SPECTRAL PROBLEMS FOR OPERATORS WITH CROSSED MAGNETIC AND ELECTRIC FIELDS

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In memory of Pierre Duclos

Abstract. We obtain a representation formula for the derivative of the spectral shift function \( \xi(\lambda; B, \epsilon) \) related to the operators \( H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x \) and \( H(B, \epsilon) = H_0(B, \epsilon) + V(x, y), B > 0, \epsilon > 0 \). We prove that the operator \( H(B, \epsilon) \) has at most a finite number of embedded eigenvalues on \( \mathbb{R} \) which is a step to the proof of the conjecture of absence of embedded eigenvalues of \( H \) in \( \mathbb{R} \). Applying the formula for \( \xi'(\lambda, B, \epsilon) \), we obtain a semiclassical asymptotics of the spectral shift function related to the operators \( H_0(h) = (hD_x - By)^2 + h^2D_y^2 + \epsilon x \) and \( H(h) = H_0(h) + V(x, y) \).

1. Introduction

Consider the two-dimensional Schrödinger operator with homogeneous magnetic and electric fields

\[
H = H(B, \epsilon) = H_0(B, \epsilon) + V(x, y), D_x = -i\partial_x, D_y = -i\partial_y,
\]

where

\[
H_0 = H_0(B, \epsilon) = (D_x - By)^2 + D_y^2 + \epsilon x.
\]

Here \( B > 0 \) and \( \epsilon > 0 \) are proportional to the strength of the homogeneous magnetic and electric fields and \( V(x, y) \) is a \( L^\infty(\mathbb{R}^2) \) real valued function satisfying the estimates

\[
|V(x, y)| \leq C(1 + |x|)^{-2-\delta} (1 + |y|)^{-1-\delta}, \delta > 0.
\]

For \( \epsilon \neq 0 \) we have \( \sigma_{\text{ess}}(H_0(B, \epsilon)) = \sigma_{\text{ess}}(H(B, \epsilon)) = \mathbb{R} \). On the other hand, for decreasing potentials \( V \) it is possible to have embedded eigenvalues \( \lambda \in \mathbb{R} \) and this situation is quite different from that with \( \epsilon = 0 \) when the spectrum of \( H(B, 0) \) is formed by eigenvalues with finite multiplicities which may accumulate only to Landau levels \( \lambda_n = (2n+1)B, n \in \mathbb{N} \) (see [7], [11], [13] and the references cited there). The analysis of the spectral properties of \( H \) and the existence of resonances have been studied in [5], [6], [3] under the assumption that \( V(x, y) \) admits a holomorphic extension in the \( x \)-variable into a domain

\[
\Gamma_{\delta_0} = \{ z \in \mathbb{C} : 0 \leq |\text{Im} \, z| \leq \delta_0 \}.
\]

On the other hand, without any assumption on the analyticity of \( V(x, y) \), it was proved in [3] that the operator \( (H - z)^{-1} - (H_0 - z)^{-1} \) for \( z \in \mathbb{C}, \text{Im} \, z \neq 0 \), is trace class. Thus, following the general setup [9], [19], we may define the spectral shift function \( \xi(\lambda) = \xi(\lambda; B, \epsilon) \) related to \( H_0(B, \epsilon) \) and \( H(B, \epsilon) \) by

\[
\langle \xi', f \rangle = \text{tr} \left( f(H) - f(H_0) \right), f \in C_0^\infty(\mathbb{R}).
\]

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By this formula $\xi(\lambda)$ is defined modulo a constant but for the analysis of the derivative $\xi' (\lambda)$ this is not important. For the analysis of the behavior of $\xi(\lambda; B, \epsilon)$ it is important to have a representation of the derivative $\xi' (\lambda; B, \epsilon)$. Such representation has been obtained in [3] for strong magnetic fields $B \to +\infty$ under the assumption that $V(x, y)$ admits an analytic continuation in $x$-direction.

In this paper we consider the operator $H$ without any assumption on the analytic continuation of $V(x, y)$ and without the restriction $B \to +\infty$. For such potentials we cannot use the techniques in [5], [6] and [3] related to the resonances of the perturbed problem. Our purpose is to study $\xi'(\lambda; B, \epsilon)$ and the existence of embedded eigenvalues of $H$. The key point in this direction is the following.

**Theorem 1.** Let $V, \partial_x V \in L^\infty(\mathbb{R}^2; \mathbb{R})$ and assume that (1.1) holds for $V$ and $\partial_x V$. Then for every $f \in C_0^\infty(\mathbb{R})$ and $\epsilon \neq 0$ we have

$$\text{tr} \left( f(H) - f(H_0) \right) = -\frac{1}{\epsilon} \text{tr} \left( (\partial_x V) f(H) \right).$$

(1.2)

Notice that in (1.2) by $\partial_x V$ we mean the operator of multiplication by $\partial_x V$. The formula (1.2) has been proved by D. Robert and X. P. Wang [17] for Stark Hamiltonians in absence of magnetic field ($B = 0$). In fact, the result in [17] says that

$$\xi'(\lambda; 0, \epsilon) = -\frac{1}{\epsilon} \int_{\mathbb{R}^2} \partial_x V(x, y) \frac{\partial e}{\partial \lambda}(x, y, x, y; \lambda, 0, \epsilon) dx dy,$$

(1.3)

where $e(\cdot, \cdot; \lambda, 0, \epsilon)$ is the spectral function of $H(0, \epsilon)$. On the other hand, the spectral shift function in [17] is related to the trace of the time delay operator $T(\lambda)$ defined via the corresponding scattering matrix $S(\lambda)$ (see [16]). The presence of magnetic filed $B \neq 0$ and Stark potential lead to some serious difficulties to follow this way. Recently, Theorem 1 has been established by the authors in [4] but the proof in [4] is technical, long and based on the trace class properties of the operators

$$\psi(H \pm i) - N, \partial_x \circ \psi(H \pm i) - N, (H \pm i)\partial_x \circ \psi(H \pm i) - N^{-2}$$

(1.4)

with $\psi \in C_0^\infty(\mathbb{R})$ and $N \geq 2$. The idea is to use the commutators with the operators $\chi_R \partial_x$, where $\chi_R(x, y) = \chi\left(\frac{x}{R}, \frac{y}{R}\right)$ and $\chi \in C_0^\infty(\mathbb{R}^2)$ is a cut-off such that $\chi = 1$ for $|(x, y)| \leq 1$. One shows that

$$\text{tr} \left( [\chi_R \partial_x, H] f(H) - [\chi_R \partial_x, H_0] f(H_0) \right) = 0$$

(1.5)

and next we are going to examine the limit $R \to \infty$ of the trace of the operators in (1.5). The commutators with $\partial_x$ and the presence of magnetic field lead to operators involving $D_x - By$ and this is one of the main difference with the case $B = 0$. To overcome this difficulty we used in [4] the trace class operators (1.4) which led to technical problems. On the other hand, the operator $\partial_x$ is often used for operators with Stark potential $\epsilon x$ and this influenced our approach in [4]. One of the goal of this work is to present a new shorter and elegant proof of Theorem 1. The new idea is to apply the shift operator $U_\tau : f(x, y) \to f(x + \tau, y)$ instead of $\partial_x$. In Proposition 1 we show that

$$\text{tr} \left( [U_\tau, H] f(H) - [U_\tau, H] f(H_0) \right) = 0.$$
The second question examined in this work is the existence of embedded real eigenvalues of $H$. In the physical literature one conjectures that for $\epsilon \neq 0$ there are no embedded eigenvalues. We established in [4] a weaker result saying that in every interval $[a, b]$ we may have at most a finite number of embedded eigenvalues with finite multiplicities. Under the assumption for analytic continuation of $V$ it was proved in [5] that in some finite interval $[a(B, \epsilon), \beta(B, \epsilon)]$ there are no resonances $z$ of $H(B, \epsilon)$ with $\text{Re} z \notin [a(B, \epsilon), \beta(B, \epsilon)]$. Since the real resonances $z$ coincide with the eigenvalues of $H(B, \epsilon)$, we obtain some information for the embedded eigenvalues. We prove in Section 3 without the condition of analytic continuation of $V(x, y)$ that $H$ has no embedded eigenvalues outside an interval $[a(B, \epsilon), \beta(B, \epsilon)]$. Combining this with the result in [4], we conclude that $H$ has at most a finite number of embedded eigenvalues. Finally, applying the representation formula for the derivative of the spectral shift function $\xi_h(\lambda) = \xi_h(\lambda, B, \epsilon)$ related to the operators $H_0(h) = (hD_x - B_y)^2 + h^2D_y^2 + \epsilon x$ and $H(h) = H_0(h) + V(x, y)$, we obtain a semiclassical asymptotics of $\xi_h(\lambda)$ as $h \searrow 0$ uniformly with respect to $\lambda \in [E_0, E_1]$ under some assumptions on the critical values of the symbol of $H(h)$.

2. REPRESENTATION OF THE SPECTRAL SHIFT FUNCTION

We suppose without loss of generality that $B = \epsilon = 1$. Set $\langle z \rangle = (1 + \langle z \rangle^2)^{1/2}$. For reader convenience we recall the following lemma proved in [4]

**Lemma 1.** Let $\delta > 0$ and let $k_j(x, y) = \langle x \rangle^{-j(1+\delta)}\langle y \rangle^{-j(1+\delta)}$, $j = 1, 2$. The operators $G_2 := k_2(H_0 + i)^{-2}$, $G_2^*$, (resp. $G_1 := k_1(H_0 + i)^{-1}$, $G_1^*$), are trace class (resp. Hilbert-Schmidt).

As an application of Lemma 1 recall that Proposition 1 in [4] says that for $g \in C_0^\infty(\mathbb{R})$ the operators $Vg(H)$ and $Vg(H_0)$ are trace class. Obviously, the same is true for $V(x + \tau, y)g(H)$ and we will use this fact below. Consider the shift operator

$$U_\tau : f(x, y) \longrightarrow f(x + \tau, y).$$

Let $H_0 = (D_x - y)^2 + D_y + x, H = H_0 + V(x, y)$. It is clear that

$$[U_\tau, H_0]u = U_\tau H_0 u - H_0 U_\tau u = U_\tau (xu) - xu U_\tau u = \tau U_\tau u,$$

hence $[U_\tau, H_0] = \tau U_\tau$. Next

$$[U_\tau, V] = U_\tau (Vu) - VU_\tau u = V(x + \tau)U_\tau u - VU_\tau u = \left(V(x + \tau, y) - V(x, y)\right)U_\tau u.$$

Thus given a function $f \in C_0^\infty(\mathbb{R})$, we get

$$[U_\tau, H]f(H) - [U_\tau, H_0]f(H_0) = \left[\tau + (V(x + \tau, y) - V(x, y))\right]U_\tau f(H) - \tau U_\tau f(H_0)$$

$$= \tau U_\tau \left(f(H) - f(H_0)\right) + \left(V(x + \tau, y) - V(x, y)\right)U_\tau f(H).$$

**Proposition 1.** We have the equality

$$\text{tr} \left([U_\tau, H]f(H) - [U_\tau, H_0]f(H_0)\right) = 0. \quad (2.1)$$

**Proof.** We write

$$\text{tr} \left[U_\tau Hf(H) - U_\tau H_0 f(H_0) + H_0 U_\tau f(H_0) - HU_\tau f(H)\right]$$

$$= \text{tr} U_\tau \left(Hf(H) - H_0 f(H_0)\right) + \text{tr} \left(H_0 U_\tau f(H_0) - HU_\tau f(H)\right) = (I) + (II).$$
For the term (I), by using the cyclicity of the trace, we have
\[
(I) = \operatorname{tr} \left( (Hf(H) - H_0f(H_0))U_\tau \right) = \operatorname{tr} \left( f(H)H - f(H_0)H_0 \right) U_\tau. \tag{2.2}
\]
On the other hand,
\[
(II) = \operatorname{tr} \left( (H_0 - H)U_\tau f(H_0) \right) + \operatorname{tr} \left[ HU_\tau \left( f(H_0) - f(H) \right) \right] = (II_1) + (II_2),
\]
and we justify below the trace class properties of the operators (II_1) and (II_2). For (II_1) we write
\[-(II_1) = VU_\tau f(H_0) = U_\tau [U_\tau^{-1} VU_\tau] f(H_0) = U_\tau V(x - \tau, y) f(H_0)\]
and the operator on the right hand side is trace class.

It is easy to see that the operator \( (f(H_0) - f(H))(H + i) \) is trace class since
\[
(f(H_0) - f(H))(H + i) = \left[ f(H_0)(H_0 + i) - f(H)(H + i) \right] + f(H_0)V,
\]
where on the right hand side we have a sum of two trace class operators. The same argument shows that the operator \( H(f(H_0) - f(H)) \) is trace class. Next we show that the operator \( H(f(H_0) - f(H))(H + i) \) is trace class. To do this, we write
\[
H(f(H_0) - f(H))(H + i) = \left( H_0 f(H_0)(H_0 + i) - H f(H)(H + i) \right) + V f(H_0)(H_0 + i)
\]
\[
+ V f(H_0)V + H_0 f(H_0)V
\]
and the four operators on the right hand side are trace class. This implies that \( HU_\tau(f(H_0) - f(H))(H + i) \) is trace class since the commutator \([H, U_\tau]\) is a bounded operator. After these preparations we write
\[
(II_2) = HU_\tau(f(H_0) - f(H)) = U_\tau H(f(H_0) - f(H)) + [H, U_\tau](f(H_0) - f(H))
\]
which obviously is trace class. Exploiting the trace class properties, we can write
\[
(II_2) = \operatorname{tr} \left[ HU_\tau(f(H_0) - f(H))(H + i)^{-1} \right]
\]
\[
= \operatorname{tr} \left[ (H + i)^{-1} HU_\tau(f(H_0) - f(H)) \right] (H + i)
\]
\[
= \operatorname{tr} \left( (f(H_0) - f(H))(H + i)(H + i)^{-1} HU_\tau \right) = \operatorname{tr} \left( (f(H_0) - f(H))HU_\tau \right).
\]
Combining the above expressions, we get
\[
(I) + (II_1) + (II_2) = \operatorname{tr} \left( (H_0 - H)U_\tau f(H_0) \right) + \operatorname{tr} \left( f(H_0)(H - H_0)U_\tau \right)
\]
\[
= \operatorname{tr} \left( -VU_\tau f(H_0) \right) + \operatorname{tr} \left( U_\tau f(H_0)V \right).
\]
It remains to show that \( \operatorname{tr}(VU_\tau f(H_0)) = \operatorname{tr}(U_\tau f(H_0)V) \). To do this, choose a function \( \chi \in C^\infty_0(\mathbb{R}^2) \) such that \( \chi = 1 \) for \( |(x, y)| \leq 1 \). For \( R > 0 \) set
\[
\chi_R(x, y) = \chi \left( \frac{x}{R}, \frac{y}{R} \right)
\]
and consider
\[
\operatorname{tr}(VU_\tau f(H_0)\chi_R) = \operatorname{tr}(U_\tau f(H_0)V\chi_R).
\]
The operator \( \chi_R \) converges strongly to identity as \( R \to \infty \) and applying the well known property of trace class operators (see for instance, Proposition 1 in [4]), we conclude that

\[
\text{tr} \left( V U_\tau f(H_0) \right) = \text{tr} \left( U_\tau f(H_0) V \right)
\]

and the proof is complete. \( \square \)

**Proof of Theorem 1.** According to Proposition 1, we have

\[
\text{tr} \left( U_\tau (f(H) - f(H_0)) \right) = -\text{tr} \left( \frac{V(x + \tau, y) - V(x, y)}{\tau} U_\tau f(H) \right) \quad (2.3)
\]

We take the limit \( \tau \to 0 \) and observe that

\[
U_\tau \to I, \quad \frac{V(x + \tau, y) - V(x, y)}{\tau} U_\tau \to \partial_x V
\]

strongly. Since \((f(H) - f(H_0))\) is a trace class operator, applying once more the property of trace class operators, we get

\[
\lim_{\tau \to 0} \text{tr} \left( U_\tau (f(H) - f(H_0)) \right) = \text{tr} \left( f(H) - f(H_0) \right).
\]

To treat the limit \( \tau \to 0 \) in the right hand term of (2.3), consider the function,

\[
g_\delta(x, y) = \langle x \rangle^{-2-\delta} \langle y \rangle^{-1-\delta}
\]

\( \delta > 0 \) being the constant of (1.1). Following Lemma 1, the operator \( g_\delta(H_0 + \mathbf{i})^{-2} \) is trace class.

Hence

\[
g_\delta f(H) = g_\delta(f(H) - f(H_0)) + g_\delta(H_0 + \mathbf{i})^{-2}(H_0 + \mathbf{i})^2 f(H_0)
\]

is also a trace class operator.

To treat the limit \( \tau \to 0 \), we use the representation

\[
\left( \frac{V(x + \tau, y) - V(x, y)}{\tau} g_\delta^{-1} \right) \left[ g_\delta U_\tau g_\delta^{-1} \right] g_\delta f(H).
\]

The operators in the brackets \( (\ldots), \ldots \) converge strongly as \( \tau \to 0 \) to \((\partial_x V)g_\delta^{-1}\) and \(I\), respectively. Letting \( \tau \to 0 \), we obtain

\[
\lim_{\tau \to 0} \text{tr} \left( \frac{V(x + \tau, y) - V(x, y)}{\tau} \right) U_\tau f(H) = \text{tr} \left( (\partial_x V) f(H) \right)
\]

and the proof is complete.

**Remark 1.** The proof of Theorem 1 works for operators \( M = (D_x - C(y))^2 + D_y^2 + \epsilon x + V(x, y) \) with non-linear \( C(y) \) assuming that we have an analog of Lemma 1 for \( H \) and \( H_0 \) replaced by \( M \) and \( M_0 = (D_x - C(y))^2 + D_y^2 + \epsilon x \), respectively. Also we may examine the operators in \( \mathbb{R}^3 \) having the form

\[
\left( D_x + \frac{B}{2} y \right)^2 + \left( D_y - \frac{B}{2} x \right)^2 + D_z^2 + \epsilon z + V(x, y, z)
\]

applying the shift operator \( U_\tau : f(x, y, z) \to f(x, y, z + \tau) \). Some operators with magnetic potentials and Stark potential in \( \mathbb{R}^n \), \( n \geq 3 \), can be investigated by the same approach.
Now consider the operators $H_0(h) = (hD_x - By)^2 + h^2D_y^2 + \epsilon x$, $H(h) = H_0(h) + V(x, y)$, $h > 0$. Under the assumption (1.1) for $V(x, y)$ we have the statement of Lemma 1 for $H_0$ replaced by $H_0(h)$. Moreover, the operators $Vg(H(h))$ and $Vg(H_0(h))$ are trace class for every $g \in C_0^\infty(\mathbb{R})$. Thus for every $f \in C_0^\infty(\mathbb{R})$ the operator $f(H(h)) - f(H_0(h))$ is trace class and we can define the spectral shift function $\xi_h = \xi_h(\lambda, \epsilon, B, \epsilon)$ by the formula

$$
\langle \xi_h, f \rangle = \text{tr} \left( f(H(h)) - f(H_0) \right), \quad f \in C_0^\infty(\mathbb{R}).
$$

Under the assumption of Theorem 1, we obtain repeating the proof of (1.2) the representation

$$
\text{tr} \left( f(H(h)) - f(H_0(h)) \right) = -\frac{1}{\epsilon} \text{tr} \left( (\partial_x V)f(H(h)) \right).
$$

3. Embedded eigenvalues of $H$

In this section we use the notation

$$
L = H(0) = (D_x - By)^2 + D_y^2 + \epsilon x.
$$

Our purpose is to prove the following

**Theorem 2.** There exists $C > 0$ such that $H$ has no eigenvalues $\lambda$, $|\lambda| \geq C$.

**Proof.** First notice that for every function $f \in C_0^\infty(\mathbb{R})$ we have

$$
f(H)[\partial_x, H]f(H) = \epsilon f^2(H) + f(H)\partial_x Vf(H).
$$

We will show the absence of embedded eigenvalues $\lambda > C > 0$. The case $\lambda < -C$ can be treated by the same argument. Assume that there exists a sequence of eigenvalues $\lambda_n \rightarrow +\infty$, $\lambda_{n+1} > \lambda_n + 1$, $\forall n$ and let $H\varphi_n = \lambda_n\varphi_n$, $n \in \mathbb{N}$ with $(\varphi_i, \varphi_j) = \delta_{i,j}$. Choose cut-off functions $f_n(t) \in C_0^\infty(\mathbb{R})$ so that $f_n(\lambda_n) = 1$, $0 \leq f_n(t) \leq 1$ and $f_n(t) = 0$ for $|t - \lambda_n| \geq 1/2$. It is clear that $f_n(H)\varphi_n = \varphi_n$ and

$$
(\varphi_n, f_n(H)[\partial_x, H]f_n(H)\varphi_n) = 0, \quad \forall n \in \mathbb{N}.
$$

We wish to prove that for $n$ large enough we have

$$
\left| (\varphi_n, f_n(H)\partial_x Vf_n(H)\varphi_n) \right| = \left| (\varphi_n, \partial_x Vf_n(H)\varphi_n) \right| \leq \epsilon/2
$$

(3.2) which leads to a contradiction with (3.1) since $(\varphi_n, f_n^2(H)\varphi_n) = 1$. Consider the operator

$$
f_n(H) = -\frac{1}{\pi} \int_{W_n} \bar{\partial}f_n(z)(z - H)^{-1}L(dz),
$$

where $\tilde{f}_n(z)$ is an almost analytic continuation of $f_n$ with supp $\tilde{f}_n(z) \subset W_n$, $W_n = \{z \in \mathbb{C} : |z - \lambda_n| \leq 2/3\}$ is a complex neighborhood of $\lambda_n$ and

$$
\bar{\partial}f_n(z) = O(|\text{Im} z|^{-\infty})
$$

uniformly with respect to $n$. Here $L(dz)$ is the Lebesgue measure in $\mathbb{C}$. We write

$$
(\varphi_n, \partial_x Vf_n(H)\varphi_n) = -\frac{1}{\pi} \int_{W_n \cap \{|\text{Im} z| \leq \eta\}} \bar{\partial}f_n(z)(\varphi_n, (\partial_x V)(z - H)^{-1}\varphi_n)L(dz)
$$

$$
-\frac{1}{\pi} \int_{W_n \cap \{|\text{Im} z| > \eta\}} \bar{\partial}f_n(z)(\varphi_n, (\partial_x V - V_0)(z - H)^{-1}\varphi_n)L(dz)
$$

$$
-\frac{1}{\pi} \int_{W_n \cap \{|\text{Im} z| \leq \eta\}} \bar{\partial}f_n(z)(\varphi_n, \epsilon V_0(z - H)^{-1}\varphi_n)L(dz)
$$

$$
-\frac{1}{\pi} \int_{W_n \cap \{|\text{Im} z| > \eta\}} \bar{\partial}f_n(z)(\varphi_n, \epsilon V_0(z - H)^{-1}\varphi_n)L(dz).
$$
\[-\frac{1}{\pi} \int_{W \cap \{|\text{Im } z| > \eta\}} \tilde{\partial} f_n(z)(\varphi_n, V_0(z - H)^{-1}\varphi_n) L(dz) = R_n + Q_n + S_n,\]

where \(V_0(x, y) \in C_0^\infty(\mathbb{R}^2)\). We choose \(\eta > 0\) small enough to arrange \(|R_n| \leq \epsilon/6\) for all \(n \in \mathbb{N}\). Next we fix \(0 < \eta < 1\) and we will estimate \(Q_n\) and \(S_n\). For the resolvent \((z - L)^{-1}\) we will exploit the following

**Proposition 2.** ([6]) Let \(f, g\) be bounded functions with compact support in \(\mathbb{R}^2\). Then for every compact \(K \subset \mathbb{R} \setminus \{0\}\) we have

\[
\lim_{\lambda \to \pm \infty} \|f(\lambda + i\gamma - L)^{-1}g\| = 0
\]

uniformly for \(\gamma \in K\).

We choose \(V_0\) so that \(\|\partial_x V - V_0\|\) is sufficiently small in order to arrange \(|Q_n| \leq \epsilon/6, \forall n \in \mathbb{N}\). Now we pass to the estimation of \(S_n\). We have

\[V_0(z - H)^{-1} = V_0(z - L)^{-1} + V_0(z - L)^{-1}(V - V_1)(z - H)^{-1} + V_0(z - L)^{-1}V_1(z - H)^{-1}.\]  (3.3)

We replace \(V_0(z - H)^{-1}\) in \(S_n\) by the right hand side (3.3) choosing \(V_1 \in C_0^\infty(\mathbb{R}^2)\). For the term involving \((V - V_1)\) in (3.3) we take \(V_1\) so that \(\|V - V_1\|\) is small enough, to obtain a term bounded by \(\epsilon/18\). Next we fix the potentials \(V_0, V_1\) with compact support. By Proposition 2 setting \(z = \lambda + i\gamma, \eta \leq |\gamma| \leq 1\), we get

\[
\|\tilde{\partial} f_n(z)V_0(\lambda + i\gamma - L)^{-1}V_1(H - z)^{-1}\| \leq C_2\eta^{-1}\|V_0(\lambda + i\gamma - L)^{-1}V_1\| \leq \frac{9}{4\pi^2} \frac{\epsilon}{18}
\]

for \(\text{Re } z = \lambda \geq C_{\epsilon, \eta}\). We choose \(n \geq n_0 = n_0(\epsilon, \eta)\), so that \(\text{Re } z \geq C_{\epsilon, \eta}\) for \(z \in W_n\) and \(n \geq n_0\). Thus we can estimate the term involving \(V_0(z - L)^{-1}V_1\) in (3.3) by \(\epsilon/18\). It remains to deal with the term containing \(V_0(z - L)^{-1}\). Let \(\psi(x, y) \in C_0^\infty(\mathbb{R}^2)\) be a cut-off function such that \(\psi = 1\) on the support of \(V_0\). We write

\[
\psi V_0(z - L)^{-1} = V_0(z - L)^{-1}\psi - V_0(z - L)^{-1}[D_x - By]^2 + D_y^2, \psi](z - L)^{-1}
\]

\[
= V_0(z - L)^{-1}\psi - V_0(z - L)^{-1}\psi_1[D_x - By]^2 + D_y^2, \psi](z - L)^{-1},
\]

where \(\psi_1\) is a cut-off function equal to 1 on the support of \(\psi\). For \(n\) large enough we will have \(\text{Re } z = \lambda \geq C_{\epsilon, \eta}\) for \(z \in \text{supp } W_n\) and can treat \(V_0(z - L)^{-1}\psi\) and \(V_0(z - L)^{-1}\psi_1\) as above. On the other hand,

\[
[D_x - By]^2 + D_y^2, \psi] = -2i\partial_x \psi(D_x - By) - 2i\partial_y \psi D_y - \Delta_{x,y} \psi
\]

(3.4)

and the operators \(\partial_x \psi(D_x - By)(z - L)^{-1}\) and \(\partial_y \psi D_y(z - L)^{-1}\) are bounded by \(C\eta^{-1}\) for \(|\text{Im } z| \geq \eta\). Indeed, we have

\[
(z - L) = (i - L)^{-1}[(1 + (i - z)(z - L)^{-1}]
\]

and it suffices to show that \(\partial_x \psi(D_x - By)(i - L)^{-1}\) and \(\partial_y \psi D_y(i - L)^{-1}\) are bounded. Next, \((i - L)^{-1}\) is a pseudodifferential operator and the symbol of the pseudodifferential operator \((D_x - By)(i - L)^{-1}\) becomes

\[
\frac{\xi - By}{(1 - (\xi - By)^2 - \eta^2 - \epsilon x)^2} - \frac{iB\eta}{(1 - (\xi - By)^2 - \eta^2 - \epsilon x)^2}.
\]

From the well known results for the \(L^2\) boundedness of pseudodifferential operators (see [1]) we deduce that (3.4) is bounded by \(C|\text{Im } z|^{-1}\). Consequently, applying Proposition 2 once more, we can arrange the norm of the operator

\[
V_0(z - L)^{-1}\psi_1[(D_x - By)^2 + D_y^2, \psi](z - L)^{-1}
\]
to be sufficiently small for \( z \in \mathcal{W}_n \), \( |\text{Im } z| \geq \eta \) and \( n \geq n_1 > n_0 \). Combining this with the previous estimates, we get \( |S_n| \leq \epsilon/6 \), hence \( |R_n + Q_n + S_n| \leq \epsilon/2 \) for \( n \) large enough. This implies (3.2) and the proof is complete.

**Corollary 1.** Assume in addition to (1.1) that \( \partial_z^2 V \in C_0(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \). Then \( H \) has at most finite number of embedded eigenvalues in every interval \([a,b] \subset \mathbb{R} \).

This result follows from Theorem 2 and Corollary 1 in [4] which guarantees that \( H \) has at most finite number of embedded eigenvalues in every interval \([a,b] \subset \mathbb{R} \). The conjecture is that \( H \) has no embedded eigenvalues on \( \mathbb{R} \).

### 4. Asymptotics of the spectral shift function

Our purpose in this section is to apply Theorem 1 and (2.4) to give a Weyl type asymptotics with optimal remainder estimates for the spectral shift function \( \xi_h(\lambda) := \xi(\lambda; H(h), H_0(h)) \) corresponding to the operators

\[
H(h) = (hD_x - y)^2 + h^2D_y^2 + x, \quad H_0(h) = H(h) + V(x, y), \quad h > 0.
\]

For simplicity of the exposition in this section we assume that \( B = \epsilon = 1 \). Let \( p_2(x, y, \zeta, \eta) = (\zeta - y)^2 + \eta^2 + x + V(x, y) \). For the analysis of \( \xi_h(\lambda) \) we need the following theorems.

**Theorem 3.** Let \( \psi \in C_0^\infty(\mathbb{R}^2) \) and \( f \in C_0^\infty([0, +\infty[; \mathbb{R}) \). Then we have

\[
\text{tr}\left[ \psi f(H(h)) \right] \sim \sum_{j=0}^{\infty} a_j h^{j-2}, \quad h \searrow 0,
\]  

with

\[
a_0 = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \psi(x, y) f(p_2(x, y, \zeta, \eta)) \ dx \ dy \ d\zeta \ d\eta.
\]

**Theorem 4.** Assume that \( \psi \in C_0^\infty(\mathbb{R}^2) \). Let \( f \in C_0^\infty([E_0, E_1]) \) and \( \theta \in C_0^\infty(]-\frac{1}{c_0}, \frac{1}{c_0}[; \mathbb{R}) \), \( \theta = 1 \) in a neighborhood of 0. Assume that if \( p_2(x, y, \zeta, \eta) = \tau, \tau \in [E_0, E_1] \), then \( dp_2 \neq 0 \). Then there exists \( C_0 > 0 \) such that for all \( N, m \in \mathbb{N} \) there exists \( h_0 > 0 \) such that

\[
\text{tr}\left( \psi \tilde{\theta}_h(\tau - H(h)) f(H(h)) \right) = (2\pi h)^{-2} \left( f(\tau) \sum_{j=0}^{N-1} \gamma_j(\tau) h^j + O(h^N(\tau)^{-m}) \right),
\]

uniformly with respect to \( \tau \in \mathbb{R} \) and \( h \in ]0, h_0[ \), where

\[
\gamma_0(\tau) = -(2\pi i)^{-1} \int_{\mathbb{R}^4} \psi(x, y) \left( (\tau + \text{i} 0 - p_2(x, y, \zeta, \eta))^{-1} - (\tau - \text{i} 0 - p_2(x, y, \zeta, \eta))^{-1} \right) \ dx \ dy \ d\zeta \ d\eta.
\]

Here

\[
\tilde{\theta}_h(\tau) = (2\pi h)^{-1} \int e^{\text{i} \tau t/h} \theta(t) \ dt.
\]

**Proof of Theorem 3 and Theorem 4.** Here and below \( \psi \prec \varphi \) means that \( \varphi(x) = 1 \) on the support of \( \psi \). Let \( G \in C_0^\infty(\mathbb{R}^2) \) with \( \psi \prec G \). Introduce the operator

\[
\tilde{H}(h) = (hD_x - G(x, y)y)^2 + h^2D_y^2 + G(x, y)x + V(x, y),
\]

and set

\[
I = \text{tr}\left[ \psi \left( f(H(h)) - f(\tilde{H}(h)) \right) \right].
\]
Let \( \tilde{f}(z) \in C_0^\infty(\mathbb{C}) \) be an almost analytic continuation of \( f \) with\( \partial_z \tilde{f}(z) = \mathcal{O}(|\text{Im} z|^{\infty}) \). From Helffer-Sjöstrand formula it follows that

\[
I = \frac{1}{\pi} \int \partial_z \tilde{f}(z) \text{tr} \left[ \psi \left( (z - \tilde{H}(h))^{-1} - (z - H(h))^{-1} \right) \right] L(dz),
\]

where \( L(dz) \) denotes the Lebesgue measure on \( \mathbb{C} \).

Let \( \psi_1 \in C_0^\infty(\mathbb{R}^2) \) be a function with \( \psi_1 = 1 \) near supp \((1 - G)\) and \( \psi_1 = 0 \) near supp \( \psi \), and let \( \tilde{\psi} \in C_0^\infty(\mathbb{R}^2) \) be equal to one near supp\((\nabla \psi_1)\) and \( \tilde{\psi} = 0 \) near supp \( \psi \). A simple calculus shows that \( \tilde{H}(h) - H(h) = \psi_1 (\tilde{H}(h) - H(h)) \) and \( \tilde{H}(h), \psi_1 = \tilde{\psi} [\tilde{H}(h), \psi_1] \tilde{H} \). Then

\[
\psi \left( (z - \tilde{H}(h))^{-1} - (z - H(h))^{-1} \right) = \psi(z - \tilde{H}(h))^{-1} \psi_1 (\tilde{H}(h) - H(h))(z - H(h))^{-1}
\]

\[
= \psi(z - \tilde{H}(h))^{-1} \tilde{\psi} \tilde{H}(h), \psi_1 (z - \tilde{H}(h))^{-1}(\tilde{H}(h) - H(h))(z - H(h))^{-1}.
\]

Let \( \chi_1, ..., \chi_N \in C_0^\infty(\mathbb{R}^2; [0, 1]) \) with \( \psi_1 < \chi_1 < ... < \chi_N \) and \( \chi_i \tilde{\psi} = 0, i = 1, ..., N \). By using the equalities \( \chi_1 \psi_1 = ... = \chi_N \psi_1 = \psi_1, \chi_k \tilde{\psi} = 0, \chi_{k-1} \chi_k, \tilde{H}(h) = 0 \) and the fact that

\[
[\chi_k, (z - \tilde{H}(h))^{-1}] = (z - \tilde{H}(h))^{-1}[\chi_k, \tilde{H}(h)](z - \tilde{H}(h))^{-1},
\]

we get

\[
\psi(z - \tilde{H}(h))^{-1} \tilde{\psi} \tilde{H}(h), \psi_1 = \psi(z - \tilde{H}(h))^{-1}[\chi_1, \tilde{H}(h)](z - \tilde{H}(h))^{-1} ... [\chi_N, \tilde{H}(h)](z - \tilde{H}(h))^{-1} \tilde{\psi} \tilde{H}(h), \psi_1 = : L_N(h).
\]

Here

\[
L_N(h) = \mathcal{O}_N(1) \left( \frac{h^N}{|\text{Im} z|^N} \right) ; H^s(\mathbb{R}^2) \rightarrow H^{s+N}(\mathbb{R}^2),
\]

where we equip \( H^N(\mathbb{R}^2) \) with the \( h \)-dependent norm \( \| (hD)^N u \|_{L^2} \). Choose \( N > 2 \) and let \( s = -N \). According to Theorem 9.4 of [1], we have

\[
\left\| \left( -h^2 \Delta + 1 \right)^{-N/2} \tilde{\psi} \right\|_{\text{tr}} = \mathcal{O}(h^{-2}).
\]

Then

\[
\left\| \psi(z - \tilde{H}(h))^{-1} \tilde{\psi} \tilde{H}(h), \psi_1 \right\|_{\text{tr}} = \left\| L_N(h) \left( -h^2 \Delta + 1 \right)^{-N/2} \left( -h^2 \Delta + 1 \right)^{-N/2} \tilde{\psi} \right\|_{\text{tr}} \leq C \left( \left( -h^2 \Delta + 1 \right)^{-N/2} \tilde{\psi} \right)_{\text{tr}} \left( \frac{h^N}{|\text{Im} z|^N} \right) \leq C_1 \left( \frac{h^{N-2}}{|\text{Im} z|^N} \right).
\]

Combining this with (4.4) and using the fact that

\[
\|(z - \tilde{H}(h))^{-1} \tilde{H}(h) - H(h))^{-1} = \|(z - \tilde{H}(h))^{-1} - (z - H(h))^{-1} = \mathcal{O}(1\text{Im} z|^{-1}),
\]

we obtain

\[
\left\| \psi \left( (z - \tilde{H}(h))^{-1} - (z - H(h))^{-1} \right) \right\|_{\text{tr}} = \mathcal{O}\left( \frac{h^{N-2}}{|\text{Im} z|^{N+1}} \right).
\]

Since \( \partial_z \tilde{f}(z) = \mathcal{O}(1\text{Im} z|^{\infty}) \), we have

\[
I = \mathcal{O}(h^{\infty}).
\]

Summing up, we have proved that

\[
\text{tr} \left( \psi f(H(h)) \right) = \text{tr} \left( \psi f(\tilde{H}(h)) \right) + \mathcal{O}(h^{\infty}).
\]
In the same way, we obtain
\[ \text{tr} \left( \psi \tilde{\theta}_h (\tau - H(h)) f(H(h)) \right) = \text{tr} \left( \psi \tilde{\theta}_h (\tau - \tilde{H}(h)) f(\tilde{H}(h)) \right) + O(h^\infty). \quad (4.7) \]

The operator \( \tilde{H}(h) \) is elliptic semi-bounded \( h \)-pseudodifferential operator, so Theorem 3 and Theorem 4 follow from the \( h \)-pseudodifferential calculus and the analysis of elliptic operators in Chapters 8, 9, 12 in [1] (see also [15]). The calculus of the leading terms is given by trivial modification of the argument of Section 7 in [2] and we omit the details. \( \square \)

**Remark 2.** Notice that \( dp_2 \neq 0 \) on \( p_2 = \tau \) is equivalent to
\[ \nabla_{x,y}(x + V(x,y)) \neq 0, \text{ on } \{(x,y); x + V(x,y) = \tau\}. \quad (4.8) \]

Now we will apply Theorem 3 and Theorem 4 to obtain a Weyl-type asymptotics for \( \xi_h(\lambda) \) when \( h \searrow 0 \).

**Theorem 5.** Assume that \( V \in C_0^\infty(\mathbb{R}^2) \) and suppose that (4.8) holds for \( \tau = \lambda_1, \lambda_2 \). Then there exists \( h_0 > 0 \) such that for \( h \in [0, h_0] \) we have
\[ \xi_h(\lambda_2) - \xi_h(\lambda_1) = (2\pi h)^{-2}(c_0(\lambda_2) - c_0(\lambda_1)) + O(h^{-1}), \quad (4.9) \]
where
\[ c_0(\lambda) = -\pi \int_{\mathbb{R}^2} \partial_x V(x,y)(\lambda - x - V(x,y))_+ dx dy. \quad (4.10) \]

**Proof.** Choose a large constant \( M \) such that
\[ M \geq \|\partial_x V\|_\infty + 1. \]
Let \( \psi \in C_0^\infty(\mathbb{R}^2; [0,1]) \) with \( \partial_x V \prec \psi^2 \). According to (2.4), by using the cyclicity of the trace, we get
\[ \langle \xi'_i, f \rangle = \text{tr} \left( f(H(h)) - f(H_0(h)) \right) = -\text{tr} \left( (\partial_x V) f(H(h)) \right) \]
\[ = \text{tr} \left( (M - \partial_x V)^{1/2} \psi f(H(h)) \psi (M - \partial_x V)^{1/2} \right) - M \text{tr} \left( \psi f(H(h)) \psi \right) \]
\[ =: \langle \xi'_1, f \rangle - \langle \xi'_2, f \rangle. \]

Since
\[ f \to \text{tr} \left( (M - \partial_{x_1} V)^{1/2} \psi f(H(h)) \psi (M - \partial_{x_1} V)^{1/2} \right) \]
and
\[ f \to M \text{tr} \left( \psi f(H(h)) \psi \right) \]
are positive functions for \( f \geq 0 \), we deduce that the functions \( \lambda \to \xi_i(\lambda), i = 1, 2 \) are monotonic.

Consequently, we may apply Tauberian arguments for the analysis of the asymptotics of \( \xi_i(\lambda), i = 1, 2 \). We treat below \( \xi_2(\lambda) \). Let \( \varphi \in C_0^\infty(\mathbb{R}), \varphi \geq 0 \), and suppose that (4.8) holds for all \( \tau \in \text{supp } \varphi \). Consider the function
\[ F_\varphi(\lambda) = \int_{-\infty}^{\lambda} \xi_2(\mu) \varphi(\mu) d\mu. \]

Applying (4.3) with \( N = 1 \) and \( m = 2 \), we obtain
\[ \frac{d}{d\lambda}(\tilde{\theta}_h * F_\varphi)(\lambda) = \int \tilde{\theta}_h(\lambda - \mu) \xi_2(\mu) \varphi(\mu) d\mu = (2\pi h)^{-2} \left( \varphi(\lambda) \gamma_0(\lambda) + O\left( \frac{h}{\lambda^2} \right) \right). \quad (4.11) \]
We integrate from \(-\infty\) to \(\lambda\) and we get
\[
\int \left( \int_{-\infty}^{\lambda} \tilde{\theta}_h(\lambda - \mu)d\lambda' \right) \xi_2'(\mu)\varphi(\mu)d\mu = \frac{1}{(2\pi h)^2} \left( \int \int_{p_2 \leq \lambda} M \psi^2(x, y)\varphi(p_2)dxdy d\eta d\zeta + \mathcal{O}(h) \right). \tag{4.12}
\]
In the following we choose \(\theta \in C^1_0(\mathbb{R})\) with \(\tilde{\theta}_h \geq 0\). Let \(h\tilde{\theta}_h(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \theta(u)du \geq 2C_1 > 0\). Therefore, it follows that there exist \(C_2 > 0\) such that
\[
|t| < \frac{h}{C_2} \implies h\tilde{\theta}_h(t) \geq C_1.
\]
Combining this with the fact that \(\tilde{\theta}_h \geq 0\), and using \(\langle \xi_2', f \rangle \geq 0\) for \(f \geq 0\), we obtain
\[
C_1 \int_{\{\lambda - \mu < \frac{C_0}{h} \}} \xi_2'(\mu)\varphi(\mu)d\mu \leq h \int_{\{\lambda - \mu < \frac{C_0}{h} \}} \tilde{\theta}_h(\lambda - \mu)\xi_2(\mu)\varphi(\mu)d\mu
\]
\[
\leq h \int_{\mathbb{R}} \tilde{\theta}_h(\lambda - \mu)\xi_2(\mu)\varphi(\mu)d\mu = h \frac{d}{d\lambda} (\tilde{\theta}_h \ast F_\varphi)(\lambda) = \mathcal{O}(h^{-1}), \tag{4.13}
\]
uniformly with respect to \(\lambda \in \mathbb{R}\). On the other hand, a simple calculus shows that
\[
\int_{-\infty}^{\lambda} \tilde{\theta}_h(\lambda - \mu)d\lambda' = \int_{-\infty}^{\lambda - \mu} \tilde{\theta}_1(t)dt = 1_{[-\infty, \lambda]}(\mu) + \mathcal{O}\left(\left(\frac{\lambda - \mu}{h}\right)^{-\infty}\right). \tag{4.14}
\]
Indeed, for \(\mu < \lambda\) and all \(k \in \mathbb{N}\) we have
\[
\int_{-\infty}^{\lambda - \mu} \tilde{\theta}_1(t)dt = 1 - \int_{\lambda - \mu}^{\infty} t^k \tilde{\theta}_1(t) \frac{1}{k!}dt
\]
and
\[
\int_{\lambda - \mu}^{\infty} t^k \tilde{\theta}_1(t) \frac{1}{k!}dt \leq \left(\frac{\lambda - \mu}{h}\right)^{-k} \int_{\mathbb{R}} t^k \tilde{\theta}_1(t)dt.
\]
A similar argument works for \(\mu > \lambda\). From (4.13) we have for \(k \geq 2\) the estimate
\[
\int_{\mathbb{R}} \left(\frac{\lambda - \mu}{h}\right)^{-k} \xi_2(\mu)\varphi(\mu)d\mu = \sum_{m=-\infty}^{\infty} \int_{\lambda - m\frac{C_0}{h} < \lambda - \mu \leq \lambda + (m+1)\frac{C_0}{h}} \left(\frac{\lambda - \mu}{h}\right)^{-k} \xi_2(\mu)\varphi(\mu)d\mu \tag{4.15}
\]
\[
\leq \sum_{m=0}^{\infty} \left(1 + \left(\frac{m+1}{C_0}\right)^2\right)^{-k/2} \int_{\lambda + m\frac{C_0}{h} \leq \lambda \leq \lambda + (m+1)\frac{C_0}{h}} \xi_2(\mu)\varphi(\mu)d\mu
\]
\[
+ \sum_{m=-\infty}^{-1} \left(1 + \left(\frac{|m+1|}{C_0}\right)^2\right)^{-k/2} \int_{\lambda + m\frac{C_0}{h} \geq \lambda \geq \lambda + (m+1)\frac{C_0}{h}} \xi_2(\mu)\varphi(\mu)d\mu \leq \sum_{m=-\infty}^{\infty} \frac{1}{(C_0 + |m|)^k} \mathcal{O}(h^{-1}),
\]
where in the last inequality at the right hand side we used the fact that (4.13) holds uniformly with respect to \(\lambda \in \mathbb{R}\) and we can estimate the integrals involving \(\xi_2(\mu)\varphi(\mu)\) by \(\mathcal{O}(h^{-1})\) uniformly with respect to \(m\).

Inserting the right hand side of (4.14) in the left hand side of (4.12) and using (4.15), we get
\[
F_\varphi(\lambda) = (2\pi h)^{-2} \left( \int \int_{p_2 \leq \lambda} M \psi^2(x, y)\varphi(p_2)dxdy d\eta d\zeta + \mathcal{O}(h) \right).
\]
We apply the same argument for $\xi_1(h)$ and introduce the function
\[
G_\varphi(\lambda) = \int_{-\infty}^{\lambda} \xi'_h(\mu) \varphi(\mu) d\mu.
\]
Replacing the function $\psi$ by $(M - \partial_x V)^{1/2} \psi$, we get
\[
G_\varphi(\lambda) = \frac{1}{(2\pi \hbar)^2} \left( \int_{p_2 \leq \lambda} (M - \partial_x V) \psi^2(x, y) \varphi(p_2) dx dy d\eta d\zeta + O(h) \right).
\]
Since $\xi_h = \xi_1 - \xi_2$, the above results yield
\[
M_\varphi(\lambda) = \int_{-\infty}^{\lambda} \xi'_h(\mu) \varphi(\mu) d\mu = \frac{1}{(2\pi \hbar)^2} \left( \int_{p_2 \leq \lambda} -\partial_x V(x, y) \varphi(p_2) dx dy d\eta d\zeta + O(h) \right).
\] (4.16)

Now, we are ready to prove Theorem 5. Assume that $\lambda_1 < \lambda_2$, and let $\epsilon > 0$ be small enough. Let $\varphi_1, \varphi_2, \varphi_3 \in C_0^\infty([\lambda_1 - \epsilon, \lambda_2 + \epsilon])$ with $\varphi_1 + \varphi_2 + \varphi_3 = 1$ on $[\lambda_1, \lambda_2]$, $\text{supp } \varphi_1 \subset [\lambda_1 - \epsilon, \lambda_1 + \epsilon]$, $\text{supp } \varphi_2 \subset [\lambda_2 - \epsilon, \lambda_2 + \epsilon]$ and $\text{supp } \varphi_3 \subset [\lambda_1, \lambda_2]$. We choose $\epsilon$ small enough so that (4.8) holds for all $\tau \in [\lambda_1 - \epsilon, \lambda_1 + \epsilon] \cup [\lambda_2 - \epsilon, \lambda_2 + \epsilon]$. We write
\[
\xi_h(\lambda_2) - \xi_h(\lambda_1) = \int_{\lambda_1}^{\lambda_2} (\varphi_1 + \varphi_2 + \varphi_3)(\lambda) \xi'_h(\lambda) d\lambda
\]
\[
= M_{\varphi_2}(\lambda_2) + M_{\varphi_1}(\lambda_2) - M_{\varphi_2}(\lambda_1) - M_{\varphi_1}(\lambda_1) - \text{tr}(\partial_x V \varphi_3(H)),
\]
where for the function $\varphi_3$ we have exploited (2.4). Next for the term involving $\varphi_3$ we apply Theorem 3 and obtain
\[
\text{tr}(\partial_x V \varphi_3(H)) = \frac{1}{(2\pi \hbar)^2} \int \int \partial_x V \varphi_3(p_2) dx dy d\zeta d\eta + O(h^{-1}).
\]
For $M_{\varphi_i}(\lambda_i)$ and $M_{\varphi_i}(\lambda_i)$, $i = 1, 2$, we exploit the above argument and we deduce the asymptotics taking into account (4.16). Summing the terms involving $\varphi_j$, $j = 1, 2, 3$, we conclude that
\[
\xi_h(\lambda_2) - \xi_h(\lambda_1) = (2\pi \hbar)^{-2} d(\lambda_2, \lambda_1) + O(h^{-1}).
\]
For the leading term we have
\[
d(\lambda_2, \lambda_1) = \int \int_{\lambda_1 \leq p_2 \leq \lambda_2} -\partial_x V(x, y) \left( \varphi_1(p_2) + \varphi_2(p_2) + \varphi_3(p_2) \right) dx dy d\zeta d\eta
\]
\[
= -\int \int_{p_2 \leq \lambda_2} \partial_x V(x, y) dx dy d\zeta d\eta + \int \int_{p_2 \leq \lambda_1} \partial_x V(x, y) dx dy d\zeta d\eta.
\]
Finally, notice that
\[
ce_0(\lambda) = -\int \int_{p_2 \leq \lambda} \partial_x V(x, y) dx dy d\zeta d\eta = -\int_{\mathbb{R}^2} \partial_x V(x, y) \left( \int_{(\zeta - y)^2 + \eta^2 \leq (\lambda - x - V(x, y))^2} d\zeta d\eta \right) dx dy
\]
\[
= -\pi \int_{\mathbb{R}^2} \partial_x V(x, y) (\lambda - x - V(x, y))_+ dx dy
\]
and the proof of Theorem 5 is complete. □
Remark 3. If $\lambda \gg 1$ is large enough (resp. $\lambda \ll -1$) then on $\text{supp} (\partial_x V)$, we have
\[(\lambda - x - V)_+ = \lambda - x - V, \quad \text{(resp. } (\lambda - x - V)_+ = 0).\]
Consequently,
\[c_0(\lambda) = -\pi \int_{\mathbb{R}^2} V(x,y) dx dy, \quad \text{for } \lambda \gg 1,
\]
and
\[c_0(\lambda) = 0, \quad \text{for } \lambda \ll -1.
\]
In particular, if we normalize $\xi_h(\lambda)$ by $\lim_{\lambda \to -\infty} \xi_h(\lambda) = 0$, we get
\[\xi_h(\lambda) = (2\pi h)^{-2} c_0(\lambda) + O(h^{-1}).\]

Remark 4. The results of this section can be generalized for potentials $V(x,y)$ for which there exists $\delta_1 \in \mathbb{R}$ such that $\text{supp } V \subset \{(x,y) \in \mathbb{R}^2 : x \geq \delta_1\}$ by using the techniques in [2]. For simplicity we treated the case of $V \in C^\infty_0(\mathbb{R}^2)$ to avoid the complications caused by the calculus of pseudodifferential operators.

References


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