H(t)P(t)\chi_a - (1 - \chi_b)\chi_a \\ &= [\Delta, \chi_b]\zeta_a H(t)P(t)\chi_a = r_{a,b}(t, x), \\ Q_a(0) &= (\partial_t Q_a)(0) = 0. \end{aligned}$$

We apply the Duhamel formula

$$Q_a(t) = - \int_t^0 U_0(t-s, x, y)r_{a,b}(s, y)dyds,$$

where $U_0(t, x, y)$ is the kernel of the operator $\frac{\sin(t\sqrt{-\Delta_0})}{\sqrt{-\Delta_0}}$. By the sharp Huyghens principle with a constant $C_a > 0$ we obtain

$$\text{supp}(Q_a(t)g) \subset \{(t, x) : -(|x| + C_a) \leq t \leq 0\}.$$

Indeed, if $(t, x) \in \text{supp} Q_a(t)g$, there exist $(s, y) \in \text{supp} r_{a,b}$ and $t \leq s \leq 0$ such that $|x - y| = -(t - s)$ and this implies

$$-t \leq |x| + |y| - s \leq |x| + 2a + T_a + 1.$$

For $U_0(t)(1 - \chi_b)\chi_a$ by a finite speed of propagation argument one obtains the same result. This proves the lower bound of t in (4.4). To establish the upper bound, we apply ones more the Duhamel formula for $W_a(t)$ writing

$$W_a(t)g = \int_t^0 U_0(t-s, x, y)(1 - \chi_b(y))F_a(s)g(y)dyds, \quad t \leq 0.$$

If $(t, x) \in \text{supp } W_a(t)g$, there exist $(s, y) \in \text{supp } (1 - \chi_b)F_a(s)g(y)$ and $t \leq s \leq 0$ such that $|x - y| = -(t - s)$. Hence

$$-t \geq |x| - |y| - s \geq |x| - |y| + |y| + T_a = |x| + T_a.$$

Choose a function $\chi_c \in C_0^\infty(B(0, a))$ which is equal to 1 near $B(0, \rho)$ and such that $\chi_b \equiv 1$ on $\text{supp } \chi_c$ and introduce the approximative resolvent

$$R_a^\sharp(\lambda) = \int_{-\infty}^0 e^{-\lambda t} \left[\zeta_a P(t)\chi_a + (1 - \chi_c)W_a(t) - B_a(t) \right] dt, \quad \text{Re } \lambda < 0.$$

The integral is well defined since for fixed x the supports with respect to t of the functions under integration are bounded. For $m > d/2 + 3$ we have $R_a^\sharp(\lambda) : H_{comp}^m(\Omega) \rightarrow H_{loc}^{m-2}(\Omega)$. We obtain

$$(\Delta - \lambda^2)R_a^\sharp(\lambda) = \chi_a(\text{Id} + L_a(\lambda)), \quad (\partial_\nu - \lambda\gamma(x))R_a^\sharp(\lambda)|_{x \in \Gamma} = 0$$

with

$$L_a(\lambda) = \int_{-\infty}^0 e^{-\lambda t} \left(-\chi_b F_a(t) - [\Delta, \chi_c]W_a(t) + \square B_a(t) \right) dt.$$

After an integration by parts with respect to t we deduce

$$L_a(\lambda) = \lambda^{-1} \int_{-\infty}^0 e^{-\lambda t} \left(-\chi_b \partial_t F_a(t) - [\Delta, \chi_c] \partial_t W_a(t) + \partial_t \square B_a(t) \right) dt.$$

For the invertibility of $\text{Id} + L_a(\lambda)$ in $H^m(\Omega \cap B(0, a))$ we must estimate the $\|\cdot\|_{m,a}$ norms of the terms under integration. For $\partial_t F_a(t)$ one observes that $\partial_t^2 P(t) = \Delta P(t)$, hence $\|\partial_t F_a(t)\|_{m,a} \leq C_{1,m} \|\chi_a\|_{m+4,a}$. Similarly, for $\square B_a(t)$ we exploit once more the property of $P(t)$ since $b_a(t, x, y) = \chi_b(x)\alpha(x)p(t, x, y)$ and conclude that $\|\partial_t \square B_a(t)\|_{m,a} \leq C_{2,m} \|\chi_a\|_{m+4,a}$. Finally, $\partial_t W_a(t)$ can be estimated by $F_a(t)$ and we deduce

$$\|L_a(\lambda)\|_{m,a} \leq D_{m+4,a} |\lambda|^{-1} \|\chi_a\|_{m+4,a}.$$

Let $\rho < a_1 < a$. We fix $m > d/2 + 3$ and fix a function $\chi_a \in C_0^\infty(B(0, a))$ such that $\chi_a \equiv 1$ for $|x| \leq a_1$. Choose $|\lambda| \geq 2D_{m+4,a} \|\chi_a\|_{m+4,a} = A_0$. Then the operator $(\text{Id} + L_a(\lambda))^{-1}$ is invertible in $H^m(\Omega \cap B(0, a))$ and for every $g \in H^m(\Omega)$ with $\text{supp } g \subset B(0, a_1)$ by an analytic continuation we get

$$R_a(\lambda)g = R_a^\sharp(\lambda)(\text{Id} + L_a(\lambda))^{-1}g = \chi_a g = g, \quad \text{Re } \lambda < 0, \quad |\lambda| \geq A_0.$$

The function $u(x, \lambda) = R_a(\lambda)g$ is analytic solution of (2.4) with $f = 0$. According to Lemma 2.1, this implies that for every $s \geq 0$ the operator $\mathcal{C}(\lambda)^{-1} : H^s(\Gamma) \rightarrow H^{s+1}(\Gamma)$ is analytic for $\text{Re } \lambda < 0$, $|\lambda| \geq A_0$ and we obtain the following

Proposition 4.1. *There exists $A_0 > 0$ such that the operator G has no eigenvalues*

$$\lambda \in \{\mathbb{C} : \text{Re } \lambda \leq -c_0\} \cup \{\mathbb{C} : \lambda \in \mathbb{C}_-, |\lambda| \geq A_0\}.$$

It is important to note that $D_{m+4,a}$ depends of constants $A_{m+4}, B_{m+4}, K_{m+4}$ in subsection 3.2.

5. ABSENCE OF EIGENVALUES $|\lambda| \leq A_0$

Consider for $\varepsilon \in [0, 1]$ the operator

$$\mathcal{C}_\varepsilon(\lambda) := N(\lambda) - \lambda \varepsilon \gamma(x) = \left(\text{Id} - \lambda \varepsilon \gamma(x) N(\lambda)^{-1} \right) N(\lambda).$$

Obviously, $\max_{x \in \Gamma} \varepsilon \gamma(x) = \varepsilon \delta < 1$. Set $\alpha_\varepsilon = 1 - \varepsilon \delta \geq \alpha$. Below for simplicity we write H^s instead of $H^s(\Gamma)$. It is easy to see that for $\mu \leq -c_0$ we have

$$-\text{Re}((N(\lambda) - \lambda \varepsilon \gamma(x))\varphi, \varphi)_0 \geq (\alpha_\varepsilon |\mu| - C_0) \|\varphi\|_0^2, \quad \forall \varphi \in C^\infty(\Gamma). \quad (5.1)$$

with constant $C_0 > 0$ independent of ε and μ . By the argument in subsection 3.1 we conclude that the operator $\mathcal{C}_\varepsilon(\lambda)^{-1} : H^s \rightarrow H^{s+1}$ is analytic for $\mu \leq -c_0$ uniformly with respect to ε . Recall the constants $c_m = \max \left\{ \frac{2C_m}{\alpha}, c_{m-1} \right\} > 0$ defined in subsection 3.1. Since C_m in (3.2) are independent of $\gamma(x)$, we obtain

$$c_{m,\varepsilon} = \max \left\{ \frac{2C_m}{\alpha_\varepsilon}, c_{m-1,\varepsilon} \right\} \leq c_m.$$

Moreover, for $\mu \leq -c_m$ we have estimates

$$\|\mathcal{C}_\varepsilon(\lambda)^{-1} h\|_s \leq \frac{B_s}{\alpha_\varepsilon |\mu|} \|h\|_s \leq \frac{B_s}{\alpha |\mu|}, \quad \forall h \in H^s(\Gamma) \quad (5.2)$$

with $B_s > 0$ independent of ε . To do this, we repeat the argument of subsection 3.1 and recall that the constants A_m depend of the norms of the derivatives of $\varepsilon \gamma(x)$ of order $j \leq m$, hence they may be chosen independent of ε . Repeating the proof of subsection 3.2, we construct an operator

$$P_\varepsilon(t) : H_{comp}^m(\Omega) \longrightarrow e^{-c_m - 3/2 t} \mathcal{H}^{m-2}(\mathbb{R}_t^- \times \Omega)$$

satisfying

$$\begin{cases} (\partial_t^2 - \Delta_x) P_\varepsilon(t) = 0, \quad P_\varepsilon(0) = 0, \quad \partial_t P_\varepsilon(0) = \text{Id}, \\ (\partial_\nu - \varepsilon \gamma(x) \partial_t) P_\varepsilon(t)|_{\mathbb{R}_t^- \times \Gamma} = 0. \end{cases} \quad (5.3)$$

The analysis of Section 4 implies that $\mathcal{C}_\varepsilon(\lambda)^{-1}$ has an analytic continuation for $\text{Re } \lambda < 0$, $|\lambda| \geq A_0$, where $A_0 > 0$ is independent of ε . This follow from the fact that the constants $A_{m+4}, B_{m+4}, K_{m+4}$ in subsection 3.2 can be chosen uniformly with respect to ε . Denote by $A(\lambda, \varepsilon)$ the compact operator

$$A(\lambda, \varepsilon) = \lambda \varepsilon \gamma(x) N(\lambda)^{-1} : H^{1/2} \rightarrow H^{3/2}.$$

We need the following

Lemma 5.1. *Let $\omega \subset \mathbb{R}$ be a closed bounded interval. Assume $\text{Id} - A(\lambda, \varepsilon)$ invertible in $H^{1/2}$ for $\forall \lambda \in \omega, \forall \varepsilon \in [0, 1]$. Then with a constant $\delta_0 > 0$ independent of λ and ε we have*

$$\|(\text{Id} - A(\lambda, \varepsilon))^{-1}\|_{H^{1/2} \rightarrow H^{1/2}} \leq \delta_0, \quad \forall \lambda \in \omega, \quad \forall \varepsilon \in [0, 1]. \quad (5.4)$$

Proof. Assume that (5.4) is not true. Then there exist sequences $\lambda_m \in \omega, \varepsilon_m \in [0, 1]$ and $u_m \in H^{1/2}$, $\|u_m\|_{1/2} = 1$ such that

$$\|u_m\|_{1/2} \geq m \|(\text{Id} - A(\lambda_m, \varepsilon_m)) u_m\|_{1/2}, \quad \forall m \in \mathbb{N}.$$

Choose a subsequence $\{u_{m_k}\}$ converging weakly to $0 \neq v \in H^{1/2}$ and sequences $\{\lambda_{m_k}\}, \{\varepsilon_{m_k}\}$ converging to $\lambda_0 \in \omega$ and $\varepsilon_0 \in [0, 1]$, respectively. Clearly, for g , $\|g\|_{H^{1/2}} = 1$ we have $A(\lambda_{m_k}, \varepsilon_{m_k})g \rightarrow A(\lambda_0, \varepsilon_0)g$ uniformly with respect to

g since $\mathcal{N}(\lambda)^{-1}$ is uniformly bounded and continuous in operator norm for $\lambda \in \omega$. This means that for fixed $\eta > 0$ and all u_{m_k} we have

$$\| [A(\lambda_{m_k}, \varepsilon_{m_k}) - A(\lambda_0, \varepsilon_0)] u_{m_k} \|_{H^{1/2}} \leq \eta,$$

if $|\lambda_{m_k} - \lambda_0|$ and $|\varepsilon_{m_k} - \varepsilon_0|$ are sufficiently small. Therefore in $H^{1/2}$ one has

$$A(\lambda_{m_k}, \varepsilon_{m_k}) u_{m_k} \rightarrow A(\lambda_0, \varepsilon_0) v$$

because $A(\lambda_0, \varepsilon_0) u_{m_k} \rightarrow A(\lambda_0, \varepsilon_0) v$ since $A(\lambda_0, \varepsilon_0)$ is compact operator. On the other hand,

$$(\text{Id} - A(\lambda_{m_k}, \varepsilon_{m_k})) u_{m_k} \rightarrow 0 \text{ in } H^{1/2}$$

and

$$\begin{aligned} \|u_{m_k} - u_{m_j}\|_{1/2} &\leq \left\| \left[u_{m_k} - A(\lambda_{m_k}, \varepsilon_{m_k}) u_{m_k} \right] - \left[u_{m_j} - A(\lambda_{m_j}, \varepsilon_{m_j}) u_{m_j} \right] \right\|_{1/2} \\ &\quad + \|A(\lambda_{m_k}, \varepsilon_{m_k}) u_{m_k} - A(\lambda_{m_j}, \varepsilon_{m_j}) u_{m_j}\|_{1/2}. \end{aligned}$$

This implies that $\{u_{m_k}\}$ is a Cauchy sequence in $H^{1/2}$ and we conclude that $u_{m_k} \rightarrow v$ in $H^{1/2}$, hence $v - A(\lambda_0, \varepsilon_0)v = 0$. This yields a contradiction with the fact that $(\text{Id} - A(\lambda_0, \varepsilon_0))$ is invertible and the proof is complete. $\square$

The above argument works if we have an closed bounded arc $\beta \in \mathbb{C}$ such that $N(\lambda)^{-1}$ is uniformly bonded for $\lambda \in \beta$ and $\text{Id} - A(\lambda, \varepsilon)$ is invertible for $\forall \lambda \in \beta, \forall \varepsilon \in [0, 1]$. Now consider the contour

$$\zeta = \{z \in \mathbb{C} : \text{Im } z = -A_0, -c_0 \leq \text{Re } z \leq 0\} \cup \{z \in \mathbb{C} : \text{Re } z = 0, |\text{Im } z| \leq A_0\}$$

$$\cup \{z \in \mathbb{C} : \text{Im } z = A_0, -c_0 \leq \text{Re } z \leq 0\} \cup \{z \in \mathbb{C} : \text{Re } z = -c_0, |\text{Im } z| \leq A_0\}.$$

On every segment of ζ the operator $\mathcal{C}_\varepsilon(\lambda)^{-1} = N(\lambda)^{-1}(\text{Id} - A(\lambda, \varepsilon))^{-1}$ is analytic and by Lemma 5.1 there exists a constant $M_0 > 0$ independent of ε such that

$$\|\mathcal{C}_\varepsilon(\lambda)^{-1}\|_{H^{1/2} \rightarrow H^{3/2}} \leq M_0, \lambda \in \zeta. \quad (5.5)$$

It is easy to see that for $\lambda \in \zeta$ the operator $N(\lambda)^{-1} : H^{1/2} \rightarrow H^{3/2}$ is trace class one. Indeed if $N(\lambda)^{-1}\psi = \eta\psi$ with $0 \neq \psi \in H^{3/2}$, $\eta \neq 0$, we obtain $N(\lambda)\psi = \frac{1}{\eta}\psi$ and $|\eta|^{-1} \leq \|N(\lambda)\|_{H^{3/2} \rightarrow H^{1/2}} \leq M_1$. Thus $N(\lambda)^{-1}$ is a finite rank operator with finite number eigenvalues.

We will prove that the trace

$$\text{tr}_{H^{1/2}(\Gamma)} \frac{1}{2\pi i} \int_{\zeta} \mathcal{C}_\varepsilon(\lambda)^{-1} \frac{d\mathcal{C}_\varepsilon(\lambda)}{d\lambda} d\lambda \quad (5.6)$$

is continuous with respect to ε . We write

$$\begin{aligned} S_{1,2}(\lambda) &= \mathcal{C}_{\varepsilon_1}(\lambda)^{-1} \frac{d\mathcal{C}_{\varepsilon_1}(\lambda)}{d\lambda} - \mathcal{C}_{\varepsilon_2}(\lambda)^{-1} \frac{d\mathcal{C}_{\varepsilon_2}(\lambda)}{d\lambda} \\ &= \mathcal{C}_{\varepsilon_1}(\lambda)^{-1} \left(\frac{d\mathcal{C}_{\varepsilon_1}(\lambda)}{d\lambda} - \frac{d\mathcal{C}_{\varepsilon_1}(\lambda)}{d\lambda} \right) + \left(\mathcal{C}_{\varepsilon_1}^{-1}(\lambda) - \mathcal{C}_{\varepsilon_2}^{-1}(\lambda) \right) \frac{d\mathcal{C}_{\varepsilon_2}(\lambda)}{d\lambda} \end{aligned}$$

and obtain

$$\begin{aligned} \mathcal{C}_{\varepsilon_1}^{-1}(\lambda) - \mathcal{C}_{\varepsilon_2}^{-1}(\lambda) &= N(\lambda)^{-1} \left((\text{Id} - A(\lambda, \varepsilon_1))^{-1} - (\text{Id} - A(\lambda, \varepsilon_2))^{-1} \right) \\ &= (\varepsilon_1 - \varepsilon_2) \lambda \mathcal{C}_{\varepsilon_1}(\lambda)^{-1} \gamma \mathcal{C}_{\varepsilon_2}(\lambda)^{-1}. \end{aligned}$$

This implies

$$S_{1,2}(\lambda) = (\varepsilon_1 - \varepsilon_2) \mathcal{C}_{\varepsilon_1}(\lambda)^{-1} \gamma \left(\text{Id} + \lambda \mathcal{C}_{\varepsilon_2}(\lambda)^{-1} \frac{d\mathcal{C}_{\varepsilon_2}(\lambda)}{d\lambda} \right).$$

The operator in bracket $(\cdot)$ is bounded in $H^{1/2}$ uniformly with respect to $\lambda \in \zeta$ and $\varepsilon \in [0, 1]$. Consequently, with some constant $M_2 > 0$, independent of λ and ε_k , $k = 1, 2$, we have

$$|\operatorname{tr} S_{1,2}(\lambda)| \leq M_2 |\varepsilon_1 - \varepsilon_2| |\operatorname{tr} N(\lambda)^{-1}|, \lambda \in \zeta.$$

To estimate $|\operatorname{tr} N(\lambda)^{-1}|$, it is convenient to use the estimate

$$|\operatorname{tr} N(\lambda)^{-1}| \leq \sum_{j=1}^{Q(\lambda)} s_j(\lambda) = \sigma(\lambda), \lambda \in \zeta, \quad (5.7)$$

where $s_j(\lambda)$ are the eigenvalues of the self-adjoint operator $\left((N(\lambda)^{-1})^* N(\lambda)^{-1}\right)^{1/2} = \mathcal{N}(\lambda)$. As above, we show that for all $\lambda \in \zeta$ the operator $\mathcal{N}(\lambda)$ has spectrum in $[\alpha, \beta] \subset \mathbb{R}^+$ with $\alpha > 0$. Changing the variable, we reduce the problem to a compact self-adjoint operator $T(\lambda) : H^{1/2} \rightarrow H^{1/2}$ depending analytically of $\lambda \in [-a, a] \subset \mathbb{R}$, $a > 0$. Moreover, with some $m_0 > 0$ we have $m_0 \|T(0)u\|_{1/2} \leq \|T(\lambda)u\|_{1/2}$, $|\lambda| \leq a$, $\forall u \in H^{1/2}$, so 0 is not an eigenvalue of $T(\lambda)$. We may apply Rellich theorem (see Chapter VII, Theorem 3.9 in [5]) and obtain the analyticity of the eigenvalues of $T(\lambda)$. This implies that the sum of eigenvalues of $T(\lambda)$ is analytic and bounded uniformly for $|\lambda| \leq a$. Going back to (5.7), one deduces $\sigma(\lambda) \leq S_0$ for $\lambda \in \zeta$, hence the trace (5.6) is continuous with respect to ε . Since (5.6) takes values in $\mathbb{N}$, we conclude that this trace is 0 because for $\varepsilon = 0$ we obtain Neumann problem for which there are no eigenvalues in $\mathbb{C}_-$. We are in position to apply Theorem C.11 in Appendix C, [1] or Proposition 2.2 in [10] for the domain $\omega \subset \mathbb{C}$ bounded by ζ . These results imply that $\mathcal{C}(\lambda)$ is invertible for $\lambda \in \omega$ and G has no eigenvalues in ω . Combining this with Proposition 4.1, we complete the proof of Theorem 1.1.

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UNIVERSITÉ DE BORDEAUX, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, 351, COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE

Email address: petkov@math.u-bordeaux.fr