Periodic locked orbit in Winfree Model with N oscillators

Mathematical approach

Walid Oukil, Philippe Thieullen & Arezki Kessi
University of Bordeaux, Institute of Mathematics (IMB)
& University of Sciences and Technology Houari-Boumedien (USTHB) Algeria

Contact Information:
Faculty of Mathematics
University of Sciences and Technology Houari-Boumedien (USTHB)
BP 32 El Alia 16111, Bab Ezzouar 16111, Algeria

Phone: +213 550 981 729
Email: oukil.walid@gmail.com

Abstract

In this poster we give a sufficient condition for the existence of locked state in finite dimensional Winfree Model independently of choice of natural frequencies and the number of oscillators, we complete our result by the existence of periodic orbit in a torus which is equivalent to the existence of rotation vectors, the proof in this paper can be applied to more generalized Winfree Model.

Introduction

The simplest synchronization model may be described by the behavior of two pendulums of equal mass coupled by an horizontal string. One notice that the two pendulums behave in the same way and begin to oscillate with the same frequency. When the frequencies of the two oscillators are identical, they are said to be "locked". This behavior seems to appears very often in biological complex systems. We will study a particular model, called the Winfree model, that may be described by N oscillators coupled uniformly.

In 1967 Winfree proposed a mean field model which describes the synchronization of a population of organisms or units that interact simultaneously. We assume that the state of each unit is described by a point on a cycle. We call natural frequency, the frequency of a unit, if it were isolated from the others. The natural frequencies are supposed to be distributed inside an interval $[1-\gamma, 1+\gamma]$ for some constant γ called "frequency width". The interaction of the rest of the population on each unit is supposed to be independent of the unit and controlled by a single parameter called the "coupling strength " κ. There exist different states: the "frequency-locked" state where all the units posses the same frequency, the "dead" state where all the states are frozen with zero frequency, the "incoherence" state where each unit oscillates at independent frequencies. There may also exist mixed states where part of the oscillators is synchronized an d the other part is dead for instance. For small values of κ, the Winfree model may be reduced to the Kuramoto model. In both models the interaction of the outside world on each unit is the same: we use the word "mean-field" to describe this kind of interaction.

The collective behavior of a population of oscillators has first been studied by Winfree in [10]: for a fixed coupling and small spectrum width, all the oscillators synchronize to a unique frequency. Kuramoto [5] extended the model by passing to the limit when the number of oscillators goes to infinity.

Winfree model is given by the following differential equation

$$\dot{x}_i = \omega_i - \kappa \frac{1}{N}\sum_{j=1}^{N} P(x_j)R(x_i) \tag{1}$$

where P and R are two periodic functions, $X(t) = (x_1(t), \ldots, x_N(t))$ is the state, and $x_i(t)$ is the phase of the i-th oscillator. Althougth $x_i(t)$ should represent a scalar in $[0, 2\pi]$, we actually consider its unique continuous lift in $\mathbb{R}$, that we continue to call $x_i(t)$. The parameter $\kappa \geq 0$ is the coupling strength; the vector of natural frequencies $\Omega := (\omega_1, \ldots, \omega_N)$ satisfy

$$1-\gamma \leq \omega_i \leq 1+\gamma, \quad \forall i = 1..N, \tag{2}$$

where $\gamma \in [0, 1[$. We actually assume a more particular form of the mean-field interaction, as in Ariaratnam and Strogatz [1, 2], we assume

$$\dot{x}_i = \omega_i - \kappa\sigma(X)\sin(x_i) \quad \text{where} \quad \sigma(X) = \frac{1}{N}\sum_{j=1}^{N}[1+\cos(x_j)]. \tag{3}$$

Notice that the mean-field interaction σ satisfies $\sigma(X) \in [0, 2]$ for every state $X = (x_1, \ldots, x_N) \in \mathbb{R}^N$.

Since the vector field is uniformly bounded, the flow is defined for all time. Because of the presence of the coupling, the instantaneous frequency $\rho_i(t) := \frac{x_i(t)}{t}$ may not be equal to ω_i. A numerical study shows that, for large t, depending on (γ, κ), three major cases occur: the "deaf state" where all the oscillators are frozen, the "locking state" where all $\rho_i(t) = \text{const} \neq 0$, and the "incoherence state" where $\rho_i(t)$ is strictly increasing in i (therefore in ω_i); in addition there are two secondary cases: the "partial death state" where some of the oscillators are frozen and the others are incoherent, and the "partial locking state" where some are locked and the others are incoherent. Intermediate cases exist numerically but are more difficult to visualize. Ariaratnam and Strogatz [1, 2] have given a precise definition of these transitions in the case $N \to +\infty$. The partial locking is still not understood very well. Giannuzzi, Marinazzo, Nardulli, Pellicoro, and Stramaglia [4] have extended Ariaratnam and Strogatz result by putting a factor in front of the mean-field σ proportional to some power of the modulus of the average phase $\frac{1}{N}\sum_{k=1}^{N}\exp(ix_k)$.

Nevertheless the fact that the instantaneous frequency $\rho_i(t)$ admits a limit (or the fact that the rotation vector exists) has never been addressed (except of course in the deaf state). Our main result is a partial result in that direction in the locking state when $\kappa \in]0, \kappa_*[$ and $\gamma \approx 0$ where κ_* is the locking bifurcation parameter for the Winfree model $\gamma = 0$ and $N = 1$ defined by

$$\kappa_* := \max\{\kappa > 0 \,:\, 1 - \kappa(1+\cos x)\sin x > 0, \ \forall x \in \mathbb{R}\}. \tag{4}$$

Main Result

We consider the Winfree model given by (3) and let $\Phi^t = (\Phi_1^t, \ldots, \Phi_N^t)$ its flow. Then, there exists a open set U in the space of parameters $(\gamma, \kappa) \in [0, 1] \times [0, 1]$, independent of N, which its adherence contains $\{0\} \times [0, \kappa_*]$ such that for every parameter $(\gamma, \kappa) \in U$ and for every choice of natural frequencies $(\omega_i)_{i=1}^N$ satisfying condition (2),

1. There exists an open set $C_{\gamma,\kappa}$ invariant by the flow Φ^t, of the form,

$$C_{\gamma,\kappa} := \Big\{ X = (x_i)_{i=1}^N \in \mathbb{R}^N \,:\, \max_{i,j}|x_j - x_i| < \Delta_{\gamma,\kappa}\Big(\frac{1}{N}\sum_{i=1}^{N} x_i\Big)\Big\}$$

   where $\Delta_{\gamma,\kappa} : \mathbb{R} \to]0, 1[$ is a 2π-periodic smooth function.

2. There exists a constant rotation number $\rho_{\gamma,\kappa} > 0$ and an initial condition $X_* \in C_{\gamma,\kappa}$ such that we have,

$$\Phi_i^t(X_*) = \rho_{\gamma,\kappa} t + \Psi_i(t), \quad \forall i = \overline{1, N}, \ \forall t \geq 0,$$

   where $\Psi_i : \mathbb{R}_+ \to \mathbb{R}$ are C^∞ and $\frac{2\pi}{\rho_{\gamma,\kappa}}$-periodic function

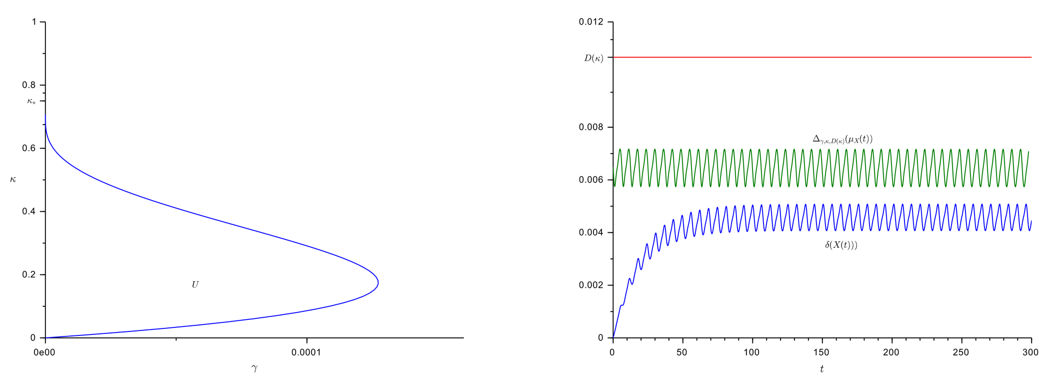

Figure 1: The left figure show the open set U of parameters (γ, κ) bonded in left by the segment $[0, \kappa_*]$ and in right by the function $\gamma = f(\kappa) = \frac{\kappa}{580}(1 - \frac{\kappa}{\kappa_*})^3$. The right figure show the phase shift $\delta(X(t)) = \max_{i,j}|x_i(t) - x_j(t)| = x_N(t) - x_1(t)$ for solution $X(t)$ of Winfree Model with uniform distribution of natural infrequences where $\kappa = 0.1$, $\gamma = f(\kappa) = 0.0001136$ and $N = 200$. Th initial condition of $X(t)$ satisfying $X(0) = (0, .., 0) \in C_{\gamma,\kappa}$, we shows that the phase shift is bounded by the dispersion curve $\Delta_{\gamma,\kappa,D(\kappa)}(\mu_X(t))$ and the horizontal axes $D = 0.0343399$.

Definitions

To be close to the physical and biological terminology, we define the locking state, as follows

Definition 1. We call *rotation number of the i-th oscillator* x_i the following limit, if it exists,

$$\rho_i := \lim_{t\to\infty}\frac{x_i(t)}{t}.$$

We call *rotation vector* of the system (3), the vector of rotation numbers of all oscillators, if they all exist.

Definition 2 (Weak Locking)**.** Two oscillators x_i and x_j of the system 3 are said to be *weakly locked* if there exists a constant $M > 0$ such that

$$|x_i(t) - x_j(t)| \leq M, \quad \forall t \geq 0$$

Definition 3 (Locking)**.** Two oscillators x_i and x_j of the system (3) are said to be *locked* if they are weakly locked and x_i or x_j has a positive rotation number.

Notice that if two oscillators x_i and x_j are locked and one of them has a rotation number ρ_i, then ρ_j exists and $\rho_j = \rho_i$.

Definition 4. The Winfree Model with $N \geq 3$ oscillators is said in *partial-locked* state if there exists tow locked oscillators and two others not locked.

Consequence of Main Result.

For all $N \geq 3$, there exists a family of density G and parameters $(\gamma, k) \in [0, 1]^2$ such that for all distribution of a natural frequencies $(\omega_i)_{i=1}^N$ with density $f \in G$, the Winfree Model (3) is partially-locked.

Generalized Winfree Model

The previous Main Result and consequence is also true for the following generalized Winfree Model :

$$\dot{x}_i = \tau_i(t) - \frac{1}{N}\sum_{j=1}^{N}\kappa_{ij}[1+\cos(x_j)]\sin(x_i)$$

where $\kappa_{ij} \in [0, 1]$ for all $i, j = \overline{1.N}$ and $\tau_i : \mathbb{R}_+ \to [1-\gamma, 1+\gamma]$ for all $i = \overline{1.N}$.

Last Researches

The last researches on the Winfree model have been published by Louca and Attay in [6], by Pazó and Montbrió in [3] and by Seung-Yeal Ha in [9].

Forthcoming Research

The notion of synchronization can be defined by different aspects; it is interesting to give a model with phase diagram who has several cases like Winfree Model and having one of the following cases :

- The oscillators diverge from their initial conditions to join them in the future; This is a example of wound recovery time.
- The oscillators evolve over time to a critical phase synchronization, with stable margins, such as a locally constant function. This phenomenon can be see in the rotation of the Earth around the Sun or in the phenomenon of leaves of a tree that changes its appearance in the winter period to the summer period . . . etc. Intuitively, one can be, to take the coupling strength in Winfree Model as a function of variable time.

References

[1] J.T. Ariaratnam and S.H. Strogatz, Phase Diagram for the Winfree Model of Coupled Nonlinear Oscillators, Phys. Rev. Lett. 86, 4278 (2001).

[2] J. Ariaratnam, Collective Dynamics of the Winfree Model of Coupled Nonlinear Oscillators. Cornell University 2002, thèse non publiée.

[3] Diego Pazó and Ernest Montbrió, Low-Dimensional Dynamics of Populations of Pulse-Coupled Oscillators Phys. Rev. X 4, 011009, 29 January 2014.

[4] F. Giannuzzi, D. Marinazzo, G. Nardulli, M. Pellicoro, S. Stramaglia, Phase Diagram of a Generalized Winfree Model, Phys. Rev. E 75 (2007) 051104.

[5] Y. Kuramoto,Self-entrainment of a population of coupled nonlinear oscillators. International Symposium on Mathematical Problems in Theoretical Physics. Lecture Notes in Physics, Vol. 39, Springer, New York, 1975, p. 420.

[6] Louca, S, Atay, F. M. Spatially Structured Networks of Pulse-Coupled Phase Oscillators on Metric Spaces. AIMS Journal of Discrete and Continuous Dynamical Systems - A, vol. 34 (2014).

[7] G.Nardulli, D. Marinazzo, M. Pellicoro, S. Stramaglia, Phase Shifts Between Synchronized Oscillators in the Winfree and Kuramoto Models (2004).

[8] D.D. Quinn, R.H. Rand, and S.H. Strogatz, Singular Unlocking Transition in the Winfree Model of Coupled Oscillators. Physical Review E 75, 036218 (2007).

[9] Seung-Yeal Ha, Jinyeong Park, Sang Woo Ryoo. (2015) Emergence of Phase-Locked States for the Winfree Model in a Large Coupling Regime. Discrete and Continuous Dynamical Systems 35, 3417-3436. . Online publication date: 1-Feb-2015.

[10] A. T. Winfree, J. Theor. Biol. 16, 15 (1967).

Acknowledgements

Our deep feelings of gratitude to Professor. Steven Strogatz who was always there to answer to our questions, and provide luminous clarifications of his experience in the non-linear, as Dr. Joel Ariaratnam for sharing his thesis, which is an artwork not published.