Truly multi-dimensional all-speed methods for the Euler equations

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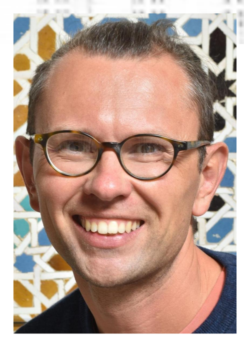

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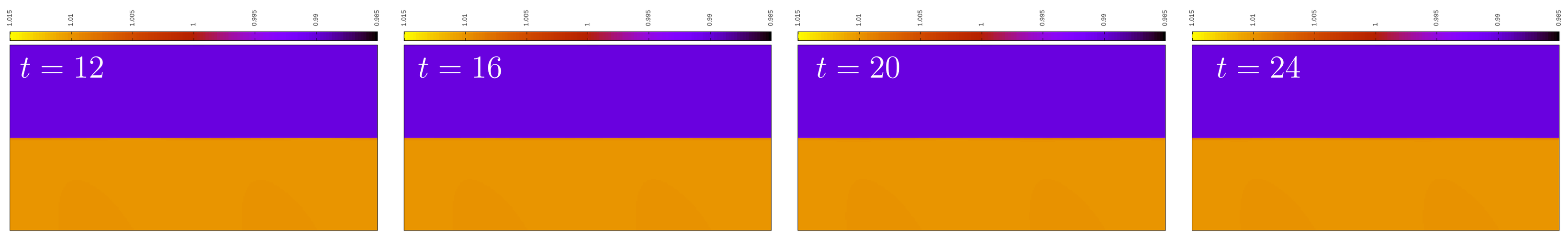
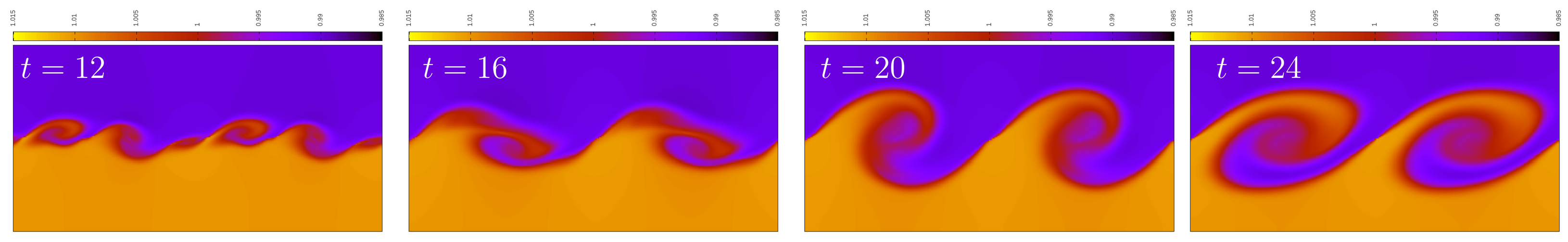
TL;DR: At low Mach number, the solutions approach the incompressible regime with $\nabla \cdot \mathbf{v} = \partial_x u + \partial_y v \in \mathcal{O}(\epsilon)$. The aim is to achieve an all-speed (high + low Mach numbers) Finite Volume method. The problematic term at low Mach number is $\frac{1}{\epsilon}\partial_x^2 u$ in the numerical diffusion, but just removing it does not work for high Mach number flow. Here, a truly multi-dimensional way without ad-hoc fixes is presented: to complement the term to $\frac{1}{\epsilon}\partial_x(\partial_x u + \partial_y v) = \frac{1}{\epsilon}\partial_x(\nabla \cdot \mathbf{v})$. The stabilizing terms for high-Mach flow are still in place, but the term is automatically $\mathcal{O}(1)$ if Mach number is low; no switch required.

Standard method (300 × 150)

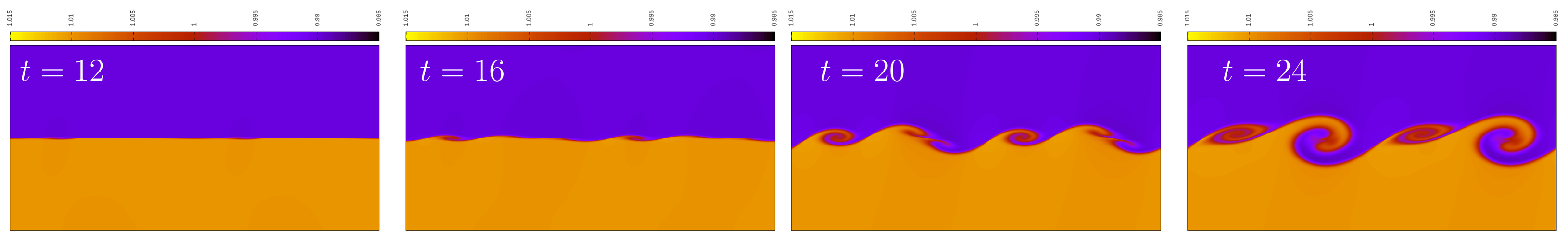
Example: Kelvin-Helmholtz instability at Mach number 0.1 and less



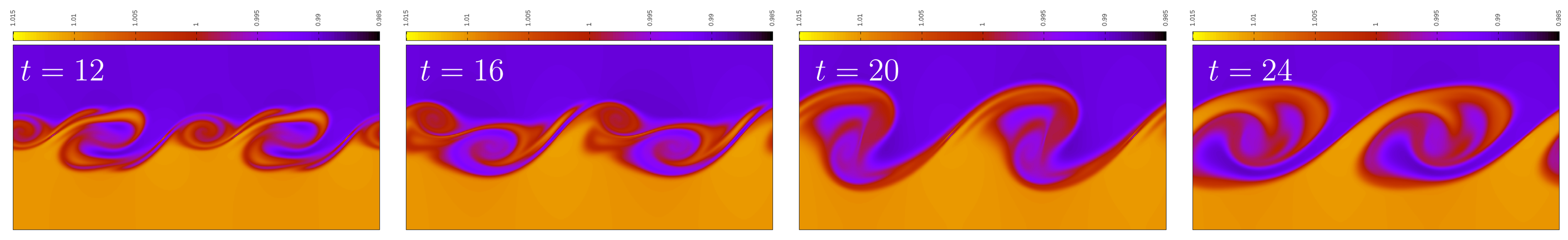
New, all-speed method (300 × 150)



Standard method (1200 × 600)



New, all-speed method (1200 × 600)



Simple way to think about the **spatial low Mach number problem**: A finite volume method that has an error $\sim \frac{\Delta x}{\epsilon} \longrightarrow$ requires unreasonable grid refinement if ϵ is small.

There is additionally a **temporal low Mach number problem** related to explicit time stepping: as $\epsilon \to 0$, due to the CFL condition $\Delta t \to 0$. A solution to this is implicit/IMEX methods. Here, let us insist on the ability to integrate explicitly in time and focus on the spatial low Mach number problem only!

Identifying the problems of standard methods:

Low Mach number limit

Take $\epsilon > 0$ and $M_{\text{loc}} = \frac{|\mathbf{v}|}{\sqrt{\gamma p/\rho}} \in \mathcal{O}(\epsilon)$ as $\epsilon \to 0$. This corresponds to solving

$$\begin{aligned} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \partial_t(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \frac{\nabla p}{\epsilon^2} &= 0 \qquad e = \frac{p}{\gamma - 1} + \frac{1}{2}\epsilon^2 \rho |\mathbf{v}|^2 \\ \partial_t e + \nabla \cdot (\mathbf{v}(e+p)) &= 0 \qquad \mathbf{v} = (u, v)^{\mathrm{T}} \quad \text{in 2d} \end{aligned}$$

Formal asymptotic analysis:

$$\begin{aligned} \mathbf{v} &= \mathbf{v}^{(0)} + \mathbf{v}^{(1)}\epsilon + \mathbf{v}^{(2)}\epsilon^2 + \dots \\ p &= p^{(0)} + p^{(1)}\epsilon + p^{(2)}\epsilon^2 + \dots \end{aligned}$$

Limit equations: $\nabla p^{(0)} = \nabla p^{(1)} = 0$, $\nabla \cdot \mathbf{v}^{(0)} = 0$.

Asymptotic analysis

E.g. for the Roe solver, as $\epsilon \to 0$, the following numerical diffusion appears

$$\partial_t(\rho u) + \dots + \frac{\partial_x p}{\epsilon^2} \simeq \Delta x \partial_x \left(\frac{c\rho}{\epsilon}\partial_x u\right) + \Delta y \partial_y \left(\frac{c\rho}{\epsilon}\partial_y v\right) + \dots$$

As $\epsilon \to 0$, we have $\partial_x p^{(0)} = 0$ (good), and either

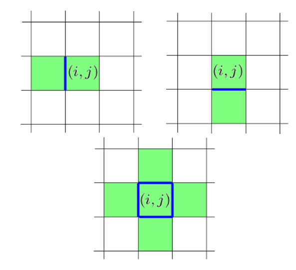
$$\partial_x p^{(1)} \neq 0 \qquad \text{or} \qquad \partial_x p^{(1)} = 0 \quad \text{and} \quad \partial_x(c^{(0)}\rho^{(0)}\partial_x u^{(0)}) + \dots = 0$$

the latter being an additional, artificial restriction on (u, v). The discrete limit states are not discretizing **all** the limit states of the PDE: instead of all those with $\partial_x u^{(0)} + \partial_y v^{(0)} = 0$, the limit states of the scheme are (at best)

$$\partial_x u^{(0)} = 0 \qquad \partial_y v^{(0)} = 0$$

The low Mach problem is **genuinely multi-dimensional**.

Directional splitting



For the conservation law $\partial_t q + \nabla \cdot \mathbf{f}(q) = 0$ (e.g. in 2d)

$$\partial_t q + \partial_x f^x(q) + \partial_y f^y(q) = 0$$

the **finite volume method** on Cartesian grids reads

$$\partial_t q + \frac{f^x_{i+\frac{1}{2},j} - f^x_{i-\frac{1}{2},j}}{\Delta x} + \frac{f^y_{i,j+\frac{1}{2}} - f^y_{i,j-\frac{1}{2}}}{\Delta y} = 0$$

Directional splitting: make the x-flux $f^x_{i+\frac{1}{2},j}$ only depend on $(q_{ij}, q_{i+1,j})$ and $f^y_{i,j+\frac{1}{2}}$ only on $(q_{ij}, q_{i,j+1})$. (This is a simplification of life!)

Directional splitting allows to immediately extend a 1-dimensional method to multi-d, but generally not in a way that takes into account subtle balances between different directions.

Solution (usual path):

- Remove the $\frac{1}{\epsilon}$-scaling from the diffusion by multiplying those terms with $f \in \mathcal{O}(\epsilon)$ (e.g $f = \min(\epsilon v/c, 1)$) (**low Mach fix**)

$$\partial_x \left(\frac{c\rho}{\epsilon}\partial_x u\right) \quad \mapsto \quad \partial_x \left(\boxed{f} \cdot \frac{c\rho}{\epsilon}\partial_x u\right) \text{ with } f \in \mathcal{O}(\epsilon)$$

  Li and Gu [2008], Thornber and Drikakis [2008], Dellacherie [2010], Rieper [2011], Li and Gu [2013], Chalons et al. [2016], Oßwald et al. [2016], Barsukow et al. [2017],...
- Remove all the diffusion in the acoustic operator → central derivatives (and retain it only for advection) (**implicit/IMEX methods**) Degond et al. [2007], Degond and Tang [2011], Cordier et al. [2012], Haack et al. [2012], Bispen et al. [2017], Boscheri et al. [2020], Thomann et al. [2020],...

Low Mach fixes come at the disadvantage of being **ad hoc**, and **reducing** stability (Birken and Meister [2005], Barsukow et al. [2017]). Implicit methods can be **expensive** for an **all-speed** regime. The actual question really is: What's wrong with Riemann solvers?

Solution (new approach):

Change the diffusive terms to $\partial_x \left(\frac{c\rho}{\epsilon}(\partial_x u \boxed{+\partial_y v})\right) + \partial_y \left(\frac{c\rho}{\epsilon}(\boxed{\partial_x u +}\partial_y v)\right)$ thus making appear the divergence $\partial_x u + \partial_y v$ each time.

This is a **low-dissipation scheme** for (asymptotically) divergence-free flows. The procedure is the opposite of low Mach fixes and central discretizations: They **remove** troublesome terms, we **complement** them to make the result disappear in the limit.

The only "magic" is how to do this discretely. Barsukow [2021]

From vorticity preserving methods (see on the bottom left) for linear acoustics, we know how to discretize divergences $\partial_x u + \partial_y v$ and gradients of the divergence (e.g. $\partial_{xx} u + \partial_{xy} v$). The only part required for nonlinear problems is a discretization of $\partial_x\Big(A(\partial_x u + \partial_y v)\Big)$, where A is any function of the dependent variables. Observe that we can rewrite $\partial_x\Big(A(\partial_x u + \partial_y v)\Big) = A(\partial_{xx} u + \partial_{xy} v) + (\partial_x A)(\partial_x u + \partial_y v)$.

Here is a discrete counterpart of this Leibniz rule (bracket-notation explained below):

$$\left[A_{\cdot,j}\left(\left[\frac{1}{4}\{\{u\}\}_{j+\frac{1}{2}}\right] + \frac{1}{4}\{[v]_{j\pm1}\}\right)\right]_{i\pm\frac{1}{2}} = \frac{1}{2}\{A\}_{i\pm\frac{1}{2},j}\left(\left[\left[\frac{1}{4}\{\{u\}\}_{j\pm\frac{1}{2}}\right]\right]_{i\pm\frac{1}{2}} + \frac{1}{4}[[v]_{i\pm1}]_{j\pm1}\right) + \frac{1}{2}[A]_{i\pm\frac{1}{2},j}\left(\left[\frac{1}{4}\{\{u\}\}_{j\pm\frac{1}{2}}\right]_{i\pm1} + \left[\frac{1}{4}\{\{v\}\}_{i\pm\frac{1}{2}}\right]_{j\pm1}\right)$$

A little bit of background:

Acoustics and vorticity preservation

$$\begin{aligned} \partial_t \mathbf{v} + \nabla p &= 0 \qquad & \mathbf{v} &: \mathbb{R}^+_0 \times \mathbb{R}^d \to \mathbb{R}^d \\ \partial_t p + \nabla \cdot \mathbf{v} &= 0 \qquad & p &: \mathbb{R}^+_0 \times \mathbb{R}^d \to \mathbb{R} \end{aligned}$$

Vorticity $\nabla \times \mathbf{v}$ is stationary: $\partial_t(\nabla \times \mathbf{v}) = 0$. Write $\mathbf{v} = (u, v)$ in 2d.
Modified equation of a **standard scheme** that is inspired by 1d-arguments:

$$\begin{aligned} \partial_t u + \partial_x p &= \frac{1}{2}\Delta x\, \partial_x^2 u + \mathcal{O}(\Delta x^2) \\ \partial_t v + \partial_y p &= \frac{1}{2}\Delta x\, \partial_y^2 v + \mathcal{O}(\Delta x^2) \\ \partial_t p + (\partial_x u + \partial_y v) &= \frac{1}{2}\Delta x\,(\partial_x^2 p + \partial_y^2 p) + \mathcal{O}(\Delta x^2) \end{aligned}$$

Vorticity is no longer stationary. A **vorticity preserving method** requires the numerical diffusion to be a gradient: Morton and Roe [2001], Sidilkover [2002], Jeltsch and Torrilhon [2006], Mishra and Tadmor [2009], Lung and Roe [2014], Barsukow [2019]

$$\begin{aligned} \partial_t u + \partial_x p &= \frac{1}{2}\Delta x\, \partial_x(\partial_x u \boxed{+\partial_y v}) + \mathcal{O}(\Delta x^2) \\ \partial_t v + \partial_y p &= \frac{1}{2}\Delta x\, \partial_y(\boxed{\partial_x u +}\partial_y v) + \mathcal{O}(\Delta x^2) \\ \partial_t p + (\partial_x u + \partial_y v) &= \frac{1}{2}\Delta x\,(\partial_x^2 p + \partial_y^2 p) + \mathcal{O}(\Delta x^2) \end{aligned}$$

The discrete version of $\partial_x u + \partial_y v = 0 \Rightarrow \partial_{xx} u + \partial_{xy} v = 0$ is not true for all discretizations of $\partial_x, \partial_y, \partial_{xx}, \partial_{xy}$, because $D_{xx} \neq D_x^2$ in general. However, the following finite differences have this property (2d Cartesian grids):

-1		1
-2		2
-1		1

1	2	1
-1	-2	-1

$\partial_x u + \partial_y v$

1	-2	1
2	-4	2
1	-2	1

-1		1
1		-1

$\partial_{xx} u + \partial_{xy} v$

Example:

Relaxation solver

Following Bouchut [2004], Chalons et al. [2010]:

$$Q\left(\frac{x}{t}; q_i, q_{i+1}\right) := \begin{cases} q_i & \frac{x}{t} \leq u^*_{i+\frac{1}{2}} - c^*_{i+\frac{1}{2}} \\ q^*_{i+\frac{1}{2},\mathrm{L}} & u^*_{i+\frac{1}{2}} - c^*_{i+\frac{1}{2}} \leq \frac{x}{t} < u^*_{i+\frac{1}{2}} \\ q^*_{i+\frac{1}{2},\mathrm{R}} & u^*_{i+\frac{1}{2}} \leq \frac{x}{t} < u^*_{i+\frac{1}{2}} + c^*_{i+\frac{1}{2}} \\ q_{i+1} & \frac{x}{t} \geq u^*_{i+\frac{1}{2}} + c^*_{i+\frac{1}{2}} \end{cases} \qquad \text{with the intermediate states} \qquad \begin{aligned} u^*_{i+\frac{1}{2}} &= \frac{\{u\}_{i+\frac{1}{2}}}{2} - \frac{1}{2a\epsilon}[p]_{i+\frac{1}{2}} \\ p^*_{i+\frac{1}{2}} &= \frac{\{p\}_{i+\frac{1}{2}}}{2} - \frac{a\epsilon}{2}[u]_{i+\frac{1}{2}} \qquad \text{etc.} \end{aligned}$$

and thus $\frac{\partial_x p^*}{\epsilon^2} \simeq \frac{\partial_x p}{\epsilon^2} - \partial_x\left(\frac{a}{\epsilon}\partial_x u\right)$. Replace it by the discrete version of $\frac{\partial_x p^*}{\epsilon^2} \simeq \frac{\partial_x p}{\epsilon^2} - \partial_x\left(\frac{a}{\epsilon}(\partial_x u \boxed{+\partial_y v})\right)$. To this end, define

$$\begin{aligned} u^*_{i+\frac{1}{2},j} &= \frac{\{\{\,\{u\}_{i+\frac{1}{2}}\}\}_{j\pm\frac{1}{2}}}{8} - \frac{1}{2a\epsilon}\frac{\{\{\,[p]_{i+\frac{1}{2}}\}\}_{j\pm\frac{1}{2}}}{4} \\ p^*_{i+\frac{1}{2},j} &= \frac{\{\{\,\{p\}_{i+\frac{1}{2}}\}\}_{j\pm\frac{1}{2}}}{8} - \frac{a\epsilon}{2}\left(\frac{\{\{\,[u]_{i+\frac{1}{2}}\}\}_{j\pm\frac{1}{2}}}{4}\boxed{+\frac{[\,\{v\}_{i+\frac{1}{2}}]_{j\pm1}}{4}}\right) \qquad \text{etc.} \end{aligned}$$

Notation: Square brackets are differences and curly brackets are sums. For example, $[a]_{i+\frac{1}{2}} = a_{i+1} - a_i$, and $\{\{a\}\}_{i\pm\frac{1}{2}} = \{a\}_{i+\frac{1}{2}} + \{a\}_{i-\frac{1}{2}} = a_{i+1} + 2a_i + a_{i-1}$. For all details, see Barsukow [2019].
Define then (the divergence in the denominator is not necessary, but seems natural)

$$\rho_{i+\frac{1}{2},j,\mathrm{L}} = \frac{\rho_{ij}}{1 + \frac{\rho_{ij}\epsilon}{2a}\left(\frac{\{\{\,[u]_{i+\frac{1}{2}}\}\}_{j\pm\frac{1}{2}}}{4}\boxed{+\frac{[\,\{v\}_{i+\frac{1}{2}}]_{j\pm1}}{4}}\right) - \frac{\rho_{ij}}{2a^2}\frac{\{\{\,[p]_{i+\frac{1}{2}}\}\}_{j\pm\frac{1}{2}}}{4}} \qquad E_{i+\frac{1}{2},j,\mathrm{L}} = \rho_{i+\frac{1}{2},j,\mathrm{L}}\left(\frac{E_{ij}}{\rho_{ij}} - \epsilon\frac{p^*_{i+\frac{1}{2},j}u^*_{i+\frac{1}{2},j} - p_{ij}u_{ij}}{a}\right) \quad \text{etc.}$$

Finally $f^x_{i+\frac{1}{2},j} := \left(\rho^*_{i+\frac{1}{2},j,\mathrm{L}}u^*_{i+\frac{1}{2},j},\, \rho^*_{i+\frac{1}{2},j,\mathrm{L}}\left(u^*_{i+\frac{1}{2},j}\right)^2,\, \rho^*_{i+\frac{1}{2},j,\mathrm{L}}u^*_{i+\frac{1}{2},j}v_{ij},\, E_{i+\frac{1}{2},j,\mathrm{L}}u^*_{i+\frac{1}{2},j}\right)$ if $u^*_{i+\frac{1}{2},j} > 0$, etc.
The method is all-speed, i.e. it is able to resolve **both low Mach number flow and shocks** in a stable way. For more details and numerical examples see Barsukow [2021].

Summary and outlook:

- Multi-dimensional extension ensures that the numerical diffusion depends on derivatives of the **divergence** in the limit of low Mach number
- Leaves the 1d method untouched: conceptually pleasing and surely a reason for the **good stability** properties
- Surprisingly **easy to achieve** all-speed property on Cartesian grids; strategy has been applied to a relaxation solver and a Lagrange-Projection scheme Barsukow [2021]
- **No ad-hoc parameters**
- Automatically **all-speed**, no switches

Future work:
- Extension to high-order and unstructured grids (see Barsukow et al. [2023])
- Showing an entropy inequality

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