LOCALIZATION OF THE INTERIOR TRANSMISSION EIGENVALUES FOR A BALL

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Abstract. We study the localization of the interior transmission eigenvalues (ITEs) in the case when the domain is the unit ball $\{x \in \mathbb{R}^d : |x| \le 1\}$, $d \ge 2$, and the coefficients $c_j(x)$, $j = 1, 2$, and the indices of refraction $n_j(x)$, $j = 1, 2$, are constants near the boundary $|x| = 1$. We prove that in this case the eigenvalue-free region obtained in [16] for strictly concave domains can be significantly improved. In particular, if $c_j(x), n_j(x), j = 1, 2$ are constants for $|x| \le 1$, we show that all (ITEs) lie in a strip $|\mathrm{Im}\,\lambda| \le C$.

Key words: Interior transmission eigenvalues, eigenvalue-free regions, Dirichlet-to-Neumann map, Bessel functions

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1. Introduction and statement of the result

Let $\Omega \subset \mathbb{R}^d$, $d \ge 2$, be a bounded, connected domain with a C^∞ smooth boundary $\Gamma = \partial\Omega$. A complex number λ with $\mathrm{Re}\,\lambda > 0$ will be called interior transmission eigenvalue (ITE) if the following problem has a non-trivial solution:

$$\begin{cases} \big(\nabla c_1(x)\nabla + \lambda^2 n_1(x)\big)\, u_1 = 0 \text{ in } \Omega, \\ \big(\nabla c_2(x)\nabla + \lambda^2 n_2(x)\big)\, u_2 = 0 \text{ in } \Omega, \\ u_1 = u_2, \;\; c_1\partial_\nu u_1 = c_2 \partial_\nu u_2 \text{ on } \Gamma, \end{cases} \tag{1.1}$$

where ν denotes the exterior Euclidean unit normal to Γ and $c_j(x), n_j(x) \in C^\infty(\overline{\Omega})$, $j = 1, 2$, are strictly positive real-valued functions.

The (ITEs) were first studied by Kirsch [6] and by Colton and Monk [2] in the context of inverse scattering problems. It was shown that the real (ITEs) correspond to the frequencies for which the reconstruction algorithm in inverse scattering based on the so-called linear sampling methods breaks down. This subject attracted the attention of many researchers and the number of publications devoted to the (ITEs) considerably increased in the recent ten years. The reader may consult the survey [1] for a complete list of references and historical remarks.

It is well-known (e.g. see [14]) that there exists a closed non-symmetric operator, A, associated in a natural way to the problem (1.1), such that the possible (ITEs) can be considered as the eigenvalues of A. The analysis of the (ITEs) leads to the following three problems:

(A) Prove the discreteness of the spectrum of A in $\mathbb{C}$;
(B) Find eigenvalue-free regions in $\mathbb{C}$;
(C) Establish a Weyl formula for the counting function of all complex (ITEs)

$$N(r) = \#\{\lambda_j \text{ is (ITE)}, \;\; |\lambda_j| \le r\}.$$

Note that the problem (A) is now relatively well studied (see [8], [13], [10], [4] and the references therein). In fact, the problem (A) is reduced to that of showing that the resolvent of A is meromorphic with residues of finite rank. On the other hand, this is true (see [14]) if the inverse of the operator $T(\lambda)$ introduced in Section 4 is meromorphic. The latter fact can be proved if the parametrix of the operator $T(\lambda)$ constructed in the deep elliptic zone is invertible.

The problems (B) and (C) are more difficult, and they are of some interest for the numerical analysis of the (ITEs). In this direction it is interesting to find an optimal eigenvalue-free region and a Weyl formula with optimal remainder (see [11], [12], [7] and the references cited there). In a recent work [14] the authors showed that (B) and (C) are closely related each other, and a *larger* eigenvalue-free region leads to a Weyl asymptotics with a *smaller* remainder term. More precisely, we proved that the remainder in the Weyl formula is $\mathcal{O}_\varepsilon(r^{d-\kappa+\varepsilon})$, $\forall\, 0<\varepsilon\ll 1$, where $0<\kappa\le 1$ is such that there are no (ITEs) in

$$\left\{\lambda\in\mathbb{C}:\ \mathrm{Re}\,\lambda>1,\ |\mathrm{Im}\,\lambda|\ge C\,(\mathrm{Re}\,\lambda)^{1-\kappa}\right\}.$$

One conjectured that the optimal value of κ must be $\kappa=1$.

The present paper is devoted to the problem (B). More precisely, we are interested in finding as small as possible neighbourhoods of the real axis containing all (ITEs). The first result of this type was obtained in [5] assuming $n_1(x)>1$ in $\bar\Omega$ and $n_2(x)\equiv 1$, $c_1(x)\equiv c_2(x)\equiv 1$. For domains Ω with *arbitrary geometry*, it has been shown in [15] that under the condition (isotropic case)

$$c_1(x)\equiv c_2(x),\quad \partial_\nu c_1(x)\equiv\partial_\nu c_2(x),\quad n_1(x)\neq n_2(x)\quad \text{on}\quad \Gamma, \tag{1.2}$$

or the condition (anisotropic case)

$$(c_1(x)-c_2(x))(c_1(x)n_1(x)-c_2(x)n_2(x))<0\quad \text{on}\quad \Gamma, \tag{1.3}$$

there are no (ITEs) in the region

$$\left\{\lambda\in\mathbb{C}:\ \mathrm{Re}\,\lambda>1,\ |\mathrm{Im}\,\lambda|\ge C_\varepsilon\,(\mathrm{Re}\,\lambda)^{\frac12+\varepsilon}\right\},\quad \forall\,0<\varepsilon\ll 1.$$

The localization of the (ITEs) has been recently studied in [16] in the case when the boundary Γ is strictly concave with respect to both Riemannian metrics $\sum_{k=1}^d \frac{n_j(x)}{c_j(x)}dx_k^2$, $j=1,2$. Under the conditions (1.2) or (1.3) it has been proved in [16] that there are no (ITEs) in the region

$$\{\lambda\in\mathbb{C}:\ \mathrm{Re}\,\lambda>1,\ |\mathrm{Im}\,\lambda|\ge C_\varepsilon\,(\mathrm{Re}\,\lambda)^{\varepsilon}\},\quad \forall\,0<\varepsilon\ll 1. \tag{1.4}$$

The approach in [15] and [16] is based on the construction of a semi-classical parametrix near the boundary for the problem

$$\begin{cases}(h^2\nabla c(x)\nabla+zn(x))u=0 & \text{in}\quad \Omega,\\ u=f\quad \text{on}\quad \Gamma,\end{cases} \tag{1.5}$$

where $0<h\ll 1$ is a semi-classical parameter and $z\in\mathbb{C}$ with $\mathrm{Re}\,z=1$. For domains with arbitrary geometry the parametrix construction for (1.5) works for $|\mathrm{Im}\,z|\ge h^{1/2-\epsilon}$, $0<\epsilon\ll 1$, while for strictly concave domains, by a more complicated construction, one can cover the region $|\mathrm{Im}\,z|\ge h^{1-\epsilon}$. It is a challenging problem to construct a semi-classical parametrix for (1.5) when $|\mathrm{Im}\,z|\ge Ch$, $C\gg 1$ being a constant.

The purpose of the present paper is to improve the eigenvalue-free region (1.4) in the case when the domain is the unit ball in $\mathbb{R}^d, d\ge 2$. Given a parameter $0<\delta\ll 1$, denote $\Omega(\delta)=\{x\in\overline{\Omega}:\mathrm{dist}(x,\Gamma)\le\delta\}$. Our main result is the following

Theorem 1.1. *Let* $\Omega = \{x \in \mathbb{R}^d : |x| \leq 1\}$, $d \geq 2$, *and suppose that there is a constant* $0 < \delta_0 \ll 1$ *such that the functions* $c_j(x)$, $n_j(x)$, $j = 1, 2$, *are constants in* $\Omega(\delta_0)$. *Assume also either the condition* (1.2) *or the condition* (1.3). *Then, there is a constant* $C > 0$ *such that there are no* $(ITEs)$ *in the region*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 1, |\operatorname{Im}\lambda| \geq C \log(\operatorname{Re}\lambda)\}. \tag{1.6}$$

If in addition the functions c_j, n_j, $j = 1, 2$, *are constants everywhere in* $\overline{\Omega}$, *then there are no* $(ITEs)$ *in a larger region of the form*

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 1, |\operatorname{Im}\lambda| \geq C\}. \tag{1.7}$$

Remark 1. The eigenvalue-free region (1.6) is still valid if we add a compact cavity $K \subset \Omega$ and consider the equation (1.1) in $\Omega \setminus K$ with Dirichlet condition on ∂K. Indeed, the only fact needed for our argument is the coercivity of the corresponding Dirichlet realization (see the operator G_D in Section 3), and this is used only in the proof of Lemma 3.4 below.

Remark 2. It is clear from the proof that the fact that the boundary Γ is a sphere is not essential. In other words, the eigenvalue-free regions (1.6) and (1.7) are still valid for any Riemannian manifold $\Omega = (0, 1) \times \Gamma$ with metric $g = dr^2 + r^2\sigma$, where $r \in (0, 1)$, and (Γ, σ) is an arbitrary $d - 1$-dimensional Riemannian manifold without boundary, the metric σ being independent of r.

In the isotropic case when $c_j \equiv 1, j = 1, 2$ and $n_1 = 1, n_2 \neq 1$ are constant, the eigenvalue-free region (1.7) has been established in the one-dimensional case $\Omega = \{x \in \mathbb{R} : |x| \leq 1\}$ (see [13], [10]). Moreover, the case of the ball $\{x \in \mathbb{R}^d : |x| \leq 1\}$, $d = 2, 3$, and radial refraction indices have been studied in [8], [3], [4], where spherical symmetric eigenfunctions depending only on the radial variable $r = |x|$ has been considered. For example, the analysis of such eigenfunctions in $\mathbb{R}^3$ leads to the following one-dimensional problem

$$\begin{cases} u'' + \frac{2}{r}u' + \lambda^2 n(r)u = 0, \quad 0 < r < 1, \\ v'' + \frac{2}{r}v' + \lambda^2 v = 0, \quad 0 < r < 1, \\ u(1) = v(1),\ u'(1) = v'(1), \end{cases} \tag{1.8}$$

where $n(r)$ is a strictly positive function. Among other things, it was shown in [4] that if $n(1) = 1$ and $n'(1)$ or $n''(1)$ is non-zero, then there may exist infinitely many complex eigenvalues of the problem (1.8) lying outside any strip parallel to the real axis. This example shows that in the isotropic case the condition $n(1) \neq 1$ (resp. (1.2)) is important to have an eigenvalue-free region like (1.7).

To study all (ITEs) and all eigenfunctions, one has to consider a family of infinitely many one-dimensional problems. Such an analysis is carried out in [10] in the isotropic case when the domain is the ball $\{x \in \mathbb{R}^d : |x| \leq 1\}$, $d \geq 1$, and

$$c_j \equiv 1, n_1 \equiv 1, n_2 \equiv \gamma \neq 1.$$

In this case the (ITEs) are the zeros in $\mathbb{C}$ of the family of functions

$$F_\nu(\lambda) = \gamma J_\nu(\lambda)J'_\nu(\gamma\lambda) - J_\nu(\gamma\lambda)J'_\nu(\lambda), \quad \nu = l + d/2 - 1, \quad l = 0, 1, 2, ...,$$

where J_ν denotes the Bessel function of order ν. It has been proved in [10] that there are infinitely many real (ITEs) whose counting function has a Weyl asymptotics. When $d = 1$ a Weyl asymptotics for the counting function of all (ITEs) is also obtained.

To prove Theorem 1.1, we follow the same strategy as in [15], [16], which consists of deriving the eigenvalue-free region from some approximation properties of the interior Dirichlet-to-Neumann map (DN). Recall that the interior DN map is a meromorphic operator-valued function, which maps $H^1(\Gamma)$ into $L^2(\Gamma)$, with poles lying on the positive real axis. Thus, the eigenvalue-free region turns out to be the region in which the DN map can be approximated by a simpler operator of the form $f(\Delta_\Gamma)$, where f is a complex-valued function and Δ_Γ denotes the negative Laplace-Beltrami operator on the boundary Γ equipped with the Riemannian metric induced by the Euclidean one. With such an approximation in hands, the problem of proving the eigenvalue-free region is transformed into the much simpler one of inverting complex-valued functions, which in turn is done using the conditions (1.2) or (1.3) (see Section 4). Therefore, a large portion of the present paper is devoted to the study of the interior DN map in the case when Ω is a ball and this analysis has an independent interest. In the case when the coefficients are constants everywhere in the domain we can express the DN map in terms of the Bessel functions. Therefore, instead of the paramatrix we have an *exact representation* of the (DN) map. Then we use the asymptotic expansions of these functions in terms of the Airy function to get the desired approximation (see Theorem 3.1). Of course, we cannot proceed in this way when the coefficients are supposed to be constants only in a neighbourhood of the boundary. In this latter case we show that the DN map can be approximated by the DN map associated to the corresponding problem with constant coefficients everywhere and for which we have an explicit expression in terms of the Bessel functions (see Lemma 3.4).

We expect that the eigenvalue-free regions (1.6) and (1.7) are still true for more general strictly concave domains, but this is hard to prove because the available semi-classical parametrix constructions for the DN map lead to the existence of smaller regions (see [16]). We also conjecture that (1.7) is optimal for a ball and constant coefficients, but this seems difficult to prove.

2. Some properties of the Bessel functions

We begin this section by recalling some basic properties of the Bessel functions $J_\nu(z)$ of real order $\nu \geq 0$ (e.g. see [9]). The function $J_\nu(z)$ satisfies the equation

$$\left(\partial_z^2 + z^{-1}\partial_z + 1 - (\nu/z)^2\right) u(z) = 0.$$

Then the function $b_\nu(z) = z^{1/2} J_\nu(z)$ satisfies the equation

$$\partial_z^2 v + \left(1 - \frac{\nu^2 - 1/4}{z^2}\right) v = 0.$$

For $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$, $\operatorname{Im} z \neq 0$, set

$$\psi_\nu(z) = \frac{J_\nu'(z)}{J_\nu(z)}, \quad \eta_\nu(z) = \frac{J_\nu(\kappa z)}{J_\nu(z)}, \quad \widetilde{\eta}_\nu(z) = \frac{J_\nu'(\kappa z)}{J_\nu(z)},$$

where $0 < \kappa < 1$ is a parameter independent of z. Set also $\rho(z) = \sqrt{z^2 - 1}$ with $\operatorname{Re} \rho > 0$. Our goal in this section is to prove the following

Theorem 2.1. *For every $0 < \delta \ll 1$, there are positive constants C_δ, C_δ' and δ_1 such that for $\operatorname{Re} \lambda \geq C_\delta$, $C_\delta' \leq |\operatorname{Im} \lambda| \leq \delta_1 \operatorname{Re} \lambda$, $\nu \geq 0$, we have the estimate*

$$(1 + \nu/|\lambda|)\, |\psi_\nu(\lambda) - \rho(\nu/\lambda)| \leq \delta. \tag{2.1}$$

There exist also positive constants C, C', C_1, C_2 and δ_1 such that for $\operatorname{Re} \lambda \geq C_1$, $C_2 \leq |\operatorname{Im} \lambda| \leq \delta_1 \operatorname{Re} \lambda$, $\nu \geq 0$, we have the estimate

$$(1 + \nu/|\lambda|)|\eta_\nu(\lambda)| + |\widetilde{\eta}_\nu(\lambda)| \leq C'|\lambda|^{1/3} e^{-C|\operatorname{Im} \lambda|}. \tag{2.2}$$

Proof. Observe first that, in view of the formula $\widetilde{\eta}_\nu(\lambda) = \psi_\nu(\kappa\lambda)\eta_\nu(\lambda)$, the bound for $|\widetilde{\eta}_\nu(\lambda)|$ follows from that one for $|\eta_\nu(\lambda)|$ and the fact that (2.1) implies the bound $|\psi_\nu(\kappa\lambda)| \le C(1+\nu/|\lambda|)$. To prove (2.1) we will consider several cases.

Case 1. $0 \le \nu \le Const$. We have $2J_\nu(\lambda) = H_\nu^+(\lambda) + H_\nu^-(\lambda)$, where $H_\nu^\pm(\lambda)$ are the Hankel functions [1] having the asymptotic expansions

$$H_\nu^\pm(\lambda) = \left(\frac{2}{\pi\lambda}\right)^{1/2} e^{\pm i(\lambda - \nu\pi/2 - \pi/4)} q_\nu^\pm(\lambda),$$

$$q_\nu^\pm(\lambda) = \sum_{s=0}^\infty \left(\frac{\pm i}{\lambda}\right)^s A_s(\nu), \tag{2.3}$$

where all $A_s(\nu)$ are real, $A_0(\nu) = 1$, $A_1(\nu) = \frac{4\nu^2-1}{8}$. Moreover, $q_{1/2}^\pm(\lambda) = 1$. All derivatives of $q_\nu^\pm(\nu)$ have asymptotic expansions obtained by differentiating (2.3). Without loss of generality, we may suppose that $\mathrm{Im}\,\lambda > 0$. For $\nu \ne 1/2$ we have

$$\left|\frac{q_\nu^+(\lambda)}{q_\nu^-(\lambda)}\right| = 1 + \mathcal{O}(|\lambda|^{-1}), \quad \left|\frac{(q_\nu^+)'(\lambda)}{(q_\nu^-)'(\lambda)}\right| = 1 + \mathcal{O}(|\lambda|^{-1}), \quad \left|\frac{(q_\nu^-)'(\lambda)}{(q_\nu^-)(\lambda)}\right| = \mathcal{O}(|\lambda|^{-2}),$$

$$\left|1 - e^{2i\lambda}\frac{q_\nu^+(\lambda)}{q_\nu^-(\lambda)}\right| \ge 1 - e^{-2\mathrm{Im}\,\lambda}\left|\frac{q_\nu^+(\lambda)}{q_\nu^-(\lambda)}\right| \ge 1 - \frac{1}{2}e^{-2\mathrm{Im}\,\lambda} \ge \frac{1}{2},$$

provided $|\lambda|$ and $\mathrm{Im}\,\lambda$ are taken large enough. We can write the function ψ_ν as follows

$$\psi_\nu(\lambda) + (2\lambda)^{-1} = i\frac{e^{i\lambda}q_\nu^+(\lambda) - e^{-i\lambda}q_\nu^-(\lambda)}{e^{i\lambda}q_\nu^+(\lambda) + e^{-i\lambda}q_\nu^-(\lambda)} + \frac{e^{i\lambda}(q_\nu^+)'(\lambda) + e^{-i\lambda}(q_\nu^-)'(\lambda)}{e^{i\lambda}q_\nu^+(\lambda) + e^{-i\lambda}q_\nu^-(\lambda)}.$$

By using the above inequalities, we get

$$|\psi_\nu(\lambda) + i| \le C|\lambda|^{-1} + Ce^{-2\mathrm{Im}\,\lambda}. \tag{2.4}$$

Since in this case $\rho(\nu/\lambda) = -i + \mathcal{O}(|\lambda|^{-2})$, the estimate (2.1) follows from (2.4). The estimate (2.2) follows in the same way from the formula

$$\frac{J_\nu(\kappa\lambda)}{J_\nu(\lambda)} = \kappa^{-1/2}\frac{e^{i\kappa\lambda}q_\nu^+(\kappa\lambda) + e^{-i\kappa\lambda}q_\nu^-(\kappa\lambda)}{e^{i\lambda}q_\nu^+(\lambda) + e^{-i\lambda}q_\nu^-(\lambda)}.$$

Indeed, as above, one can easily see that $\eta_\nu = \mathcal{O}\left(e^{-2(1-\kappa)\mathrm{Im}\,\lambda}\right)$ if $0 < \kappa < 1$.

Case 2. $\nu \gg 1$. We set $z = \lambda/\nu$. Then $1/\nu \ll |\mathrm{Im}\,z| \ll \mathrm{Re}\,z$. In this case we will use the asymptotic expansions of the Bessel functions in terms of the Airy function $\mathrm{Ai}(\sigma)$. Recall first that $\mathrm{Ai}(\sigma)$ has the expansion

$$\mathrm{Ai}(\sigma) = \sigma^{-1/4}e^{-\frac{2}{3}\sigma^{3/2}}\sum_{\ell=0}^\infty \beta_\ell\,\sigma^{-3\ell/2} \tag{2.5}$$

for $|\sigma| \gg 1$, $\sigma \in \Lambda_\varepsilon := \{\sigma \in \mathbb{C} : |\arg\sigma| \le \pi - \varepsilon\}$, $0 < \varepsilon \ll 1$, where β_ℓ are real numbers and the fractional powers of σ take their principal values. The expansion (2.5) implies

$$F(\sigma) := \frac{\mathrm{Ai}'(\sigma)}{\mathrm{Ai}(\sigma)} = -\sigma^{1/2}\sum_{\ell=0}^\infty \widetilde{\beta}_\ell\,\sigma^{-3\ell/2}, \tag{2.6}$$

where $\widetilde{\beta}_0 = 1$, $\widetilde{\beta}_1 = 1/4$. The behavior of the function F in $\mathbb{C} \setminus \Lambda_\varepsilon$ is more complicated and is given by the following

[1] $H^\pm$ are the Hankel functions of first and second kind

Lemma 2.2. *For $\sigma \in \mathbb{C} \setminus \Lambda_\varepsilon$, $\operatorname{Im}\sigma \neq 0$, we have the bounds*

$$|F(\sigma)| \leq C|\sigma|^{1/2} + C|\operatorname{Im}\sigma|^{-1}, \tag{2.7}$$

$$|\operatorname{Ai}(\sigma)| \leq C\langle\sigma\rangle^{-1/4} e^{\frac{2}{3}|\operatorname{Re}\sigma^{3/2}|}, \tag{2.8}$$

$$|\operatorname{Ai}(\sigma)|^{-1} \leq C\langle\sigma\rangle^{-1/4}\left(|\sigma|^{1/2} + |\operatorname{Im}\sigma|^{-1}\right) e^{-\frac{2}{3}|\operatorname{Re}\sigma^{3/2}|}. \tag{2.9}$$

For $\sigma \in \mathbb{C} \setminus \Lambda_\varepsilon$, $|\sigma| \gg 1$, $|\operatorname{Re}\sigma^{3/2}| \gg 1$, we have the bound

$$\left|F(\sigma) + \sigma^{1/2} + \frac{1}{4\sigma}\right| \leq C|\sigma|^{1/2} e^{-|\operatorname{Re}\sigma^{3/2}|}. \tag{2.10}$$

Proof. The bound (2.7) is proved in [16] (see Lemma 3.1). To prove the other bounds, we will use that $\operatorname{Ai}(-\sigma) = \operatorname{Ai}_+(\sigma) + \operatorname{Ai}_-(\sigma)$, where $\operatorname{Ai}_\pm(\sigma) = e^{\pm\pi i/3}\operatorname{Ai}\left(\sigma e^{\pm\pi i/3}\right)$. By (2.5), for $|\arg\sigma| \leq \varepsilon$, $|\sigma| \gg 1$, we have

$$\operatorname{Ai}_\pm(\sigma) = \sigma^{-1/4} e^{\pm i\frac{2}{3}\sigma^{3/2}} a_\pm(\sigma), \quad a_\pm(\sigma) = \sum_{\ell=0}^{\infty} \beta_\ell^\pm \sigma^{-3\ell/2} \tag{2.11}$$

with $|\beta_\ell^\pm| = |\beta_\ell|$. In particular, this implies

$$|\operatorname{Ai}_\pm(\sigma)| \leq C\langle\sigma\rangle^{-1/4} e^{\mp\frac{2}{3}\operatorname{Im}\sigma^{3/2}}, \quad |\operatorname{Ai}'_\pm(\sigma)| \leq C\langle\sigma\rangle^{1/4} e^{\mp\frac{2}{3}\operatorname{Im}\sigma^{3/2}}. \tag{2.12}$$

Since $|\operatorname{Im}\sigma^{3/2}| = |\operatorname{Re}(-\sigma)^{3/2}|$, we get (2.8) from (2.12). The bound (2.9) follows from (2.7), (2.12) and the identity

$$\operatorname{Ai}(-\sigma)^{-1} = c_\pm F(-\sigma)\operatorname{Ai}_\pm(\sigma) + \widetilde{c}_\pm \operatorname{Ai}'_\pm(\sigma), \tag{2.13}$$

where $c_\pm$ and $\widetilde{c}_\pm$ are some constants. To prove (2.10), observe that, if $|\arg\sigma| \leq \varepsilon$, $\operatorname{Im}\sigma > 0$, we have $\operatorname{Im}\sigma^{3/2} > 0$, and we can write

$$\begin{aligned} -F(-\sigma) + i\sigma^{1/2} + \frac{1}{4\sigma} &= 2i\sigma^{1/2} \frac{e^{i\frac{2}{3}\sigma^{3/2}} a_+(\sigma)}{e^{i\frac{2}{3}\sigma^{3/2}} a_+(\sigma) + e^{-i\frac{2}{3}\sigma^{3/2}} a_-(\sigma)} \\ &+ \frac{e^{i\frac{2}{3}\sigma^{3/2}} a'_+(\sigma) + e^{-i\frac{2}{3}\sigma^{3/2}} a'_-(\sigma)}{e^{i\frac{2}{3}\sigma^{3/2}} a_+(\sigma) + e^{-i\frac{2}{3}\sigma^{3/2}} a_-(\sigma)}. \end{aligned} \tag{2.14}$$

The above expansions imply

$$\left|\frac{a_-(\sigma)}{a_+(\sigma)}\right| = 1 + \mathcal{O}(|\sigma|^{-1}), \quad \left|\frac{a'_-(\sigma)}{a'_+(\sigma)}\right| = 1 + \mathcal{O}(|\sigma|^{-1}), \quad \left|\frac{a'_+(\sigma)}{a_+(\sigma)}\right| = \mathcal{O}(|\sigma|^{-1}).$$

Therefore in this case (2.10) follows from (2.14) after making a change of variables $\sigma \to -\sigma$ and using that if $|\arg\sigma| \leq \varepsilon$, $\operatorname{Im}\sigma > 0$, then $-\sigma \in \mathbb{C} \setminus \Lambda_\varepsilon$ and $(-\sigma)^{1/2} = -i\sigma^{1/2}$. The analysis of the case $\operatorname{Im}\sigma < 0$ is similar. □

Define the functions $\varphi(z)$ and $\zeta(z)$ by

$$\varphi = \frac{2}{3}\zeta^{3/2} = \ln\frac{1 + (1 - z^2)^{1/2}}{z} - (1 - z^2)^{1/2}, \qquad |\arg z| < \pi,$$

where the branches take their principal values when $z \in (0, 1)$, $\varphi, \zeta \in (0, +\infty)$, and are continuous elsewhere. It is well-known (e.g. see pages 420-422 of [9]) that the function $\zeta(z)$ is holomorphic for $|\arg z| < \pi$, $\zeta(z)$ takes real values for $z \in (0, +\infty)$, and $\zeta(z) = 2^{1/3}(1 - z) + \mathcal{O}(|1 - z|^2)$

in a neighbourhood of $z = 1$. Moreover, $\zeta(z) \to -\infty$ as $z \to +\infty$ and $\zeta(z) \to +\infty$ as $z \to 0^+$. The first derivatives of $\varphi(z)$ and $\zeta(z)$ satisfy

$$\zeta(z)^{1/2}\zeta'(z) = \varphi'(z) = -\frac{(1-z^2)^{1/2}}{z}. \tag{2.15}$$

One can easily see that for $0 < \pm\mathrm{Im}\, z \ll \mathrm{Re}\, z$ we have

$$\mathrm{Re}\,\varphi'(z) < 0, \quad \pm\mathrm{Im}\,\varphi'(z) > 0. \tag{2.16}$$

In particular, this implies that the function ρ defined above satisfies

$$\rho\left(\frac{1}{z}\right) = \frac{(1-z^2)^{1/2}}{z}. \tag{2.17}$$

Given parameters $0 < \delta, \delta_1 \ll 1$, set

$$\Theta_1(\delta,\delta_1) = \left\{\mathrm{Re}\, z \geq 1+\delta^2,\, 0 < |\mathrm{Im}\, z| \leq \delta_1 \mathrm{Re}\, z\right\},$$

$$\Theta_2(\delta,\delta_1) = \left\{0 < \mathrm{Re}\, z \leq 1-\delta^2,\, 0 < |\mathrm{Im}\, z| \leq \delta_1 \mathrm{Re}\, z\right\},$$

$$\Theta_0(\delta,\delta_1) = \left\{1-\delta^2 \leq \mathrm{Re}\, z \leq 1+\delta^2,\, 0 < |\mathrm{Im}\, z| \leq \delta_1 \mathrm{Re}\, z\right\}.$$

The next lemma is more or less well-known and follows from the properties of the functions φ and ζ studied in [9]. We will sketch the proof for the sake of completeness.

Lemma 2.3. *For every $0 < \delta \ll 1$ there is $\delta_1 = \delta_1(\delta) > 0$ such that the following properties hold: For $z \in \Theta_1(\delta,\delta_1)$ we have $|\arg\zeta(z)| = \pi - \mathcal{O}(\delta)$, and*

$$2|\zeta(z)|^{1/2}|\mathrm{Im}\,\zeta(z)| \geq |\mathrm{Re}\,\varphi(z)| \geq C|\mathrm{Im}\, z| \tag{2.18}$$

with a constant $C > 0$ depending on δ. For $z \in \Theta_2(\delta,\delta_1)$ we have $|\arg\zeta(z)| = \mathcal{O}(\delta)$. For $z \in \Theta_0(\delta,\delta_1)$ we have

$$|\mathrm{Im}\,\zeta(z)| \geq |\mathrm{Im}\, z|. \tag{2.19}$$

Proof. We will use the formula

$$\varphi(z) - \varphi(\mathrm{Re}\, z) = \int_0^1 \frac{d}{d\tau}\varphi(\mathrm{Re}\, z + i\tau\mathrm{Im}\, z)d\tau = i\mathrm{Im}\, z\int_0^1 \varphi'(\mathrm{Re}\, z + i\tau\mathrm{Im}\, z)d\tau. \tag{2.20}$$

Let $z \in \Theta_1(\delta,\delta_1)$. Then

$$\mathrm{Re}\,\varphi(\mathrm{Re}\, z) = 0, \quad \mathrm{Im}\,\varphi(\mathrm{Re}\, z) \geq C_\delta \mathrm{Re}\, z.$$

In this case we also have

$$\varphi'(\mathrm{Re}\, z + i\tau\mathrm{Im}\, z) = \mathcal{O}_\delta(1)$$

and, in view of (2.16), if $\pm\mathrm{Im}\, z > 0$,

$$\pm\mathrm{Im}\,\varphi'(\mathrm{Re}\, z + i\tau\mathrm{Im}\, z) \geq C_\delta - \mathcal{O}_\delta(\delta_1) \geq C_\delta/2 > 0,$$

provided δ_1 is taken small enough. Thus, by (2.20) we get

$$-\mathrm{Re}\,\varphi(z) \geq C_\delta|\mathrm{Im}\, z|,$$

$$\pm\mathrm{Im}\,\varphi(z) \geq (C_\delta - \mathcal{O}_\delta(\delta_1))\mathrm{Re}\, z \geq 2^{-1}C_\delta \mathrm{Re}\, z, \quad \pm\mathrm{Im}\, z > 0.$$

This yields $\mathrm{Re}\,(\mp i\varphi(z)) > 0$, $\pm\mathrm{Im}\,(\mp i\varphi(z)) > 0$, and hence $0 < \pm\arg(\mp i\varphi(z)) = \mathcal{O}_\delta(\delta_1) = \mathcal{O}(\delta)$ if δ_1 is small enough. Since

$$\varphi = \frac{2}{3}\zeta^{3/2} = \pm i\frac{2}{3}(-\zeta)^{3/2},$$

we have

$$0 < \pm\arg(-\zeta(z)) = \frac{2}{3}\arg(\mp i\varphi(z)) = \mathcal{O}(\delta)$$

and
$$|\mathrm{Re}\,\varphi(z)| = \frac{2}{3}\left|\mathrm{Im}\,(-\zeta(z))^{3/2}\right| = |\mathrm{Im}\,\zeta(z)||\zeta(z)|^{1/2}(1+\mathcal{O}(\delta)).$$
Let $z \in \Theta_2(\delta,\delta_1)$. Then
$$\mathrm{Im}\,\varphi(\mathrm{Re}\,z) = 0, \quad \mathrm{Re}\,\varphi(\mathrm{Re}\,z) \geq C_\delta > 0,$$
$$\mathrm{Im}\,\varphi'(\mathrm{Re}\,z + i\tau\mathrm{Im}\,z) = \mathcal{O}_\delta(1),$$
$$-\mathrm{Re}\,\varphi'(\mathrm{Re}\,z + i\tau\mathrm{Im}\,z) \geq (C_\delta - \mathcal{O}_\delta(\delta_1))(\mathrm{Re}\,z)^{-1} \geq 2^{-1}C_\delta(\mathrm{Re}\,z)^{-1},$$
provided δ_1 is taken small enough. Therefore, by (2.20) we get
$$|\mathrm{Im}\,\varphi(z)| \leq C_\delta\frac{|\mathrm{Im}\,z|}{\mathrm{Re}\,z} = \mathcal{O}_\delta(\delta_1),$$
$$\mathrm{Re}\,\varphi(z) = \mathrm{Re}\,\varphi(\mathrm{Re}\,z) + \mathcal{O}_\delta(|\mathrm{Im}\,z|) \geq C_\delta - \mathcal{O}_\delta(\delta_1) \geq C_\delta/2.$$
Hence, $\arg\varphi(z) = \mathcal{O}_\delta(\delta_1) = \mathcal{O}(\delta)$, which yields
$$\arg\zeta(z) = \frac{2}{3}\arg\varphi(z) = \mathcal{O}(\delta).$$
Let $z \in \Theta_0(\delta,\delta_1)$. Then we have $\zeta'(z) = -2^{1/3} + \mathcal{O}(|1-z|)$ at $z = 1$. To prove (2.19) we will use the formula
$$\zeta(z) - \zeta(\mathrm{Re}\,z) = \int_0^1 \frac{d}{d\tau}\zeta(\mathrm{Re}\,z + i\tau\mathrm{Im}\,z)d\tau = i\mathrm{Im}\,z\int_0^1 \zeta'(\mathrm{Re}\,z + i\tau\mathrm{Im}\,z)d\tau$$
$$= -i2^{1/3}\mathrm{Im}\,z(1+\mathcal{O}(\delta)). \tag{2.21}$$
Since $\mathrm{Im}\,\zeta(\mathrm{Re}\,z) = 0$, we deduce from (2.21),
$$\mathrm{Im}\,\zeta(z) = -2^{1/3}\mathrm{Im}\,z(1+\mathcal{O}(\delta)),$$
which clearly implies (2.19). □

For $|\arg z| \leq \varepsilon$, $\nu \to +\infty$, we have the expansion (see [9]):
$$J_\nu(\nu z) = 2^{1/2}\nu^{-1/3}\left(\frac{\zeta}{1-z^2}\right)^{1/4}\left(\mathrm{Ai}(\nu^{2/3}\zeta)A(\zeta) + \nu^{-4/3}\mathrm{Ai}'(\nu^{2/3}\zeta)B(\zeta) + \mathcal{E}_1(\zeta)\right),$$
where
$$A(\zeta) = \sum_{s=0}^M \frac{A_s(\zeta)}{\nu^{2s}}, \qquad B(\zeta) = \sum_{s=0}^M \frac{B_s(\zeta)}{\nu^{2s}},$$
for every integer $M \gg 1$, where the functions $A_s(\zeta)$, $B_s(\zeta)$ are smooth and bounded with their derivatives, $A_0(\zeta) = 1$, $B_s(\zeta) = \mathcal{O}(\langle\zeta\rangle^{-1/2})$. The error term satisfies the bounds
$$\left|\partial_\zeta^\ell\mathcal{E}_1(\zeta)\right| \leq C_M\nu^{-2M}\langle\zeta\rangle^{(\ell-1)/4}e^{\frac{2\nu}{3}|\mathrm{Re}\,\varphi(z)|}, \quad \ell = 0,1. \tag{2.22}$$
We will derive now a similar expansion for the first derivative of J_ν. To this end, observe first that by (2.15) we have
$$\left(\frac{\zeta}{1-z^2}\right)^{1/4}\zeta'(z) = -\frac{1}{z}\left(\frac{\zeta}{1-z^2}\right)^{-1/4},$$
$$\frac{\partial}{\partial z}\left(\frac{\zeta}{1-z^2}\right)^{1/4} = -\frac{1}{z}\left(\frac{\zeta}{1-z^2}\right)^{-1/4}\phi(z),$$
where
$$\phi(z) = \frac{1}{4\zeta} - \frac{\zeta^{1/2}z^2}{2(1-z^2)^{3/2}}.$$

Since $|\zeta| \sim |z|$ as $|z| \to \infty$, $|\zeta| \sim \log(|z|^{-1})$ as $|z| \to 0$, $\zeta(z) = 2^{1/3}(1-z) + \mathcal{O}(|1-z|^2)$ as $z \to 1$, we have

$$\zeta^{-1/2}\left(\phi(z) - \frac{1}{4\zeta}\right) = \begin{cases} \mathcal{O}_\epsilon(|z|^{2-\epsilon}), \ \forall 0 < \epsilon \ll 1, & |z| \to 0, \\ \mathcal{O}(\langle\zeta\rangle^{-1}), & |z| \to \infty, \\ \mathcal{O}(|\zeta|^{-3/2}), & z \to 1. \end{cases}$$

Differentiating the expansion of J_ν above with respect to the variable z and using that $\mathrm{Ai}''(\sigma) = \sigma \mathrm{Ai}(\sigma)$, we get

$$z(J_\nu)'(\nu z) = -2^{1/2}\nu^{-2/3}\left(\frac{\zeta}{1-z^2}\right)^{-1/4}\left(\mathrm{Ai}'(\nu^{2/3}\zeta)C(\zeta) + \nu^{-2/3}\mathrm{Ai}(\nu^{2/3}\zeta)D(\zeta) + \mathcal{E}_2(\zeta)\right),$$

where

$$C = A + \nu^{-2}(\partial_\zeta B + \phi B), \quad D = \partial_\zeta A + \phi A + \zeta B, \quad \mathcal{E}_2 = \nu^{-2/3}(\partial_\zeta \mathcal{E}_1 + \phi \mathcal{E}_1).$$

Then we have the identity

$$\begin{aligned} &\psi_\nu(\nu z) - \frac{(1-z^2)^{1/2}}{z} \\ &= -\left(\frac{(1-z^2)^{1/2}}{z}\right)\frac{\Phi(\zeta)(1+P_1(\zeta)) + P_2(\zeta) + P_3(\zeta)}{1 + Q_1(\zeta) + \nu^{-1/3}\zeta^{-1/2}F(\nu^{2/3}\zeta)Q_2(\zeta) + Q_3(\zeta)}, \end{aligned}$$

where

$$\begin{aligned} \Phi(\zeta) &= \nu^{-1/3}\zeta^{-1/2}F(\nu^{2/3}\zeta) + 1 + (4\nu\zeta^{3/2})^{-1}, \\ Q_1(\zeta) &= A(\zeta) - 1 = \mathcal{O}(\nu^{-2}), \\ Q_2(\zeta) &= \nu^{-1}\zeta^{1/2}B(\zeta) = \mathcal{O}(\nu^{-1}w(\zeta)^{1/2}), \\ Q_3(\zeta) &= \mathcal{E}_1(\zeta)\mathrm{Ai}(\nu^{2/3}\zeta)^{-1}, \\ P_1(\zeta) &= C(\zeta) - 1 + \nu^{-1}\zeta^{1/2}B(\zeta) \\ &= A(\zeta) - 1 + \nu^{-1}\zeta^{1/2}B(\zeta) + \nu^{-2}(\partial_\zeta B(\zeta) + \phi B(\zeta)) = \mathcal{O}(\nu^{-1}), \\ P_2(\zeta) &= \left(1 + (4\nu\zeta^{3/2})^{-1}\right)(A - C) - (4\nu\zeta^{3/2})^{-1}A - \left(1 + (4\nu\zeta^{3/2})^{-1}\right)Q_2 + \nu^{-1}\zeta^{-1/2}D \\ &= \nu^{-2}\left(1 + (4\nu\zeta^{3/2})^{-1}\right)(\partial_\zeta B(\zeta) + \phi B(\zeta)) - (4\nu\zeta^{3/2})^{-1}(A(\zeta) - 1) \\ &\quad -\nu^{-1}(4\nu\zeta^{3/2})^{-1}\zeta^{1/2}B(\zeta) + \nu^{-1}\zeta^{-1/2}(\partial_\zeta A(\zeta) + \phi(A(\zeta) - 1)) \\ &\quad +\nu^{-1}\zeta^{-1/2}\left(\phi - (4\zeta)^{-1}\right) = \mathcal{O}\left(\nu^{-1}w(\zeta)^{-3/2}w(z)^{2-\epsilon}\right) + \mathcal{O}\left(\nu^{-2}\right), \\ P_3(\zeta) &= \nu^{-1/3}\left(\zeta^{-1/2}\mathcal{E}_2(\zeta) + \mathcal{E}_1(\zeta)\right)\mathrm{Ai}(\nu^{2/3}\zeta)^{-1} \\ &= \nu^{-1}\left(\zeta^{-1/2}(\partial_\zeta \mathcal{E}_1(\zeta) + \phi \mathcal{E}_1(\zeta)) + \nu^{1/3}\mathcal{E}_1(\zeta)\right)\mathrm{Ai}(\nu^{2/3}\zeta)^{-1}, \end{aligned}$$

uniformly for $|\zeta| \geq \nu^{-1}$, where $w(\sigma) = |\sigma|/\langle\sigma\rangle$. We will consider now three cases.

a) $z \in \Theta_1(\delta, \delta_1)$. Then $|\zeta| \geq C_\delta > 0$, and by Lemma 2.3 we have $|\arg \zeta(z)| = \pi - \mathcal{O}(\delta)$ and $|\mathrm{Im}\, z| \gg \nu^{-1}$ implies $\nu|\mathrm{Re}\,\varphi(z)| \gg 1$. Therefore, in this case we can use the estimates (2.9), (2.10) and (2.18) to obtain

$$|\mathrm{Ai}(\nu^{2/3}\zeta)|^{-1} \leq C\nu^{1/6}|\zeta|^{1/4}e^{-\frac{2\nu}{3}|\mathrm{Re}\,\varphi(z)|}, \tag{2.23}$$

$$|\Phi(\zeta)| \leq Ce^{-\nu|\mathrm{Re}\,\varphi(z)|}. \tag{2.24}$$

b) $z \in \Theta_2(\delta, \delta_1)$. Then $|\zeta| \geq C_\delta > 0$, and by Lemma 2.3 we have $|\arg\zeta(z)| = \mathcal{O}(\delta)$. Hence in this case we can use the expansions (2.5) and (2.6) to obtain

$$|\mathrm{Ai}(\nu^{2/3}\zeta)|^{-1} \leq C\nu^{1/6}|\zeta|^{1/4}e^{-\frac{2\nu}{3}|\mathrm{Re}\,\varphi(z)|}, \tag{2.25}$$

$$|\Phi(\zeta)| \leq C\nu^{-2}. \tag{2.26}$$

c) $z \in \Theta_0(\delta, \delta_1)$. Then we have

$$\nu^{-1} \leq |\mathrm{Im}\, z| \leq |z-1| \leq |\zeta| \leq 2|z-1| \leq 2\delta^2$$

and by (2.19), $|\mathrm{Im}\,\zeta| \geq |\mathrm{Im}\, z|$. Note also that in view of the expansions (2.5) and (2.6), the bounds (2.7) and (2.9) hold for all $\sigma \in \mathbb{C} \setminus (-\infty, 0)$. Using this fact together with (2.19) we obtain in this case

$$|\mathrm{Ai}(\nu^{2/3}\zeta)|^{-1} \leq C\nu^{1/3}e^{-\frac{2\nu}{3}|\mathrm{Re}\,\varphi(z)|}, \tag{2.27}$$

$$\begin{aligned}\left|\nu^{-1/3}\zeta^{-1/2}F(\nu^{2/3}\zeta)\right| &\leq C + C|\zeta|^{-1/2}(\nu|\mathrm{Im}\,\zeta|)^{-1}\\ &\leq C + Cw(\zeta)^{-1/2}(\nu|\mathrm{Im}\, z|)^{-1}\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}|\Phi(\zeta)| &\leq C + C|\zeta|^{-1/2}(\nu|\mathrm{Im}\,\zeta|)^{-1} + (4\nu|\zeta|^{3/2})^{-1}\\ &\leq C + Cw(\zeta)^{-1/2}(\nu|\mathrm{Im}\, z|)^{-1} + C\nu^{-1}w(\zeta)^{-3/2}.\end{aligned} \tag{2.29}$$

It follows from the above bounds that in all three cases we have, for $|\mathrm{Im}\, z| \gg \nu^{-1}$,

$$\nu^{-1/3}|\zeta|^{-1/2}|F(\nu^{2/3}\zeta)| \leq Cw(\zeta)^{-1/2}, \tag{2.30}$$

$$|\mathrm{Ai}(\nu^{2/3}\zeta)|^{-1} \leq C\nu^{1/3}\langle\zeta\rangle^{1/4}e^{-\frac{2\nu}{3}|\mathrm{Re}\,\varphi(z)|}. \tag{2.31}$$

In view of (2.8) we also have

$$|\mathrm{Ai}(\nu^{2/3}\zeta)| \leq C\langle\zeta\rangle^{-1/4}e^{\frac{2\nu}{3}|\mathrm{Re}\,\varphi(z)|}. \tag{2.32}$$

By (2.22) and (2.31), we get, for $|\mathrm{Im}\, z| \gg \nu^{-1}$,

$$|P_3(\zeta)| + |Q_3(\zeta)| \leq C_M\nu^{-2M+1}. \tag{2.33}$$

By (2.30) and (2.33), for $|\mathrm{Im}\, z| \gg \nu^{-1}$ and ν large enough, we get

$$\begin{aligned}&\left(1 + \frac{1}{|z|}\right)\left|\psi_\nu(\nu z) - \frac{(1-z^2)^{1/2}}{z}\right|\\ &\leq \frac{2}{w(z)}\left|\frac{(1-z^2)^{1/2}}{z}\right|\left(|\Phi(\zeta)| + \widetilde{C}\nu^{-1}w(z)^{2-\epsilon}w(\zeta)^{-3/2} + \widetilde{C}\nu^{-2}\right)\\ &\leq 4w(\zeta)^{1/2}w(z)^{-2}\left(|\Phi(\zeta)| + \widetilde{C}\nu^{-1}w(z)^{2-\epsilon}w(\zeta)^{-3/2} + \widetilde{C}\nu^{-2}\right)\\ &\leq 4w(\zeta)^{1/2}w(z)^{-2}\,|\Phi(\zeta)| + 4\widetilde{C}\nu^{-1}w(z)^{-\epsilon}w(\zeta)^{-1} + 4\widetilde{C}\nu^{-2}w(z)^{-2}.\end{aligned}$$

Taking into account that $w(z) \sim 1$, $w(\zeta) \sim 1$, $|\mathrm{Re}\,\varphi(z)| \geq C|\mathrm{Im}\, z|$ in case a), $w(\zeta) \sim 1$, $w(z) \sim |z|$, $|z| \gg \nu^{-1}$ in case b), and $w(z) \sim 1$, $w(\zeta) \sim |\zeta| \leq 2\delta^2$, $|\zeta| \geq |\mathrm{Im}\, z|$ in case c), we deduce from the above estimate combined with (2.24), (2.26) and (2.29),

$$\left(1 + \frac{1}{|z|}\right)\left|\psi_\nu(\nu z) - \frac{(1-z^2)^{1/2}}{z}\right| \leq \begin{cases} C_\delta e^{-C_\delta\nu|\mathrm{Im}\, z|} + C_\delta\nu^{-1}, & \text{in case a)},\\ C_\delta(\nu|z|)^{-2} + C_{\epsilon,\delta}\nu^{-1+\epsilon}, & \text{in case b)},\\ C\delta + C_\delta(\nu|\mathrm{Im}\, z|)^{-1}, & \text{in case c)},\end{cases} \tag{2.34}$$

where the constant $C > 0$ is independent of δ. Now, we can make the LHS of (2.34) less than $(C+1)\delta$ by taking $\nu|\mathrm{Im}\, z|$ and ν large enough. This implies (2.1) in view of (2.17) after making the change $(C+1)\delta \to \delta$.

Given $0 < \kappa < 1$, define the functions $\varphi_\kappa(z)$ and $\zeta_\kappa(z)$ by $\varphi_\kappa(z) = \varphi(\kappa z)$ and $\zeta_\kappa(z) = \zeta(\kappa z)$. To bound the function $\eta_\nu(\nu z)$, we write it in the form

$$\eta_\nu(\nu z) = \frac{\mathrm{Ai}\left(\nu^{2/3}\zeta_\kappa\right)}{\mathrm{Ai}\left(\nu^{2/3}\zeta\right)} \frac{\left(1 + Q_1(\zeta_\kappa) + \nu^{-1/3}\zeta_\kappa^{-1/2} F(\nu^{2/3}\zeta_\kappa) Q_2(\zeta_\kappa) + Q_3(\zeta_\kappa)\right)}{\left(1 + Q_1(\zeta) + \nu^{-1/3}\zeta^{-1/2} F(\nu^{2/3}\zeta) Q_2(\zeta) + Q_3(\zeta)\right)}.$$

As above, using (2.30)-(2.33), we have, for $\nu \gg 1$, $|\mathrm{Im}\, z| \gg \nu^{-1}$,

$$|\eta_\nu(\nu z)| \le 2\left|\frac{\mathrm{Ai}\left(\nu^{2/3}\zeta_\kappa\right)}{\mathrm{Ai}\left(\nu^{2/3}\zeta\right)}\right| \le C\nu^{1/3}\left(\frac{\langle\zeta\rangle}{\langle\zeta_\kappa\rangle}\right)^{1/4} e^{-\frac{2\nu}{3}\mathrm{Re}(\varphi_\kappa(z)-\varphi(z))}$$

$$\le C\nu^{1/3}\left(\frac{\langle\varphi\rangle}{\langle\varphi_\kappa\rangle}\right)^{1/6} e^{-\frac{2\nu}{3}\mathrm{Re}(\varphi_\kappa(z)-\varphi(z))}. \tag{2.35}$$

On the other hand, in view of (2.15), we have the formula

$$\varphi_\kappa(z) - \varphi(z) = -\int_\kappa^1 \frac{\varphi_\tau(z)}{d\tau} d\tau = -z\int_\kappa^1 \varphi'(\tau z) d\tau = \int_\kappa^1 \sqrt{1-(\tau z)^2}\, d\tau. \tag{2.36}$$

It follows from (2.36) that

$$|\varphi_\kappa(z) - \varphi(z)| \le C_1\langle z\rangle$$

which in turn implies

$$\frac{\langle\varphi\rangle}{\langle\varphi_\kappa\rangle} \le 1 + C_2\frac{\langle z\rangle}{\langle\varphi_\kappa\rangle} \le C_3 \tag{2.37}$$

since $\langle\varphi_\kappa\rangle \sim \kappa|z|$ as $|z| \to +\infty$. Set $\Theta_j := \Theta_j(\delta, \delta_1)$, $j = 0, 1, 2$, for some fixed, sufficiently small constants $\delta, \delta_1 > 0$. It is easy to see that

$$\mathrm{Re}\sqrt{1-(\tau z)^2} \ge \begin{cases} C|\mathrm{Im}\, z|, & z \in \Theta_1 \cup \Theta_0, \\ C, & z \in \Theta_2, \end{cases}$$

for all $\kappa \le \tau \le 1$, with a constant $C > 0$ independent of z and τ. Hence, by (2.36),

$$\mathrm{Re}\left(\varphi_\kappa(z) - \varphi(z)\right) \ge \begin{cases} \widetilde{C}|\mathrm{Im}\, z|, & z \in \Theta_1 \cup \Theta_0, \\ \widetilde{C}, & z \in \Theta_2, \end{cases} \tag{2.38}$$

with a constant $\widetilde{C} > 0$ independent of z. By (2.35), (2.37) and (2.38), we conclude

$$|\eta_\nu(\nu z)| \le \begin{cases} C'\nu^{1/3} e^{-C\nu|\mathrm{Im}\, z|}, & z \in \Theta_1 \cup \Theta_0, \\ C' e^{-C\nu}, & z \in \Theta_2, \end{cases} \tag{2.39}$$

with constants $C, C' > 0$ independent of z and ν. In particular, (2.39) implies

$$\left(1 + \frac{1}{|z|}\right)|\eta_\nu(\nu z)| \le C''(\nu|z|)^{1/3} e^{-C\nu|\mathrm{Im}\, z|}$$

for all z such that $\nu^{-1} \ll |\mathrm{Im}\, z| \ll \mathrm{Re}\, z$, which is the desired bound (2.2). □

3. Some properties of the Dirichlet-to-Neumann map

Let $\Omega = \{x \in \mathbb{R}^d : |x| \le 1\}$, $\Gamma = \partial\Omega$ and let $\lambda \in \mathbb{C}$ with $1 \ll |\mathrm{Im}\,\lambda| \ll \mathrm{Re}\,\lambda$. Given a function $f \in H^{s+1}(\Gamma)$, let u solve the equation

$$\begin{cases} \left(\Delta + \lambda^2\right) u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \Gamma, \end{cases} \tag{3.1}$$

where Δ is the negative Euclidean Laplacian. Then the Dirichlet-to-Neumann (DN) map

$$\mathcal{N}_0(\lambda) : H^{s+1}(\Gamma) \to H^s(\Gamma)$$

is defined by

$$\mathcal{N}_0(\lambda) f := \lambda^{-1} \partial_\nu u|_\Gamma$$

ν being the unit normal to Γ. Let Δ_Γ be the negative Laplace-Beltrami operator on the boundary Γ equipped with the Riemannian metric induced by the Euclidean one. In what follows the Sobolev space $H^1(\Gamma)$ will be equipped with the semi-classical norm $\|f\|_{H^1(\Gamma)} = \|(I - |\lambda|^{-2}\Delta_\Gamma)^{1/2} f\|_{L^2(\Gamma)}$, where I denotes the identity. For $\sigma \ge 0$, set

$$\rho_0(\sigma) = \sqrt{\left(\sigma + \left(\frac{d-2}{2}\right)^2\right)\lambda^{-2} - 1} \quad \text{with} \quad \mathrm{Re}\,\rho_0 > 0.$$

Theorem 3.1. *For every $0 < \delta \ll 1$, independent of λ, there are positive constants C_δ, $\widetilde{C}_\delta$ and $\delta_1 = \delta_1(\delta)$ such that for $\mathrm{Re}\,\lambda \ge \widetilde{C}_\delta$, $C_\delta \le |\mathrm{Im}\,\lambda| \le \delta_1 \mathrm{Re}\,\lambda$, we have the estimate*

$$\left\| \mathcal{N}_0(\lambda) - \rho_0(-\Delta_\Gamma) + \frac{d-2}{2\lambda} I \right\|_{L^2(\Gamma) \to H^1(\Gamma)} \le \delta. \tag{3.2}$$

Proof. We will express the DN map in terms of the Bessel functions. If $r = |x|$ is the radial variable, we have

$$r^{\frac{d-1}{2}} \Delta r^{-\frac{d-1}{2}} = \partial_r^2 + \frac{\Delta_\Gamma - (d-1)(d-3)/4}{r^2}. \tag{3.3}$$

Let $\{\mu_j^2\}$ be the eigenvalues of $-\Delta_\Gamma$ repeated with their multiplicities and let $\{e_j\}$, $\|e_j\| = 1$, be the corresponding eigenfunctions, that is, $-\Delta_\Gamma e_j = \mu_j^2 e_j$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the scalar product and the norm in $L^2(\Gamma)$. If the functions u and f satisfy equation (3.1), we write

$$f = \sum_j f_j e_j, \quad f_j = \langle f, e_j \rangle, \quad \|f\|^2 = \sum_j f_j^2,$$

$$u = \sum_j u_j(r) e_j, \quad u_j(r) = \langle u(r, \cdot), e_j(\cdot) \rangle.$$

In view of (3.3), $w_j(r) = r^{\frac{d-1}{2}} u_j(r)$ and f_j satisfy the equation

$$\begin{cases} \left(\partial_r^2 - (\nu_j^2 - 1/4) r^{-2} + \lambda^2\right) w_j = 0 & \text{in} \quad (0,1), \\ w_j = f_j & \text{at} \quad r = 1, \end{cases} \tag{3.4}$$

where

$$\nu_j = \sqrt{\mu_j^2 + \left(\frac{d-2}{2}\right)^2}.$$

The solution of (3.4) is given by the formula

$$w_j(r) = \frac{b_{\nu_j}(r\lambda)}{b_{\nu_j}(\lambda)} f_j = r^{1/2}\frac{J_{\nu_j}(r\lambda)}{J_{\nu_j}(\lambda)} f_j,$$

where b_ν and J_ν are the functions introduced in the previous section. Hence

$$u_j(r) = r^{-\frac{d-2}{2}}\frac{J_{\nu_j}(r\lambda)}{J_{\nu_j}(\lambda)} f_j$$

and we have

$$\mathcal{N}_0(\lambda)f = \sum_j \lambda^{-1}\partial_r u_j|_{r=1} f_j = \sum_j \left(\psi_{\nu_j}(\lambda) - \frac{d-2}{2\lambda}\right) f_j,$$

where $\psi_\nu = J'_\nu/J_\nu$. This implies

$$\begin{aligned}
&\left\|\left(I - |\lambda|^{-2}\Delta_\Gamma\right)^{1/2}\left(\mathcal{N}_0(\lambda) - \rho_0(-\Delta_\Gamma) + \frac{d-2}{2\lambda}I\right) f\right\|^2_{L^2(\Gamma)}\\
&= \sum_j \left(1+|\lambda|^{-2}\mu_j^2\right)\left|\psi_{\nu_j}(\lambda) - \rho_0(\mu_j^2)\right|^2 |f_j|^2\\
&\le \sup_{\nu\ge 0}\left(1+|\lambda|^{-2}\nu^2\right)|\psi_\nu(\lambda) - \rho(\nu/\lambda)|^2\,\|f\|^2, \qquad (3.5)
\end{aligned}$$

where the function ρ is as in the previous section. Now (3.2) follows from (3.5) and Theorem 2.1. □

Let $0 < \kappa_1 < \kappa_2 < 1$ be constants and let $\phi(r) \in C_0^\infty([\kappa_1, \kappa_2])$. Then the function $\chi(x) = \phi(|x|)$ vanishes near Γ. Denote by $H_r^1(\Omega)$ the space equipped with the norm

$$\|u\|_{H_r^1(\Omega)} = \|u\|_{L^2(\Omega)} + |\lambda|^{-1}\|\partial_r u\|_{L^2(\Omega)},$$

where $r = |x|$ is the radial variable. It is easy to see that the estimate (2.2) implies the following

Lemma 3.2. *There exist positive constants C and $\widetilde{C}$ such that the solution u of equation* (3.1) *satisfies the estimate*

$$\|\chi u\|_{H_r^1(\Omega)} \le \widetilde{C}|\lambda|^{1/3}e^{-C|\operatorname{Im}\lambda|}\|f\|_{L^2(\Gamma)}. \qquad (3.6)$$

We will now study the DN map in a more general situation. Let $c(x), n(x) \in C^\infty(\overline{\Omega})$ be strictly positive functions and define the DN map associated to these functions by

$$\mathcal{N}(\lambda)f := \lambda^{-1}\partial_\nu u|_\Gamma,$$

where u is the solution to the equation

$$\begin{cases} \left(\nabla c(x)\nabla + n(x)\lambda^2\right) u = 0 & \text{in} \quad \Omega,\\ u = f & \text{on} \quad \Gamma. \end{cases} \qquad (3.7)$$

We suppose that there exist a constant $0 < \delta_0 \ll 1$ and positive constants $\widetilde{c}$ and $\widetilde{n}$ such that $c(x) = \widetilde{c}$, $n(x) = \widetilde{n}$ in $\Omega(\delta_0)$. Set

$$\widetilde{\rho}(\sigma) = \sqrt{\left(\sigma + \left(\frac{d-2}{2}\right)^2\right)\lambda^{-2} - \widetilde{n}/\widetilde{c}} \quad \text{with} \quad \operatorname{Re}\widetilde{\rho} > 0.$$

Theorem 3.3. *For every $0<\delta \ll 1$, independent of λ, there are positive constants C_δ, $\widetilde{C}_\delta$ and $\delta_1 = \delta_1(\delta)$ such that for* $\mathrm{Re}\,\lambda \geq \widetilde{C}_\delta$, $C_\delta \log|\lambda| \leq |\mathrm{Im}\,\lambda| \leq \delta_1 \mathrm{Re}\,\lambda$, *we have the estimate*

$$\left\| \mathcal{N}(\lambda) - \widetilde{\rho}(-\Delta_\Gamma) + \frac{d-2}{2\lambda} I \right\|_{L^2(\Gamma)\to H^1(\Gamma)} \leq \delta. \tag{3.8}$$

Proof. We will compare $\mathcal{N}(\lambda)$ with the DN map $\widetilde{\mathcal{N}}(\lambda)$ defined by

$$\widetilde{\mathcal{N}}(\lambda) f := \lambda^{-1}\partial_\nu u|_\Gamma,$$

where u is the solution of the equation

$$\begin{cases} \left(\widetilde{c}\Delta + \widetilde{n}\lambda^2\right) u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \Gamma. \end{cases} \tag{3.9}$$

Clearly, we have

$$\widetilde{\mathcal{N}}(\lambda) = \left(\frac{\widetilde{n}}{\widetilde{c}}\right)^{-1/2} \mathcal{N}_0\left(\lambda\left(\frac{\widetilde{n}}{\widetilde{c}}\right)^{1/2}\right).$$

In other words, the estimate (3.2) holds true with $\mathcal{N}_0$ and ρ_0 replaced by $\widetilde{\mathcal{N}}$ and $\widetilde{\rho}$, respectively. Therefore, one can easily see that Theorem 3.3 follows from Theorem 3.1 and the following

Lemma 3.4. *There exist positive constants C and $\widetilde{C}$ such that we have the estimate*

$$\|\mathcal{N}(\lambda) - \widetilde{\mathcal{N}}(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \widetilde{C}|\lambda| e^{-C|\mathrm{Im}\,\lambda|}. \tag{3.10}$$

Proof. Denote by G_D the Dirichlet self-adjoint realization of the operator $-n^{-1}\nabla c \nabla$ on the Hilbert space $L^2(\Omega, n(x)dx)$. Let χ_1 be a smooth function depending only on the radial variable such that $\chi_1 = 1$ in $\Omega(\delta_0/3)$, $\chi_1 = 0$ in $\Omega \setminus \Omega(\delta_0/2)$. Let u_1 be the solution to (3.7) and u_2 the solution to (3.9), $u_1 = u_2 = f$ on Γ. We have

$$\left(n^{-1}\nabla c \nabla + \lambda^2\right)\chi_1 u_2 = \left(\widetilde{n}^{-1}\widetilde{c}\Delta + \lambda^2\right)\chi_1 u_2 = \widetilde{n}^{-1}\widetilde{c}[\Delta, \chi_1]u_2$$

and $u_1 - \chi_1 u_2 = 0$ on Γ. Hence

$$u_1 - \chi_1 u_2 = -(G_D - \lambda^2)^{-1}\widetilde{n}^{-1}\widetilde{c}[\Delta, \chi_1]u_2$$

which implies

$$\mathcal{N}(\lambda)f - \widetilde{\mathcal{N}}(\lambda)f = -\lambda^{-1}\gamma\partial_\nu(G_D - \lambda^2)^{-1}\widetilde{n}^{-1}\widetilde{c}[\Delta, \chi_1]u_2, \tag{3.11}$$

where γ denotes the restriction on Γ. Let χ_2 be a smooth function depending only on the radial variable such that $\chi_2 = 1$ on $\mathrm{supp}[\Delta, \chi_1]$ and $\chi_2 = 0$ in $\Omega(\delta_0/4)$. By (3.11) we obtain

$$\|\mathcal{N}(\lambda)f - \widetilde{\mathcal{N}}(\lambda)f\|_{L^2(\Gamma)} \leq C \left\|\gamma\partial_\nu(G_D - \lambda^2)^{-1}\right\|_{L^2(\Omega)\to L^2(\Gamma)} \|\chi_2 u_2\|_{H^1_r(\Omega)}. \tag{3.12}$$

On the other hand, by the trace theorem and the coercitivity of G_D we have

$$\left\|\gamma\partial_\nu(G_D - \lambda^2)^{-1}\right\|_{L^2(\Omega)\to L^2(\Gamma)} \leq \left\|(G_D - \lambda^2)^{-1}\right\|_{L^2(\Omega)\to H^{3/2}(\Omega)} \leq \frac{C|\lambda|^{1/2}}{|\mathrm{Im}\,\lambda|}, \tag{3.13}$$

where the Sobolev space $H^{3/2}(\Omega)$ is equipped with the usual norm. Now (3.10) follows from (3.6), (3.12) and (3.13).

□

4. Eigenvalues-free regions

In this section we derive Theorem 1.1 from Theorems 3.1 and 3.3. Let $c_j(x), n_j(x) \in C^\infty(\overline{\Omega})$, $j = 1, 2$, be strictly positive functions such that $c_j(x) = \widetilde{c}_j$, $n_j(x) = \widetilde{n}_j$ in $\Omega(\delta_0)$, where $\widetilde{c}_j$, $\widetilde{n}_j$ are positive constants satisfying either the condition

$$\widetilde{c}_1 = \widetilde{c}_2, \quad \widetilde{n}_1 \neq \widetilde{n}_2, \tag{4.1}$$

or the condition

$$(\widetilde{c}_1 - \widetilde{c}_2)(\widetilde{c}_1\widetilde{n}_1 - \widetilde{c}_2\widetilde{n}_2) < 0. \tag{4.2}$$

Denote by $\mathcal{N}_j(\lambda)$ the DN map associated to the pair (c_j, n_j) defined in Section 3 and introduce the operator

$$T(\lambda) = \widetilde{c}_1\mathcal{N}_1(\lambda) - \widetilde{c}_2\mathcal{N}_2(\lambda).$$

Clearly, to prove Theorem 1.1 one has to show that, under the conditions (4.1) or (4.2), $T(\lambda)f = 0$ implies $f = 0$ for $\lambda \in \Lambda_\ell$, $\ell = 1, 2$, where

$$\Lambda_1 = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \gg 1,\, 1 \ll |\operatorname{Im}\lambda| \ll \operatorname{Re}\lambda\}$$

when the functions c_j, n_j are constants in Ω,

$$\Lambda_2 = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \gg 1,\, \log(\operatorname{Re}\lambda) \ll |\operatorname{Im}\lambda| \ll \operatorname{Re}\lambda\}$$

when the functions c_j, n_j are constants in $\Omega(\delta_0)$, only. Denote by $\widetilde{\rho}_j$, $j = 1, 2$, the functions obtained by replacing the pair (c, n) by (c_j, n_j) in the definition of the function $\widetilde{\rho}$ introduced in Section 3. If $T(\lambda)f = 0$, $\lambda \in \Lambda_\ell$, $\ell = 1, 2$, by Theorems 3.1 and 3.3, respectively, we have for all $\delta > 0$,

$$\left\| (1 - |\lambda|^{-2}\Delta_\Gamma)^{1/2} \left(\widetilde{\rho}_1(-\Delta_\Gamma) - \widetilde{\rho}_2(-\Delta_\Gamma)\right) f \right\|_{L^2(\Gamma)} \leq \delta \|f\|_{L^2(\Gamma)} \tag{4.3}$$

if (4.1) holds, and

$$\|(\widetilde{c}_1\widetilde{\rho}_1(-\Delta_\Gamma) - \widetilde{c}_2\widetilde{\rho}_2(-\Delta_\Gamma))f\|_{L^2(\Gamma)} \leq \delta\|f\|_{L^2(\Gamma)} \tag{4.4}$$

if (4.2) holds. On the other hand, one gets

$$g(\sigma) := \widetilde{c}_1\widetilde{\rho}_1(\sigma) - \widetilde{c}_2\widetilde{\rho}_2(\sigma) = \frac{(\widetilde{c}_1^2 - \widetilde{c}_2^2)\sigma/\lambda^2 - (\widetilde{c}_1\widetilde{n}_1 - \widetilde{c}_2\widetilde{n}_2)}{\widetilde{c}_1\widetilde{\rho}_1 + \widetilde{c}_2\widetilde{\rho}_2}.$$

Hence, under the above conditions, $g(\sigma) \neq 0$, $\forall \sigma \geq 0$, and we have the bound

$$|g(\sigma)|^{-1} \leq C \left\langle \frac{\sigma}{|\lambda|^2} \right\rangle^{k/2}, \tag{4.5}$$

where $k = 1$ if (4.1) holds and $k = -1$ if (4.2) holds. This implies that the operator

$$(1 - |\lambda|^{-2}\Delta_\Gamma)^{-k/2} g(-\Delta_\Gamma)^{-1}$$

is bounded on $L^2(\Gamma)$ uniformly in λ. Therefore, in both cases by (4.3) and (4.4) we conclude

$$\|f\|_{L^2(\Gamma)} \leq C\delta\|f\|_{L^2(\Gamma)}, \quad \forall \delta > 0, \quad \lambda \in \Lambda_\ell, \tag{4.6}$$

with a constant $C > 0$ independent of δ. Hence, taking δ small enough we deduce from (4.6) that $\|f\| = 0$, which is the desired result. □

References

[1] F. Caconi and H. Haddar, *Transmission eigenvalues in inverse scattering theory*, in Inverse Problems and Applications, Inside Out 2 G. Uhlmann, editor, MSRI Publications, **60**, Cambridge University Press, Cambridge, 2013, 529-580.

[2] D. Colton and P. Monk *The inverse scattering problem for time-harmonic acoustic waves in an inhomogeneous medium*, Quart. J. Mech. Appl. Math. **41** (1988), 97-125.

[3] D. Colton and Y.-J. Leung, *Complex eigenvalues and the inverse spectral problem for transmission eigenvalues*, Inverse Problems **29** (2013), 104008.

[4] D. Colton, Y.-J. Leung and S. Meng, *Distribution of complex transmission eigenvalues for spherically stratified media*, Inverse Problems **31** (2015), 035006.

[5] M. Hitrik, K. Krupchyk, P. Ola and L. Päivärinta, *The interior transmission problem and bounds of transmission eigenvalues*, Math. Res. Lett. **18** (2011), 279-293.

[6] A. Kirsch, *The denseness of the far field patterns for the transmission problem*, IMA J. Appl. Math. **37** (1986), 213-225.

[7] E. Lakshtanov and B. Vainberg, *Application of elliptic theory to the isotropic interior transmission eigenvalue problem*, Inverse Problems **29** (2013), 104003.

[8] Y.-J. Leung and D. Colton, *Complex transmission eigenvalues for spherically stratified media*, Inverse Problems **28** (2012), 075005.

[9] F. Olver, *Asymptotics and Special Functions*, Academic Press, New York, London, 1974.

[10] H. Pham and P. Stefanov, *Weyl asymptotics of the transmission eigenvalues for a constant index of refraction*, Inverse problems and imaging **8**(3) (2014), 795-810.

[11] L. Robbiano, *Spectral analysis of interior transmission eigenvalues*, Inverse Problems **29** (2013), 104001.

[12] L. Robbiano, *Counting function for interior transmission eigenvalues*, Mathematical Control and Related Fields, to appear.

[13] J. Sylvester, *Transmission eigenvalues in one dimension*, Inverse Problems **29** (2013), 104009.

[14] V. Petkov and G. Vodev, *Asymptotics of the number of the interior transmission eigenvalues*, J. Spectral Theory, to appear.

[15] G. Vodev, *Transmission eigenvalue-free regions*, Comm. Math. Phys. **336** (2015), 1141-1166.

[16] G. Vodev, *Transmission eigenvalues for strictly concave domains*, Math. Ann., to appear, DOI 10.1007/s00208-015-1329-2.

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