Sujets de stage M2 2026

Development and analysis of new Active Flux methods

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1 Abstract

These master's projects explore the development of Active Flux methods for solving evolutionary partial differential equations (PDEs). Active Flux is a modern approach that blends ideas from Finite Volume and Finite Element methods, using both cell averages and point values at interfaces without relying on Riemann solvers. Students can choose from three research directions: extending Active Flux to non-hyperbolic equations like the heat equation, studying its ability to preserve asymptotic regimes in singular limits (e.g., the Korteweg–de Vries equation), or improving the method's stability by aligning it with physical time step constraints. The projects involve both theoretical analysis and numerical implementation, and are suited for students with a background in numerical analysis and PDEs.

2 Context

The projects presented here are concerned with the numerical approximation of various types of **evolutionary partial differential equations** (PDEs). The natural setting is that of an initial-(boundary-)value problem, with the solution to the PDE sought in a domain $\Omega \subset \mathbb{R}^d$ for times $t > 0$. The numerical approach uses grid-based methods at the frontier between **Finite Volume and Finite Element methods**.

A long-standing paradigm in the development of numerical methods for hyperbolic PDEs has been that they need to be based on discontinuous reconstructions (see e.g. [LeV02]). This is very different from e.g. classical Finite Element approximations, where continuity across element boundaries is enforced. Finite Volume methods evolve the degrees of freedom using so-called Riemann solvers: short-time evolutions of a discontinuity. Piecing together these short-time evolutions as building blocks at every cell interface gives viable methods. The most well-known Finite Element approach to conservation laws (Discontinuous Galerkin methods) also introduces jumps at cell interfaces.

Active Flux methods (e.g. [ER13, WB21, DBK25]) take a radically different approach: additionally to storing one average per cell, a point value is stored at every cell interface. The reconstruction thus must be continuous and no Riemann solvers are needed. One requires instead an update procedure for the point values. Here the approach of Active Flux is *not* to use a variational formulation, as Finite Element methods would. Instead, the point values are evolved directly, as a short-time evolution of (continuous) data. These building blocks are referred to as evolution operators. Active Flux thus blends ideas from Finite Volume and Finite Element methods, and has been shown to have many favorable properties.

For linear hyperbolic equations (e.g. linear advection) the time evolution can be found by tracing the characteristics. For nonlinear problems, and for problems in multiple space dimensions finding adequate updates for the point values is an active area of research. The choice of the update has direct impact on the properties of the resulting method, most importantly on its stability.

3 List of projects

Many questions still remain unanswered for Active Flux today, but the novelty of Active Flux can make significant progress easier to achieve. The following master projects are available:

1. **Active Flux for non-hyperbolic equations**

   Hyperbolic PDEs are characterized by a finite speed of information propagation, i.e. the solution at $x \in \mathbb{R}^d$ at time $t > 0$ depends only on the initial data in some compact set around x. The actual domain of dependence can be just one point or e.g. a disk around x. This is the reason why short-time evolutions can be assembled, as long as the time step is short enough for them not to interact with each other. For non-hyperbolic equations, the domain of dependence is no longer compact. For example, the heat equation involves a convolution with all the initial datum. The aim of the project will be to find suitable approximations, to implement them, and to study their order of accuracy and stability both theoretically and experimentally.

2. **Active Flux and the preservation of asymptotic regimes**

   "When you've got a hammer, everything looks like a nail". An alternative approach to non-hyperbolic equations is to write them as (singular) limits $\nu \to \infty$ of hyperbolic systems. Today, a huge number of such rewritings is known; the focus here will be on those for the Korteweg-de-Vries equation. This equation is of great interest due to the existence of solitons: solutions which travel without changing shape due to a balance between nonlinearity and dispersion. However, there is a price to pay when dealing numerically with a singular limit: One might suffer from a very restrictive time step condition, and the numerical method in the limit needs to be a consistent discretization of the (non-hyperbolic) limit equation. This property is generally not guaranteed. Methods that remain consistent in the limit are called *asymptotic preserving*, and are an active area of research. The aim of this project is to implement an Active Flux method for several hyperbolic variants of the KdV equation, to study theoretically and experimentally its asymptotic properties, and possibly to modify the method to make it asymptotic preserving.

3. **Stability of Active Flux**

   Stability of a numerical method is among the most important properties that need to be guaranteed. A number of results (e.g. [CHK21, BKKL24]) have established the largest possible time step possible (the so-called CFL condition) for various variants of the method. From the way how the method is constructed, a time step restriction also arises by requiring that neighbouring short-time evolutions do not interact with each other; this is sometimes referred to as the "physical stability condition". In certain cases, the actual stability bound is stricter, which means that simulations need to be run with a small time step and thus are not as efficient as they could be. The aim of the project is to derive numerical methods whose actual stability corresponds to the physical stability condition. Starting from known examples in one spatial dimension, new Active Flux methods will be systematically implemented and studied both theoretically and experimentally, possibly thus uncovering underlying patterns. Starting from existing results in one spatial dimension, multi-d methods on Cartesian meshes will be the particular focus of this project.

4 Requirements and further details

The student should have an understanding of the basics of numerical analysis (interpolation, numerical integration and differentiation). Knowledge on numerical approximation to initial-value problems for ordinary differential equations as well as some previous experience with Finite Volume and/or Finite Element methods will be helpful. The student will also be expected to implement and test the new methods in a programming language of the student's choice (such as matlab, py, F90, C/C++, . . .).

Communication languages: English, French, or German.

For further inquires do not hesitate to contact Dr Wasilij Barsukow via the email address `wasilij.barsukow@math.u-bordeaux.fr`.

References

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