# von Neumann stability without computing eigenvalues

Wasilij Barsukow

DFG postdoctoral fellow

University of Zurich

April 6, 2020

# Stability via Fourier

Consider a linear finite difference method ($q^n : \mathbb{Z} \to \mathbb{R}^m$)

$$q_i^{n+1} = L(\dots, q_{i-1}^n, q_i^n, q_{i+1}^n, \dots) \tag{1}$$

Inserting

$$q_i^n = \sum_k \hat{q}^n(k) \exp(\mathbb{i} k i \Delta x) \tag{2}$$

results in

$$\sum_k \hat{q}^{n+1}(k) \exp(\mathbb{i} k i \Delta x) = \sum_k \hat{q}^n(k) \exp(\mathbb{i} k i \Delta x) L\Big( \dots, \exp(-\mathbb{i} k \Delta x), 1, \exp(\mathbb{i} k \Delta x), \dots \Big)$$

$$\hat{q}^{n+1}(k) = L\Big( \dots, \exp(-\mathbb{i} k \Delta x), 1, \exp(\mathbb{i} k \Delta x), \dots \Big) \hat{q}^n(k) =: \tilde{L}(k) \hat{q}^n(k) \tag{3}$$

# Stability via Fourier

$$\hat{q}^{n+1}(k) = \tilde{L}(k)\hat{q}^n(k) \tag{4}$$

$\tilde{L}(k) \in \mathbb{C}^{m\times m}$ is a complex-valued matrix with eigenvalues $\{\lambda_j(k)\}_{j=1,\ldots,m}$.

Definition 1 (Von Neumann stability)

The numerical method (1) is *von Neumann stable* if $|\lambda_j(k)| \leq 1 \ \forall k, \forall j = 1, \ldots, m$.

The eigenvalues are **zeros of the polynomial** $\det(\tilde{L} - \lambda \mathbb{1}) = 0$.

Notation: $f(z) \equiv 0$ means $f(z) = 0 \, \forall z \in \mathbb{C}$.

**Definition 2**

A polynomial $f \in \mathbb{C}[z]$

$$f(z) = a_0 + a_1 z + \ldots + a_n z^n \tag{5}$$

is called

- *Schur*, if all its zeros $z_0$ (s.t. $f(z_0) = 0$) are contained in the open unit disc $D$: $|z_0| < 1$.
- *von Neumann*, if all its zeros $z_0$ are contained in $\overline{D}$: $|z_0| \leq 1$, or $f$ is non-vanishing constant.

# The theorem

# The theorem

Consider $f(z) = \sum_{j=0}^{n} a_j z^j$ a polynomial of degree $n$ ($a_n \neq 0$) with $f(0) \neq 0$. (E.g. $f(z) = z + c$, $f'(z) = 1$)

**Definition 3**

1. $f^*(z) := \sum_{j=0}^{n} \bar{a}_{n-j} z^j$ $\qquad f^*(z) = 1 + \bar{c}z$
2. $f_1(z) := \frac{f^*(0) f(z) - f(0) f^*(z)}{z}$ $\qquad f_1(z) = \frac{1 \cdot (z + c) - c(1 + \bar{c}z)}{z} = 1 - c\bar{c}$

**Theorem 4 (Miller, 1971)**

*$f$ ($n \geq 1$) is von Neumann iff either*

- $|f^*(0)| > |f(0)|$ *and $f_1$ is von Neumann, or*
- $f_1 \equiv 0$ *and $f'$ is von Neumann*

Here: $1 = |f^*(0)| \overset{?}{>} |f(0)| = |c|$, i.e. if $|c| < 1$, or if $|c| = 1$.

# The proof

# The proof

Recall the theorem:

**Theorem 5 (Miller, 1971)**

*$f$ $(n \geq 1)$ is von Neumann iff either*

- *$|f^*(0)| > |f(0)|$ and $f_1$ is von Neumann, or*
- *$f_1 \equiv 0$ and $f'$ is von Neumann*

We will focus here on proving the first statement only.

# Rouché's theorem

**Theorem 6 (Rouché, 1862)**

*For any two holomorphic $f$ and $g$ inside some region $K$ with closed contour $\partial K$, if $|g(z)| < |f(z)|$ on $\partial K$, then $f$ and $f + g$ have the same number of zeros inside $K$.*

Define $D = \{z : |z| < 1\}$ and $S = \{z : |z| = 1\}$.

**Lemma 7**

*For* $f \in \mathbb{C}[z]$, $|f^*(z)| = |f(z)|\ \forall z \in S$

**Proof**: If $z = \mathrm{e}^{\mathbb{i}\vartheta}$, then $f^*(\mathrm{e}^{\mathbb{i}\vartheta}) = \sum_{j=0}^n \bar{a}_{n-j} \mathrm{e}^{\mathbb{i}\vartheta j} = \mathrm{e}^{\mathbb{i}n\vartheta} \sum_{j=0}^n \bar{a}_j \mathrm{e}^{-\mathbb{i}\vartheta j} = \mathrm{e}^{\mathbb{i}n\vartheta} \overline{f(\mathrm{e}^{\mathbb{i}\vartheta})}$. Thus $|f^*(z)| = |f(z)|\ \forall z \in S$.

□

**Definition 8 (Self-inversive)**

$f$ is *self-inversive* if $f$ and $f^* = \sum_{j=0}^n \bar{a}_{n-j} z^j$ have the same set of zeros.

Recall $f_1(z) := \dfrac{f^*(0)f(z) - f(0)f^*(z)}{z}$.

**Theorem 9**

i) *$f$ is self-inversive iff $f_1(z) \equiv 0$, i.e. $f^*(0)f(z) \equiv f(0)f^*(z)$.*

ii) *If $f$ is self-inversive, then $|f^*(z)| \equiv |f(z)|$.*

**Proof**: ($\Rightarrow$) If $f$ and $f^*$ have the same set of zeros, then, for some constant $c$

$$f^*(z) \equiv cf(z) \tag{6}$$

Taking $z = \mathrm{e}^{\mathring{\imath}\vartheta}$, we learn by Lemma 7 that $|c| = 1$ and $|f^*(z)| = |f(z)|$ $\forall z \in \mathbb{C}$. Use now $z = 0$, then $c = f^*(0)/f(0)$ which gives $f^*(z) = \frac{f^*(0)}{f(0)} f(z)$ $\forall z$.

($\Leftarrow$) Trivial. $\square$

Let $f = \psi g$ with $\psi$ its maximal self-inversive factor $\psi$ and a completely non self inversive factor $g$. (By splitting the set of zeros into two halves.)

**Theorem 10**

*If $f = \psi g$ as above, then*

i) $f_1(z) \equiv \psi^*(0)\psi(z)g_1(z)$

ii) $|f^*(z)| - |f(z)| \equiv |\psi(z)|\Big(|g^*(z)| - |g(z)|\Big)$

**Proof**:

i) $\psi$ is self-inversive, and thus by Thm. 9i $\psi^*(0)\psi(z) \equiv \psi(0)\psi^*(z)$.

$$f_1(z) = \frac{\psi^*(0)\psi(z)g(z)g^*(0) - \psi(0)\psi^*(z)g(0)g^*(z)}{z} \tag{7}$$

$$= \psi^*(0)\psi(z)\frac{g(z)g^*(0) - g(0)g^*(z)}{z} \tag{8}$$

ii) $\psi$ is self-inversive and by Thm. 9ii $|\psi^*(z)| \equiv |\psi(z)|$. Using $(fg)^* = f^*g^*$ one obtains

$$|f^*(z)| - |f(z)| = |\psi^*(z)g^*(z)| - |\psi(z)g(z)| = |\psi(z)|\Big(|g^*(z)| - |g(z)|\Big)$$

$\square$

**Definition 11**

If a polynomial $f$ of degree $n$ has (with multiplicities) $p_1$ zeros in the open unit disc $D$, $p_2$ on its boundary $S$, $p_3$ outside ($p_1 + p_2 + p_3 = n$), we say that it is of *type* $(p_1, p_2, p_3)$.

# The proof

**Lemma 12**

*Suppose $f$ is a polynomial of degree $n$ such that $|f^*(0)| > |f(0)|$. Then $f$ is of type $(p_1, p_2, p_3)$ iff $f_1$ is of type $(p_1 - 1, p_2, p_3)$.*

**Proof**: $f$ is not self-inversive (Thm. 9ii). Let $f = \psi g$ with $\psi$ its maximal self-inversive factor $\psi$. Then, $\forall z \in S$ by Lemma 7

$$|g^*(0)g(z)| = |g^*(0)||g^*(z)| \tag{9}$$

By Thm. 10ii

$$|f^*(z)| - |f(z)| \equiv |\psi(z)|\Big(|g^*(z)| - |g(z)|\Big) \tag{10}$$

$|f^*(0)| > |f(0)|$ implies $|g^*(0)| > |g(0)|$ and thus

$$|g^*(0)g(z)| > |g(0)g^*(z)| \quad \forall z \in S \tag{11}$$

Thus

$$g_1(z) = \frac{g^*(0)g(z) - g(0)g^*(z)}{z} \neq 0 \quad \forall z \in S \tag{12}$$

**Proof**: (Cont'd.) We have shown that $g_1(z) \neq 0\ \forall z \in S$. Also $g$ does not have any zeros on $S$ (otherwise $\psi$ would not be a maximally self-inversive factor).
Eqn. (11) allows to apply Rouché's theorem: $g^*(0)g(z)$ and $g^*(0)g(z) - g(0)g^*(z)$ have the same number of zeros in $D$. Hence $g$ has one more zero in $D$ than does

$$g_1(z) = \frac{g^*(0)g(z) - g(0)g^*(z)}{z} \tag{13}$$

because $g^*(0)g(z) - g(0)g^*(z)$ has the trivial zero $z = 0$. As $g_1$ is of degree one less than $g$, then $g_1$ and $g$ must have the same number of zeros outside $S$.
Note that $f(z) \equiv \psi(z)g(z)$ and (Lemma 10i) $f_1(z) \equiv \psi^*(0)\psi(z)g_1(z)$, i.e. $\psi$ is a factor of both $f$ and $f_1$. Thus $f$ is of type $(p_1, p_2, p_3)$ iff $f_1$ is of type $(p_1 - 1, p_2, p_3)$.

$\square$

# The proof

This allows to prove the theorem:

**Theorem 13 (Miller, 1971)**

*$f$ $(n \geq 1)$ is von Neumann iff either*

- $|f^*(0)| > |f(0)|$ *and* $f_1$ *is von Neumann, or*
- $f_1 \equiv 0$ *and* $f'$ *is von Neumann*

**Proof**: Suppose $f$ is von Neumann (of type $(p, n-p, 0)$). Then the absolute value of the product of its zeros is $\leq 1$. Thus $\left|\frac{f(0)}{f^*(0)}\right| = \left|\frac{a_0}{a_n}\right| \leq 1$. Thus either

- $|f^*(0)| > |f(0)|$. Then by Lemma 12 $f_1$ is of type $(p-1, n-p, 0)$ and hence von Neumann.
- or $|f^*(0)| = |f(0)|$. (We skip this part of the proof.)

$\square$

The part omitted here can be found in Schur, 1918, Journ. f. Math. **148**, 122-145

Unser Satz gestattet auch, die notwendigen und hinreichenden Bedingungen dafür anzugeben, daß die *Wurzeln* $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ *einer Gleichung n-ten Grades*

$$F(x) = c_0 + c_1 x + \cdots + c_n x^n = 0$$

*voneinander verschieden und sämtlich vom absoluten Betrage* 1 *seien.* Dies ist, wie ich zeigen will, dann und nur dann der Fall, *wenn erstens*

$$c_0 \bar{c}_{n-\nu} = \bar{c}_n c_\nu \qquad (\nu = 1, 2, \dots, n) \tag{43.}$$

*ist, und zweitens die Wurzeln* $\varepsilon_1', \varepsilon_2', \dots, \varepsilon_{n-1}'$ *der derivierten Gleichung* $F'(x) = 0$ *sämtlich im Innern des Einheitskreises liegen* †), *d. h.* $F'(x)$ *den Bedingungen des Satzes* XVII *genügt.*

Sind nämlich alle Zahlen $|\varepsilon_\nu|$ gleich 1 und setzt man

$$F^*(x) = x^n \bar{F}(x^{-1}) = \bar{c}_n + \bar{c}_{n-1}' x + \cdots + \bar{c}_0 x^n,$$

so unterscheiden sich $F$ und $F^*$ voneinander nur um einen konstanten

Faktor. Durch Vergleichen der von $x$ freien Glieder erhält man $c_0 F^* = \bar{c}_n F$, und das liefert die Bedingungen (43.). Liegen ferner alle Wurzeln von $F = 0$ in oder auf dem Rande $\mathfrak{R}$ eines konvexen Bereiches $\mathfrak{K}$, so gilt dasselbe nach einem bekannten Satze von *Gauß**) auch für die Wurzeln von $F' = 0$. Die auf $\mathfrak{R}$ liegenden Wurzeln von $F' = 0$ sind hierbei nichts anderes als die auf $\mathfrak{R}$ gelegenen *mehrfachen* Wurzeln von $F = 0$. In unserem Fall kann für $\mathfrak{K}$ der Einheitskreis gewählt werden, und da $F = 0$ keine mehrfachen Wurzeln besitzen soll, so müssen die Zahlen $\varepsilon'_\lambda$ im *Innern* von $\mathfrak{K}$ liegen.

Sind umgekehrt die Gleichungen (43.) erfüllt, so ist insbesondere $|c_0| = |c_n|$. Ist dann $\zeta$ einer der beiden Werte von $\sqrt{\frac{\bar{c}_n}{c_0}}$ und setzt man

$$P(x) = \zeta x F'(x), \qquad Q(x) = x^n \bar{P}(x^{-1}) = \zeta^{-1} x^{n-1} \bar{F}'(x^{-1}),$$

so wird, wie eine einfache Rechnung zeigt,

$$P(x) + Q(x) = n\zeta F(x).$$

Liegen nun alle Wurzeln von $F' = 0$ oder, was dasselbe ist, von $P = 0$ im Innern des Einheitskreises, so folgt aus dem P. I, S. 230 bewiesenen Satze XII, daß die Gleichung $F = 0$ die verlangte Eigenschaft besitzt.

# History

It seems that the entire theory has been given in [Schur, 1917]/[Schur, 1918].

An algorithmic review of these results is given already in [Cohn, 1922], and then in [Miller, 1971] who emphasizes its use for finite differences and stability analysis.

If you like this kind of mathematics, have a look into this book: [Marden, 1966].

# Practical use

# Implementation for parameter studies

```
boolean vonNeumann(double[][] f){
  double f0abs2 = |f(0)|^2;
  double[][] fStar = starPolynomial(f);
  double fStar0abs2 = |fStar(0)|^2;
  double[][] fOne = onePolynomial(f, fStar);
  double[][] fPrime = primePolynomial(f);

  return (
    (isConstant(f)) ||
    ((isZero(fOne)) && (vonNeumann(fPrime)) ||
    ((fStar0abs2 > f0abs2) && (vonNeumann(fOne)))
  );
}
```

# Implementation for parameter studies

```
double[][] starPolynomial(double[][] poly){
  for (int i = 0; i < poly.length; i++){
    res[i][Complex.RE] =  poly[poly.length-1-i][Complex.RE];
    res[i][Complex.IM] = -poly[poly.length-1-i][Complex.IM];
  }
  return res;
}
```

# Summary

- Whenever a von Neumann analysis is interesting, Miller's (Schur's) theorem is a shortcut
- Analytical statements about stability conditions:
  - possible without computing eigenvalues (important for DG/AF/... and/or for systems)
  - algorithm consists of easy steps
  - there is no fundamental difficulty in multi-d
- Analytical statements might be complicated because the stability region might be complicated
- Semi-analytical parameter studies
  - very quick with respect to experimental studies (and no influence of the initial data)
  - easier than stability analysis without Fourier (and no influence of the grid size)
  - no information on the location of the eigenvalue inside $\bar{D}$, but info on which mode explodes: test case for checks

Cohn, A. (1922).

Über die Anzahl der Wurzeln einer algebraischen Gleichung in einem Kreise.
*Mathematische Zeitschrift*, 14(1):110–148.

Marden, M. (1966).

Geometry of polynomials, providence, ri.
*American Mathematical Society.*

Miller, J. J. (1971).

On the location of zeros of certain classes of polynomials with applications to numerical analysis.
*IMA Journal of Applied Mathematics*, 8(3):397–406.

Schur, I. (1917).

Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind.
*Journal für die reine und angewandte Mathematik*, 147:205–232.

Schur, I. (1918).

Über Potenzreihen, die im Innern des Einheitskreises beschränkt sind. part ii.
*J. Reine Angew. Math*, 148:122–145.