Algant Master Thesis

On a deformation problem in a peu ramifiée case

Canditate:
Dario Antolini

Supervisor:
Prof. Dr.
Vytautas Paškūnas

Università degli Studi
di Padova

Universität
Duisburg–Essen

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Introduction

Let p be an odd prime number, $\mathbb{Q}_p$ the field of p-adic numbers and E a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field $\mathbb{F}$.

Let $\bar{\rho}\colon G_p := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{F})$ be a continuous representation such that $\mathrm{End}_{\mathbb{F}[G_p]}(\bar{\rho}) = \mathbb{F}$. A *deformation of* $\bar{\rho}$ to a local artinian $\mathcal{O}$-algebra R with residue field $\mathbb{F}$ is a continuous representation $\rho\colon G_p \to \mathrm{GL}_2(R)$ which is (an equivalence class of) liftings of $\bar{\rho}$. The study of deformation theory of Galois representations was initiated by Mazur ([Maz89]) and Ramakrishna ([Ram93]) who showed that there exists a unique local noetherian complete $\mathcal{O}$-algebra R^{univ} with residue field $\mathbb{F}$, called *(framed) universal deformation ring*, together with a universal deformation ρ^{univ}, which parametrizes all the deformations of $\bar{\rho}$ to any local artinian $\mathcal{O}$-algebra with residue field $\mathbb{F}$. After the work of Wiles ([Wil95]) and the conjectures of Fontaine–Mazur ([FM95]), it became clear that for arithmetic applications it was important to understand certain quotients of R^{univ} corresponding to deformations satisfying certain conditions.

The starting point of this thesis is the study of one of these quotients. Fix an integer $k > 1$ and a representation $\tau\colon \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\mathrm{unr}}) \to \mathrm{GL}_2(E)$ with open kernel which can be extended on the whole Galois group. The "deformation problem" we will study is to find the shape of the quotient $R(k,\tau,\bar{\rho})$ of R^{univ} which parametrizes all the deformations ρ of $\bar{\rho}$ to a finite extension of $\mathcal{O}$ satisfying the following conditions:

(a) $\rho \otimes \overline{\mathbb{Q}_p}$ is potentially semistable with Hodge–Tate weights $(0, k-1)$;

(b) the restriction to the inertia subgroup of the Weil–Deligne representation associated to $\rho \otimes \overline{\mathbb{Q}_p}$ is isomorphic to τ;

(c) $\det \rho$ is the $(k-1)$th power of the p-adic cyclotomic character times a finite character coprime to p.

For the exact definition of $R(k,\tau,\bar{\rho})$, we refer the reader to section 5.1.

This problem was studied by Breuil and Mézard in [BM02], where the authors generalized a conjecture given in [BCDT01]. The two authors proved the cited conjecture when $\tau = triv$ is the trivial representation and k even, $1 < k < p$. In order to do this, Breuil and Mézard also described a way to find the above ring $R(k,\tau,\bar{\rho})$ in that cases.

Later, Kisin pointed out (see footnote of [Kis08, p. 514]) that in the particular case when the residual representation $\bar{\rho}$ is a peu ramifiée extension (following the notation of [Ser87]) the universal deformation ring has a different property from the previous one. Moreover, developing other tools, Kisin was able to prove the conjecture in a more general setting, and in the last decade there were other proofs from other authors, such as Hu-Tan ([HT13]) and Paškūnas ([Paš15]). In particular, a joint work of Hu and

Paškūnas ([HP17]) shows the real form of the universal deformation ring in the peu ramifiée case; however, the two authors use a theory developed in the last 15 years. Thus, the aim of this work is to give another proof of this fact by using the original machinery developed by Breuil and Mézard.

The approach the two authors used in their paper requires the p-adic Hodge theory. This consists in describing some categories of p-adic semistable representations with prescribed Hodge–Tate weights as semilinear algebra objects, the so-called *weakly admissible* (ϕ, N)*-modules.* However, the deformation theory studies $\mathcal{O}$-lattices stable under the action of G_p inside representations over the base field E, so we need an integral version of the p-adic Hodge theory. This was developed at the end of the century by Breuil and it defines an equivalence between G_p-stable $\mathcal{O}$-lattices and the so-called *strongly divisible module over* $\mathcal{O}$ only if k is small, i.e., when $1 < k < p$.

The rough idea of the proof is to start with the classification of 2-dimensional p-adic semistable representation over E with fixed Hodge–Tate and describe all the semilinear algebra objects linked with them. Then, after parametrized these representations and their invariant lattices, we look at their reduction modulo ϖ and deduce what kind of residual representations they get. Finally, writing down a 'special' strongly divisible module, we are able to prove that the candidate ring is actually the universal deformation ring for our deformation problem.

The content of this dissertation is the following.

In section 1, we will recall, in order to fix also the notation, some Galois theory of finite extension K of $\mathbb{Q}_p$, the characters of $\mathrm{Gal}(\overline{K}/K)$ and the definition of peu and très ramifiées extensions.

In section 2, we will study the main theorem of deformation theory of Galosi representations: the prorepresentability of the deformation functors, both the classical and the framed one.

In section 3, we will develop the p-adic Hodge theory, the construction of the Fontaine's period rings and the categories of representations and of semilinear algebra objects.

In section 4, we will give an outline of the integral p-adic Hodge theory given by Breuil in [Bre97], [Bre98] and [Bre99], taking a look at Fontaine–Laffaile modules and at strongly divisible modules with coefficients.

In section 5, we will treat the deformation problem described in [BM02], we will construct a 'universal' strongly divisible module over our candidate ring and then conclude the proof.

Notation

Fix a prime number $p \neq 0$: in most of cases, it can be equal to 2, we are interested on odd primes.

Once for all, fix an algebraic closure $\overline{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ with ring of integers $\overline{\mathbb{Z}_p}$ and residue field $\overline{\mathbb{F}_p}$, which is an algebraic closure of the finite field with p

elements $\mathbb{F}_p$, thus every extension of $\mathbb{Q}_p$ (resp. of $\mathbb{F}_p$) will be a subfield of $\overline{\mathbb{Q}_p}$ (resp. $\overline{\mathbb{F}_p}$).

Let k (resp. $\mathbb{F}$) denote a finite extension of $\mathbb{F}_p$, $W(k)$ (resp. $W(\mathbb{F})$) the ring of Witt vectors of k (resp. $\mathbb{F}$), $K_0 = W(k)[1/p]$ (resp. $E_0 = W(\mathbb{F})[1/p]$) and K/K_0 (resp. E/E_0) a finite totally ramified extension with ring of integers $\mathcal{O}_K$ (resp. $\mathcal{O}_E$) and uniformiser π.

We will denote by val_p the valuation on $\mathbb{Q}_p$ satisfying $val_p(p) = 1$ and with the same symbol, by abuse of notations, the (unique) valuation val_p on K (resp. E) normalized by $val_p(p) = 1$ (every time there will not be confusion).

We will use the aboslute Galois group $G_p := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ of $\mathbb{Q}_p$ and its inertia subgroup $I_p := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\mathrm{unr}})$. We will denote by $\chi\colon G_p \to \mathbb{Z}_p^\times$ the p-adic cyclotomic character, by ω its reduction modulo $\mathfrak{m}_{\overline{\mathbb{Z}_p}}$ and by $\omega_2\colon I_p \to \mathbb{F}_{p^2}^\times$ the fundamental character of level 2.

All the rings are meant to be commutative with the identity $1 \neq 0$.

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1 Basics of Galois theory

In this section, we will recall some basic properties of Galois theory in order to fix the notations. First, we recall the definition of a Galois group for an infinite extension, then we will describe some properties of the absolute Galois group of the p-adic field $\mathbb{Q}_p$ (or of a finite extension K) and of its subgroups.

1.1 Infinite Galois theory

In this section, we will recall some notions on infinite Galois theory and we will deal with some basic but important examples.

A field extension F/K, not necessarily finite, is called *Galois extension* if it is algebraic, normal and separable.

Assume F/K is a Galois extension and denote the group of automorphisms of F fixing K by

$$\mathrm{Gal}(F/K) := \{\tau \in \mathrm{Aut}F \mid \tau(a) = a \quad \forall a \in K\}.$$

Let $\mathcal{F} = \{L \text{ field} \mid K \subseteq L \subseteq F, \ L/K \text{ is finite and Galois}\}$ be the set of all finite Galois extensions of K inside F. We have a natural map $\alpha_L \colon \mathrm{Gal}(F/K) \to \mathrm{Gal}(L/K)$, $\tau \mapsto \tau|_L$. Its kernel $\ker \alpha_L = \mathrm{Gal}(F/L)$ is normal in $\mathrm{Gal}(F/K)$ and has finite index:

$$|\,\mathrm{Gal}(F/K) : \mathrm{Gal}(F/L)| = |\,\mathrm{Gal}(L/K)| = [L : K] < \infty.$$

Therefore, the family $\{\mathrm{Gal}(F/L) \mid L \in \mathcal{F}\}$ is a basis of open neighbourhoods of the identity for $\mathrm{Gal}(F/K)$. With this topology, called *Krull topology*, $\mathrm{Gal}(F/K)$ becomes a topological group, and in particular:

Proposition 1.1.1. *With the notation as above,* $\mathrm{Gal}(F/K) \simeq \varprojlim_{L\in\mathcal{F}} \mathrm{Gal}(L/K)$ *is a profinite group.*

Example 1.1.2. Consider $K = \mathbb{F}_p$ and $F = \overline{\mathbb{F}_p}$ a fixed algebraic closure. Thus:

$$G_{\mathbb{F}_p} := \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \simeq \varprojlim_n \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \simeq \varprojlim_n \mathrm{Gal}\,\mathbb{Z}/n\mathbb{Z} =: \widehat{\mathbb{Z}}$$

In particular, since $\widehat{\mathbb{Z}}$ is (topologically) generated by one element, the corresponding element in the absolute Galois group $G_{\mathbb{F}_p}$ is the Frobenius map $\mathrm{Frob}_p \colon x \mapsto x^p$. More in general, if k is a finite field with $q = p^r$ elements, its absolute Galois group $G_k \simeq \widehat{\mathbb{Z}}$ is generated by the Frobenius map $\mathrm{Frob}_k \colon x \mapsto x^q$.

Consider $K = \mathbb{Q}$ and $F = \mathbb{Q}(\zeta_n | n \in \mathbb{N})$ its maximal abelian extension. Hence:

$$\begin{aligned}\mathrm{Gal}(F/\mathbb{Q}) \simeq \varprojlim_n \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) \simeq \varprojlim_n (\mathbb{Z}/n\mathbb{Z})^\times \simeq \\ \simeq \widehat{\mathbb{Z}}^\times \simeq \mathbb{Z}/2\mathbb{Z} \times \prod_{p \text{ odd}} \mathbb{Z}/(p-1)\mathbb{Z} \times \widehat{\mathbb{Z}}.\end{aligned}$$

Now, we state the infinite Galois correspondence. This theorem shows why it is so important to deal with topological groups; notice that if the extension is finite, the topology on its Galois group is discrete, so the theorem is precisely a generalisation of the finite case.

Theorem 1.1.3 (Krull)**.** *Let F/K be a Galois extension. Then the contravariant functor $\mathrm{Gal}(F/\cdot)$ is an equivalence of categories between sub-extensions (not necessarily Galois) L/K inside K and closed subgroups of $\mathrm{Gal}(F/K)$, with quasi-inverse the functor sending a closed subgroup H to the subfield $F^H := \{x \in F | \sigma(x) = x \; \forall \sigma \in H\}$ of H-invariants.*

In particular, finite extensions correspond to open subgroups.

1.2 Ramification of local fields

Let p be a prime number, let K be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}_K$, $\mathfrak{m}$, k and π denote its ring of integers, maximal ideal, residue field and a uniformiser, respectively. Let K^{unr} denote the maximal unramified extension of K and let K^{tame} denote the maximal tamely ramified extension of K.

We have a short exact sequence

$$0 \to I_K \to G_K \to \mathrm{Gal}(K^{\mathrm{unr}}/K) \to 0$$

where $\mathrm{Gal}(K^{\mathrm{unr}}/K) = \mathrm{Gal}(\overline{\mathbb{F}_p}/k) = \widehat{\mathbb{Z}}$ canonically and $I_K = \mathrm{Gal}(\overline{\mathbb{Q}_p}/K^{\mathrm{unr}})$ is the *inertia subgroup*. Likewise, we have

$$K^{\mathrm{tame}} = \bigcup_{(d,p)=1} K^{\mathrm{unr}}(\pi^{1/d})$$

so that $\mathrm{Gal}(K^{\mathrm{tame}}/K^{\mathrm{unr}}) = \varprojlim_{(d,p)=1} \mu_d$, where the map is given by

$$g \mapsto \{g(\pi^{1/d})/\pi^{1/d}\}_{d \geq 1}.$$

In particular, if $\alpha \in \mathrm{Gal}(K^{\mathrm{tame}}/K^{\mathrm{unr}})$ and the image of $\sigma \in \mathrm{Gal}(K^{\mathrm{tame}}/K)$ in $\mathrm{Gal}(\overline{\mathbb{F}_p}/k)$ is Frob_k, then $\sigma\alpha\sigma^{-1} = \alpha^q$ with $q := \#k$. Finally, $I_K^{\mathrm{wild}} = \mathrm{Gal}(\overline{K}/K^{\mathrm{tame}})$ is the p-Sylow subgroup of I_K, called the *wild inertia subgroup*; therefore, we can consider the quotient group $I_t := I_K/I_K^{\mathrm{wild}}$, called *moderate inertia subgroup*, which can be identified with $\mathrm{Gal}(K^{\mathrm{tame}}/K^{\mathrm{unr}})$.

1.3 The p-adic cyclotomic character and its twists

Let's choose a compatible sequence of primitive p^nth roots of unity $\varepsilon^{(n)} \in \mu_{p^n} \subset \overline{K}$ such that $\varepsilon^{(0)} = 1$, $\varepsilon^{(1)} \neq 1$ and $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$. Let $K_\infty := \bigcup_{n\geq 1} K(\varepsilon^{(n)})$. The *$p$-adic cyclotomic character* $\chi\colon G_K \to \mathbb{Z}_p^\times$ is defined by the formula $g(\zeta) = \zeta^{\chi(g)}$ for all $g \in G_K$ and $\zeta \in \varprojlim_n \mu_{p^n}$. This defines an action of G_K on the so-called *p-adic Tate module* $\mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}(\overline{K}) \simeq \mathbb{Z}_p^\times$, where the isomorphism is given by a choice of basis of $\mathbb{Z}_p(1)$, for instance $\varepsilon := (\varepsilon^{(n)})_{n\geq 0}$.

For any $r \geq 0$ define $\mathbb{Z}_p(r) = \mathbb{Z}_p(1)^{\otimes r}$ and $\mathbb{Z}_p(-r) = \mathbb{Z}_p(r)^\vee$ (linear dual: $M^\vee = \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ for any finite free $\mathbb{Z}_p$-module M) with the naturally associated G_K-actions (from functoriality of tensor powers and duality), so upon fixing a basis of $\mathbb{Z}_p(1)$ we identify $\mathbb{Z}_p(r)$ with the $\mathbb{Z}_p$-module $\mathbb{Z}_p$ endowed with the G_K-action χ^r for all $r \in \mathbb{Z}$. If M is an arbitrary $\mathbb{Z}_p[G_K]$-module, we let $M(r) = \mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} M$ with its natural G_K-action, so upon fixing a basis of $\mathbb{Z}_p(1)$ this is simply M with the modified G_K-action $g.m = \chi(g)^r g(m)$ for $g \in G_K$ and $m \in M$. Elementary isomorphisms such as $(M(R))(r') \simeq M(r + r')$ (with evident transitivity behaviour) for $r, r' \in \mathbb{Z}$ and $(M(r))^\vee \simeq M^\vee(-r)$ for $r \in \mathbb{Z}$ and M finite free over $\mathbb{Z}_p$ will be used without comment.

1.4 Mod p characters of $\mathrm{Gal}(\overline{K}/K)$

Keep the notations as in §102. If $\lambda\colon \mathrm{Gal}(\overline{K}/K) \to \overline{\mathbb{F}_p}^\times$ is an unramified character, i.e., $\lambda(I_K) = 1$, it is uniquely determined by the image of the Frobenius map Frob_k. For this reason, whenever $\alpha \in \overline{\mathbb{F}_p}^\times$, we will denote by $\lambda(\alpha)$ the unique unramified character sending Frob_k to α.

Above, we stated the isomorphism $I_t \simeq \varprojlim_{(d,p)=1} \mu_d$, where the projective system is given by $\mu_{dd'} \to \mu_d$, $a \mapsto a^{d'}$. In particular, if π is a uniformiser of K (and so K^{unr}), the field $K_d := K^{\mathrm{unr}}(\pi^{1/d})$ is totally ramified, moderate and of degree d, so it induces an isomorphism $\theta_d\colon \mathrm{Gal}(K_d/K^{\mathrm{unr}}) \simeq \mu_d$.

Let $q = p^n$, then $\mathbb{F}_q^\times = \mu_{q-1}$. Moreover, the numbers of the form $q - 1$ are cofinite in the set of all integers coprime to p: indeed, if d is such an integer, there exists $n \geq 1$ such that $p^n \equiv 1 \mod (d)$, for example $n = \varphi(d)$. Hence, the projective system (μ_d) is equivalent to the projective system given by the $\mathbb{F}_q^\times$ with the norm maps $N\colon \mathbb{F}_{q^m}^\times \to \mathbb{F}_q^\times$, $N(\alpha) = \alpha^{1+q+\cdots+q^{m-1}}$, as transition maps. Therefore, the isomorphisms θ_{q-1} define an isomorphism $\theta\colon I_t \to \varprojlim_q \mathbb{F}_q^\times$.

In virtue of this isomorphism, a mod p character of I_t is called *of level* n and denoted by ω_n if it factors through $\mathbb{F}_{p^n}$ but does not through $\mathbb{F}_{p^m}$ for every m strict divisor of n. Hence, for any n there exist exactly n characters of I_t called *fundamental characters of level* n, and fixed one of those, say $\psi\colon I_t \to \mathbb{F}_{p^n}$, the others are given by the composition of ψ with the n-embeddings of $\mathbb{F}_{p^n}$ inside $\overline{\mathbb{F}_p}$:

$$\psi_i \colon I_t \xrightarrow{\psi} \mathbb{F}_{p^n} \hookrightarrow \overline{\mathbb{F}_p}.$$

Since the embeddings are the p-power of the Frobenius, given a fundamental character ψ, the others are $\psi, \psi^p, \dots, \psi^{p^{n-1}}$.

Example 1.4.1. For $n = 1$ there exists a unique fundamental character $\omega_1 \colon I_t \to \mathbb{F}_p$. It corresponds to the p-adic cyclotomic character modulo $\mathfrak{m}_{\mathbb{Z}_p}$. We will denote this character simply by ω.

1.5 Peu and très ramfiée extensions

For $p \geq 3$, the extension of a mod p character ψ by the trivial representation is an $\mathbb{F}_p$ vector space of dimension 1 if $\psi \neq \omega$, otherwise it is 2-dimensional. This fact can be seen as follows.

Starting with a short exact sequence

$$1 \to \mu_p(\overline{\mathbb{Q}_p}) \to \overline{\mathbb{Q}_p}^\times \xrightarrow{.p} \overline{\mathbb{Q}_p}^\times \to 1$$

of $G_{\mathbb{Q}_p}$-modules and taking its invariants, we get

$$1 \to \mathbb{Q}_p^\times \xrightarrow{.p} \mathbb{Q}_p^\times \to H^1(G_{\mathbb{Q}_p}, \mathbb{F}_p(1)) \to H^1(G_{\mathbb{Q}_p}, \overline{\mathbb{Q}_p}^\times).$$

By Hilbert 90 theorem ([Gru67, §2.7]), the latter group is trivial, then $H^1(G_{\mathbb{Q}_p}, \omega) \simeq \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p$, which is a 2-dimensional $\mathbb{F}_p$-vector space (generated, for instance, by p and $1-p$). The isomorphism is given explicitly by $u \mapsto [g \mapsto g(\sqrt[p]{u})/\sqrt[p]{u}]$. Following the notation given by Serre in [Ser87], we say that an extension is *peu ramifié* if it corresponds to the image of $\mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^p$; otherwise, it is called *très ramifié*.

1.6 Weil group and Weil-Deligne representations

In this section we conclude the description of all basic objects we will need.

Although we will not need it, in the following definition we want to treat also a case when the residue field k is infinite

Definition 1.6.1. The *Weil group relative to* $\overline{K}/K$ is the subgroup of G_K given by all the elements which image inside $\mathrm{Gal}(\overline{k}/k)$ is an integer power of the absolute Frobenius Frob_k over k.

In order to define a topology on the Weil group, we do not consider the subspace topology on G_K, rather we set on W_K the coarsest topology for which I_K is open. With this topology, the Weil group becomes a locally profinite group (i.e., a locally compact Hausdorff topological group) and the inclusion $W_K \subset G_K$ is dense.

Thus, we have a commutative diagram with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & I_K \arrow[r] & G_K \arrow[r, "v_K"] & \widehat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & I_K \arrow[r] \arrow[u, equal] & W_K \arrow[r] \arrow[u, hook] & \mathbb{Z} \arrow[r] \arrow[u, hook] & 1
\end{tikzcd}$$

The map v_K is called *valuation of* G_K.

Remark 1.6.2. If k is not finite, then we have $G_K = W_K = I_K$ and $v_K(w) = 0$ for any $w \in W$.

Let $q := \#k$. If we consider W_K as a group scheme over $\mathbb{Q}$, we can put the following definition:

Definition 1.6.3. The *Weil–Deligne group* WD_K *relative to* $\overline{K}/K$ is the group scheme over $\mathbb{Q}$ which is the semi-direct product of the Weil group W_K and the additive group $\mathbb{G}_{\mathrm{a}}$, on which W_K acts by $wxw^{-1} = q^{v_K(w)}x$ for any $w \in W_K$, $x \in \mathbb{G}_{\mathrm{a}}$.

Remark 1.6.4. If k is not finite, the Weil–Deligne group is the direct product of I_K by $\mathbb{G}_{\mathrm{a}}$.

Definition 1.6.5. Let E be a field of characteristic 0. A *representation of* WD_K is a E-linear representation of $WD_K \otimes E$ of finite dimension.

In other words, we can consider such object to be a triple (Δ, ρ_0, N) where Δ is a finite-dimensional E-vector space, $\rho_0 \colon W_K \to \mathrm{Aut}_E(\Delta)$ a morphism with kernel an open subgroup of I_K and $N \colon \Delta \to \Delta$ a linear application satisfying:

$$\rho_0(w)N\rho_0(w)^{-1} = q^{v_K(w)}N$$

for any $w \in W_K$.

2 Deformation theory of Galois representation

In this section, we will define the notion of a deformation of a representation of the absolute Galois group of a finite extension K of $\mathbb{Q}_p$.

More generally, for a finite field $\mathbb{F}$, we will start with a profinite group G and a $\mathbb{F}[G]$-module $V_{\mathbb{F}}$ on which G acts continuously and with $d := \dim_{\mathbb{F}} V_{\mathbb{F}}$ a finite number.

In the sequel, we will denote by $W = W(\mathbb{F})$ the ring of Witt vectors of $\mathbb{F}$.

2.1 Deformation functors

Let $\widehat{\mathfrak{Ar}}_W$ denote the category of complete noetherian local W-algebras with residue field $\mathbb{F}$, and $\mathfrak{Ar}_W$ denote the full subcategory of finite local artinian W-algebras. The maximal ideal of $A \in \widehat{\mathfrak{Ar}}_W$ is denoted by $\mathfrak{m}_A$. Note that, via the W-structure, the residue field $A/\mathfrak{m}_A$ of any $A \in \widehat{\mathfrak{Ar}}_W$ is canonically identified with $\mathbb{F}$.

Definition 2.1.1. Let A be in $\mathfrak{Ar}_W$. A *deformation of* $V_{\mathbb{F}}$ *to* A is a pair (V_A, ι_A) such that:

- V_A is a $A[G]$-module which is finite free over A and on which G acts continuously on V_A;
- ι_A is a G-equivariant isomorphism $\iota_A \colon V_A \otimes_A \mathbb{F} \to V_{\mathbb{F}}$.

Fixing an $\mathbb{F}$-basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$, a *framed deformation of* $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ *to* A is a triple (V_A, ι_A, β_A), where (V_A, ι_A) is a deformation of $V_{\mathbb{F}}$ to A and β_A is a basis of V_A which reduces to $\beta_{\mathbb{F}}$ under ι_A.

One defines functors $D_{V_{\mathbb{F}}}, D^{\square}_{V_{\mathbb{F}}} \colon \mathfrak{Ar}_W \to \mathbf{Set}$ by setting, for all $A \in \mathfrak{Ar}_W$,

$$D_{V_{\mathbb{F}}}(A) = \{\text{isomorphism classes of deformations of } V_{\mathbb{F}} \text{ to } A\},$$
$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\text{isomorphism classes of framed deformations of } V_{\mathbb{F}} \text{ to } A\},$$

and with the obvious extension to morphisms.

Remark 2.1.2. (a) The fixed basis $\beta_{\mathbb{F}}$ identifies the vector space underlying $V_{\mathbb{F}}$ with $\mathbb{F}^d$ and thus allows us to view $V_{\mathbb{F}}$ as a representation $\bar{\rho} \colon G \to \mathrm{GL}_d(\mathbb{F})$. Then $D^{\square}_{V_{\mathbb{F}}}(A)$ is the set of continuous representations:

$$\rho \colon G \to \mathrm{GL}_d(A)$$

lifting $\bar{\rho}$. In terms of representations, $D_{V_{\mathbb{F}}}(A)$ is the set of such representations modulo the action by conjugation of $\ker(\mathrm{GL}_d(A) \to \mathrm{GL}_d(V_{\mathbb{F}}))$.

(b) It is often useful to consider deformation functors on $\mathfrak{Ar}_{\mathcal{O}}$, where $\mathcal{O}$ is the ring of integers of a finite totally ramified extension of $W(\mathbb{F})[1/p]$, so that $\mathbb{F}$ is still the residue field of $\mathcal{O}$, and where $\mathfrak{Ar}_{\mathcal{O}}$ is the category of local artinian $\mathcal{O}$-algebras with residue field $\mathbb{F}$.

2.2 A finiteness condition

Definition 2.2.1. A profinite group G satisfies the *finiteness condition* Φ_p if, for all open subgroups $G' \leq G$, the $\mathbb{F}_p$-vector space $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p)$ of continuous group homomorphisms is finite dimensional.

By the Burnside's basis theorem, the group G' satisfies $\dim \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < \infty$ if and only if the maximal pro-p quotient of G' is topologically finitely generated.

Example 2.2.2. The group $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p)$ is isomorphic to $\mathrm{Hom}_{\mathrm{cont}}(G'^{\mathrm{ab}}, \mathbb{F}_p)$.
Thus class field theory shows that the following groups satisfy Condition Φ_p:

(a) the absolute Galois group of a finite extension of $\mathbb{Q}_p$;

(b) the Galois group $G_{K,S} = \mathrm{Gal}\,(K_S/K)$, where K is a number field, S is a finite set of places of K, and $K_S \subset \overline{K}$ denotes the maximal extension of K unramified outside S.

2.3 Representability

Proposition 2.3.1. *Assume that G satisfies Condition Φ_p. Then:*

(a) $D^{\square}_{V_{\mathbb{F}}}$ is pro-representable by some $R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_W$;

(b) if $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $D_{V_{\mathbb{F}}}$ is pro-representable by some $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_W$.

One calls $R^{\square}_{V_{\mathbb{F}}}$ the *universal framed deformation ring* and $R_{V_{\mathbb{F}}}$ the *universal deformation ring* of $V_{\mathbb{F}}$.

Remark 2.3.2. (a) Recall that pro-representability (e.g., for $D^{\square}_{V_{\mathbb{F}}}$) means that there exists an isomorphism

$$D^{\square}_{V_{\mathbb{F}}}(A) \simeq \mathrm{Hom}_W(R^{\square}_{V_{\mathbb{F}}}, A)$$

which is functorial in $A \in \mathfrak{Ar}_W$. This universal property implies that $R^{\square}_{V_{\mathbb{F}}}$ is unique up to unique isomorphism. Moreover, the identity map in $\mathrm{Hom}(R^{\square}_{V_{\mathbb{F}}}, R^{\square}_{V_{\mathbb{F}}})$ gives rise to a universal framed deformation over $R^{\square}_{V_{\mathbb{F}}}$.

(b) Originally, Mazur considered the functor $D_{V_{\mathbb{F}}}$. It describes representations lifting $V_{\mathbb{F}}$ up to isomorphism. The additional choice of basis is not a very interesting datum. However, the functor $D_{V_{\mathbb{F}}}$ is not always representable. A good way to remedy this problem is to rigidify the situation by adding a choice of basis to a given representation and thus to consider the functor $D^{\square}_{V_{\mathbb{F}}}$ instead. This is important for residual representations $V_{\mathbb{F}}$ of the absolute Galois group of a number field K, in the sense that $V_{\mathbb{F}}$ may have trivial centralizer as a representation of G_K and yet the restriction of $V_{\mathbb{F}}$ to a decomposition group may no longer share this property.

(c) Without condition Φ_p, the universal ring $R^{\square}_{V_{\mathbb{F}}}$ still exists (as an inverse limit of artinian rings), but it may no longer be noetherian.

(d) Due to the canonical homomorphism $\mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}})$, it is justified to write "=" in $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

Proof. We prove part (a).

Suppose first that G is finite, say with a presentation $G =< g_1, \ldots, g_s \mid r_1(g_1, \ldots, g_s), \ldots, r_t(g_1, \ldots, g_s) >$. Define

$$\mathcal{R} = W[X^k_{i,j} \mid i, j = 1, \ldots, d; k = 1, \ldots, s]/\mathfrak{a}$$

where $\mathfrak{a}$ is the ideal generated by the coefficients of the matrices

$$r_l(X^1, \ldots, X^s) - \mathrm{id},\ l = 1, \ldots, t,$$

with X^k the matrix $(X^k_{i,j})$. Let $\mathcal{J}$ be the kernel of the homomorphism $\mathcal{R} \to \mathbb{F}$ defined by mapping X^k to $\bar{\rho}(g_k)$ for $k = 1, \ldots, s$, with $\bar{\rho}$ as in Remark 2.1.2(a). Then $R^{\square}_{V_{\mathbb{F}}}$ is the $\mathcal{J}$-adic completion of $\mathcal{R}$ and $\rho^{\square}_{V_{\mathbb{F}}}$ is the unique representation $G \to \mathrm{GL}_d(R^{\square}_{V_{\mathbb{F}}})$ mapping g_k to the image of X^k in $\mathrm{GL}_d(R^{\square}_{V_{\mathbb{F}}})$.

We may write any profinite group G as a filtered inverse limit $G = \varprojlim_i (G/H_i)$ over some index set I of open normal subgroups $H_i \subset \mathrm{Ker}(\bar{\rho})$. For each i the above construction yields a universal pair $(R^{\square}_i, \rho^{\square}_i)$. By the universality of thee pairs, one can form their inverse limit over the index set I. This yields

$$(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}) = \varprojlim_i (R^{\square}_i, \rho^{\square}_i),$$

which clearly satisfies the required universal property. By definition, $R^{\square}_{V_{\mathbb{F}}}$ lies in $\widehat{\mathfrak{Ar}}_W$. It remains to show that $R = R^{\square}_{V_{\mathbb{F}}}$ is Noetherian. Since R is complete, it suffices to show that $\mathfrak{m}_R/(\mathfrak{m}_R^2, p)$ is finite-dimensional as a vector space over $\mathbb{F}$. It is most natural to prove the latter using tangent spaces. We refer to the proof of Lemma 2.4.3, where we shall see how Condition Φ_p is used.

The proof of part (b) in [Maz89, Thm. 1.7.2] uses Schlessinger's representability criterion. $\square$

2.4 The tangent space

Let $\mathbb{F}[\epsilon] = \mathbb{F}[X]/(X^2)$ denote the ring of dual numbers. The set $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ is naturally isomorphic to $\mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}})$, as an element of $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ gives rise to an (continuous) extension

$$0 \to V_{\mathbb{F}} \to V_{\mathbb{F}[\epsilon]} \to V_{\mathbb{F}} \to 0$$

where we have identified $\epsilon \cdot V_{\mathbb{F}}$ with $V_{\mathbb{F}}$, and, conversely, any extension of one copy of $V_{\mathbb{F}}$ by another $V_{\mathbb{F}}$ can be viewed as an $\mathbb{F}[\epsilon]$-module, with multiplication by ϵ identifying the two copies of $V_{\mathbb{F}}$. In particular, $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ is naturally an $\mathbb{F}$-vector space.

Definition 2.4.1. The $\mathbb{F}$-vector space $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ is called the *Zariski tangent space* of $D_{V_{\mathbb{F}}}$.(The same terminology will be used for $D^{\square}_{V_{\mathbb{F}}}$ and other deformation functors.)

Remark 2.4.2. Recall that, for any $A \in \widehat{\mathfrak{Ar}}_W$, its (mod p) Zariski tangent space is the $\mathbb{F}$-vector space $\mathfrak{t}_A = \mathrm{Hom}_W(A, \mathbb{F}[\epsilon])$. Thus, if $D_{V_{\mathbb{F}}}$ is prorepresentable, then the tangent spaces of $D_{V_{\mathbb{F}}}$ and of the universal ring representing $D_{V_{\mathbb{F}}}$ agree.

Lemma 2.4.3. *(a) Defining* $\mathrm{ad}V_{\mathbb{F}}$ *as the G-representation* $\mathrm{End}_{\mathbb{F}}(V_{\mathbb{F}})$*, there is a canonical isomorphism*

$$D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) \xrightarrow{\simeq} H^1(G, \mathrm{ad}V_{\mathbb{F}})$$

(b) If G satisfies Condition Φ_p, then $D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon])$ is a finite dimensional $\mathbb{F}$-vector space.

(c) One has $\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\epsilon]) + d^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$.

Remark 2.4.4. The symbol $h^*(\dots)$ always denotes $\dim_{\mathbb{F}} H^*(\dots)$.

Proof. It's enough to show the isomorphism $\mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \simeq H^1(G, \mathrm{ad}V_{\mathbb{F}})$. Consider an element $E \in \mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}})$, so in other words we have a short exact sequence

$$0 \to V_{\mathbb{F}} \to E \to V_{\mathbb{F}} \to 0$$

as $\mathbb{F}[G]$-modules. Tensoring by $V_{\mathbb{F}}^{\vee}$ we get another extension:

$$0 \to V_{\mathbb{F}} \otimes V_{\mathbb{F}}^{\vee} \to E' \to \mathbb{F} \to 0$$

Since $V_{\mathbb{F}} \otimes V_{\mathbb{F}}^{\vee} = \mathrm{End}_{\mathbb{F}}(V_{\mathbb{F}})$ canonically, we get the desired isomorphism.

We now prove part (b), thereby completing the proof of Proposition 2.3.1(a). Let $G' = \mathrm{Ker}(\bar{\rho})$, which is an open subgroup of G. The inflation-restriction sequence ([AW67, Prop. 4]) yields the left exact sequence

$$0 \to H^1(G/G', \mathrm{ad}V_{\mathbb{F}}) \to H^1(G, \mathrm{ad}V_{\mathbb{F}}) \to (\mathrm{Hom}(G', \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{ad}V_{\mathbb{F}})^{G/G'}.$$

The term on the left is finite because G/G' and $\mathrm{ad}V_{\mathbb{F}}$ are finite. The term on the right is finite because of Condition Φ_p for G. Hence (b) is proved.

To prove part (c), fix a deformation $V_{\mathbb{F}[\epsilon]}$ of $V_{\mathbb{F}}$ to $\mathbb{F}[\epsilon]$. The set of $\mathbb{F}[\epsilon]$ bases of $V_{\mathbb{F}[\epsilon]}$ lifting a fixed basis of $V_{\mathbb{F}}$ is an $\mathbb{F}$-vector space of dimension d^2.

Let β' and β'' be two such bases. Then there is an isomorphism of framed deformations

$$(V_{\mathbb{F}[\epsilon]}, \beta') \simeq (V_{\mathbb{F}[\epsilon]}, \beta'')$$

if and only if there is an automorphism $1+\epsilon\alpha$ of $V_{\mathbb{F}[\epsilon]}$, where $\alpha \in \mathrm{ad}V_{\mathbb{F}}$, which takes β'' to β', so that $\alpha \in \mathrm{ad}V_{\mathbb{F}}^G$. Thus the fibers of

$$D^{\square}_{V_{\mathbb{F}}}(V_{\mathbb{F}[\epsilon]}) \to D_{V_{\mathbb{F}}}(V_{\mathbb{F}[\epsilon]})$$

are a principal homogeneous space under $\mathrm{ad}V_{\mathbb{F}}/(\mathrm{ad}V_{\mathbb{F}})^G$. □

Definition 2.4.5. Let $\varphi\colon D' \to D$ be a natural transformation of functors from $\mathfrak{Ar}_W$ to **Set**. The map φ will be called *formally smooth* if, for any surjection $A \to A' \in \mathfrak{Ar}_W$, the map

$$D'(A) \to D'(A') \times_{D(A')} D(A)$$

is surjective.

Essentially the same proof as that of Lemma 2.4.3(c) implies the following:

Corollary 2.4.6. *The natural transformation $D^{\square}_{V_{\mathbb{F}}} \to D_{V_{\mathbb{F}}}$, $(V_A, \beta_A) \mapsto V_A$ is formally smooth. Thus, if $R_{V_{\mathbb{F}}}$ is representable, then $R^{\square}_{V_{\mathbb{F}}}$ is a power series ring over $R_{V_{\mathbb{F}}}$ of relative dimension $d^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$.*

2.5 Schlessinger's criterion

In order to complete Proposition 2.3.1(b), we introduce the Schlessinger's theorem; for a proof, we refer to [Sch68].

Let $D\colon \mathfrak{Ar}_W \to \mathbf{Set}$ be a functor such that $D(\mathbb{F})$ is a point.

We say that an epimorphism $A'' \to A$ in $\mathfrak{Ar}_W$ is *small surjective* if its kernel is a principal ideal which is annihilated by $\mathfrak{m}_{A''}$.

For any $A, A', A'' \in \mathfrak{Ar}_W$ with morphisms $A' \to A$ and $A'' \to A$, we have a natural map (in **Set**):

$$D(A' \times_A A'') \to D(A') \times_{D(A)} D(A''). \tag{2.1}$$

The axioms of Schlessinger are as follows:

(H1) If $A'' \to A$ is small surjective, then (2.1) is surjective.

(H2) If $A'' \to A$ is $\mathbb{F}[\epsilon] \to \mathbb{F}$, then (2.1) is bijective.

(H3) $\dim_{\mathbb{F}} D(\mathbb{F}[\epsilon])$ is finite.

(H4) If $A'' \to A$ is small surjective and $A' = A''$, then (2.1) is bijective.

Note that with condition (H2), $D(\mathbb{F}[\epsilon])$ carries a natural structure of $\mathbb{F}$-vector space. The following is a main theorem in *loc.cit.*:

Theorem 2.5.1. *If D satisfies (H1), (H2), (H3) and (H4), then D is prorepresentable.*

3 p-adic Hodge theory

In this section, we will develop some important equivalence of categories between particular representations and (semi-)linear algebra objects.

Let K and E be two finite field extensions of $\mathbb{Q}_p$ and fix an algebraic closure $\overline{K}$ of K. In this section, we want to introduce the notions of crystalline and semistable representations with values in E of the absolute Galois group $G_K := \mathrm{Gal}(\overline{K}/K)$ of K.

3.1 The cohomology of $\mathbb{C}_K$

Let K as above. Since the algebraic closure $\overline{K}$ is not complete, we define by $\mathbb{C}_K$ the completion $\widehat{\overline{K}}$ of $\overline{K}$ endowed with its unique valuation extending the given valuation val_p on K.

Of course, $\mathbb{C}_K$ is complete and fortunately it is also algebraically closed:

Proposition 3.1.1. *The field $\mathbb{C}_K$ is algebraically closed.*

Proof. By scaling the variable suitably, it suffices to construct roots for monic non-constant polynomials over $\mathcal{O}_{\mathbb{C}_K}$. Write such a polynomial as

$$P = X^N + a_1 X^{N-1} + \cdots + a_N \in \mathcal{O}_{\mathbb{C}_K}[X]$$

with $N > 0$. We can make a sequence of degree-N monic polynomials $P_n \in \mathcal{O}_{\overline{K}}[X]$ converging to P term-wise in coefficients. More specifically, for each $n \geq 0$ choose

$$P_n = X^N + a_{1,n} X^{N-1} + \cdots + a_{N,n} \in \mathcal{O}_{\overline{K}}[X]$$

with $P - P_n \in p^{Nn}\mathcal{O}_{\mathbb{C}_K}[X]$. By monicity, each P_n splits over $\mathcal{O}_{\overline{K}}$; let $\alpha_n \in \mathcal{O}_{\overline{K}}$ be a root of P_n.

Since $P_{n+1} - P_n \in p^{Nn}\mathcal{O}_{\mathbb{C}_K}[X]$, we have $P_{n+1}(\alpha_n) \in pNn\mathcal{O}_{\mathbb{C}_K}$ for all n. Expanding P_{n+1} as $\prod_{i=0}^{N}(X - \rho_{i,n+1})$ with roots $\rho_{i,n+1} \in \mathcal{O}_{\overline{K}}$, the product of the N differences $\alpha_n - \rho_{i,n+1}$ is divisible by p^{Nn}, so for some root α_{n+1} of P_{n+1} we must have that $\alpha_{n+1} - \alpha_n$ is divisible by p^n. In this way, proceeding by induction on n we have constructed a Cauchy sequence $\{\alpha_n\}$ in $\mathcal{O}_{\overline{K}}$ such that $P_n(\alpha_n) = 0$ for all n. Hence, if $\alpha \in \mathcal{O}_{\mathbb{C}_K}$ is the limit of the α_n's then $P(\alpha) = 0$ by continuity (since $P_n \to P$ coefficient-wise). □

Since $G_K = \mathrm{Gal}(\overline{K}/K)$ acts on $\overline{K}$ by isometries, this action uniquely extends to an action on the field $\mathbb{C}_K$ by isometries, and so identifies G_K with the isometric automorphism group of $\mathbb{C}_K$ over K. It is the natural to ask if there is a kind of "completed" Galois theory: how does $\mathbb{C}_K^H$ compare with $\overline{K}^H$ for a closed subgroup $H \subseteq G_K$? Since G_K acts by isometries, $\mathbb{C}_K^H$ is a closed subfield of $\mathbb{C}_K$, so it contains the closure of $\overline{K}^H$. Is it any bigger? By Galois theory we have $\mathbb{C}_K^H \cap \overline{K} = K$, so another way to put the question is :

are there transcendental invariants? The following proposition shows that there are none.

Proposition 3.1.2. *Let H be a closed subgroup of G_K. Then $\mathbb{C}_K^H$ is the completion $\widehat{L}$ of $L = \overline{K}^H$ for the valuation val_p. In particular, if H is an open subgroup of G_K, then $\mathbb{C}_K^H$ is the finite extension $\overline{K}^H$ of K, and $\widehat{L} \cap \overline{K} = L$.*

Proof. Choose $x \in \mathbb{C}_K^H$, so we want to show that x is a limit of points in $\overline{K}^H$. To do this, we approximation x by the algebraic elements and then try to modify the approximating sequence by using that assumed H-invariance of x. Pick a sequence $\{x_n\}_{n\geq 0}$ in $\overline{K}$ with $x_n \to x$; more specifically, arrange that $val_p(x - x_n) \geq n$ for all n. For $g \in H$, we have:

$$\begin{aligned} val_p(g(x_n) - x_n) &= val_p(g(x_n - x) - (x_n - x)) \geq \\ &\geq \min(val_p(g(x_n - x)), val_p(x_n - x)) = val_p(x_n - x) \geq n. \end{aligned}$$

Since $x_n \in \overline{K}$ is close to its entire H-orbit (as made precise above), it is natural to guess that this may be explained by x being essentially as close to an algebraic H-invariant element. This is indeed true: by [Ax70, Prop. 1], for each n there exists $y_n \in \overline{K}^H$ such that $val_p(x_n - y_n) \geq n - p/(p-1)^2$. But $x_n \to x$, so we conclude that likewise $y_n \to x$. That is, x is a limit of points in $\overline{K}^H$, as desired. □

To state the following theorem, we recall the notation $\mathbb{C}_K(r) := \mathbb{Z}_p(r) \otimes_{\mathbb{Z}_p} \mathbb{C}_K$ with the G_K-action on both sides of the tensor product, i.e., $g(z \otimes c) = g(z) \otimes g(c)$ for all $g \in G_K$.

Theorem 3.1.3 (Tate–Sen theorem)**.** *For any finite extension K of $\mathbb{Q}_p$, we have $K = \mathbb{C}_K^{G_K}$ (i.e., there are no transcendental invariants) and $\mathbb{C}_K(r)^{G_K} = 0$ for $r \neq 0$ (i.e., if $x \in \mathbb{C}_K$ and $g(x) = \chi(g)^{-r}x$ for all $g \in G_K$ and some $r \neq 0$ then $x = 0$). Moreover:*

$$\dim_F \mathrm{H}^1_{\mathrm{cont}}(G_K, \mathbb{C}_K(r)) = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{if } r \neq 0. \end{cases}$$

For a proof of Theorem 3.1.3 we refer to [BC09, §14].

3.2 Rings of periods: formalism

Before introducing the Fontaine's rings of periods, we describe the idea behind the construction of those rings. As above, we will consider the field K and its Galois group G_K.

Definition 3.2.1. Let B denote an $\mathbb{Q}_p$-algebra domain equipped with a G_K-action and, possibly, some supplementary structures compatible with the action of G_K (for example, a filtration, a Frobenius map, a monodromy map, etc.). We say that B is $(\mathbb{Q}_p, G_K)$-*regular* if:

(i) $\mathrm{Frac}(B)^{G_K} = B^{G_K}$;

(ii) if $b \in B$ is such that its $\mathbb{Q}_p$-linear span $\mathbb{Q}_p \cdot b$ is G_K-stable, then $b \in B^\times$.

Note that if B is a field then the conditions in the definition are obviously satisfied. Moreover, if property (i) holds, then the invariant subalgebra $F := B^{G_K}$ is automatically a field.

Now we introduce one of the principal objects we will use in this dissertation:

Definition 3.2.2. A *p-adic representation* of a profinite group Γ is a representation $\rho\colon \Gamma \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$ of Γ on a finite-dimensional $\mathbb{Q}_p$-vector space V such that ρ is continuous. The category of such representations is denoted by $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma)$.

A p-adic representation *with coefficient in* E of Γ, where E is a finite extension of $\mathbb{Q}_p$, is a p-adic representation V which is also an E-vector space for which the Γ-action is E-linear. In this case, the category is denoted by $\mathrm{Rep}_E(\Gamma)$.

Coming back to the general axiomatic setting, for any p-adic representation V of G_K we define

$$D_B(V) = (B \otimes_{\mathbb{Q}_p} V)^{G_K}$$

so $D_B(V)$ is a $F = B^{G_K}$-vector space equipped with a canonical map

$$\alpha_V\colon B \otimes_F D_B(V) \to B \otimes_F (B \otimes_{\mathbb{Q}_p} V) = (B \otimes_F B) \otimes_{\mathbb{Q}_p} V \to B \otimes_{\mathbb{Q}_p} V.$$

This is a B-linear G_K-equivariant map (where G_K acts trivially on $D_B(V)$ in the right tensor factor of the source).

For a p-adic representation V of G_K, it is true (see below) the $\dim_F D_B(V) \leq \dim_{\mathbb{Q}_p} V$; in case equality holds we call V a *B-admissible* representation.

Theorem 3.2.3. *Fix $V \in \mathrm{Rep}_E(G_K)$.*

1. *The map α_V is always injective and $\dim_F D_B(V) \leq \dim_E V$, with equality if and only if α_V is an isomorphism.*

2. *Let $\mathrm{Rep}_E^B(G_K) \subseteq \mathrm{Rep}_E(G_K)$ be the full subcategory of B-admissible representations. The covariant functor $D_B\colon \mathrm{Rep}_E^B(G_K) \to \mathrm{Vec}_F$ to the category of finite-dimensional F-vector spaces is exact and faithful, and any subrepresentation or quotient of a B-admissible representation is B-admissible.*

3. *If $V_1, V_2 \in \mathrm{Rep}_E^B(G_K)$ then there is a natural isomorphism*

$$D_B(V_1) \otimes_F D_B(V_2) \simeq D_B(V_1 \otimes_E V_2),$$

so $V_1 \otimes_E V_2 \in \mathrm{Rep}_E^B(G_K)$. If $V \in \mathrm{Rep}_E^B(G_K)$ then its dual representation $V^\vee$ lies in $\mathrm{Rep}_E^B(G_K)$ and the natural map

$$D_B(V) \otimes_F D_B(V^\vee) \simeq D_B(V \otimes_E V^\vee) \to D_B(E) = F$$

is a perfect duality between $D_B(V)$ and $D_B(V^\vee)$.

In particular, $\mathrm{Rep}_E^B(G_K)$ is stable under the formation of duals and tensor products in $\mathrm{Rep}_E(G_K)$, and D_B naturally commutes with the formation of these constructions in $\mathrm{Rep}_E^B(G_K)$ and in Vec_F.

Proof. See [BC09, Thm. 5.2.1]. □

If B has some supplementary structures, then these descend to $D_B(V)$ and in this way we obtain some non-trivial invariants of B-admissible representations, which can then be used to classify them.

3.3 Ring of periods: B_{HT} and the Hodge-Tate weights

Let V be a p-adic representation of G_K and consider its extension by scalar $W := \mathbb{C}_K \otimes V$ on which G_K acts continuously on both sides of the tensor product, i.e., by $g(c \otimes v) = g(c) \otimes g(v)$ for $c \in \mathbb{C}_K$ and $v \in V$.

Definition 3.3.1. A $\mathbb{C}_K$*-representation* of G_K is a finite-dimensional $\mathbb{C}_K$-vector space W equipped with a continuous G_K-action map $G_K \times W \to W$ that is semilinear (i.e., $g(cw) = g(c)g(w)$ for all $c \in \mathbb{C}_K$ and $w \in W$). The category of such object (using $\mathbb{C}_K$-linear G_K-equivariant morphism) is denoted by $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$.

For $W \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ and $q \in \mathbb{Z}$, consider the K-vector space

$$W\{q\} := W(q)^{G_K} \simeq \{w \in W \mid g(w) = \chi(g)^{-q} w \text{ for all } g \in G_K\},$$

where the isomorphism rests on a choice of basis of $\mathbb{Z}_p(1)$. In particular, $W\{q\}$ is not a $\mathbb{C}_K$-subspace of $W(q)$ when it is nonzero.

We have a natural G_K-equivariant K-linear multiplication map

$$K(-q) \otimes_K W\{q\} \hookrightarrow K(-q) \otimes_K W(q) \simeq W,$$

so extending scalars defines maps

$$\mathbb{C}_K(-q) \otimes_K W\{q\} \to W$$

for all $q \in \mathbb{Z}$.

The following lemma is due to Serre-Tate:

Lemma 3.3.2 (Serre–Tate)**.** *For a $\mathbb{C}_K$-representation W of G_K as above, the natural $\mathbb{C}_K$-linear G_K-equivariant map*

$$\alpha_W \colon \bigoplus_q (\mathbb{C}_K(-q) \otimes_K W\{q\}) \to W$$

is injective. In particular, $W\{q\} = 0$ for all but finitely many q and $\dim_K W\{q\} < \infty$, with $\sum_q W\{q\} \leq \dim_{\mathbb{C}_K} W$; equality holds here if and only if α_W is an isomorphism.

With the formalism of the previous section, we can reduce the proof of this lemma to construct a $(\mathbb{Q}_p, G_K)$-regular domain B: the perfect candidate for this role is the so-called *Hodge–Tate ring* $B_{\mathrm{HT}} := \bigoplus_{q\in\mathbb{Z}} \mathbb{C}_K(q)$.

Proof. Non-canonically, $B_{\mathrm{HT}} = \mathbb{C}_K[T, 1/T]$ with G_K acting through the p-adic cyclotomic character $\chi \colon G_K \to \mathbb{Z}_p^\times$ via $g(\sum a_n T^n) = \sum g(a_n)\chi(g)^n T^n$. Obviously, in this case $C := \mathrm{Frac}(B_{\mathrm{HT}}) = \mathbb{C}_K(T)$ and $B_{\mathrm{HT}}^{G_K} = K$.

By the Tate–Sen theorem 3.1.3, $B_{\mathrm{HT}}^{G_K} = \oplus \mathbb{C}_K(q)^{G_K} = K$. To compute that C^{G_K} is also equal to K, consider the G_K-equivariant inclusion of $C = \mathbb{C}_K(T)$ into the formal Laurent series field $\mathbb{C}_K(\!(T)\!)$ equipped with its evident G_K-action. It suffices to show that $\mathbb{C}_K(\!(T)\!)^{G_K} = K$. The action of $g \in G_K$ on a formal Laurent series $\sum c_n T^n$ is given by $\sum c_n T^n \mapsto \sum g(c_n)\chi(g)^n T^n$, so G_K-invariance amounts to the condition $c_n \in \mathbb{C}_K(q)^{G_K}$ for all $q \in \mathbb{Z}$. Hence, by the Tate–Sen theorem we get $c_n = 0$ for $n \neq 0$ and $c_0 \in K$, as desired.

In order to verify the second property of being $(\mathbb{Q}_p, G_K)$-regular, we proceed in a similar way: if $b \in B_{\mathrm{HT}} \setminus \{0\}$ spans a G_K-stable $\mathbb{Q}_p$-line then G_K acts on the line $\mathbb{Q}_p b$ by some character $\psi \colon G_K \to \mathbb{Q}_p^\times$. From the continuity of the G_K-action on each direct summand $\mathbb{C}_K(q)$ of B_{HT} we have that ψ must be continuous (so it takes values in $\mathbb{Z}_p^\times$). Writing the Laurent polynomial b as $b = \sum c_j T^j$, we have $\psi(g)b = g(b) = \sum g(c_j)\chi(g)^j T^j$, so for each j we have $(\psi^{-1}\chi^j)(g) \cdot g(c_j)$ for all $g \in G_K$. But by the Tate–Sen theorem, for a $\mathbb{Z}_p^\times$-valued continuous character η of G_K, if $\mathbb{C}_K(\eta)$ has a nonzero G_K-invariant element then $\eta|_{I_K}$ has finite order. Hence, $(\psi^{-1}\chi^j)|_{I_K}$ has finite order whenever $c_j \neq 0$. It follows that we cannot have $c_j, c_{j'} \neq 0$ for some $j \neq j'$, for otherwise taking the ratio of the associated finite-order characters would give that $\chi^{j-j'}|_{I_K}$ cuts out an infinitely ramified extension of K. It follows that there is at most one j such that $c_j \neq 0$, and there is a nonzero c_j since $b \neq 0$. Hence, $b = cT^j$ for some j and some $c \in \mathbb{C}_K^\times$, so $b \in B_{\mathrm{HT}}^\times$. $\square$

Definition 3.3.3. A representation W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is *Hodge–Tate* if α_W is an isomorphism, or equivalently if W is B_{HT}-admissible.

A representation V in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is *Hodge–Tate* if $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V \in \mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge–Tate.

For any Hodge–Tate object W we define the *Hodge–Tate weights* of W to be those $q \in \mathbb{Z}$ such that $W\{q\} := (\mathbb{C}_K(q) \otimes_{\mathbb{C}_F} W)^{G_K}$ is nonzero, and then we call $h_q := \dim_F W\{q\} \geq 1$ the *multiplicity* of q as a Hodge–Tate weight of W. Attention: with this definition $\mathbb{C}_K(q)$ has $-q$ as its unique Hodge–Tate weight. Obviously, if W is Hodge–Tate then so is $W^\vee$, with negated Hodge–Tate weights (compatibility with multiplicity), so it is harmless to change the definition of "Hodge–Tate weight" by a sign. In terms of p-adic Hodge theory, this confusion about sign comes down to later choosing to use covariant or contravariant functors when passing between p-adic representations and semilinear algebra objects.

3.4 Formalism of Hodge–Tate representations

For Serre–Tate's lemma 3.3.2, any W in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ is Hodge–Tate if and only if it is a (finite) direct sum of finite-dimensional vector spaces, so direct sum, tensor product and dual of two Hodge–Tate representations is again Hodge–Tate.

Therefore, we may define $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ to be the full subcategory of objects V that are Hodge–Tate (i.e., $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ is Hodge–Tate in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$). Our results in the $\mathbb{C}_K$-representations show that $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$ is stable under tensor product, duality, subrepresentations and quotients in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$.

For any V object in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, we look at the K-vector space

$$D_{\mathrm{HT}}(V) = (B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V)^{G_K}$$

but it carries over also a grading structure coming from the ring B_{HT}: for this reason, it is useful to introduce some terminology.

Definition 3.4.1. A *graded vector space* over a field F is a F-vector space D equipped with direct sum decomposition $\oplus_{q\in\mathbb{Z}} D_q$ for F-subspaces $D_q \subseteq D$ (and we define the *qth graded piece of* D). Morphisms $T\colon D' \to D$ between graded F-vector spaces are F-linear maps that respect the grading (i.e., $T(D'_q) \subseteq D_q$). The category of these is denoted Gr_F; we let $\mathrm{Gr}_{F,f}$ denote the full subcategory of D for which $\dim_F D$ is finite.

Thus, the above K-vector space $D_{\mathrm{HT}}(V)$ is actually an element of Gr_K. In general, we may define a functor $D_{\mathrm{HT}}\colon \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Gr}_K$, which actually takes values in $\mathrm{Gr}_{K,f}$ by Serre–Tate lemma. Now, we show some properties of this functor.

Lemma 3.4.2. *If* $0 \to V' \to V \to V'' \to 0$ *is a short exact sequence in* $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ *and* V *is Hodge–Tate then so are* V' *and* V''*, in which case the sequence*

$$0 \to D_{\mathrm{HT}}(V') \to D_{\mathrm{HT}}(V) \to D_{\mathrm{HT}}(V'') \to 0$$

in $\mathrm{Gr}_{K,f}$ *is short exact (so the multiplicities for each Hodge–Tate weight are additive in short exact sequences of Hodge–Tate representations).*

Proof. It's enough to show the result in $\mathrm{Rep}_{\mathbb{C}_K}(G_K)$ with $W := \mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ and similar for W' and W''.

By left-exactness of taking invariants, we have

$$0 \to D_{\mathrm{HT}}(W') \to D_{\mathrm{HT}}(W) \to D_{\mathrm{HT}}(W'') \tag{3.1}$$

with $\dim_K D_{\mathrm{HT}}(W') \le \dim_{\mathbb{C}_K} W'$ and similar for W and W''. But equality holds for W by the Hodge–Tate property, so

$$\begin{aligned}\dim_{\mathbb{C}_K} W = \dim_K D_{\mathrm{HT}}(W) &\le \dim_K D_{\mathrm{HT}}(W') + \dim_K D_{\mathrm{HT}}(W'') \\ &\le \dim_{\mathbb{C}_K} W' + \dim_{\mathbb{C}_K} W'' \\ &= \dim_{\mathbb{C}_K} W,\end{aligned}$$

forcing equality throughout. In particular, W' and W'' are Hodge–Tate and so for K-dimension reasons the left-exact sequence (3.1) is right-exact too. $\square$

Therefore, our functor D_{HT} on $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$ is exact and compatible with tensor products and duality (by the general formalism in §3.2). Furthermore, the comparison morphism $\alpha_V \colon B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$ for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is an isomorphism precisely when V is Hodge–Tate, and hence $D_{\mathrm{HT}} \colon \mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K) \to \mathrm{Gr}_{K,f}$ is a faithful functor.

Remark 3.4.3. One can show that the functor $D_{\mathrm{HT}} \colon \mathrm{Rep}_{\mathbb{C}_K}(G_K) \to \mathrm{Gr}_{K,f}$ is a fully faithful. However, our functor on the category $\mathrm{Rep}^{\mathrm{HT}}_{\mathbb{Q}_p}(G_K)$ of Hodge–Tate representations of G_K over $\mathbb{Q}_p$ is not fully faithful.

To improve this, we need to introduce a nice category of p-adic representations of G_K into a category of semilinear algebra objects.

3.5 Reminder: Witt vectors

In this section, I will recall some basic properties of the Witt vectors, rather than their construction. A classical reference is [Ser79], while a more detailed treatment is given in [BC09, §4.2].

We start with basic definitions:

Definition 3.5.1. A *p-ring* is a ring B that is separated and complete for the topology defined by a specified decreasing collection of ideals $\mathfrak{b}_1 \supseteq \mathfrak{b}_2 \supseteq \dots$ such that $\mathfrak{b}_n \mathfrak{b}_m \subseteq \mathfrak{b}_{n+m}$ for all $n, m \ge 1$ and $B/\mathfrak{b}_1$ is a perfect $\mathbb{F}_p$-algebra.

A *strict p-ring* is a p-ring B such that $\mathfrak{b}_i = p^i B$ for all $i \ge 1$ (i.e., B is p-adically separated and complete with B/pB a perfect $\mathbb{F}_p$-algebra) and $p \colon B \to B$ is injective.

Remark 3.5.2. The topological ring B, with a topology induced by a sequence of ideals as above, is said separated (or Hausdorff) and complete if, respectively, $\bigcap_{i\geq 1} \mathfrak{b}_i = 0$ and $B \simeq \varprojlim_i B/\mathfrak{b}_i$.

Starting with a perfect ring A of characteristic p (i.e., an $\mathbb{F}_p$-algebra for which $a \mapsto a^p$ is an automorphism of A), the aim is to construct a strict p-ring B with characteristic 0 and residue field A. In this case, the usual construction given by Witt works well in the sense that the ring of Witt vectors $W(A)$ of A is "the" desired strict p-ring in the following sense:

Proposition 3.5.3. *If A is a perfect $\mathbb{F}_p$-algebra and B is a p-ring, then the natural "reduction" map* $\mathrm{Hom}(W(A), B) \to \mathrm{Hom}(A, B/\mathfrak{b}_1)$ *(which makes sense since $A = W(A)/(p)$ and $p \in \mathfrak{b}_1$) is bijective. More generally, for any strict p-ring $\mathcal{B}$, the natural map*

$$\mathrm{Hom}(\mathcal{B}, B) \to \mathrm{Hom}(\mathcal{B}/(p), B/\mathfrak{b}_1)$$

is bijective for every p-ring B.

In particular, strict p-rings $\mathcal{B}$ are precisely the rings of the form $W(A)$ for perfect $\mathbb{F}_p$-algebras A.

An useful tool of this construction is the so-called *Teichmüller map*:

Lemma 3.5.4. *Let B be a p-ring. There is a unique set-theoretic section $[\cdot]\colon B/\mathfrak{b}_1 \to B$ to the reduction map such that $[x^p] = [x]^p$ for all $x \in B/\mathfrak{b}_1$. Moreover, $[\cdot]$ is multiplicative and $[1] = 1$.*

An immediate consequence of this lemma is that in a strict p-ring B endowed with the p-adic topology (relative to which it is separated and complete), each element $b \in B$ has the unique form $b = \sum_{n\geq 0} [b_n] p^n$ with $b_n \in B/\mathfrak{b}_1 = B/pB$.

Example 3.5.5. (i) $W(\mathbb{F}_p) = \mathbb{Z}_p$.

(ii) If $A = k$ is the finite field with p^n elements, then $W(k)$ is the ring of integers of the unique unramified extension of $\mathbb{Z}_p$ of degree n. If we consider a finite extension K of $\mathbb{Q}_p$ with residue field k, we will denote by $K_0 = W(k)[1/p]$ the maximal unramified extensions of $\mathbb{Q}_p$ inside K.

(iii) If $\overline{k}$ is the (fixed) algebraic closure of the residue field k of K, then $W(\overline{k}) = \mathcal{O}_{\widehat{K^{\mathrm{unr}}}}$ is the valuation ring of the completion $\widehat{K^{\mathrm{unr}}}$ of the maximal unramified extension of K. In particular, $\mathcal{O}_{\mathbb{C}_K}/(p) = \mathcal{O}_{\overline{K}}/(p)$ is not only an algebra over $W(k)/(p) = \mathbb{F}$ in a canonical manner, but also over $W(\overline{k})/(p)$.

3.6 The ring R

To the ring $\mathcal{O}_{\mathbb{C}_K}/(p) = \mathcal{O}_{\overline{K}}/(p)$ we can associate a perfect $\mathbb{F}_p$-algebra

$$R := \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/(p) = \left\{ (x_0, x_1, \dots) \in \prod_{n \geq 0} \mathcal{O}_{\mathbb{C}_K}/(p) \;\middle|\; x_{i+1}^p = x_i \text{ for all } i \right\}$$

with the product ring structure and endowed with a natural G_K-action. This is perfect because the additive pth power map on R is surjective by construction and is injective since if $(x_i) \in R$ satisfies $(x_i)^p = 0$ then $x_{i-1} = x_i^p = 0$ for all $i \geq 0$, so $(x_i) = 0$.

Every element of R can be uniquely lifted to a p-power compatible sequence in $\mathcal{O}_{\mathbb{C}_K}$ (but possibly not in $\mathcal{O}_{\overline{K}}$) in the following way. Fix an element $x = (x_n)_{n \geq 0} \in R$ and for every n choose a lift $\hat{x}_n$ of x_n inside $\mathcal{O}_{\mathbb{C}_K}$ (or inside $\mathcal{O}_{\overline{K}}$), then the sequence $\widehat{x_{n+m}}^{p^m} \xrightarrow{m \to \infty} x^{(n)} \in \mathcal{O}_{\mathbb{C}_K}$ which does not depend on the choice of the lifting $\hat{x}_n$ ([BC09, Prop. 4.3.1]). Moreover, the map $R \to \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}$, $x \mapsto (x^{(n)})_n$ defines a bijection on these sets and we will identify R in this way by setting the following ring structure on the second set: for $x = (x^{(n)})$ and $y = (y^{(n)})$ in R, define

$$\begin{cases} xy = (x^{(n)} y^{(n)})_{n \geq 0} \\ x + y = (z^{(n)})_{n \geq 0} \text{ with } z^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}. \end{cases}$$

An element $x \in R$ is a unit if and only if the component $x_0 \in \mathcal{O}_{\overline{K}}/(p)$ is a unit, so R is a local ring. Also, since every element of $\mathcal{O}_{\overline{K}}$ is a square, it follows that the nonzero maximal ideal $\mathfrak{m}$ of R satisfies $\mathfrak{m} = \mathfrak{m}^2$. In particular, R is not noetherian. If val_p denotes the valuation of $\mathcal{O}_{\mathbb{C}_K}$ normalized by $val_p(p) = 1$, then we may define a G_K-equivariant valuation v_R on R by setting $v_R(x) = val_p(x^{(0)})$ for all $x = (x^{(n)})_{n \geq 0}$. This valuation makes R into a valuation ring for which it is v_R-adically separated and complete, integrally closed in $\mathrm{Frac}(R)$ with residue field $\overline{k}$.

An important example of an element of R is

$$\varepsilon = (\varepsilon^{(n)})_{n \geq 0} = (1, \zeta_p, \zeta_{p^2}, \dots)$$

with $\varepsilon^{(0)} = 1$ but $\varepsilon^{(1)} \neq 1$ (so $\varepsilon^{(1)} = \zeta_p$ is a primitive pth root of unity and hence $\varepsilon^{(n)}$ is a primitive pth root of unity for all $n \geq 0$). Any two such elements are $\mathbb{Z}_p^\times$-power of each other and for any such choice of element we have $v_R(\varepsilon - 1) = p/(p-1)$.

If $\alpha \in \overline{k}$, then $([\alpha^{1/p^n}])_{n \geq 0} \in R$, and this gives an injective map $\overline{k} \to R$. If we define $\bar{\pi} = \varepsilon - 1$, we get $\overline{k}[\![\bar{\pi}]\!] \subset R$ independently of ε and:

Theorem 3.6.1. *The field* $\mathrm{Frac}(R) = R[1/\bar{\pi}]$ *of characteristic p is algebraically closed. In particular, it is the completion of the algebraic closure* $\overline{k}(\!(\bar{\pi})\!)$.

3.7 Ring of periods: B_{dR}

Once one constructs the ring R, one can define the other three Fontaine's ring of periods; we start, as usual in the literature, with B_{dR}.

Since the ring R is perfect of characteristic p, we will take its ring of Witt vectors, but in the same time we do not want to loose much informations about its construction, in particular its G_K-equivariant surjective reduction map $\theta_0 \colon R \to \mathcal{O}_{\mathbb{C}_K}/(p)$, $(x_i) \mapsto x_0$. In order to get a lifting of θ_0, one is tented to apply 3.5.3, but unfortunately the target ring is not perfect, although $\mathcal{O}_{\mathbb{C}_K}$ is p-adically separated and complete. Nonetheless, we can construct such lifting $\theta \colon W(R) \to \mathcal{O}_{\mathbb{C}_K}$ in a canonical and G_K-equivariant manner.

First, we note that every element of $W(R)$ can be written in a unique as $\sum [c_n] p^n$ with $c_n = (c_n^{(m)})_{m \geq 0} \in R$ (since $W(R)$ is a strict p-ring and $W(R)/(p) = R$). Then, we define θ in an explicit way as:

$$\theta(\sum [c_n] p^n) = \sum c_n^{(0)} p^n$$

By definition, θ is G_K-equivariant and surjective; further, it is a ring homomorphism ([BC09, Lemma 4.4.1]).

Inverting p in both sides, we can extend θ by a G_K-equivariant surjective ring homomorphism

$$\theta_{\mathbb{Q}} \colon W(R)[1/p] \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}[1/p] = \mathbb{C}_K$$

but the source ring is not a complete discrete valuation ring. We shall replace $W(R)[1/p]$ with its $\ker \theta_{\mathbb{Q}}$-adic completion, and the reason this works is that $\ker \theta_{\mathbb{Q}} = (\ker \theta)[1/p]$ turns out to be a principal ideal. In fact, if we pick $\underline{p} \in R$ such that $\underline{p}^{(0)} = p$, then $\xi = [\underline{p}] - p \in W(R)$ is a generator for $\ker \theta \subseteq W(R)$. Moreover, $W(R) \cap (\ker \theta_{\mathbb{Q}})^j = (\ker \theta)^j$ and $\cap_j (\ker \theta_{\mathbb{Q}})^j = 0$. We conclude that $W(R)[1/p]$ injects into the inverse limit

$$B_{\mathrm{dR}}^{+} := \varprojlim_j W(R)[1/p]/(\ker \theta_{\mathbb{Q}})^j \tag{3.2}$$

whose transition maps are G_K-equivariant, so B_{dR}^{+} has a natural G_K-action that is compatible with the action on its subring $W(R)[1/p]$. (Beware that in (3.2) we cannot move the p-localization outside of the inverse limit). The inverse limit B_{dR}^{+} map G_K-equivariantly onto each quotient $W(R)[1/p]/(\ker \theta_{\mathbb{Q}})^j$ via the evident natural map, and in particular, for $j = 1$, the map $\theta_{\mathbb{Q}}$ induces a natural G_K-equivariant surjective map $\theta_{\mathrm{dR}}^{+} \colon B_{\mathrm{dR}}^{+} \twoheadrightarrow \mathbb{C}_K$.

Proposition 3.7.1. *The ring B_{dR}^{+} is a complete discrete valuation ring with residue field $\mathbb{C}_K$, and any generator of $\ker \theta_{\mathbb{Q}}$ in $W(R)[1/p]$ is a uniformiser of B_{dR}^{+}.*

Proof. See [BC09, p. 4.4.6]. ∎

The Frobenius automorphism ϕ_R of $W(R)[1/p]$ does not naturally extend to B_{dR}^+ since it does not preserve $\ker\theta_{\mathbb{Q}}$, so there is no natural Frobenius structure on B_{dR}^+. Nevertheless, we do have a filtration via powers of the maximal ideal, and this is a G_K-stable filtration. We get the same on the fraction field:

Definition 3.7.2. The *field of p-adic periods* (or the *de Rham period ring*) is $B_{\mathrm{dR}} := \mathrm{Frac}(B_{\mathrm{dR}}^+)$ equipped with its natural G_K-action and G_K-stable filtration via the $\mathbb{Z}$-powers of the maximal ideal of B_{dR}^+.

Another way to see the filtration steps inside B_{dR}^+ and B_{dR} is to define an element t which is a generator for the maximal ideal $\ker\theta_{\mathbb{Q}}$ and also a uniformiser for B_{dR}^+ (by Proposition 3.7.1), so $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/t]$; in particular, this construction shows that the filtered field B_{dR} is actually an appropriate refinement of B_{HT}:

$$\mathrm{Fil}^i B_{\mathrm{dR}} := (\ker\theta_{\mathbb{Q}})^i = t^i B_{\mathrm{dR}}^+.$$

Now we construct such element. Take the above element $\varepsilon \in R$, so $\theta([\varepsilon]-1) = \varepsilon^{(0)} - 1 = 0$. Hence, $[\varepsilon]-1 \in \ker\theta \subseteq \ker\theta_{\mathrm{dR}}^+$, so $[\varepsilon] = 1 + ([\varepsilon]-1)$ is a 1-unit in the complete discrete valuation ring B_{dR}^+ over K. We can therefore make sense of the logarithm

$$t := \log([\varepsilon]) = \log(1 + ([\varepsilon]-1)) = \sum_{n\geq 1}(-1)^{n+1}\frac{[\varepsilon]-1}{n} \in B_{\mathrm{dR}}^+.$$

This lies in the maximal ideal of B_{dR}^+. Note that if we make another choice ε' then $\varepsilon' = \varepsilon^a$ for a unique $a \in \mathbb{Z}_p^\times$ using the natural $\mathbb{Z}_p$-module structure on 1-units in R. Hence, we have $t' := \log([\varepsilon']) = a\log([\varepsilon]) = at$.

In other words, the line $\mathbb{Z}_p t$ in the maximal ideal of B_{dR}^+ is intrinsic (i.e., independent of the choice of ε) and making a choice of $\mathbb{Z}_p$-basis of this 'line' is the same as making a choice of ε. Also, choosing ε is literally a choice of $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(1) = \varprojlim \mu_{p^n}(\overline{K})$. For any $g \in G_K$, we have $g(\varepsilon) = \varepsilon^{\chi(g)}$ in R since $g(\varepsilon^{(n)}) = (\varepsilon^{(n)})^{\chi(g)}$ for the primitive p^nth root of unity $\varepsilon^{(n)} \in \mathcal{O}_{\overline{K}}$ for all $n \geq 0$. Thus, by the G_K-equivariance of the logarithm on 1-units of B_{dR}^+,

$$g(t) = \log(g([\varepsilon])) = \log([g(\varepsilon)]) = \log([\varepsilon^{\chi(g)}]) = \log([\varepsilon]^{\chi(g)}) = \chi(g)t.$$

We conclude that $\mathbb{Z}_p t$ is a canonically copy of $\mathbb{Z}_p(1)$ as a G_K-stable line in B_{dR}^+.

Remark 3.7.3. The ring B_{dR}^+ is isomorphic to $\mathbb{C}_K[\![t]\!]$ only as abstract rings, and there is no such isomorphism which is compatible with the action of G_K.

In order to conclude this section, we note some other properties of B_{dR}.

If $P(X) \in K[X]$ is a polynomial with simple roots, then it splits completely in $\mathbb{C}_K$ and hence, by Hensel's lemma, it also splits completely in B_{dR}^+, since $B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+ = \mathbb{C}_K$. In this way we see that $\overline{K} \subset B_{\mathrm{dR}}^+$.

The last property we mention is the determination of its subfield of G_K-invariants. The above G_K-equivariant embedding $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ gives us an inclusion $K \subseteq B_{\mathrm{dR}}^{G_K}$.

Proposition 3.7.4. *The inclusion $K \subseteq B_{\mathrm{dR}}^{G_K}$ is an equality.*

Proof. Consider the short exact sequence

$$0 \to t^{h+1}B_{\mathrm{dR}}^+ \to t^h B_{\mathrm{dR}}^+ \to \mathbb{C}_K(h) \to 0$$

and for $h = 0$ apply Tate–Sen Theorem 3.1.3 on the invariants. □

3.8 de Rham representations

Since B_{dR} is $(\mathbb{Q}_p, G_K)$-regular with $B_{\mathrm{dR}}^{G_K} = K$, the general formalism of admissible representations provides a good class of p-adic representations: the B_{dR}-admissible ones. More precisely, we define the covariant functor $D_{\mathrm{dR}} \colon \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vec}_K$ valued in the category of finite-dimensional K-vector spaces by

$$D_{\mathrm{dR}}(V) := (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)^{G_K},$$

so $\dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$. In case this inequality is an equality we say that V is a *de Rham representation* (i.e., V is B_{dR}-admissible). Let $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ denote the full subcategory of de Rham representations.

By the general formalism from §3.2, for $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K)$ we have a B_{dR}-linear G_K-compatible comparison isomorphism

$$\alpha_V \colon B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

and the subcategory $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \subseteq \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is stable under passage to subquotients, tensor products, and duals, and moreover the functor

$$D_{\mathrm{dR}} \colon \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(G_K) \to \mathrm{Vec}_K$$

is faithful and exact and commutes with the formation of duals and tensor powers.

Since duality does not affect whether or not the de Rham property holds, working with D_{dR} is equivalent to working with the contravariant functor

$$D_{\mathrm{dR}}^*(V) := D_{\mathrm{dR}}(V^\vee) \simeq \mathrm{Hom}_{\mathbb{Q}[G_K]}(V, B_{\mathrm{dR}}).$$

The output of the functor D_{dR} has extra K-linear structure (arising from additional structure on the K-algebra B_{dR}), namely a K-linear filtration arising from the canonical K-linear filtration on the fraction field B_{dR} of the complete discrete valuation ring B_{dR}^+ over K. Before we explain this, we review some terminology from linear algebra.

Definition 3.8.1. A *filtered module* over a commutative ring R is an R-module D endowed with a collection $\{\mathrm{Fil}^i D\}_{i\in\mathbb{Z}}$ of submodules that is decreasing in the sense that $\mathrm{Fil}^{i+1}D \subseteq \mathrm{Fil}^i D$ for all $i \in \mathbb{Z}$. If $\cup_i \mathrm{Fil}^i D = D$ then the filtration is *exhaustive*, if $\cap_i \mathrm{Fil}^i D = 0$ then the filtration is *separated*. For any filtered R-module D, the *associated graded module* is $\mathrm{gr}^\bullet(D) = \oplus_i(\mathrm{Fil}^i D/\mathrm{Fil}^{i+1}D)$

Of course, if $(D, \mathrm{Fil}^i D)$ is a finite-dimensional filtered vector space then the filtration is exhaustive if and only if $\mathrm{Fil}^i D = D$ for $i \ll 0$ and it is separated if and only if $\mathrm{Fil}^i = 0$ for $i \gg 0$. Let Fil_K denote the category of finite-dimensional filtered vector spaces $(D, \mathrm{Fil}^i D)$ over K equipped with an exhaustive and separated filtration, where a *morphism* between such objects is a linear map $T\colon D' \to D$ that is filtration-compatible in the sense that $T(\mathrm{Fil}^i D') \subseteq \mathrm{Fil}^D$ for all i.

In the category Fil_K there are good functorial notions of kernel and cokernel of a map, tensor product (for $D, D' \in \mathrm{Fil}_K$ the tensor product $D \otimes D'$ has underlying K-vector space $D \otimes_K D'$ and filtration $\mathrm{Fil}^n(D \otimes D') = \displaystyle\sum_{p+q=n} \mathrm{Fil}^p D \otimes_K \mathrm{Fil}^q D'\Big)$ and dual. A *short exact sequence* $0 \to D' \to D \to D'' \to 0$ in Fil_K is a short exact sequence of K-vector spaces $0 \to D' \to D \to D'' \to 0$ such that the sequence $0 \to \mathrm{Fil}^i D' \to \mathrm{Fil}^i D \to \mathrm{Fil}^i D'' \to 0$ is exact as K-vector spaces for all i. Beware this, the category Fil_K is not abelian, so in general coimage and image of a map cannot coincide; we refer to *strict morphism* when the latter happens.

Finally, there is a natural functor $\mathrm{gr} = \mathrm{gr}^\bullet\colon \mathrm{Fil}_K \to \mathrm{Gr}_{K,f}$ to the category of finite-dimensional graded K-vector spaces via $\mathrm{gr}(D) = \oplus_i \mathrm{Fil}^i D/\mathrm{Fil}^{i+1}D$. This functor is dimension-preserving and exact. By choosing bases compatible with filtrations we see that the functor gr is compatible with tensor products in the sense that there is a natural isomorphism $\mathrm{gr}(D) \otimes \mathrm{gr}(D') \simeq \mathrm{gr}(D \otimes D')$ in $\mathrm{Gr}_{K,f}$ for any $D, D' \in \mathrm{Fil}_K$.

For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, the K-vector space $D_{\mathrm{dR}}(V) = (B_{\mathrm{dR}} \otimes V)^{G_K} \in \mathrm{Vec}_K$ has a natural structure of object in Fil_K: since B_{dR} has an exhaustive and separated G_K-stable K-linear filtration via $\mathrm{Fil}^i B_{\mathrm{dR}} = t^i B_{\mathrm{dR}}^+$, we get an evident K-linear G_K-stable filtration $\{\mathrm{Fil}^i(B_{\mathrm{dR}}) \otimes_{\mathbb{Q}_p} V\}_i$ on $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$, so this induces an exhaustive and separate filtration on the *finite-dimensional* K-subspace $D_{\mathrm{dR}}(V)$ of G_K-invariant elements. Explicitly,

$$\mathrm{Fil}^i D_{\mathrm{dR}}(V) = (t^i B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

The finite-dimensionality of $D_{\mathrm{dR}}(V)$ is what ensures that this filtration fills up all of $D_{\mathrm{dR}}(V)$ for sufficiently negative filtration degrees and vanishes for sufficiently positive filtration degrees.

Proposition 3.8.2. *If V is de Rham then V is Hodge–Tate and* $\mathrm{gr}(D_{\mathrm{dR}}(V)) = D_{\mathrm{HT}}(V)$ *as graded K-vector spaces. In general, there is an injection* $\mathrm{gr}(D_{\mathrm{dR}}) \hookrightarrow D_{\mathrm{HT}}(V)$ *and it is an equality of $\mathbb{C}_K$-vector spaces when V is de Rham.*

Proof. By left exactness of the formation of G_K-invariants, we get a natural K-linear injection

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow D_{\mathrm{HT}}(V)$$

for all $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ because $\mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ as graded $\mathbb{C}_K$-algebras with G_K-action.
Thus,

$$\dim_K D_{\mathrm{dR}}(V) = \dim_K \mathrm{gr}(D_{\mathrm{dR}}(V)) \leq \dim_K D_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V$$

for all V. In the de Rham case the outer terms are equal, so the inequalities are all equalities. □

We say that the *Hodge–Tate weights* of a de Rham representation V are those i for which the filtration on $D_{\mathrm{dR}}(V)$ "jumps" from degree i to degree $i+1$, which is to say $\mathrm{gr}^i(D_{\mathrm{dR}}(V)) \neq 0$. This is exactly that $\mathbb{C}_K \otimes_{\mathbb{Q}_p} V$ has i as a Hodge–Tate weight. The multiplicity of such an i as a Hodge–Tate weight is the K-dimension of the filtration jump, which is to say $\dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V))$.

Example 3.8.3. $D_{\mathrm{dR}}(\mathbb{Q}_p(n))$ is a line with nontrivial gr^{-n}, so $\mathbb{Q}_p(n)$ has Hodge–Tate weight $-n$ (with multiplicity 1).

Sometimes it is more convenient to define Hodge–Tate weights using the same filtration condition ($\mathrm{gr}^i \neq 0$) applied to the contravariant functor $D^*_{\mathrm{dR}}(V) = D_{\mathrm{dR}}(V^\vee) = \mathrm{Hom}_{\mathbb{Q}[G_K]}(V, B_{\mathrm{dR}})$ so as to negate things (so that $\mathbb{Q}_p(n)$ acquires Hodge–Tate weight n instead).

The general formalism of §3.2 tells us that D_{dR} on the full subcategory $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ is exact and respects tensor products and duals when viewed with values in Vec_K, but it is a stronger property to ask if the same is true as a functor valued in Fil_K. Fortunately, such good behaviour of isomorphisms relative to filtrations does hold:

Proposition 3.8.4. *The faithful functor $D_{\mathrm{dR}}\colon \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K) \to \mathrm{Fil}_K$ carries short exact sequences to short exact sequences and is compatible with the formation of tensor products and duals. In particular, if V is a de Rham representation and*

$$0 \to V' \to V \to V'' \to 0$$

is a short exact sequence in $\mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$ (so V' and V'' are de Rham) then $D_{\mathrm{dR}}(V') \subseteq D_{\mathrm{dR}}(V)$ has the subspace filtration and the linear quotient $D_{\mathrm{dR}}(V'')$ of $D_{\mathrm{dR}}(V)$ has the quotient filtration.

Proof. See [BC09, Prop. 6.3.3]. □

Corollary 3.8.5. *For $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $n \in \mathbb{Z}$, V is de Rham if and only if $V(n)$ is de Rham.*

An important refinement of Proposition 3.8.4 is that the de Rham comparison isomorphism is also filtration-compatible:

Proposition 3.8.6. *For $V \in \mathrm{Rep}^{\mathrm{dR}}_{\mathbb{Q}_p}(G_K)$, the G_K-equivariant B_{dR}-linear comparison isomorphism*

$$\alpha\colon B_{\mathrm{dR}} \otimes_K D_{\mathrm{dR}}(V) \simeq B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

respects the filtrations and its inverse does too.

Proof. By construction α is filtration-compatible, so the problem is to prove that its inverse is as well. It is equivalent to show that the induced B_{HT}-linear map $\mathrm{gr}(\alpha)$ on associated graded objects is an isomorphism. On the right side the associated graded object is naturally identified with $B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$. For the left side, we first recall that (by a calculation with filtration-adapted bases) the formation of the associated graded space of an arbitrary filtered K-vector space (of possibly infinite dimension) is naturally compatible with the formation of tensor products (in the graded and filtered sense), so the associated graded object for the left side is naturally identified with $B_{\mathrm{HT}} \otimes_K \mathrm{gr}(D_{\mathrm{dR}}(V))$.

By Proposition 3.8.2, the de Rham representation V is Hodge–Tate and there is a natural graded isomorphism $\mathrm{gr}(D_{\mathrm{dR}}(V)) \simeq D_{\mathrm{HT}}(V)$. in this manner, $\mathrm{gr}(\alpha)$ is naturally identified with the graded comparison morphism

$$\alpha_{\mathrm{HT}}\colon B_{\mathrm{HT}} \otimes_K D_{\mathrm{HT}}(V) \to B_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$$

that is a graded isomorphism because V is Hodge–Tate. □

3.9 Ring of periods: B_{cris}

One defect of B^+_{dR} is that the Frobenius automorphism of $W(R)[1/p]$ does not preserve $\ker\theta_{\mathbb{Q}}$, so there is no natural Frobenius endomorphism of B_{dR}. In fact, $\theta([\underline{p}^{1/p}] - p) \neq 0$, so that $[\underline{p}^{1/p}] - p$ is invertible in B^+_{dR}, and so $1/([\underline{p}^{1/p}] - p) \in B^+_{\mathrm{dR}}$. But if ϕ was a natural extension of $\phi_R\colon W(R)[1/p] \to W(R)[1/p]$, then one should have $\phi(1/([\underline{p}^{1/p}] - p)) = 1/([\underline{p}] - p)$, and since $\theta([\underline{p}] - p) = 0$, $1/([\underline{p}] - p) \notin B^+_{\mathrm{dR}}$.

To remedy this defect, we will introduce an auxiliary subring $A^0_{\mathrm{cris}} \subseteq W(R)[1/p]$ that is Frobenius-stable and gives rise to a large subring $B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$ on which there is a natural Frobenius endomorphism.

Let A^0_{cris} denote the divided power envelope of $W(R)[1/p]$ with respect to $\ker\theta$, which in concrete terms means that it is the G_K-stable $W(R)$-subalgebra

$$W(R)[\alpha^m/m!]_{m\geq 1, \alpha\in\ker\theta} = W(R)[\xi^m/m!]_{m\geq 1}$$

in $W(R)[1/p]$ generated by "divided powers" of all elements of $\ker\theta$, where $\xi := [\underline{p}] - p$ is a generator for this ideal. Since A^0_{cris} is a $\mathbb{Z}$-flat domain, if we define

$$A_{\mathrm{cris}} = \varprojlim_n A^0_{\mathrm{cris}}/p^n A^0_{\mathrm{cris}}$$

to be the p-adic completion of A^0_{cris}, then A_{cris} is p-adically separated and complete and the natural map $A^0_{\mathrm{cris}}/p^nA^0_{\mathrm{cris}} \to A_{\mathrm{cris}}/p^nA_{\mathrm{cris}}$ is an isomorphism for all $n \geq 1$. In particular, it follows that A_{cris} is $\mathbb{Z}_p$-flat. However, it is not at all evident and difficult to show that $A^0_{\mathrm{cris}} \to A_{\mathrm{cris}}$ is injective and that there exists a unique continuous injective map $j\colon A_{\mathrm{cris}} \hookrightarrow B^+_{\mathrm{dR}}$ (so that it is G_K-equivariant and that A_{cris} is a domain). We address these properties to the literature (most of them are inside [Fon82]).

Concretely, the image of A_{cris} in B^+_{dR} is the subring of elements

$$\left\{ \sum_{n\geq 0} a_n \frac{\xi^n}{n!} \;\middle|\; a_n \in W(R),\ a_n \to 0 \text{ for the } p\text{-adic topology} \right\}$$

in which the infinite sums are taken with respect to the discretely-valued topology of B^+_{dR}; such sums converge since ξ lies in the maximal ideal of B^+_{dR}. Further, the G_K-action on A_{cris} is continuous for the p-adic topology.

Define the G_K-stable $W(R)[1/p]$-subalgebra

$$B^+_{\mathrm{cris}} := A_{\mathrm{cris}}[1/p] \subseteq B^+_{\mathrm{dR}}.$$

Lemma 3.9.1. *We have $t \in A_{\mathrm{cris}}$.*

Proof. Choose a generator ξ of $\ker\theta$. Since $[\varepsilon]-1 \in \ker\theta = \xi W(R)$, we have $[\varepsilon]-1 = w\xi$ for some $w \in W(R)$. Thus, in B^+_{dR} we have

$$t = \sum_{n\geq 1}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} = \sum_{n\geq 1}(-1)^{n+1}(n-1)!w^n \cdot \frac{\xi^n}{n!}$$

with $(n-1)!w^n \to 0$ in $W(R)$ relative to the p-adic topology. Hence, $t \in A_{\mathrm{cris}}$ inside of B^+_{dR}. $\square$

In a similar way, one can show that $t^{p-1} \in pA_{\mathrm{cris}}$.

Definition 3.9.2. The *crystalline period ring* B_{cris} for K is the G_K-stable $W(R)[1/p]$-subalgebra $B^+_{\mathrm{cris}}[1/t] = A_{\mathrm{cris}}[1/t]$ inside of $B^+_{\mathrm{dR}}[1/t] = B_{\mathrm{dR}}$. (Since $t^{p-1} \in A_{\mathrm{cris}}$, inverting t makes p into a unit.)

Since $W(k) \subseteq W(R) \subseteq A_{\mathrm{cris}}$, we have $K_0 = W(k)[1/p] \subseteq B_{\mathrm{cris}}$, so $K_0 \subseteq B_{\mathrm{cris}}^{G_K} \subseteq B_{\mathrm{dR}}^{G_K} = K$. We claim that $B_{\mathrm{cris}}^{G_K} = K_0$. This is immediate from the following non-obvious crucial fact.

Theorem 3.9.3. *The natural G_K-equivariant map $K \otimes_{K_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective, and if we give $K \otimes_{K_0} B_{\mathrm{cris}}$ the subspace filtration then the induced map between the associated graded algebras is an isomorphism.*

Proof. The proof is entirely given via the construction of the map $j\colon A_{\mathrm{cris}} \to B^+_{\mathrm{dR}}$, so we refer as above to [Fon82]. $\square$

As for the isomorphism property on associated graded objects, since $t \in B_{\mathrm{cris}}$ and A_{cris} map to onto $\mathcal{O}_{\mathbb{C}_K}$, we get the isomorphism result since $\mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ has its graded components of dimension 1 over $\mathrm{gr}^0(B_{\mathrm{dR}}) = \mathbb{C}_K$. $\square$

Since B_{dR} is a field, it follows from Theorem 3.9.3 that $K \otimes_{K_0} \mathrm{Frac}(B_{\mathrm{cris}}) \to B_{\mathrm{dR}}$ is injective. Hence, we likewise deduce that $\mathrm{Frac}(B_{\mathrm{cris}})^{G_K} = K_0$. This proves part of:

Proposition 3.9.4. *The domain B_{cris} is $(\mathbb{Q}_p, G_K)$-regular.*

Proof. It remains to show that if $b \in B_{\mathrm{cris}}$ is nonzero and $\mathbb{Q}_p b$ is G_K-stable then $b \in B_{\mathrm{cris}}^{\times}$. Since $t \in B_{\mathrm{cris}}^{\times}$, if the nonzero b has exact filtration degree i in B_{dR} then by replacing b with $t^{-i}b$ we can arrange that $n \in B_{\mathrm{dR}}^{+}$ and b is not in the maximal ideal. Let $\eta \colon G_K \to \mathbb{Q}_p^{\times}$ be the abstract character on the line $\mathbb{Q}_p b$. Thus, the residue class $\bar{b}$ in $\mathbb{C}_K$ spans a $\mathbb{Q}_p$-line in $\mathbb{C}_K$ with G_K-action by η. This forces η to be continuous and hence $\mathbb{Z}_p^{\times}$-valued, with $\mathbb{C}_K(\eta^{-1})^{G_K} \neq 0$. By Theorem 3.1.3 we conclude that $\eta(I_F)$ is finite. But $I_F = G_{\widehat{K^{\mathrm{unr}}}}$, so again by using Theorem 3.1.3 (for the absence of transcendental invariants, applied over a finite extension of $\widehat{K^{\mathrm{unr}}}$ splitting η), we deduce that the element $\bar{b} \in \mathbb{C}_K$ is algebraic over $\widehat{K^{\mathrm{unr}}} = W(\bar{k})[1/p] \subseteq B_{\mathrm{dR}}^{+}$.

Such an element $\bar{b}$ in the residue field $\mathbb{C}_K$ of the $\widehat{K^{\mathrm{unr}}}$-algebra B_{dR}^{+} uniquely lifts to an element $\beta \in B_{\mathrm{dR}}^{+}$ that is algebraic over $\widehat{K^{\mathrm{unr}}}$ by Hensel's lemma for the complete discrete valuation ring B_{dR}^{+} with the residue characteristic 0, so $b - \beta \in \mathrm{Fil}^1(B_{\mathrm{dR}}^{+})$. The G_K-action on B_{dR}^{+} restricted to β is given by the $\mathbb{Q}_p^{\times}$-valued η due to the uniqueness of β as a lifting of $\bar{b}$ that is algebraic over $\widehat{K^{\mathrm{unr}}}$. Hence, $b - \beta$ spans a G_K-stable $\mathbb{Q}_p$-line in $\mathrm{Fil}^1(B_{\mathrm{dR}}^{+})$ with character η if $b - \beta \neq 0$. If there is such a $\mathbb{Q}_p$-line, then its nonzero elements live in some exact filtration degree $r \geq 1$ and so passing to the quotient by the next filtered piece would give a nonzero element in $\mathbb{C}_K(r)$ on which G_K acts through η. In other words, $\mathbb{C}_K(\chi^r \cdot \eta)$ has a nonzero G_K-invariant element. But by Theorem 3.1.3 this forces $\chi^r \eta(I_F)$ to be finite, which is a contradiction since $\eta(I_F)$ is finite and $r > 0$. We conclude that $b - \beta = 0$, so $b = \beta$ is algebraic over $\widehat{K^{\mathrm{unr}}}$.

Thus, $L := \widehat{K^{\mathrm{unr}}}(b) \subseteq B_{\mathrm{cris}}$ is a finite extension of $\widehat{K^{\mathrm{unr}}}$, and its maximal unramified subfield L_0 must be $\widehat{K^{\mathrm{unr}}}$. By applying Theorem 3.9.3 over the ground field L (in the role of K in that theorem) we get that the map of rings $L \otimes_{L_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$ is injective. Hence, the subring $L \otimes_{L_0} L$ is a domain (as B_{dR} is a domain), so $L = L_0$ and therefore $b \in L_0^{\times} = \widehat{K^{\mathrm{unr}}}^{\times} \subseteq B_{\mathrm{cris}}^{\times}$. $\square$

Now, we describe the construction of the injective G_K-equivariant endomorphism of B_{cris} that extends the Frobenius automorphism ϕ_R of $W(R)[1/p]$.

Fix $\underline{p} \in R$ such that $\underline{p}^{(0)} = p$, so for $\xi = [\underline{p}] - p \in \ker\theta$ we have that $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ with A_{cris} defined to be the p-adic completion of A^0_{cris}. The key point is:

Lemma 3.9.5. *The $W(R)$-subalgebra $A^0_{\mathrm{cris}} \subseteq W(R)[1/p]$ is ϕ_R-stable.*

Proof. We compute $\phi_R(\xi) = [\underline{p}^p] - p = [\underline{p}]^p - p = (\xi + p)^p - p = \xi^p + pw$ for some $w \in W(R)$. Thus,

$$\phi_R(\xi) = p \cdot (w + (p-1)! \cdot (\xi^p/p!)),$$

so $\phi_R(\xi^m) = p^m(w + (p-1)! \cdot (\xi^p/p!))^m$ for all $m \geq 1$. But $p^m/m! \in \mathbb{Z}_p$ for all $m \geq 1$, so $\phi_R(\xi^m/m!) \in A^0_{\mathrm{cris}}$ for all $m \geq 1$. $\square$

The endomorphism of A^0_{cris} induced by ϕ_R on $W(R)[1/p]$ extends uniquely to a continuous endomorphism of the p-adic completion A_{cris}, and hence an endomorphism ϕ of $B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ that extends the Frobenius automorphism ϕ_R of the subring $W(R)[1/p]$. We claim that for $t \in A_{\mathrm{cris}}$ (inside of B^+_{dR}) we have $\phi(t) = pt$ with $p \in (B^+_{\mathrm{cris}})^\times$, so ϕ extends uniquely to an endomorphism of $B_{\mathrm{cris}} = B^+_{\mathrm{cris}}[1/t]$. Recall that the element t, initially defined in B^+_{dR} as $\sum_{n\geq 1}(-1)^{n+1}([\varepsilon]-1)^n/n$, makes sense in A_{cris} as a convergent sum in the p-adic topology. Thus, we may use p-adic continuity to compute

$$\phi(t) = \sum_{n\geq 1}(-1)^{n+1}\frac{(\phi([\varepsilon])-1)^n}{n} = \sum_{n\geq 1}(-1)^{n+1}\frac{([\varepsilon^p]-1)}{n}$$

since ϕ on A_{cris} extends the usual Frobenius map on $W(R)$. Thus, $\phi(t) = \log([\varepsilon^p]) = pt$, where the last equality is computed in a previous section.

Corollary 3.9.6. *The Frobenius endomorphism $\phi\colon A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ is injective. In particular, the induced Frobenius endomorphism of $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ is injective.*

Give B_{cris} the subspace filtration from $F \otimes_{K_0} B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}$, i.e., define

$$\mathrm{Fil}^i B_{\mathrm{cris}} = B_{\mathrm{cris}} \cap \mathrm{Fil}^i B_{\mathrm{dR}}.$$

Beware that (since there is no Frobenius on B_{dR}) this is not ϕ-stable. We require a fundamental property of the filtration on B_{cris}.

Theorem 3.9.7. *The space $(\mathrm{Fil}^0 B_{\mathrm{cris}})^{\phi=1} := \{b \in \mathrm{Fil}^0 B_{\mathrm{cris}} \mid \phi(b) = b\}$ of ϕ-invariant elements in the 0th filtered piece of B_{cris} is equal to $\mathbb{Q}_p$.*

Proof. This is a difficult result, we refer to [Fon94a]. $\square$

3.10 Weakly admissible filtered (ϕ, N)-modules

By general formalism §3.2 we will consider the functor $D = D_{\text{cris}} \colon \text{Rep}_{\mathbb{Q}_p}(G_K) \to \text{Vec}_{K_0}$ defined by

$$D_{\text{cris}}(V) := (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}.$$

This finite-dimensional $B_{\text{cris}}^{G_K} = K_0$-vector space has two kinds of structure:
(i) an *injective* Frobenius-semilinear endomorphism induced by the G_K-equivariant injective Frobenius ϕ_{cris} on B_{cris} (so this is bijective since the Frobenius map on K_0 is an automorphism);
(ii) an exhaustive and separated K-linear filtration on the scalar extension

$$D_{\text{cris}}(V)_K = ((K \otimes_{K_0} B_{\text{cris}}) \otimes_{\mathbb{Q}_p} V)^{G_K}$$

via the G_K-stable filtration on $K \otimes_{K_0} B_{\text{cris}}$.

So, first of all we shall study a suitable subcategory of Fil_K.

Definition 3.10.1. A *filtered ϕ-module over K* is a triple $(D, \phi_D, \text{Fil}^\bullet)$ where D is a finite-dimensional K_0-vector space, $\phi_D \colon D \to D$ is a bijective Frobenius-semilinear endomorphism and $(D_K := K \otimes_{K_0} D, \text{Fil}^\bullet)$ is an object in Fil_K (i.e., $\{\text{Fil}^i\}_i$ is a decreasing exhaustive and separated filtration on D_K).

A *morphism* $D' \to D$ between two filtered ϕ-modules is a K_0-linear map $D' \to D$ compatible with both $\phi_{D'}$ and ϕ_D and has K-linear extension $D'_K \to D_K$ that is a morphism in Fil_K. The category of triples $(D, \phi, \text{Fil}^\bullet)$ is denoted MF_K^ϕ.

Thus, it is clear that $D_{\text{cris}}(V)$ has a structure of object in MF_K^ϕ. Let $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_K) \in \text{Rep}_{\mathbb{Q}_p}(G_K)$ denote the full subcategory of the B_{cris}-admissible representations. By §3.2, this full subcategory is stable under duality and tensor products. Moreover, the above argument shows that the functor $D_{\text{cris}} \colon \text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_K) \to \text{Vec}_{K_0}$ has values in MF_K^ϕ and it is faithful since so is the forgetful functor $\text{MF}_K^\phi \to \text{Vec}_{K_0}$. Somewhat deeper is the fact that it is actually *fully faithful*, so we want to specialize our target category on finding its essential image.

To do this, we need to introduce two important invariants of a filtered ϕ-module.

Definition 3.10.2. Let $D \in \text{MF}_K^\phi$.

The *Hodge number* of D is:

$$t_H(D) := \begin{cases} 0 & \text{if } D = 0; \\ i & \text{if } \dim_{K_0} D = 1, \text{Fil}^i D = D \text{ and } \text{Fil}^{i+1} D = 0; \\ t_H(\bigwedge^n D) & \text{if } n := \dim_{K_0} D \geq 2. \end{cases}$$

Denoting by val_p the valuation on K normalized by $val_p(p) = 1$, the *Newton number* of D is:

$$t_N(D) := \begin{cases} 0 & \text{if } D = 0; \\ val_p(a) & \text{if } D = K_0 d \text{ and } \phi(d) = a \cdot d; \\ t_N(\bigwedge^n D) & \text{if } n := \dim_{K_0} D \geq 2. \end{cases}$$

Remark 3.10.3. Note that the Newton number is well-defined: if $D = K_0 d = K_0 d'$, one has $d' = c \cdot d$ with $c \in K_0$; if $\phi(d) = ad$ and $\phi(d') = a'd'$, one has $a' = c^{-1}.\tau(c)a$, where τ is the Frobenius on K_0, and $val_p(a') = val_p(a)$.

It is immediate from the definition that t_H and t_N are additive in the sense that if we have a short exact sequence $0 \to D' \to D \to D'' \to 0$ in MF_K^ϕ, then $t_H(D) = t_H(D') + t_H(D'')$ and $t_N(D) = t_N(D') + t_N(D'')$.

Now we are ready to define our candidate category of semilinear objects in MF_K^ϕ:

Definition 3.10.4. A filtered ϕ-module D over K is *weakly admissible* if $t_H(D) = t_N(D)$ and if, for any subobject $D' \subseteq D$ in MF_K^ϕ, we have $t_H(D') \leq t_N(D')$.

The full subcategory of MF_K^ϕ consisting of weakly admissible objects is denoted $\mathrm{MF}_K^{\phi,\mathrm{w.a.}}$.

An easy property of weakly admissible modules is the following:

Lemma 3.10.5. *If $D \in \mathrm{MF}_K^\phi$, then D is weakly admissible if and only if its dual $D^\vee$ is weakly admissible.*

Proof. Since t_H and t_N are negated under duality, it suffices to show that in the definition of weak admissibility it is equivalent to work with the alternative condition that for all quotients $D \twoheadrightarrow D''$ we have $t_H(D) = t_N(D)$ and $t_H(D'') \geq t_N(D'')$. For any D in MF_K^ϕ there is a natural bijective correspondence between subobjects $D' \subseteq D$ and quotient objects $\pi \colon D \twoheadrightarrow D''$ (up to isomorphism), namely $D' \mapsto D'' := D/D'$ and $D'' \mapsto \ker \pi$. Since $t_H(D') + t_H(D/D') = t_H(D)$ and $t_N(D') + t_N(D/D') = t_N(D)$ with the values $t_H(D)$ and $t_N(D)$ fixed and independent of D', we are done. $\square$

It is a remarkable fact that $\mathrm{MF}_K^{\phi,\mathrm{w.a.}}$ is an abelian category (using kernels and cokernels as in the additive category MF_K^ϕ that is not abelian), and more specifically that any morphism between weakly admissible filtered ϕ-modules is strict with respect to filtrations over K. To avoid later duplication of effort, rather than prove these properties for $\mathrm{MF}_K^{\phi,\mathrm{w.a.}}$ now, we prefer to establish such a result for a larger category of structures beyond MF_K^ϕ.

Definition 3.10.6. A *filtered (ϕ, N)-module over K* is a quadruple $(D, \phi_D, N_D, \mathrm{Fil}^\bullet)$ where $(D, \phi_D, \mathrm{Fil}^\bullet)$ is a filtered ϕ-module over K equipped with a K_0-linear endomorphism $N_D\colon D \to D$ (called the *monodromy operator*) such that $N_D\phi_D = p\phi_D N_D$. The notion of morphism between such objects is the evident one, and the category of these is denoted $\mathrm{MF}_K^{\phi,N}$.

In this definition we do not assume N_D is nilpotent; it can be deduced later thanks to the finite-dimensionality of the space ([BC09, Lemma 8.2.8]). In particular, if $D \in \mathrm{MF}_K^{\phi,N}$ is an 1-dimensional object, then $N_D = 0$.

Example 3.10.7. We write $K_0[0]$ to denote the 1-dimensional unit object of MF_K^{ϕ} (i.e., $D = K_0$ with $\mathrm{gr}^0(D_F) \neq 0$ and ϕ equal to the Frobenius automorphism); this is a "unit object" for the tensor product. Upon endowing it with the monodromy operator $N = 0$ it likewise becomes the unit object of MF_K^{ϕ}.

Note that in general MF_K^{ϕ} is exactly the full subcategory of $\mathrm{MF}_K^{\phi,N}$ consisting of objects whose monodromy operator vanishes.

The category $\mathrm{MF}_K^{\phi,N}$ has evident notions of short exact sequence, kernel, cokernel, image and coimage. We also define duals and tensor products in the evident manner, with monodromy operators

$$N_{D\otimes D'} = N_D \otimes id_{D'} + id_D \otimes N_{D'} \qquad \text{and} \qquad N_{D^\vee} = -N_D^\vee.$$

Definition 3.10.4 now extends to incorporate a monodromy operator:

Definition 3.10.8. An object $D \in \mathrm{MF}_K^{\phi,N}$ is *weakly admissible* if $t_H(D) = t_N(D)$ and for all subobjects $D' \subseteq D$ in MF_K^{ϕ} (so D' is required to be N-stable in D too) we have $t_H(D') \leq t_N(D')$. Equivalently, if $t_H(D) = t_N(D)$ and for all quotient objects $D \twoheadrightarrow D''$ in $\mathrm{MF}_K^{\phi,N}$ we have $t_H(D'') \geq t_N(D'')$.

These objects constitute a full subcategory $\mathrm{MF}_K^{\phi,N,\text{w.a.}}$ of $\mathrm{MF}_K^{\phi,N}$. (clearly $\mathrm{MF}_K^{\phi,\text{w.a.}}$ consists of objects in $\mathrm{MF}_K^{\phi,N,\text{w.a.}}$ for which $N = 0$.)

Weak admissibility is a very subtle link between three structures: the Frobenius, the filtration and the monodromy operator (whose only role here is to constrain the possible subobjects in $\mathrm{MF}_K^{\phi,N}$ via the N-stability condition). Since $N_{D^\vee} = -N_D^\vee$, we see as in the case $N = 0$ that D in $\mathrm{MF}_K^{\phi,N}$ is weakly admissible if and only if $D^\vee$ is weakly admissible.

The next two results in $\mathrm{MF}_K^{\phi,N}$ could have been proved much earlier in MF_K^{ϕ}, but we waited so that we could handle $\mathrm{MF}_K^{\phi,N}$ in general.

Proposition 3.10.9. *If $0 \to D' \to D \to D'' \to 0$ is a short exact sequence in $\mathrm{MF}_K^{\phi,N}$ and any two of the three terms are weakly admissible then so is the third.*

Proof. If D is weakly admissible then for any subobject D_1' of D' we may view D_1' as a subobject of D and hence $t_H(D_1') \leq t_N(D_1')$. If in addition D'' is weakly admissible then $t_H(D'') = t_N(D'')$, so $t_H(D') = t_H(D) - t_H(D'') =$

$t_N(D) - t_N(D'') = t_N(D')$. Thus, D' is weakly admissible when D and D'' are so. Applying these considerations after dualizing the original exact sequence and using the general identity that t_H and t_N negate under duality, we conclude that if D and D' are weakly admissible then so is D''.

Now suppose that D' and D'' are weakly admissible. By additivity in short exact sequences we see that $t_H(D) = t_N(D)$ due to the analogous such equalities for D' and D''. It remains to prove $t_H(D_1) \leq t_N(D_1)$ for all subobjects $D_1 \subseteq D$. We let $D_1' := D' \cap D_1$ and give $(D_1')_K$ the subspace filtration from either $(D_1)_K$ or D_K' (these subspace filtrations coincide!), and let $D_1'' := D_1/D_1'$ with the quotient filtration on $(D_1'')_K$. There is a natural injective map $j\colon D_1'' \hookrightarrow D'' = D/D'$ in $\mathrm{MF}_K^{\phi,N}$, but a priori it may not be strict (i.e., the quotient filtration on $(D_1'')_K$ from $(D_1)_K$ may be finer than the subspace filtration from D_K''). Since D_1' is a subobject of the weakly admissible D', $t_H(D_1') \leq t_N(D_1')$. Thus,

$$t_H(D_1) = t_H(D_1') + t_H(D_1'') \leq t_N(D_1') + t_H(D_1'')$$

and $t_N(D_1) = t_N(D_1') + t_N(D_1'')$, so it suffices to prove that $t_H(D_1'') \leq t_N(D_1'')$.

Let $j(D_1'')$ denote D_1'' endowed with the subspace filtration from D'', so the natural map $D_1'' \to j(D_1'')$ in $\mathrm{MF}_K^{\phi,N}$ is a linear isomorphism. We have $t_N(D_1'') = t_N(j(D_1''))$ since j is an isomorphism in the category ϕ-modules over K_0 (without filtrations). Hence, it is enough to prove $t_H(D_1'') \leq t_N(j(D_1''))$. But $j(D_1'')$ is a subobject of the weakly admissible D''. so $t_H(j(D_1'')) \leq t_N(j(D_1''))$ and hence our problem reduces to proving the inequality $t_H(D_1'') \leq t_H(j(D_1''))$ between Hodge numbers for the bijective morphism $j\colon D_1'' \to j(D_1'')$ in $\mathrm{MF}_K^{\phi,N}$.

In general, if $h\colon \Delta' \to \Delta$ is a bijective morphism in Fil_K then we claim that $t_H(\Delta') \leq t_H(\Delta)$ with equality if and only if h is an isomorphism in Fil_K (i.e., it is a strict morphism). To prove this, first note that $t_H(\Delta) = t_H(\det \Delta)$ and $t_H(\Delta') = t_H(\det \Delta')$, and a consideration of bases adapted to filtrations shows that a bijective morphism in Fil_K is an isomorphism in Fil_K if and only if the induced map on top exterior powers is an isomorphism in Fil_K. Thus, by passing to $\det h\colon \det \Delta' \to \det \Delta$ we reduce to the 1-dimensional case, for which t_H is the unique i such that $\mathrm{gr}^i \neq 0$. This concludes the argument. $\square$

We now come to the remarkable fact that in the presence of the weak admissibility condition the filtration structures behave as in an abelian category:

Theorem 3.10.10. *Let $h\colon D \to D'$ be a map in $\mathrm{MF}_K^{\phi,N,\mathrm{w.a.}}$. The map h is strict (i.e., $D/\ker h \to \mathrm{im} h$ is an isomorphism in $\mathrm{MF}_K^{\phi,N}$), and $\ker h$ and $\mathrm{coker} h$ with their respective subspace and quotient filtration structures are weakly admissible. In particular, the object $\mathrm{im} h \simeq D/\ker h$ is weakly admissible and the category $\mathrm{MF}_K^{\phi,N}$ is abelian.*

Proof. See [BC09, Thm. 8.2.11]. □

3.11 Crystalline representations

We have constructed our candidate (abelian) category $\mathrm{MF}_K^{\phi,\mathrm{w.a.}}$ inside MF_K^{ϕ}, so now we will show that it is actually the valued category of the functor $D_{\mathrm{cris}}\colon \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}} \to \mathrm{MF}_K^{\phi}$ restricted to those p-adic representations that are *crystalline* (i.e., B_{cris}-admissible).

By the general formalism §3.2, recall that the functor D_{cris} is faithful, exact and naturally commutes with the formation of tensor products and duals.

Proposition 3.11.1. *If $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K)$ then the natural map $j_V\colon K \otimes_{K_0} D_{\mathrm{cris}}(V) \to D_{\mathrm{dR}}(V)$ in Fil_K is an isomorphism. In particular, crystalline representations are de Rham.*

Moreover, the B_{cris}-linear Frobenius-compatible G_K-equivariant crystalline comparison isomorphism

$$\alpha\colon B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V) \simeq B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$$

satisfies the property that α_K is a filtered isomorphism.

Proof. The natural map j_V is a subobject inclusion in Fil_K by definition of the filtration structure on $D_{\mathrm{cris}}(V)_K$, so the problem is one of comparing K-dimensions. The crystalline condition says $\dim_{K_0} D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$, and since $\dim_K D_{\mathrm{dR}}(V) \le \dim_{\mathbb{Q}_p} V$ we must have equality, so V is de Rham. To verify that the K-linear inverse α_K^{-1} is filtration-compatible too, or in other words that the filtration-compatible α_K is a filtered isomorphism, it is equivalent to show that $\mathrm{gr}(\alpha_K)$ is an isomorphism. Since j_V is an isomorphism and $\mathrm{gr}(K \otimes_{K_0} B_{\mathrm{cris}}) = \mathrm{gr}(B_{\mathrm{dR}}) = B_{\mathrm{HT}}$ by Theorem 3.9.3, the method of proof of Proposition 3.8.6 adapts to show that $\mathrm{gr}(\alpha_K)$ is identified with the Hodge–Tate comparison isomorphism for V. □

The Theorem 3.9.7 underlies the key to the full faithfulness properties for D_{cris}. The reason for the importance of this theorem is that it shows how to extract $\mathbb{Q}_p$ out of B_{cris} using only its "linear structures": the G_K-action, the Frobenius operator, and the filtration. To see how useful it is, we finally come to the key point of the story: we can recover V from $D_{\mathrm{cris}}(V)$ when V is crystalline.

Indeed, consider the crystalline comparison isomorphism

$$\alpha\colon B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V) \simeq B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V \tag{3.3}$$

for $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K)$. We have seen that not only α is B_{cris}-linear, G_K-equivariant, and Frobenius-compatible, but α_K is a filtered isomorphism too.

Hence, by intersecting with the 0th filtered parts after scalar extension to K we get a G_K-equivariant K_0-linear isomorphism

$$\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V)) \simeq \mathrm{Fil}^0(B_{\mathrm{cris}}) \otimes_{\mathbb{Q}_p} V$$

that is compatible with the Frobenius actions on both sides (within the ambient B_{cris}-modules as in (3.11)). Passing to ϕ-fixed parts therefore gives a $\mathbb{Q}_p[G_K]$-linear isomorphism

$$\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D_{\mathrm{cris}}(V))^{\phi=1} \simeq V. \tag{3.4}$$

In other words, if we define the covariant functor

$$V_{\mathrm{cris}} \colon \mathrm{MF}_K^{\phi} \to \mathbb{Q}_p[G_K]\text{-mod}$$

by $D \mapsto \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} D)^{\phi=1}$ then $V \simeq V_{\mathrm{cris}}(D_{\mathrm{cris}}(V))$ for crystalline representations V of G_K. Hence, modulo the issue that $V_{\mathrm{cris}}(D)$ may not be finite-dimensional over $\mathbb{Q}_p$, with continuous G_K-action for arbitrary D in MF_K^{ϕ}, the functor V_{cris} provides an inverse to D_{cris} (or rather, D_{cris} restricted to $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K)$). Most importantly, we have almost shown:

Proposition 3.11.2. *The exact tensor-functor $D_{\mathrm{cris}} \colon \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_K) \to \mathrm{MF}_K^{\phi}$ is fully faithful, with inverse on its essential image given by V_{cris}. The same holds for the contravariant D_{cris}^* using the contravariant functor $V_{\mathrm{cris}}^*(D) = \mathrm{Hom}_{\mathrm{Fil},\phi}(D, B_{\mathrm{cris}})$.*

Proof. The fully faithfulness needs further discussion. Suppose that V and V' are crystalline p-adic representations of G_K and let $D = D_{\mathrm{cris}}(V)$ and $D' = D_{\mathrm{cris}}(V)$ in MF_K^{ϕ}. If $T \colon D' \to D$ is a map in MF_K^{ϕ} then via the crystalline comparison isomorphism as in (3.11) for V and V', the B_{cris}-linear extension $1 \otimes T \colon B_{\mathrm{cris}} \otimes_{K_0} D' \to B_{\mathrm{cris}} \otimes_{K_0} D$ of T is identified with a B_{cris}-linear, G_K- and Frobenius-compatible, and filtration-compatible isomorphism $\widetilde{T} \colon B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V' \simeq B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$.

Explicitly, $\widetilde{T} = \alpha_{\mathrm{cris}}(V) \circ T \circ \alpha_{\mathrm{cris}}(V')^{-1}$. The map $\widetilde{T}$ respects the formation of the ϕ-fixed part in filtration degree 0, which is to say (by (3.4)) that this B_{cris}-linear isomorphism must carry V' into V by a G_K-equivariant map. Hence, $\widetilde{T}$ is the B_{cris}-scalar extension of some map $V' \to V$ in $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$, so by functoriality of the crystalline comparison isomorphism we see that this map $V' \to V$ between Galois representations induces the given map $T \colon D_{\mathrm{cris}}(V') = D' \to D = D_{\mathrm{cris}}(V)$. This gives full faithfulness as desired. $\square$

Since we usually work with twisted object, we introduce the notation in the category $\mathrm{MF}_K^{\phi,N}$:

Definition 3.11.3. For $D \in \mathrm{MF}_K^{\phi,N}$, the *i-fold Tate twist of* D is the object $D\langle i\rangle$ whose underlying K_0-vector space is D, monodromy operator $N_{D\langle i\rangle}$ is N_D, Frobenius operator $\phi_{D\langle i\rangle}$ is p^{-i} and filtration structure over K is $\mathrm{Fil}^r(D\langle i\rangle_K) = \mathrm{Fil}^{i+r}(D_K)$.

Beware that this definition is adapted to the use of *contravariant* Fontaine functors $D^*_{\mathrm{cris}}(\cdot) = \mathrm{Hom}_{\mathbb{Q}[G_K]}(\cdot, B_{\mathrm{cris}})$ (and similarly for the future D^*_{st}). In this way, for $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and $i \in \mathbb{Z}$ we have $D^*_{\mathrm{cris}}(V(i)) \simeq D^*_{\mathrm{cris}}(V)\langle i\rangle$.

We conclude with a basic calculation.

Example 3.11.4. Let's calculate $D^*_{\mathrm{cris}}(\mathbb{Q}_p(r)) = \mathrm{Hom}_{\mathbb{Q}[G_K]}(\mathbb{Q}_p(r), B_{\mathrm{cris}})$. Given any $\mathbb{Q}_p[G_K]$-linear map $\mathbb{Q}_p(r) \to B_{\mathrm{cris}}$, if we multiply it by t^{-r} then we get a $\mathbb{Q}_p[G_K]$-linear map $\mathbb{Q}_p \to B_{\mathrm{cris}}$. In other words, $D = D^*_{\mathrm{cris}}(\mathbb{Q}_p(r)) = B_{\mathrm{cris}}^{G_K} \cdot t^r = K_0 t^r$. This has Frobenius action $\phi(ct^r) = \sigma(c)(\phi t)^r = p^r \sigma(c) t^r$, and the unique filtration jump for D_F happens in degree r (i.e., $\mathrm{gr}^r(D_F) \neq 0$). In other words, $D^*_{\mathrm{cris}}(\mathbb{Q}_p(r))$ is the Tate twist $(K_0[0])\langle r\rangle$ (notation as in Example 3.10.7).

Let's push this further and compute $V^*_{\mathrm{cris}}(D^*_{\mathrm{cris}}(\mathbb{Q}_p(R))) = V^*_{\mathrm{cris}}((K_0[0])\langle r\rangle)$. This consists of K_0-linear maps $T\colon K_0 \to \mathrm{Fil}^r B_{\mathrm{cris}}$ that satisfy $\phi(T(c)) = T(p^r\sigma(c))$ for all $c \in K_0$, or in other words $\sigma(c) \cdot \phi(T(1)) = p^r \sigma(c) T(1)$ for all $c \in K_0$. This says $\phi(T(1)) = p^r T(1)$ with $T(1) \in \mathrm{Fil}^r B_{\mathrm{cris}}$, and if we write $T(1) = bt^r$ with $b \in \mathrm{Fil}^0 B_{\mathrm{cris}}$ (as we may since $t \in B^\times_{\mathrm{cris}}$) then the condition on b is exactly $b \in (\mathrm{Fil}^0 B_{\mathrm{cris}})^{\phi=1} = \mathbb{Q}_p$. Hence, $V^*_{\mathrm{cris}}(D^*_{\mathrm{cris}}(\mathbb{Q}_p(r))) = \mathbb{Q}_p t^r$ is the canonical copy of $\mathbb{Q}_p(r)$ inside of B_{cris}. This illustrates in a special (but important!) case of the general fact that V^*_{cris} is "inverse" to D^*_{cris} restricted to crystalline representations.

This is not an isolate case but corresponds to a more general statement on 1-dimensional crystalline representations (see [BC09, Prop. 8.3.4]):

Proposition 3.11.5. *The functor D^*_{cris} is an equivalence of categories between 1-dimensional crystalline representations of G_K and 1-dimensional weakly admissible filtered (ϕ, N)-modules over K. The characters arising in this way are precisely the Tate twist of the $\mathbb{Z}_p^\times$-valued unramified characters of G_K.*

The next step in the development of D_{cris} is to show that it takes values in the full subcategory of *weakly admissible* filtered ϕ-modules over K, as suggest by the 1-dimensional case. Rather than prove this result now, we shall first digress to develop the theory of of another $(\mathbb{Q}_p, G_K)$-regular period ring B_{st} containing B_{cris} whose associated theory of admissible representations (to be called *semistable*) generalizes the theory of crystalline representations. The desired weak admissibility property for D_{cris} with crystalline V will be a special case of a more general weak admissibility property that we will prove for $D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K} \in \mathrm{MF}_K^{\phi,N}$ for semistable V.

3.12 Ring of periods: B_{st}

The period ring B_{st} will be a canonical extension ring of B_{cris} endowed with compatible Galois and Frobenius structures, as well as a filtration on $K \otimes_{K_0} B_{\mathrm{st}}$, but there will not be a canonical injective map $B_{\mathrm{st}} \to B_{\mathrm{dR}}$ as B_{cris}-algebras with G_K-action. Instead, such a map will depend on a certain non-canonical choices, but the image of the map will be independent of the choices.

Fix a choice of q in the maximal ideal of $\mathcal{O}_K$ and pick $\underline{q} \in \mathfrak{m}_R \setminus \{0\}$ such that $\underline{q}^{(0)} = q$. Define

$$\log \frac{[\underline{q}]}{q} := \sum_{n \geq 1} (-1)^{n+1} \frac{([\underline{q}]/q - 1)^n}{n} \in B_{\mathrm{dR}}^+$$

Definition 3.12.1. Define the subring B_{st}^+ to be the subring of B_{dR}^+ generated by B_{cris}^+ and $\log([\underline{q}]/q)$. The *semistable period ring* B_{st} is the ring $B_{\mathrm{st}} := B_{\mathrm{st}}^+[1/t]$.

The element $\log([\underline{q}]/q) \in B_{\mathrm{dR}}^+$ is transcendental over B_{cris}, so non-canonically we have $B_{\mathrm{st}}^+ \simeq B_{\mathrm{cris}}^+[X]$ and $B_{\mathrm{st}} \simeq B_{\mathrm{cris}}[X]$. In this identification, G_K acts as usual on B_{cris} and on X by the formula $g(X) = X + \log([\varepsilon_{\underline{q}}(g)])$, where $\varepsilon_{\underline{q}}(g) = g(\underline{q})/\underline{q} \in (R)^\times$ is a compatible sequence of (possibly non-primitive) p^nth roots of unity (so $\log([\varepsilon_{\underline{q}}(g)])$ lies in the canonical $\mathbb{Z}_p(1)$ in A_{cris}). Moreover, we extend the injective Frobenius ϕ on B_{cris}^+ to a (visibly injective) Frobenius ϕ on B_{st}^+ and B_{st} via the requirement $\phi(X) = pX$.

The ring B_{st} admits an additional structure, a monodromy operator N defined to be $N = \mathrm{d}/\mathrm{d}X$ on $B_{\mathrm{st}}^+ = B_{\mathrm{cris}}^+[X]$. Note that this operator interacts with ϕ by the formula $N\phi = p\phi N$.

Finally, to define a filtration on $K \otimes_{K_0} B_{\mathrm{st}}$ extending the one in $K \otimes_{K_0} B_{\mathrm{cris}}$, we seek to construct a G_K-equivariant B_{cris}-algebra embedding $B_{\mathrm{st}} \to B_{\mathrm{dR}}$ carrying B_{st}^+ into B_{dR}^+. The image of such a map will be canonical but the actual map will depend on a choice: the standard convention is to take $q = p$ and to fix the value of its logarithm $\log(p) = c \in F$, in general $c = 0$. Under this assumptions, we put $\mathrm{Fil}^i B_{\mathrm{st}} := B_{\mathrm{st}} \cap \mathrm{Fil}^i B_{\mathrm{dR}}$ (and the same for B_{st}^+ and B_{dR}^+). In particular, the (non-canonical) embedding $K \otimes_{K_0} B_{\mathrm{st}} \to B_{\mathrm{dR}}$ is injective as $K[G_K]$-algebra, so the inclusion $K_0 \subseteq B_{\mathrm{st}}^{G_K}$ is an equality.

Now we want to sum up all the non-canonical choices in order to use the isomorphism $B_{\mathrm{st}} \simeq B_{\mathrm{cris}}[X]$ from now on.

Fix $q = p$ (inside $\mathcal{O}_K$) and take $\underline{p} \in \mathfrak{m}_R \setminus \{0\}$. Moreover, fix the value $\log(p) = 0$, so that we get an isomorphism $B_{\mathrm{st}}^+ \simeq B_{\mathrm{cris}}^+[X]$ and a B_{cris}^+-algebra map $B_{\mathrm{st}}^+ \to B_{\mathrm{dR}}^+$ carries X to $\log([\underline{p}]) := \log([\underline{p}]/p)$. As the choice of $\underline{p}$ may vary by $\mathbb{Z}_p(1)^\times$-elements, we fix a $\mathbb{Z}_p$-basis ε, so $g(\underline{p}) = \underline{p} \cdot \varepsilon^{\eta_{\underline{p}}(g)}$ for a unique $\eta_{\underline{p}}(g) \in \mathbb{Z}_p$. Letting $t = \log([\varepsilon])$, the G_K-action is given on X by $g(X) = X + \eta_{\underline{p}}(g)t$.

Proposition 3.12.2. *The ring $B_{\rm st}$ is $(\mathbb{Q}_p, G_K)$-regular.*

Proof. It remains to prove that if $b \in B_{\rm st}$ is nonzero and $\mathbb{Q}_p b$ is G_K-stable then $b \in B_{\rm st}^\times$. It is harmless for this purpose to replace K with $\widehat{K^{\rm unr}}$, which is to say that k is algebraically closed. We shall use the concrete description $B_{\rm st} = B_{\rm cris}[X]$ with $g(X) = X + \eta(g)t$ where $t = log([\varepsilon])$ is a fixed choice and the continuous $\eta\colon G_K \to \mathbb{Z}_p$ is defined by $g(\pi) = \pi\varepsilon^{\eta(g)}$ for a fixed $\pi \in R$ such that $\pi^{(0)} = p$. Let $\psi\colon G_K \to \mathbb{Q}_p^\times$ be the character on the line $\mathbb{Q}_p b$ in $B_{\rm st} = B_{\rm cris}[X]$. We may write $b = b_0 + \cdots + b_r X^r$ with $b_i \in B_{\rm cris}$ and $b_r \neq 0$. Our goal is to show $r = 0$, as then $b = b_0$ spans a G_K-stable $\mathbb{Q}_p$-line in $B_{\rm cris}$, whence $b \in B_{\rm cris}^\times = B_{\rm st}^\times$ due to the known $(\mathbb{Q}_p, G_K)$-regularity of $B_{\rm cris}$.

Consider the identity

$$\psi(g)b = g(b) = g(b_0) + g(b_1)(X + \eta(g)t) + \cdots + g(b_r)(X + \eta(g)t)^r$$

in $B_{\rm st}$ for $g \in G_K$. Comparing top-degree terms in X gives $\psi(g)b_r = g(b_r)$, so b_r spans a G_K-stable $\mathbb{Q}_p$-line in $B_{\rm cris}$. The character ψ is continuous, by the same trick with $t^{\mathbb{Z}}$-scaling and projection into $\mathbb{C}_K$ as in the proof of $(\mathbb{Q}_p, G_K)$-regularity of $B_{\rm cris}$ in Proposition 3.9.4. Hence, ψ is a continuous character that appears in $B_{\rm cris}$, so it is a crystalline character of G_K. As such ψ is Hodge–Tate, so it has some Hodge–Tate weight $n \in \mathbb{Z}$. Thus, $\chi^{-n}\psi$ is a crystalline character with Hodge–Tate weight 0. Therefore, by Proposition 3.11.5, $\chi^{-n}\psi$ is a Tate twist of an unramified character of G_K. But $G_K = I_F$ since now k is algebraically closed, and so the vanishing of the Hodge–Tate weight means that there is no Tate twist at all: $\chi^{-n}\psi = 1$.

We may now replace b with $t^{-n}b$ (as $t \in B_{\rm cris}^\times$) to reduce to the case $n = 0$, so $\psi = 1$. In particular, $g(b_r) = \psi(g)b_r = b_r$ for all $g \in G_K$, so $b_r \in (B_{\rm cris}^\times)^{G_K} = K_0^\times$. Assuming $r > 0$, we seek a contradiction. Consideration of terms in X-degree $r - 1$ in our formula for $\psi(g)b$ gives

$$b_{r-1} = \psi(g)b_{r-1} = g(b_{r-1}) + g(b_r)r\eta(g)t = g(b_{r-1}) + b_r r\eta(g)t.$$

Thus, $g(b_{r-1}) - b_{r-1} = -rb_r\eta(g)t$ with $c := -rb_r \in K_0^\times$ and any $g \in G_K$. Hence,

$$g(b_{r-1}/c) - b_{r-1}/c = \eta(g)t = g(X) - X,$$

so $X - b_{r-1}/c \in B_{\rm st}^{G_K} = K_0 \subseteq B_{\rm cris}$. But $b_{r-1} \in B_{\rm cris}$ and $X \notin B_{\rm cris}$, so we have a contradiction. $\square$

3.13 Semistable representations

Now we are ready to treat the last (but not least) type of p-adic representations:

Definition 3.13.1. A *semistable* representation of G_K is a p-adic one that is $B_{\rm st}$-admissible, and the full subcategory of these is denoted $\mathrm{Rep}_{\mathbb{Q}_p}^{\rm st}(G_K)$.

We may apply the formalism §3.2 to the functor $D_{\mathrm{st}}\colon \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \to \mathrm{Vec}_{K_0}$ defined by

$$D_{\mathrm{st}}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_K},$$

so $\dim_{K_0} D_{\mathrm{st}}(V) \le \dim_{\mathbb{Q}_p} V$ for all V and equality holds precisely when V is semistable. By using the additional structures on B_{st} (including the subspace filtration on $K \otimes_{K_0} B_{\mathrm{st}}$ from B_{dR}), we see that D_{st} is naturally valued in $\mathrm{MF}_K^{\phi,N}$.

Much like in our analysis of D_{cris}, we also see that the faithful functor

$$D_{\mathrm{st}}\colon \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(G_K) \to \mathrm{MF}_K^{\phi,N}$$

is an exact functor compatible with tensor products and duals (endowed with their natural filtrations). Likewise, the B_{st}-linear G_K-equivariant Frobenius-compatible and N-compatible semistable comparison isomorphism

$$\alpha\colon B_{\mathrm{st}} \otimes_{K_0} D_{\mathrm{st}}(V) \simeq B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V$$

is seen to be an isomorphism with respect to the filtration structures after scalar extension to K. (i.e., α_K and α_K^{-1} are filtration-compatible).

Lemma 3.13.2. *Crystalline representations are semistable, and* $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)$ *in* $\mathrm{MF}_K^{\phi,N}$ *for all* V*. If* V *is semistable and* D_{st} *has vanishing monodromy operator then* V *is crystalline.*

Proof. Since $B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$, we see that $D_{\mathrm{st}}(V)^{N=0} = D_{\mathrm{cris}}(V)$ in $\mathrm{MF}_K^{\phi,N}$ for every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$. In particular, if V is crystalline then for dimension reasons the K_0-linear inclusion $D_{\mathrm{cris}}(V) \subseteq D_{\mathrm{st}}(V)$ is an isomorphism in $\mathrm{MF}_K^{\phi,N}$. Thus, crystalline representations are semistable.

If V is semistable but $D_{\mathrm{st}}(V)$ has vanishing monodromy operator then $D_{\mathrm{cris}}(V) = D_{\mathrm{st}}(V)$ and this has K_0-dimension $\dim_{\mathbb{Q}_p} V$, so V is crystalline. □

It follows from this lemma that by working in the generality of semistable representations we can keep track of crystalline objects simply by observing whether or not N vanishes.

Lemma 3.13.3. *Semistable representations are de Rham, and if* V *is semistable then the natural injective map* $K \otimes_{K_0} D_{\mathrm{st}}(V) \to D_{\mathrm{dR}}(V)$ *is an isomorphism in* Fil_K*.*

Proof. If V is semistable then the natural injective map $K \otimes_{K_0} D_{\mathrm{st}}(V) \to D_{\mathrm{dR}}(V)$ has source with K-dimension $\dim_{\mathbb{Q}_p} V$ that is an upper bound on the K-dimension of the target, so it is a K-linear isomorphism. In particular, V is de Rham. By the definition of the filtration structure on $K \otimes_{K_0} B_{\mathrm{st}}$, this natural injective map is always a subobject inclusion in Fil_K, so when it is an isomorphism as K-vector space it must be an isomorphism in Fil_K. □

To summarize:

$$\text{crystalline} \Longrightarrow \text{semistable} \Longrightarrow \text{de Rham} \Longrightarrow \text{Hodge–Tate}.$$

As with crystalline representations in Proposition 3.11.2, there is a full faithfulness result for D_{st} on semistable representations and we can write down an inverse functor on the essential image of D_{st} on semistable representations, as follows. The equality $(\mathrm{Fil}^0 B_{\mathrm{st}})^{N=0,\phi=1} = (\mathrm{Fil}^0 B_{\mathrm{cris}})^{\phi=1} = \mathbb{Q}_p$ implies that the functor

$$V_{\mathrm{st}} \colon \mathrm{MF}_K^{\phi,N} \to \mathbb{Q}_p[G_K]\text{-mod}$$

defined by

$$\begin{aligned} V_{\mathrm{st}}(D) &= \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\phi=1} \\ &:= \ker(\delta(D) \colon (B_{\mathrm{st}} \otimes_{K_0} D)^{N=0,\phi=1} \to (B_{\mathrm{dR}} \otimes_K D_K)/\mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K)) \end{aligned}$$

provides an inverse to the functor D_{st} on semistable representations: there is a natural $\mathbb{Q}_p[G_K]$-linear isomorphism $V \simeq V_{\mathrm{st}}(D_{\mathrm{st}}(V))$ for all $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}$. (If we use the contravariant functor $D_{\mathrm{st}}^*(V) = \mathrm{Hom}_{\mathbb{Q}[G_K]}(V, B_{\mathrm{st}})$, then the inverse is given by the contravariant functor $V_{\mathrm{st}}^*(D) = \mathrm{Hom}_{Fil,\phi,N}(D, B_{\mathrm{st}})$.) In particular, as in the crystalline case in Proposition 3.11.2, we deduce via the comparison isomorphism:

Proposition 3.13.4. *The functor $D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}} \to \mathrm{MF}_K^{\phi,N}$ is fully faithful, with quasi-inverse on its essential image given by V_{st}.*

Note also that if $D \in \mathrm{MF}_K^{\phi,N}$ with $N_D = 0$ (i.e., $D \in \mathrm{MF}_K^{\phi}$) then $V_{\mathrm{st}}(D) = V_{\mathrm{cris}}(D)$ because $B_{\mathrm{st}}^{N=0} = B_{\mathrm{cris}}$.

The most important property concerns an intrinsic characterization of the essential images of these fully faithful functors. Unfortunately, the result can only be proven with finer properties, so we refer the proof of the following to [BC09, Thm. 9.3.4].

Theorem 3.13.5. *If $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}$ then $D_{\mathrm{st}} \in \mathrm{MF}_K^{\phi,N}$ is weakly admissible. In particular, if V is crystalline then $D_{\mathrm{cris}} \in \mathrm{MF}_K^{\phi}$ is weakly admissible.*

In the section on de Rham representations, we saw that the functor D_{dR} is not fully faithful, due to the de Rham property being insensitive to replacing G_K with $G_{K'}$ for a finite extension K'/K. This is best explained by a fundamental result independently due to Berger and André-Kedlaya-Mebkhout that relates p-adic differential equations with de Rham representations to prove Fontaine's potential semistability conjecture:

Theorem 3.13.6. *A p-adic representation V of G_K is de Rham if and only if it is potentially semistable in the sense that V is a semistable $G_{K'}$-representation for some finite extension K'/K.*

This theorem implies that although we cannot invert the functor D_{dR}, the gap between de Rham representations and semistable representations amounts to an insensitivity to finite extensions of K. However, keep in mind that $D_{\mathrm{dR}}(V)$ contains too little informations even to recover V as a $G_{K'}$-representation for some unknown finite extension of K'.

Finally, a fundamental result of Colmez and Fontaine [CF00, Thm. A] is that the fully faithful and exact tensor functor $D_{\mathrm{st}} \colon \mathrm{Rep}_{\mathbb{Q}_p}^{st} \to \mathrm{MF}_K^{\phi,N,\mathrm{w.a.}}$ is an *equivalence of categories*. That is, every weakly admissible filtered (ϕ, N)-module D over K is isomorphic as such to $D_{\mathrm{st}}(V)$ for a semistable p-adic representation V of G_K.

3.14 Weakly admissible filtered (ϕ, N)-modules with coefficients

We will conclude the section introducing the coefficients in our filtered (ϕ, N)-modules.

Let K and E be two finite extensions of $\mathbb{Q}_p$ (inside $\overline{\mathbb{Q}_p}$) and fix a uniformiser π of K (so we fix an embedding $K \otimes_K B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$). Let K_0 denote the maximal unramified extension of $\mathbb{Q}_p$ inside K, by $\sigma \colon K_0 \to K_0$ its Frobenius automorphism and by val_p its valuation normalized by $val_p(p) = 1$.

Most of p-adic representations we worked with were $\mathbb{Q}_p$-linear with a continuous G_K-action. Now, we will extend these when the *coefficients* are in E, i.e., they are E-vector space.

Definition 3.14.1. A *filtered* (ϕ, N, K, E)*-module* D is a finite free $K_0 \otimes_{\mathbb{Q}_p} E$-module of endowed with:

(a) a K_0-semilinear and E-linear automorphism ϕ;

(b) a (nilpotent) $K_0 \otimes_{\mathbb{Q}_p} E$-linear endomorphism N such that $N\phi = p\phi N$;

(c) extending by scalars $D_K := K \otimes_{K_0} D$ to K, a decreasing exhaustive and separated filtration $\{\mathrm{Fil}^i D_K\}_i$ of $K \otimes_{\mathbb{Q}_p} E$-modules.

Any filtered (ϕ, N, K, E)-module is also a filtered $(\phi, N, K, \mathbb{Q}_p)$-module (in the previous notation: filtered (ϕ, N)-module over K) forgetting its E-vector space structure. The category of filtered (ϕ, N, K, E)-modules has evident notions of short exact sequence, kernel, cokernel, tensor product and dual.

Let D be a filtered (ϕ, N, K, E)-module. If we denote by $d := \dim_{K_0} D$ its K_0-dimension, then $\bigwedge_{K_0}^d D$ is of dimension 1 over K_0. As in the classical case, we define the *Hodge number* of D to be

$$t_H(D) := \max\left\{ i \in \mathbb{Z} \;\middle|\; \mathrm{Fil}^i(K \otimes_{K_0} \bigwedge^d{}_{K_0} D \neq 0) \right\}$$

and the *Newton number* of D to be

$$t_N(D) := val_p\left(\frac{\phi(x)}{x}\right)$$

where $x \in \bigwedge_{K_0}^d D$, $x \neq 0$ and $\phi(x)/x \in K_0^\times$.

Definition 3.14.2. A filtered (ϕ, N, K, E)-module D is *weakly admissible* if so is the underlying filtered $(\phi, N, K, \mathbb{Q}_p)$-module.

For every filtered (ϕ, N, K, E)-module D, we associate

$$V_{\mathrm{st}} = (B_{\mathrm{st}} \otimes_{K_0} D)^{\phi=1, N=0} \bigcap \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_K D_K)$$

where ϕ on $B_{\mathrm{st}} \otimes_{K_0} D$ is defined as $\phi \otimes \phi$, N as $N \otimes id + id \otimes N$ and Fil^0 is the "tensor product filtration" on K.

Thanks to [CF00], we know that $V_{\mathrm{st}}(D)$ is a E-linear representation of G_K via its action on B_{st} and that the functor $D \mapsto V_{\mathrm{st}}(D)$ is an equivalence of categories between weakly admissible filtered (ϕ, N, K, E)-modules and the semistable p-adic representations of G_K with coefficients in E.

Let $k \geq 1$ be an integer, D a filtered (ϕ, N, K, E)-module and put:

$$V_{\mathrm{st},k}(D) := (B_{\mathrm{st}} \otimes_{K_0} D)^{\phi=p^{k-1}, N=0} \bigcap \mathrm{Fil}^{k-1}(B_{\mathrm{dR}} \otimes_K D_K).$$

This is again an E-linear representation of G_K.

Lemma 3.14.3. *For any filtered (ϕ, N, K, E)-module D, we have an isomorphism of $E[G_K]$-modules*

$$V_{\mathrm{st},k} \simeq V_{\mathrm{st}}(D)(k-1)$$

Remark 3.14.4. The above notation $V_{\mathrm{st}}(D)(k-1)$ denotes the $k-1$ twist of $V_{\mathrm{st}}(D)$.

Proof. an element of $V_{\mathrm{st}}(D)$ (resp. $V_{\mathrm{st},k}$) is sent to an element of $V_{\mathrm{st},k}(D)$ (resp. $V_{\mathrm{st}}(D)$) by multiplication (resp. division) by t^{k-1}, where t is a generator of $\mathbb{Z}_p(1)$ in B_{st}. □

Thanks to the above lemma and that the Hodge–Tate weights of $V_{\mathrm{st}}(D)$ are the opposite of the integers i such that $\mathrm{Fil}^i D_K \neq \mathrm{Fil}^{i+1} D_K$, we get:

Corollary 3.14.5. *The functor $D \mapsto V_{\mathrm{st},k}(D)$ is an equivalence of categories between weakly admissible filtered (ϕ, N, K, E)-modules D such that $\mathrm{Fil}^0 D_K = D_K$ and $\mathrm{Fil}^k D_K = D_K$ and p-adic semistable representation of G_K with coefficients E and Hodge–Tate weights inside $\{0, 1, \ldots, k-1\}$.*

Remark 3.14.6. Let D be a weakly admissible filtered (ϕ, N, K, E)-module. It is not true that in general $\mathrm{Fil}^i(D_K := K \otimes_{K_0} D)$ is a free $K \otimes_{K_0} E$-module.

When we are working with filtered (ϕ, N, K, E)-modules, to check the weak admissibility we need (by definition) to forget the E-vector space structure. However, for these modules there is an equivalent and more efficient criterion:

Proposition 3.14.7. *Let D be a filtered (ϕ, N, K, E)-module. Then D is weakly admissible if and only if $t_H(D) = t_N(D)$ and for any $K_0 \otimes_{\mathbb{Q}_p} E$-submodule D' of D stable under ϕ and N we have $t_H(D') \leq t_N(D')$, in which $K \otimes_{K_0} E$ is equipped with the induced filtration.*

Proof. We note that a such D' is in particular a filtered (ϕ, N, K, E)-module, thus the statement makes sense. The condition is certainly necessary, the point is to show that it is sufficient to have $t_H(D') \leq t_N(D')$ for any $K \otimes_{K_0} E$-submodule of D stable under ϕ and N to deduce the same inequality for any K_0-subspace stable under ϕ and N.

Recall that the category of filtered (ϕ, N, K, E)-modules such that any K_0-subspace stable under ϕ and N (with the induced filtration) verifies the inequality $t_H \leq t_N$ is closed by direct sum. Suppose the statement and let D' be counter-example with minimal dimension over K_0, i.e., a K_0-subspace of D stable under ϕ and N such that $t_H(D') > t_N(D')$ and $t_H(D'') \leq t_N(D'')$ for any $D'' \subsetneq D'$ stable under ϕ and N. Let $d := [E : \mathbb{Q}_p]$, x a primitive element of E over $\mathbb{Q}_p$ and $D'_{sat} = \sum_{i=0}^{d-1} x^i D' \subset D$: it is a $K_0 \otimes_{\mathbb{Q}_p} E$-submodule of D stable under ϕ and N with $t_H(D'_{sat}) \leq t_N(D'_{sat})$. Let $i_1, \ldots, i_r \in \{0, \ldots, d-1\}$ such that $D'_{sat} = \sum_{j=1}^{r} x^{i_j} D'$ with R minimal. Denote D_1 to be the kernel of $\oplus_{j=1}^{r} x^{i_j} D' \to D'_{sat}$ (with $K \otimes_{K_0} D_1$ equipped with the induced filtration) and $D_2 = (\oplus_{j=1}^{r} x^{i_j} D')/D_1$ with $K \otimes_{K_0} D_2$ equipped with the quotient filtration. Remark that $D_2 \simeq D'_{sat}$ if we forget the filtration but $\mathrm{Fil}^i(K \otimes_{K_0} D_2) \hookrightarrow \mathrm{Fil}^i(K \otimes_{K_0} D'_{sat})$, from which we get $t_H(D_2) \leq t_H(D'_{sat}) \leq t_N(D'_{sat}) = t_N(D_2)$. Let D'_j be the image of D_1 inside $x^{i_j} D'$ (via the projection on $x^{i_j} D'$) and endowed $K \otimes D'_j$ with the filtration induced by $K \otimes_{K_0} x^{i_j} D'$. We have $D'_j \subsetneq x^{i_j} D'$ except r otherwise it would not be minimal since we can omit $x^{i_j} D'$. Since multiplication by $x^{i_j} : D' \to x^{i_j} D'$ is an isomorphism of filtered (ϕ, N, K, E)-modules, we get that all the K_0-subspace of D'_j stable under ϕ and N (with the induced filtration) verify the inequality $t_H \leq t_N$, so does $\oplus_{j=1}^{r} x^{i_j} D'_j$ which leads to $t_H(D_1) \leq t_N(D_1)$. The additive properties of t_H and t_N imply both

$$t_H(\oplus_{j=1}^{r} x^{i_j} D'_j) = t_H(D_1) + t_H(D_2) \leq t_N(D_1) + t_N(D_2)$$

and

$$t_H(\oplus_{j=1}^{r} x^{i_j} D'_j) = r t_H(D') > r t_N(D') = r t_N(\oplus_{j=1}^{r} x^{i_j} D'_j) = t_N(D_1) + t_N(D_2)$$

which is a contradiction. $\square$

Now, we specialize in the case of semistable representations of $G_{\mathbb{Q}_p}$, i.e., $K = K_0 = \mathbb{Q}_p$, and we will denote this absolute Galois group by G_p.

In this case, a filtered $(\phi, N, \mathbb{Q}_p, E)$-module is simply a finite dimensional E-vector space D endowed with an E-linear automorphism ϕ, a nilpotent E-linear endomorphism N satisfying $N\phi = p\phi N$ and a decreasing exhaustive and separated filtration of E-subspaces $\{\mathrm{Fil}^i D\}_i$.

If D is a filtered $(\phi, N, \mathbb{Q}_p, E)$-module, so is $\bigwedge_E^d D$ for all nonzero integers d and put $t_H^E(D) := \max\{i \in \mathbb{Z} \mid \mathrm{Fil}^i(\bigwedge_E^{\dim_E D}) \neq 0\}$ and $t_N^E(D) := val(\frac{\phi(x)}{x})$, where $x \in \bigwedge_E^{\dim_E D} D$, $x \neq 0$ and $\phi(x)/x \in E^\times$.

Corollary 3.14.8. *(i) Let D be a filtered $(\phi, N, \mathbb{Q}_p, E)$-module. Then D is weakly admissible if and only if $t_H^E(D) = t_N^E(D)$ and for any E-subspace $D' \subseteq D$ stable under ϕ and N equipped with the induced filtration we have $t_H^E(D') \leq t_N^E(D')$.*

(ii) The functor $D \mapsto V_{\mathrm{st},k}(D) = \mathrm{Fil}^{k-1}(B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} D)^{\phi = p^{k-1}, N=0}$ is an equivalence of categories between weakly admissible filtered $(\phi, N, \mathbb{Q}_p, E)$-module D such that $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^k D = D$ and p-adic semistable representations of $G_{\mathbb{Q}_p}$ with coefficients in E and Hodge–Tate weights inside $\{0, \ldots, k-1\}$.

Proof. Point (i) follows from Proposition 3.14.7 and formulas $t_H^E = [E : \mathbb{Q}_p] t_H$, $t_N^E = [E : \mathbb{Q}_p] t_N$.
Point (ii) is a particular case of Corollary 3.14.5. □

Example 3.14.9. We give an example of semistable representations of G_p of dimension 2 over $\overline{\mathbb{Q}_p}$. Up to twist by a power of the cyclotomic character χ, we may assume that the Hodge–Tate weights are $(0, k-1)$ for some integer $k \geq 1$. The weakly admissible filtered modules associated to these representations corresponding via the functor $V_{\mathrm{st},k}$ are exactly of the form $D = \overline{\mathbb{Q}_p} e_1 \oplus \overline{\mathbb{Q}_p} e_2$ with $\mathrm{Fil}^i D = D$ for $i \leq 0$, $\mathrm{Fil}^{k-1} D \neq 0$ and $\mathrm{Fil}^i D = 0$ for $i \geq k$.

If $k = 1$ then $N = 0$ and, up to base change, ϕ is written in a unique way either as $\phi(e_1) = \mu e_1$, $\phi(e_2) = \mu e_2$ with $\mu \in \overline{\mathbb{Z}_p}^\times$ or as $\phi(e_1) = \mu_1 e_1 + e_2$, $\phi(e_2) = \mu_2 e_2$ with $\mu_1, \mu_2 \in \overline{\mathbb{Z}_p}^\times$.

If $k \geq 2$ then $\mathrm{Fil}^1 D = \cdots = \mathrm{Fil}^{k-1} D$ is a line and, up to base change, we have one and only one of the following (pairwise non-isomorphic) possibilities:

(1) $N = 0$ and the representation is reducible split:

$$\begin{cases} \phi(e_1) & = p^{k-1} \mu_1 e_1 \\ \phi(e_2) & = \mu_2 e_2 \\ \mathrm{Fil}^{k-1} D & = \overline{\mathbb{Q}_p} e_1 \\ \mu_1, \mu_2 & \in \overline{\mathbb{Z}_p}^\times \end{cases}$$

(2) $N = 0$ and the representation is reducible non-split:

$$\begin{cases} \phi(e_1) & = p^{k-1}(\mu_1 e_1 + e_2) \\ \phi(e_2) & = \mu_2 e_2 \\ \mathrm{Fil}^{k-1} D & = \overline{\mathbb{Q}_p} e_1 \\ \mu_1, \mu_2 & \in \overline{\mathbb{Z}_p}^{\times} \end{cases}$$

(3) $N = 0$ and the representation is irreducible:

$$\begin{cases} \phi(e_1) & = p^{k-1}\mu e_1 \\ \phi(e_2) & = -e_1 + \nu e_2 \\ \mathrm{Fil}^{k-1} D & = \overline{\mathbb{Q}_p} e_1 \\ \mu & \in \overline{\mathbb{Z}_p}^{\times} \\ \nu & \in \mathfrak{m}_{\overline{\mathbb{Z}_p}} \end{cases}$$

(4) $N \neq 0$. Fix $\pi \in \overline{\mathbb{Z}_p}$ such that $\pi^2 = p$:

$$\begin{cases} \phi(e_1) & = \pi^k \mu e_1 \\ \phi(e_2) & = \pi^{k-2}\mu e_2 \\ \mathrm{Fil}^{k-1} D & = \overline{\mathbb{Q}_p}(e_1 + \mathfrak{L} e_2) \\ N(e_1) & = e_2 \\ N(e_2) & = 0 \\ \mu & \in \overline{\mathbb{Z}_p}^{\times} \\ \mathfrak{L} & \in \overline{\mathbb{Q}_p} \end{cases}$$

This result can be extrapolated from Corollary 3.14.8, from [FM95, Thm. A] and [Bre98, §6.11]. In the rest of the thesis, we will denote $D = D(\mu_1, \mu_2)$ for $(\mu_1, \mu_2) \in \overline{\mathbb{Z}_p}^{\times} \times \overline{\mathbb{Z}_p}^{\times}$ if D comes from (1) or (2), $D = D(\mu, \nu)$ for $(\mu, \nu) \in \overline{\mathbb{Z}_p}^{\times} \times \mathfrak{m}_{\overline{\mathbb{Z}_p}}$ if D comes from (3) and $D = D(\mu, \mathfrak{L})$ for $(\mu, \mathfrak{L}) \in \overline{\mathbb{Z}_p}^{\times} \times \overline{\mathbb{Q}_p}$ if D comes from (4).

4 Integral p-adic Hodge theory

Since our aim is to deal with Galois deformation theory with artinian coefficients, it is useful to have a finer theory in which p-adic vector spaces are replaced with lattices or torsion modules.

For the entire section, we fix a choice of uniformiser π of K and let $E \in W(\mathbb{F})[u]$ be the minimal polynomial of π over K_0 (it is an Eisenstein polynomial of degree $e := [K : K_0]$). As we can do, we try to expose all the definition in greater generality, but sometimes we will deal directly assuming $K = K_0$, which is our final goal.

4.1 The ring $\widehat{A_{\mathrm{st}}}$

In order to produce G_K-stable lattices in p-adic semistable representations, we need the ring $\widehat{A_{\mathrm{st}}}$ which involves the Fontaine's rings A_{cris} and B_{st}.

In [Bre97, §2], Breuil defines the ring $\widehat{A_{\mathrm{st}}}$ as the p-adic completion of the divided power envelope of A_{cris}, in other words:

$$\widehat{A_{\mathrm{st}}} := \widehat{A_{\mathrm{cris}}\langle X\rangle} := \left\{ \sum_{n=0}^{+\infty} a_n \frac{X^n}{n!} \mid a_n \in A_{\mathrm{cris}},\ a_n \to 0 \text{ as } n \to +\infty \right\}.$$

For $n \in \mathbb{Z}$, let $\widehat{A_{\mathrm{st}}}^{\geq n} \subset \widehat{A_{\mathrm{st}}}$ be the subring of elements such that $a_0 = \cdots = a_{n-1} = 0$. We endow the ring $\widehat{A_{\mathrm{st}}}$ with the following structures:

- $\mathrm{Fil}^i \widehat{A_{\mathrm{st}}} := \sum_{j=0}^{i} \mathrm{Fil}^{i-j} A_{\mathrm{cris}} \cdot \frac{u^i}{i!} + \widehat{A_{\mathrm{st}}}^{\geq i+1}$;
- for $g \in G_K$, $g(\sum a_n \frac{X^n}{n!}) = \sum g(a_n) \frac{g(X)}{n!}$, where $g(X) := [\varepsilon_{\underline{q}}(g)]X + [\varepsilon_{\underline{q}}(g)] - 1$ with $\varepsilon_{\underline{q}}(g) = g(\underline{q})/\underline{q}$ is as 3.12 and the choice $\underline{q}^{(0)} = \pi$;
- $\phi(\sum a_n \frac{X^n}{n!}) = \sum \phi(a_n) \frac{\phi(X)^n}{n!}$, where $\phi(X) := (1+X)^p - 1$;
- $N(\sum a_n \frac{X^n}{n!}) = \sum a_n \frac{N(X)^n}{n!}$, where $N(X) := 1 + X$.

The relations between these structures are the following:
(i) G_K preserves the filtration and commutes with ϕ and N;
(ii) $N\phi = p\phi N$;
(iii) $\phi(\mathrm{Fil}^i \widehat{A_{\mathrm{st}}}) \subset p^i \widehat{A_{\mathrm{st}}}$ if $i \leq p-1$.

A result due to Kato in [Kat94, §3] links this ring with the semistable period ring B_{st}:

Lemma 4.1.1.
- *We have* $B^+_{\mathrm{cris}}[\log(1+X)] \simeq \{x \in \widehat{A_{\mathrm{st}}}[1/p] | N^n(x) = 0$ *for some* $n > 0\}$.
- *The map* $\widehat{A_{\mathrm{st}}}[1/p] \to B^+_{\mathrm{dR}}$, $X \mapsto \frac{[\pi]}{\pi} - 1$ *induces an isomorphism* $B^+_{\mathrm{cris}}[\log(1+X)] \xrightarrow{\simeq} B^+_{\mathrm{st}}$ *which is compatible with* ϕ, N *and* G_K *(but only induces inclusions* $\mathrm{Fil}^i(B^+_{\mathrm{cris}}[\log(1+X)]) \subsetneq \mathrm{Fil}^i B^+_{\mathrm{st}}$*).*

As a summary, B_{st}^+ consists of all elements in $\widehat{A_{\mathrm{st}}}[1/p]$ for which N is nilpotent, except that its filtration is finer than the induced filtration.

4.2 The Breuil's ring S

As with the other period's rings, one needs to compute the G_K-invariant elements of $\widehat{A_{\mathrm{st}}}$. Before of this, we introduce the Breuil's ring S ([Bre97]) which will play a big role in the rest of the dissertation.

Let $W(\mathbb{F})[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1}$ be the subring of $K_0[u]$ generated over $W(\mathbb{F})[u]$ (u is a indeterminate) by the set $\{E^i/i!\}_{i\geq 1}$ (this is the *divided power envelope* of $W(\mathbb{F})[u]$ with respect to the ideal $E(u)\cdot W(\mathbb{F})[u]$). Clearly this is ring is $W(\mathbb{F})$-flat. Further, there is an evident surjective map

$$W(\mathbb{F})[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1} \twoheadrightarrow \mathcal{O}_K \tag{4.1}$$

defined via $u \mapsto \pi$ with kernel generated by all $E^i/i!$. Let S be the p-adic completion of $W(\mathbb{F})[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1}$ and let $\mathrm{Fil}^1 S \subseteq S$ be the ideal that is (topologically) generated by all $E^i/i!$. We view S as a topological ring via its (separated and complete) p-adic topology. The ring S is local and $W(\mathbb{F})$-flat (but not noetherian) and the map (4.1) induces an isomorphism

$$S/\mathrm{Fil}^1 S \xrightarrow{\simeq} \mathcal{O}_K.$$

Denoting by σ the Frobenius automorphism on $W(\mathbb{F})$, we endow S with the following structures:

- a continuous σ-linear Frobenius denoted $\phi\colon S \to S$ such that $\phi(u) = u^p$;
- a continuous linear derivation $N\colon S \to S$ such that $N(u) = -u$;
- a decreasing filtration $\{\mathrm{Fil}^i S\}_{i\geq 0}$, where $\mathrm{Fil}^i S$ is the p-adic completion of $\sum_{j\geq i} S\frac{E(u)^j}{j!}$.

Note that $N\phi = p\phi N$, $N(\mathrm{Fil}^{i+1} S) \subset \mathrm{Fil}^i S$ and $\phi(\mathrm{Fil}^i S) \subset p^i S$ for $i \in \{0,\dots,p-1\}$. As one can note the similarity with $\widehat{A_{\mathrm{st}}}$, we mention the following lemma (which is due to Breuil, see [Bre97, §4.2]):

Lemma 4.2.1. *The continuous $W(\mathbb{F})$-linear map $S \to \widehat{A_{\mathrm{st}}}$ defined by $u \mapsto [\underline{\pi}](1+X)^{-1}$ induces a G_K-equivariant ϕ and N-stable filtration-invariant isomorphism $S \xrightarrow{\simeq} \widehat{A_{\mathrm{st}}}^{G_K}$.*

4.3 Strongly divisible modules

Let D be an object in $\mathrm{MF}_K^{\phi,N,\text{w.a.}}$ and assume $\mathrm{Fil}^0 D_F = D_F$ (this is harmless since, up to twist, one can always assume Galois representations have positive Hodge–Tate weights). In this case, we say that D is *effective*. Let

$$\mathcal{D} := S[1/p] \otimes_{K_0} D$$

be an $S[1/p]$-module endowed with endomorphisms $\phi := \phi_S \otimes \phi_D$, $N := N \otimes id + id \otimes N$ and a decreasing filtration given inductively by $\mathrm{Fil}^0\mathcal{D} := \mathcal{D}$ and

$$\mathrm{Fil}^{i+1}\mathcal{D} := \{x \in \mathcal{D} | N(x) \in \mathrm{Fil}^i\mathcal{D} \text{ and } f_\pi(x) \in \mathrm{Fil}^{i+1}D_F\}$$

where $f_\pi \colon \mathcal{D} \twoheadrightarrow D_F$ is the surjective map defined by $s(u) \otimes x \mapsto s(\pi)x$.

Example 4.3.1. In the following examples, we consider $F = K_0 = \mathbb{Q}_p$ and we choose $\pi = p$ (so that our Eisenstein polynomial is $E(u) = u-p$). Keeping in mind the example 3.14.9, we compute the module $\mathcal{D}$ for each case.

Assume D is as in Example 3.14.9 (1), (2) or (3) (corresponding to the crystalline case), then one finds:

$$\begin{aligned} \mathrm{Fil}^i\mathcal{D} &= \mathrm{Fil}^i S[1/p] \otimes D + S[1/p] \cdot e_1 & &\text{if } i \leq k-1; \\ \mathrm{Fil}^i\mathcal{D} &= \mathrm{Fil}^i S[1/p] \otimes D + \mathrm{Fil}^{i-k+1} S[1/p] \cdot e_1 & &\text{if } i > k-1. \end{aligned}$$

Similarly, if D is as in Example 3.14.9 (4) (corresponding to the semistable non-crystalline case) and $k = 2$, we have:

$$\begin{aligned} \mathrm{Fil}^i\mathcal{D} &= \mathrm{Fil}^i S[1/p] \otimes D + S[1/p] \cdot (e_1 + \mathfrak{L} e_2) & &\text{if } i \leq 1; \\ \mathrm{Fil}^i\mathcal{D} &= \mathrm{Fil}^i S[1/p] \otimes D + \mathrm{Fil}^{i-k+1} S[1/p] \cdot (e_1 + \mathfrak{L} e_2) & &\text{if } i > 1. \end{aligned}$$

More in general, for $k-1 \geq 3$, the filtrations in the semistable case are more involved.

The filtered module $\mathcal{D}$ has the advantage over the filtered module D that all of its data are defined at the same level (no need to extends scalars to K).

Now we are ready to define integral structures inside the $\mathcal{D}$'s.

Definition 4.3.2. Let $D \in \mathrm{MF}_K^{\phi,N,\text{w.a.}}$ such that $\mathrm{Fil}^k D_F = 0$ with $k \leq p$. A *strongly divisible module* in $\mathcal{D}$ is a finite free S-submodule $\mathcal{M}$ of $\mathcal{D}$ endowed with an S-submodule $\mathrm{Fil}^{k-1}\mathcal{M} := \mathcal{M} \cap \mathrm{Fil}^{k-1}\mathcal{D}$ such that:

(1) $\mathcal{M}[1/p] \xrightarrow{\simeq} \mathcal{D}$;

(2) $\mathcal{M}$ is stable under ϕ and N;

(3) $\phi(\mathrm{Fil}^{k-1}\mathcal{M}) \subseteq p^{k-1}\mathcal{M}$.

We will spend the next section giving non-trivial examples of such modules. So, here is a trivial one.

Example 4.3.3. Let $D = K_0[0]$ be the trivial filtered module (as in 3.10.7). Then S is a strongly divisible module in $\mathcal{D} = S[1/p]$.

There is also an alternative definition of strongly divisible module, from which we derive the definition of 'torsion analogue of a strongly divisible module'.

First, in [Bre98, §2.1.2] Breuil gives the definition of the following category:

Definition 4.3.4. Let $1 \leq k \leq p-1$. An object in the category $'\mathfrak{M}^{k-1}$ is an S-module $\mathcal{M}$ endowed with an S-submodule $\mathrm{Fil}^{k-1}\mathcal{M} \subset \mathcal{M}$ and endomorphisms $\phi_{k-1}\colon \mathrm{Fil}^{k-1} \to \mathcal{M}$, $N\colon \mathcal{M} \to \mathcal{M}$ such that:

- $\mathrm{Fil}^{k-1}\mathcal{M}$ contains $(\mathrm{Fil}^{k-1}S)\mathcal{M}$;
- $\phi_{k-1}(sx) = \phi_S(s)\phi_{k-1}(x)$ for all $s \in S$, $x \in \mathrm{Fil}^{k-1}\mathcal{M}$ and $\phi_{k-1}(sx) = \frac{\frac{\phi_S}{p^{k-1}}(s)}{\frac{\phi_S}{p^{k-1}}(E(u)^{k-1})}\phi_{k-1}(E(u)^{k-1}x)$ for all $s \in \mathrm{Fil}^{k-1}S$, $x \in \mathcal{M}$;
- $N(sx) = N(s)x + sN(x)$, $(\mathrm{Fil}^1 S)N(\mathrm{Fil}^{k-1}\mathcal{M}) \subset \mathrm{Fil}^{k-1}\mathcal{M}$ and $\phi_{k-1} \circ (E(u)N|_{\mathrm{Fil}^{k-1}}) = \frac{\phi_S}{p}(E(u))N \circ \phi_{k-1}$.

Morphisms in $'\mathfrak{M}^{k-1}$ are S-linear maps preserving all the structures.

For any $\mathcal{M} \in' \mathfrak{M}^{k-1}$, define $\phi\colon \mathcal{M} \to \mathcal{M}$ by $\phi(x) := \frac{p^{k-1}}{\phi_S(E(u)^{k-1})}\phi_{k-1}(E(u)^{k-1}x)$. If $\mathcal{M}$ has no p-torsion, the knowledge of $\phi_{k-1}\colon \mathrm{Fil}^{k-1}\mathcal{M} \to \mathcal{M}$ is equivalent to that of ϕ (via $\phi_{k-1} = \frac{\phi}{p^{k-1}}|_{\mathrm{Fil}^{k-1}}$).

Example 4.3.5. $\widehat{A_{\mathrm{st}}}$, $\widehat{A_{\mathrm{st}}}/p^n\widehat{A_{\mathrm{st}}}$, $\widehat{A_{\mathrm{st}}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ and $\widehat{A_{\mathrm{st}}} \otimes_{W(\mathbb{F})} K_0/W(\mathbb{F})$ are objects of $'\mathfrak{M}^{k-1}$ for any $k-1 < p$.

Now, we are ready to (re-)define what a strongly divisible module is ([Bre98, Def. 4.1.1.1])

Definition 4.3.6. A *strongly divisible module* is an object $\mathcal{M}$ in $'\mathfrak{M}^{k-1}$ satisfying in addition:

- the S-module $\mathcal{M}$ is free of finite rank;
- $\mathrm{Fil}^{k-1}\mathcal{M} \cap p\mathcal{M} = p\mathrm{Fil}^{k-1}\mathcal{M}$;
- $\phi_{k-1}(\mathrm{Fil}^{k-1}\mathcal{M})$ spans $\mathcal{M}$.

Of course, the first definition implies the second, but it is not immediate to see the converse. In fact, given a strongly divisible module $\mathcal{M}$ as in the second definition, one may consider $\mathcal{D} := K_0 \otimes_{W(\mathbb{F})} \mathcal{M}$ endowed with the same structure as above (see [Bre98, §4.1.1] for more detail). The question is whether there exists a weakly admissible module D such that $\mathcal{D} = S[1/p] \otimes_{K_0} D$, but this is actually what Breuil ensures us in [Bre97, Thm. 6.1.1], proving that the "category of $\mathcal{D}$'s" is equivalent to the category $\mathrm{MF}_K^{\phi,N}$.

These objects are the integral counterpart of weakly admissible modules for G_K-stable lattices (at least for low weights and for $K = K_0$). We will see this equivalence in the following, when we specialize to strongly divisible modules with coefficients. Before of this, we introduce a torsion analogue of these modules.

4.4 Torsion version of strongly divisible modules

As with Galois lattices, it is tempting to reduce strongly divisible modules modulo p. As in [Bre98, §2.1.2], we define them as follows.

Definition 4.4.1. For $1 \leq k \leq p-1$, an object in the full subcategory $\mathfrak{M}^{k-1} \subset {}'\mathfrak{M}^{k-1}$ is an S-module $\mathcal{N}$ satisfying in addition:

- $\mathcal{N}$ is of the form $\oplus_{n\geq 1}(S/p^nS)^{r_n}$ for integers r_n almost all zero;
- $\phi_{k-1}(\mathrm{Fil}^{k-1}\mathcal{N})$ spans $\mathcal{N}$.

Remark 4.4.2. If $K = K_0$, one can prove that the category $\mathfrak{M}^{k-1}$ is abelian, see [Bre98, Thm. 2.1.2.2].

If $\mathcal{M}$ is a strongly divisible module of weight $\leq k-1$ then for any $n \in \mathbb{Z}$ the S-module $\mathcal{M}/p^n\mathcal{M}$ is naturally an object of $\mathfrak{M}^{k-1}$ endowed with the quotient filtration $\mathrm{Fil}^{k-1}(\mathcal{M}/p^n\mathcal{M}) = \mathrm{Fil}^{k-1}\mathcal{M}/p^n\mathrm{Fil}^{k-1}\mathcal{M}$ and ϕ and N the reduction modulo p^n.

In [Bre98, §3.1.3] Breuil defines a contravariant exact and faithfully flat functor $V_{\mathrm{st}}\colon \mathfrak{M}^{k-1} \to \mathrm{Rep}^f_{\mathbb{Z}_p}(G_K)$ into the category of finite length $\mathbb{Z}_p$-modules endowed with a linear and continuous action of G_K defined by

$$V_{\mathrm{st}}(\mathcal{N}) = \mathrm{Hom}_{Fil^{k-1},\phi_{k-1},N}(\mathcal{N}, \widehat{A_{\mathrm{st}}} \otimes_{\mathbb{Z}_p} K_0/W(\mathbb{F}))$$

with action $(g.f)(x) = g(f(x))$ for $g \in G_K, f \in V_{\mathrm{st}}(\mathcal{N})$.

By construction, the obtained representations are annihilated by the same powers of p which annihilate $\mathcal{N}$. The following theorem is the goal of [Bre98, Thm. 3.1.3.1]:

Theorem 4.4.3. *Suppose $K = K_0$ and $k-1 \leq p-2$. Then the functor V_{st} is exact and fully faithful. In particular, if $\mathcal{N} = \oplus_{i\geq 1}S/p^{n_i}S$ then $V_{\mathrm{st}}(\mathcal{N}) = \oplus_{i\geq 1}\mathbb{Z}_p/p^{n_i}\mathbb{Z}_p$.*

If $\mathcal{M}$ is a strongly divisible module in $\mathfrak{M}^{k-1}$, put

$$V_{\mathrm{st}}(\mathcal{M}) = \mathrm{Hom}_{Fil^{k-1},\phi_{k-1},N}(\mathcal{M}, \widehat{A_{\mathrm{st}}}).$$

As corollary of the previous theorem, we easily get:

Corollary 4.4.4. *Suppose $K = K_0$ and $k-1 \leq p-2$. If $\mathcal{M}$ is a strongly divisible module of weight $\leq k-1$, then*

$$V_{\mathrm{st}}(\mathcal{M})/p^nV_{\mathrm{st}}(\mathcal{M}) \simeq V_{\mathrm{st}}(\mathcal{M}/p^n\mathcal{M})$$

4.5 Fontaine-Laffaile modules

If we restrict to the case $k = 2$ and $K = K_0$, the objects of $\mathfrak{M}^1$ killed by p can be easier described as simple objects in another category, the category of Fontaine-Laffaille modules.

Suppose $K = K_0$ and let D be a weakly admissible filtered (ϕ, N, F, E)-module which is effective ($\mathrm{Fil}^0 D = D$). Instead of balancing the filtration with the ring S, we can directly consider its lattices, but we shall keep the original structures.

Definition 4.5.1. A *strongly divisible* $\mathcal{O}_E$*-lattice of Fontaine-Laffaille* in D is a free $\mathcal{O}_K \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-submodule $\Lambda \subset D$ endowed with the induced filtration such that:

(a) $\Lambda[1/p] = D$;

(b) $\phi(\mathrm{Fil}^i \Lambda) \subset p^i \Lambda$ (so, in particular, Λ is ϕ-stable by taking $i = 0$);

(c) $\sum_{i \geq 0} p^{-i} \phi(\mathrm{Fil}^i \Lambda) = \Lambda$.

These strongly divisible lattices are useful because Fontaine and Laffaile proved in [FL82] the following

Theorem 4.5.2. *There are exact quasi-inverse anti-equivalences between the category of strongly divisible lattices* Λ *with* $\mathrm{Fil}^p \Lambda = 0$ *and the category of* $\mathbb{Z}_p[G_K]$*-lattices* T *in crystalline* G_K*-representations with Hodge–Tate weights inside* $\{0, \ldots, p-1\}$

We now wish to apply this theory to torsion representations. To do this, we need a torsion replacement for strongly divisible lattices.

Definition 4.5.3. A *Fontaine-Laffaile module* over $\mathcal{O}_K$ is a finite length $\mathcal{O}_K$-module M equipped with a finite and separated decreasing filtration $\{\mathrm{Fil}^i M\}_i$ and a σ-semilinear endomorphism $\phi^i_M \colon \mathrm{Fil}^i M \to M$ such that

(a) $\phi^i_M|_{\mathrm{Fil}^{i+1} M} = p\phi^{i+1}_M$ for all $i \geq 0$;

(b) $\sum_i \phi^i_M(\mathrm{Fil}^i M) = M$;

(c) $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^k M = 0$ for some $k > 1$.

A morphism between two objects is an $\mathcal{O}_E$-linear homomorphism compatible with the filtrations and commuting with the ϕ^i_M's. We denote the category by $\mathrm{MF}^{f,k-1}_{\mathrm{tor}}$.

In [FL82, 1.8 and 3.2], they show that the category $\mathrm{MF}^{f,k-1}_{\mathrm{tor}}$ is abelian. The simple objects of $\mathrm{MF}^{f,k-1}_{\mathrm{tor}}$ are those modules which are killed by p, in other words they are $\mathbb{F}_p$-vector spaces; we denote this category by $\widetilde{\mathrm{MF}}^{f,k-1}_{\mathrm{tor}}$.

Example 4.5.4. If Λ is a strongly divisible lattice, then for each $n > 0$ we obtain a Fontaine-Laffaille module M by setting $M = \Lambda/p^n\Lambda$, taking $\mathrm{Fil}^i M$ to be the image of $\mathrm{Fil}^i\Lambda$ under the natural quotient map, and letting $\phi^i_M := p^{-i}\phi_\Lambda|_{\mathrm{Fil}^i M}$.

Let M be an object in $\mathrm{MF}^{f,k-1}_{\mathrm{tor}}$, we may associate an object $\mathcal{F}^{k-1}(M)$ of $\mathfrak{M}^{k-1}$ by setting:

$$\Bigg(\mathcal{F}^{k-1}(M) := S \otimes_{\mathcal{O}_K} M, \mathrm{Fil}^{k-1}\mathcal{F}^{k-1}(M) := \sum_{j=0}^{k-1} \mathrm{Fil}^{k-1-j} S \otimes_{\mathcal{O}_K} \mathrm{Fil}^j M,$$
$$\phi_{k-1} = \sum_{j=0}^{k-1} \phi_{k-1-j} \otimes \phi_j,\ N = N \otimes id\Bigg)$$

Thus, we have defined a functor $\mathcal{F}^{k-1}\colon \mathrm{MF}^{f,k-1}_{\mathrm{tor}} \to \mathfrak{M}^{k-1}$ which satisfies the following properties ([Bre98, Prop. 2.4.1.1]):

Proposition 4.5.5. *The functor $\mathcal{F}^{k-1}$ is exact and fully faithful.*

Hence, we may consider the category $\mathrm{MF}^{f,k-1}_{\mathrm{tor}}$ as a full subcategory of $\mathfrak{M}^{k-1}$. Another useful consequence of this Proposition is that the simple objects $\widetilde{\mathrm{MF}}^{f,k-1}_{\mathrm{tor}}$ are equal to those simple objects in $\mathfrak{M}^{k-1}$ ([Bre98, Cor. 2.4.2.2]).

Thanks to the full faithfulness of V_{st}, we want to find an analogue of $H^1_{\mathbb{F}_p}(G_p, \omega) = \mathrm{Ext}^1_{\mathbb{F}_p[G_p]}(1, \omega)$ in the subcategory of simple objects in $\mathfrak{M}^1$, so equivalently in $\widetilde{\mathrm{MF}}^{f,1}_{\mathrm{tor}}$. However, to do this, I need to enlarge the category considering also a monodromy operator.

Definition 4.5.6. We denote by NMF^1 the category of objects in $\mathrm{MF}^{f,1}_{\mathrm{tor}}$ which are (finite-dimensional) $\overline{\mathbb{F}_p}$-vector spaces equipped with, in addition, a linear map $N\colon M \to M$ satisfying $N\phi^0_M = 0$ and $N\phi^1_M = \phi^0_M N$.

As above, we have that NMF^1 is equivalent to the subcategory of objects in $\mathfrak{M}^1$ killed by p. In this category we have two distinguish objects:

- the object $\overline{\mathbb{F}_p}(1) := (M = \overline{\mathbb{F}_p}e_1, \mathrm{Fil}^1 M = \overline{\mathbb{F}_p}e_1, \phi_0(e_1) = 0, \phi_1(e_1) = e_1, N = 0)$ corresponds via the functor V_{st} to the Galois character which factors through $\mathrm{Gal}(\mathbb{Q}_p[x]/\mathbb{Q}_p)$ with $x \neq 0$ and $x^p = -px$, which is exactly the mod p cyclotomic character ω (see §1.5).
- the object $\overline{\mathbb{F}_p}(2) := (M = \overline{\mathbb{F}_p}e_2, \mathrm{Fil}^1 M = 0, \phi_0(e_2) = e_2, 0, N = 0)$ is sent to the trivial character.

Lemma 4.5.7. $\dim_{\overline{\mathbb{F}_p}} \mathrm{Ext}^1_{\mathrm{NMF}^1}(\overline{\mathbb{F}_p}(1), \overline{\mathbb{F}_p}(2)) = 2.$

Proof. Any extension $0 \to \overline{\mathbb{F}_p}(2) \to M \to \overline{\mathbb{F}_p}(1) \to 0$ can be written $M = \overline{\mathbb{F}_p}e_1 + \overline{\mathbb{F}_p}e_2$ with $\mathrm{Fil}^1 M = \overline{\mathbb{F}_p}e_1$, $\phi_0(e_2) = e_2$, $\phi_1(e_1) = e_1 + \lambda e_2$, $N(e_1) = \mu e_2$ and $N(e_2) = 0$. Hence we see they are parametrized by $(\lambda, \mu) \in \overline{\mathbb{F}_p}^2$. $\square$

By full faithfulness of V_{st} we get

$$\mathrm{Ext}^1_{\mathrm{NMF}^1}(\overline{\mathbb{F}_p}(1), \overline{\mathbb{F}_p}(2)) \xrightarrow{\simeq} \mathrm{Ext}^1_{\overline{\mathbb{F}_p}[G_p]}(1, \omega)$$

Lemma 4.5.8. *The extension with $N = 0$ correspond to the peu ramifiées ones. The extensions with $N \neq 0$ correspond to the très ramifiées ones.*

Proof. By the full faithfulness, it is enough to prove the first statement. We can assume that the extension is not trivial. Up to isomorphism, we then have $M = \overline{\mathbb{F}_p}e_1 + \overline{\mathbb{F}_p}e_2$, $\mathrm{Fil}^1 M = \overline{\mathbb{F}_p}e_1$, $\phi_0(e_2) = e_2$, $\phi_1(e_1) = e_1 + e_2$ and $N = 0$. A careful analysis of $V_{\mathrm{st}}(M)$ shows that the Galois action on it factors through $\mathrm{Gal}(K/\mathbb{Q}_p)$ where K is the compositum of $\mathbb{Q}_p[x_1, x_2]$ for all $x_1, x_2 \in \overline{\mathbb{Q}_p}$ such that $x_1^p = (-p)(x_1 + x_2)$ and $x_2^p = x_2$. If $x_2 = 0$, we have $\mathbb{Q}_p[x_1] = \mathbb{Q}_p[\sqrt[p]{1}]$. If $x_2 \neq 0$, the equations imply $x_2 \in \mathbb{Z}_p^\times$ and, replacing x_1 by x_1/x_0, we have $(x_1 + 1)^p = (1 - p)w$ where $w \in 1 + px_1^2\mathbb{Z}_p[x_1] = (1 + z_1\mathbb{Z}_p[x_1])^p$. Hence $\mathbb{Q}_p[x_1]$ contains $\mathbb{Q}_p[\sqrt[p]{1-p}]$ if $x_2 \neq 0$. Since these two extensions have degree p, they are equal and $K = \mathbb{Q}_p[\sqrt[p]{1}, \sqrt[p]{1-p}]$, as in the peu ramifié case. $\square$

Therefore, when we will deal with strongly divisible modules and we are interested on their torsion analogue, we will use their easier description as objects in $\widetilde{\mathrm{MF}}^{f,k-1}_{\mathrm{tor}}$ "without the complication given by the ring S".

4.6 Some strongly divisible modules with coefficients

In this section, our goal is not to introduce a systematic theory of strongly divisible modules with coefficients, but only to recall the strictly needed for our results, as Breuil and Mézard did in §3.2 of [BM02, §3]. So, we will only describe the theory without prove the statements, we invite to refer the reader to *loc. cit.* for the proofs.

Put $K = K_0 = \mathbb{Q}_p$, let $\mathcal{O}$ be the ring of integers of a finite extension E of $\mathbb{Q}_p$ and denote by $\mathbb{F}$ its residue field. Fix moreover an integer $k \in \{1, \dots, p-1\}$ and an uniformiser p of $\mathbb{Q}_p$ (with Eisenstein polynomial $E(u) = u - p$) in order to use B_{st} and $\widehat{A_{\mathrm{st}}}$.

Fix R a local noetherian flat $\mathcal{O}$-algebra complete for the topology of the maximal ideal $\mathfrak{m}_R$ with residue field $\mathbb{F}$. Let S_R be the $\mathfrak{m}_R$-adic completion of the divided power envelope of $R[u]$ with respect to the ideal $u \cdot R[u]$, in other words:

$$S_R := \widehat{R\langle u \rangle} = \left\{ \sum_{j=0}^{+\infty} r_j \frac{u^j}{j!} \;\middle|\; r_j \in R, r_j \to 0 \text{ as } j \to +\infty \text{ inside } R \right\}.$$

We endow S_R with a positive decreasing filtration of S_R-submodules $\mathrm{Fil}^i S_R$ (i integer), with a Frobenius operator ϕ and an R-linear derivation N as follows:

$$\mathrm{Fil}^i S_R = \left\{ \sum_{j=i}^{+\infty} r_j \frac{(u-p)^j}{j!} \;\middle|\; r_j \in R, r_j \to 0 \text{ as } j \to +\infty \right\}$$
$$\phi\left(\sum_{j=0}^{\infty} r_j \frac{u^j}{j!}\right) = \sum_{j=0}^{\infty} r_j \frac{u^{jp}}{j!}$$
$$N\left(\sum_{j=0}^{\infty} r_j \frac{u^j}{j!}\right) = -\sum_{j=0}^{\infty} r_j \frac{u^j}{(j-1)!}$$

Remark 4.6.1. The Breuil's ring S corresponds to the ring $S_{\mathbb{Z}_p}$ with the above notation.

Thus, we have $N\phi = p\phi N$, $N(\mathrm{Fil}^i S_R) \subset \mathrm{Fil}^{i-1} S_R$ for any i and $\phi(\mathrm{Fil}^i S_R) \subset p^i S_R$ for any $i \leq p-1$ (note that $\phi((u-p)^i) \in p^i S_R^\times$ for $0 \leq i \leq p-1$. If I is an ideal of R, note that $IS_R = \{\sum_{j=0}^{\infty} r_j \frac{u^j}{j!} | r_j \in I, r_j \to 0 \text{ as } j \to \infty\}$ by Artin-Rees' lemma. Now, we generalize the concept of strongly divisible module over any local complete noetherian flat $\mathcal{O}$-algebra R.

Definition 4.6.2. A *strongly divisible module over* R is a finite free S_R-module $\mathcal{M}$ endowed with a S_R-submodule $\mathrm{Fil}^{k-1}\mathcal{M}$ and with endomorphisms $\phi, N \colon \mathcal{M} \to \mathcal{M}$ satisfying:

(i) $\mathrm{Fil}^{k-1}\mathcal{M}$ contains $(\mathrm{Fil}^{k-1} S_R)\mathcal{M}$;

(ii) $\mathrm{Fil}^{k-1}\mathcal{M} \cap I\mathcal{M} = I\mathrm{Fil}^{k-1}\mathcal{M}$ for any ideal $I \subset R$;

(iii) $\phi(sx) = \phi(s)\phi(x)$ for any $s \in S_R, x \in \mathcal{M}$;

(iv) $\phi(\mathrm{Fil}^{k-1}\mathcal{M})$ is contained in $p^{k-1}\mathcal{M}$ and generates it over S_R;

(v) $N(sx) = N(s)x + sN(x)$ for any $s \in S_R, x \in \mathcal{M}$;

(vi) $N\phi = p\phi N$;

(vii) $(\mathrm{Fil}^1 S_R)N(\mathrm{Fil}^{k-1}\mathcal{M}) \subset \mathrm{Fil}^{k-1}\mathcal{M}$.

If $\mathcal{M}$ is a strongly divisible R-module then $\phi \colon \mathcal{M} \to \mathcal{M}$ is automatically injective (regarded over $\mathcal{M}[1/p]$).

Example 4.6.3. 1. Let M be a strongly divisible $\mathbb{Z}_p$-lattice of Fontaine-Laffaile (see Definition 4.5.1) equipped with an $\mathcal{O}$-action (so that M

is also a free $\mathcal{O}$–module) and an $\mathcal{O}$-linear endomorphism N such that $N(\mathrm{Fil}^i M) \subset \mathrm{Fil}^{i-1} M$ and $N\phi = p\phi N$, then

$$\Big(\mathcal{M} = M \otimes_{\mathcal{O}} S_R,\ \mathrm{Fil}^{k-1}\mathcal{M} = \sum_{i=0}^{k-1} \mathrm{Fil}^i M \otimes \mathrm{Fil}^{k-1-i} S_R,$$
$$\phi = \sum_{i=0}^{k-1} p^i \phi_i \otimes \phi,\ N = N \otimes id + id \otimes N\Big)$$

is a strongly divisible R-module.

2. Let $h \in \{0, \ldots, k-1\}$ and $r \in R^\times$ then

$$\Big(\mathcal{M} = S_R e_1,\ \mathrm{Fil}^{k-1}\mathcal{M} = (\mathrm{Fil}^{k-1} S_R)e_1 + S_R(u-p)^h e_1,$$
$$\phi(e_1) = p^{k-1-h} w e_1,\ N(e_1) = 0\Big)$$

is a strongly divisible R-module of rank 1. In particular, if R is an integral domain then all the strongly divisible R-modules of rank 1 are of this form.

3. In [BM02] one can find non-trivial examples of strongly divisible modules of rank 2 (see for instance proofs of Prop. 4.1.1 and 4.2.1 or §5.2). In the next section, we will describe some of them and we will compute a new important strongly module divisible module.

Let I be an ideal of R containing a power of $\mathfrak{m}_R^n$ (so that R/I is a $\mathbb{Z}_p$-module of finite length). By definition, we say that an object $\mathcal{N} \in \mathfrak{M}^{k-1}$ is endowed with an action of R/I if $\mathcal{N}$ has an algebra morphism $R/I \to \mathrm{End}_{\mathfrak{M}^{k-1}}(\mathcal{N})$. In particular, $\mathcal{N}$ is an S_R/IS_R-module.

For every ideal $I \subset R$ and every strongly divisible R-module $\mathcal{M}$, we define $\mathrm{Fil}^{k-1}(\mathcal{M}/I\mathcal{M}) = \mathrm{Fil}^{k-1}\mathcal{M}/I\mathrm{Fil}^{k-1}\mathcal{M} \hookrightarrow \mathcal{M}/I\mathcal{M}$, $\phi_{k-1}\colon \mathrm{Fil}^{k-1}(\mathcal{M}/I\mathcal{M}) \to \mathcal{M}/I\mathcal{M}$ to be the reduction modulo I of $\frac{\phi}{p^{k-1}}|_{\mathrm{Fil}^{k-1}\mathcal{M}}$ and ϕ, N the reduction modulo I of ϕ, N. If R/I is flat over $\mathcal{O}$ then $\mathcal{M}/I\mathcal{M}$ (endowed with $\mathrm{Fil}^{k-1}(\mathcal{M}/I\mathcal{M})$, ϕ and N) is a strongly divisible R/I-module. If R/I is artinian then $\mathcal{M}/I\mathcal{M}$ (endowed with $\mathrm{Fil}^{k-1}(\mathcal{M}/I\mathcal{M}), \phi$ and N) is an object of $\mathfrak{M}^{k-1}$ with an action of R/I.

Lemma 4.6.4. *Let I be an ideal of R containing $\mathfrak{m}_R^n$ for $n \gg 0$, R' an artinian local $\mathcal{O}$-algebra with residue field a finite extension of $\mathbb{F}$, $R/I \to R'$ a local $\mathcal{O}$-algebra morphism and $\mathcal{N}$ an object of $\mathfrak{M}^{k-1}$ endowed with an action of R/I. Then $\mathcal{N} \otimes_{R/I} R'$ has a natural structure of object of $\mathfrak{M}^{k-1}$ endowed with an action of R' such that the canonical map $\mathcal{N} \to \mathcal{N} \otimes_{R/I} R'$ is a morphism in $\mathfrak{M}^{k-1}$.*

Remark 4.6.5. Let R' be a complete local noetherian flat $\mathcal{O}$-algebra with residue field a finite extension of $\mathbb{F}$, $R \to R'$ be a local $\mathcal{O}$-algebra morphism and $\mathcal{M}$ a strongly divisible R-module. By using previous lemma and a suitable passage at limit, we can show that $\mathcal{M} \otimes_R R'$ endowed with $\phi \otimes id$, $N \otimes id$ and with the image of $\mathrm{Fil}^{k-1}\mathcal{M} \otimes_R R'$ is always a strongly divisible R'-module. We have seen that when $R \to R'$ is surjective, which is the only case we are interested.

Recall that in Theorem 4.4.3 we have a contravariant exact and fully faithful functor V_{st}. For any object $\mathcal{N}$ of $\mathfrak{M}^{k-1}$, we set $T_{\mathrm{st},k}(\mathcal{N}) = V_{\mathrm{st}}(\mathcal{N})^*(k-1)$ where the dual is given as $\mathbb{Z}_p$-morphisms to $\mathbb{Q}_p/\mathbb{Z}_p$ and then one takes its $(k-1)$-twist. For any ideal I of R containing $\mathfrak{m}_R^n$ for $n \gg 0$ and for any object $\mathcal{N}$ of $\mathfrak{M}^{k-1}$ endowed with an action of R/I, we give to $T_{\mathrm{st},k}$ a structure of R/I by setting: $(\lambda \cdot F)(f) = F(\lambda \cdot f)$ for $F \in T_{\mathrm{st},k}(\mathcal{N})$, $f \in V_{\mathrm{st}}(\mathcal{N})$, $\lambda \in R/I$ and $(\lambda \cdot f)(x) = f(\lambda x)$ if $x \in \mathcal{N}$. The action of G_p is then well defined and R/I-linear. The advantage of $T_{\mathrm{st},k}$ with respect to V_{st} is on its covariantness, which makes easier the presence of coefficients.

Lemma 4.6.6. *Let I be an ideal of R containing $\mathfrak{m}_R^n$ for $n \gg 0$.*
(i) Let $\mathcal{N}$ be an object of $\mathfrak{M}^{k-1}$ endowed with an action of R/I and I' an ideal of R containing I, then the map $T_{\mathrm{st},k}(\mathcal{N}) \to T_{\mathrm{st},k}(\mathcal{N}/I'\mathcal{N})$ is surjective.
(ii) Let $\mathcal{N}$ be an object of $\mathfrak{M}^{k-1}$ endowed with an action of R/I which is a free S_R/IS_R-module of rank d, then the R/I-module $T_{\mathrm{st},k}(\mathcal{N}/I\mathcal{N})$ is free of rank d.
(iii) Let $\mathcal{N}$ and $\mathcal{N}'$ be two objects in $\mathfrak{M}^{k-1}$ endowed with an action of R/I then

$$\mathrm{Hom}_{R,\mathfrak{M}^{k-1}}(\mathcal{N},\mathcal{N}') \xrightarrow{\simeq} \mathrm{Hom}_{R[G_p]}(T_{\mathrm{st},k}(\mathcal{N}), T_{\mathrm{st},k}(\mathcal{N}'))$$

Lemma 4.6.7. *Let I be an ideal of R containing $\mathfrak{m}_R^n$ for $n \gg 0$, R' a local artinian $\mathcal{O}$-algebra with residue field a finite extension of $\mathbb{F}$, $R/I \to R'$ a local $\mathcal{O}$-algebra homomorphism and $\mathcal{N}$ be an object of $\mathfrak{M}^{k-1}$ endowed with an action of R/I which is finite free S_R/IS_R-module. Then*

$$T_{\mathrm{st},k}(\mathcal{N}) \otimes_R R' \xrightarrow{\simeq} T_{\mathrm{st},k}(\mathcal{N} \otimes_R R')$$

If $\mathcal{M}$ is a strongly divisible R-module, set

$$T_{\mathrm{st},k}(\mathcal{M}) := \varprojlim_n T_{\mathrm{st},k}(\mathcal{M}/\mathfrak{m}_r^n\mathcal{M}).$$

Of course, this is an $R[G_p]$module.

Corollary 4.6.8. *1. Let $\mathcal{M}$ be a strongly divisible R-module of rank d, then the R-module $T_{\mathrm{st},k}(\mathcal{M})$ is free of rank d, the G_p-action is continuous for the $\mathfrak{m}_R$-adic topology and for $n \in \mathbb{N}$ we have*

$$T_{\mathrm{st},k}(\mathcal{M})/\mathfrak{m}_R^n \xrightarrow{\simeq} T_{\mathrm{st},k}(\mathcal{M}/\mathfrak{m}_R^n\mathcal{M})$$

2. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two strongly divisible R-modules, then*

$$\mathrm{Hom}_{R,Fil^{k-1},\phi,N}(\mathcal{M},\mathcal{M}') \xrightarrow{\simeq} \mathrm{Hom}_{R[G_p]}(T_{\mathrm{st},k}(\mathcal{M}), T_{\mathrm{st},k}(\mathcal{M}')).$$

3. *Let $\mathcal{M}$ be a strongly divisible R-module, R' a local complete noetherian $\mathcal{O}$-algebra of residue field a finite extension of $\mathbb{F}$ and $R \to R'$ a local $\mathcal{O}$-algebra homomorphism, then*

$$T_{\mathrm{st},k}(\mathcal{M}) \otimes_R R' \xrightarrow{\simeq} T_{\mathrm{st},k}(\mathcal{M} \otimes_R R')$$

where $T_{\mathrm{st},k}(\mathcal{M} \otimes_R R') := \varprojlim_n T_{\mathrm{st},k}(\mathcal{M}/\mathfrak{m}_R^n\mathcal{M} \otimes R'/\mathfrak{m}_{R'}^n)$.

Now suppose $R = \mathcal{O}$, i.e., R is the ring of integers of a finite extension E of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}_p}$. Let $\mathcal{M}$ be a strongly divisible R-module. According with [Bre98, §4.1.1], there exists a weakly admissible filtered $(\phi, N, \mathbb{Q}_p, E)$-module D such that $\mathrm{Fil}^0 D = D$ and that $\mathcal{M} \otimes_{\mathcal{O}} E \xrightarrow{\simeq} S_{\mathcal{O}}[1/p] \otimes_E D$ is an $S_{\mathcal{O}}[1/p]$-linear isomorphism compatible with every structure.

Lemma 4.6.9. *With the previous notation, there is an isomorphism $T_{\mathrm{st},k}(\mathcal{M})[1/p] \xrightarrow{\simeq} V_{\mathrm{st},k}(D)$ as $E[G_p]$-modules.*

Proof. Let π be a uniformiser of $\mathcal{O}$, there exists an isomorphism of $E[G_p]$-modules:

$$\begin{aligned} T_{\mathrm{st},k}(\mathcal{M})[1/p] &\simeq (\varprojlim_n \mathrm{Fil}^{k-1}(\mathcal{M}/\pi^n\mathcal{M} \otimes_S \widehat{A_{\mathrm{st}}})^{\phi_{k-1}=1,N=0})[1/p] \\ &\simeq \mathrm{Fil}^{k-1}(\mathcal{M} \otimes_{S_{\mathcal{O}}} \widehat{A_{\mathrm{st}}})^{\phi_{k-1}=1,N=0}[1/p] \\ &\simeq \mathrm{Fil}^{k-1}(D \otimes_{\mathbb{Q}_p} \widehat{A_{\mathrm{st}}}[1/p])^{\phi=p^{k-1},N=0} \\ &\simeq \mathrm{Fil}^{k-1}(D \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}^+)^{\phi=p^{k-1},N=0}. \end{aligned}$$

By Corollary 4.6.8 point 1, $\mathrm{Fil}^{k-1}(D \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}^+)^{\phi=p^{k-1},N=0}$ is a E-vector space of dimension $\dim_E D$. Since D is weakly admissible, $V_{\mathrm{st},k}(D) = \mathrm{Fil}^{k-1}(D \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\phi=p^{k-1},N=0}$ is an E-vector space too which contains the previous one, so the result. □

In particular, by Corollary 4.6.8 point 1, $T_{\mathrm{st},k}(\mathcal{M})$ is a $\mathcal{O}$-lattice G_p-stable in $V_{\mathrm{st},k}(D)$.

Proposition 4.6.10. *Let D be a weakly admissible filtered $(\phi, N, \mathbb{Q}_p, E)$-module, then every $\mathcal{O}$-lattice stable under G_p inside $V_{\mathrm{st},k}(D)$ is isomorphic to $T_{\mathrm{st},k}(\mathcal{M})$ for a strongly divisible $\mathcal{O}$-module $\mathcal{M}$ inside $S_{\mathcal{O}}[1/p] \otimes_E D$.*

Proof. See [BM02, Prop. 3.2.3.2]. □

5 The deformation problem

In this section we will compute the semistable deformation ring when the Hodge–Tate weights are $(0,1)$ and the residual representation is peu remifié, following the method of Breuil and Mézard in [BM02].

The notation is similar to that used in previous sections. In particular, we will deal with $K = K_0 = \mathbb{Q}_p$, so with (continuous and linear) representations of the absolute Galois group $G_p := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ of $\mathbb{Q}_p$; we denote by E a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$, uniformiser ϖ and residue field $\mathbb{F}$. As usual, p will denote an odd prime number.

5.1 Deformations of type (k, τ)

Definition 5.1.1. A *Galois type of degree* 2, or simply a *Galois type*, is an equivalence class of a two-dimensional representation $\tau\colon I_p := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{\mathrm{unr}}) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p})$ of the inertia subgroup of G_p with open kernel which can be extended to a representation of the Weil group W_p.

Fix a continuous representation $\bar{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{F}})$ such that it is Schur, i.e., $\mathrm{End}_{\overline{\mathbb{F}}[G_p]}(\bar{\rho}) = \overline{\mathbb{F}}$. We have seen in the section 2 that for any finite extension E of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}_p}$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$ such that $\bar{\rho}(G_p) \subset \mathrm{GL}_2(\mathbb{F})$, there exists a universal framed deformation ring R^{univ} which is a local complete noetherian $\mathcal{O}$-algebra with residue field $\mathbb{F}$.

Fix an integer $k > 1$, a Galois type τ of degree 2 and a finite extension E of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}_p}$ with ring of integers $\mathcal{O}$ and residue field $\mathbb{F}$. Suppose that τ is rational on E (i.e., $\tau\colon I_p \to \mathrm{GL}_2(E) \hookrightarrow \mathrm{GL}_2(\overline{\mathbb{Q}_p})$) and that $\bar{\rho}$ is rational on $\mathbb{F}$ (up to extending one of the two).

Definition 5.1.2. A *deformation* ρ of $\bar{\rho}$ to the ring of integers $\mathcal{O}'$ of a finite extension E' of E inside $\overline{\mathbb{Q}_p}$ *is of type* (k, τ) if:

(a) $\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p}$ is potentially semistable with Hodge–Tate weights $(0, k-1)$;

(b) $WD(\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p})|_{I_p} \simeq \tau$, where $\mathrm{WD}(\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p})$ denotes the Weil–Deligne representation associated to $\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p}$ (as in § 1.6);

(c) $(\chi^{-k+1} \det \rho)(G_p)$ is of finite order and $p \nmid \frac{\#(\chi^{-k+1} \det \rho)(G_p)}{\#(\chi^{-k+1} \det \rho)(I_p^{\mathrm{wild}})}$.

Remark 5.1.3. Note that:

$$\begin{aligned}(\chi^{-k+1} \det(\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p}))|_{I_p} &= \mathrm{WD}(\det(\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p}))|_{I_p} \\ &= \det(\mathrm{WD}(\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p}))|_{I_p} \\ &= \det(\mathrm{WD}(\rho \otimes_{\mathcal{O}'} \overline{\mathbb{Q}_p})|_{I_p}) = \det \tau\end{aligned}$$

so the third condition is equivalent to

(c)$_{\mathrm{bis}}$ $(\chi^{-k+1} \det \rho)(G_p)$ is of finite order and $p \nmid \frac{\#(\chi^{-k+1} \det \rho)(G_p)}{\#(\det \tau(I_p^{\mathrm{wild}}))}$.

In particular, if $\det\tau$ is moderate then the last condition can be formulated as
(c)$_{\text{mod}}$ $(\chi^{-k+1}\det\rho)(G_p)$ is of finite order coprime to p.

Definition 5.1.4. A *prime ideal* $\mathfrak{p}$ *of* R^{univ} *is of type* (k,τ) if there exists an $\mathcal{O}$-algebra homomorphism $R^{\text{univ}} \to \overline{\mathbb{Z}_p}$ with kernel $\mathfrak{p}$ such that the deformation $\rho\colon G_p \to \mathrm{GL}_2(\overline{\mathbb{Z}_p})$ given by composition with the universal deformation $G_p \to \mathrm{GL}_2(R^{\text{univ}})$ is of type (k,τ).

Since τ is defined over on E, if $\mathfrak{p}$ is such a prime ideal then for every $\mathcal{O}$-algebra homomorphism $R^{\text{univ}} \to \overline{\mathbb{Z}_p}$ with kernel $\mathfrak{p}$ the correspondent deformation $G_p \to \mathrm{GL}_2(\overline{\mathbb{Z}_p})$ is of type (k,τ).

We define:

$$R(k,\tau,\bar{\rho}) := \begin{cases} 0 & \text{if there are no such prime } \mathfrak{p} \text{ of type } (k,\tau) \\ R^{\text{univ}}/\bigcap_{\mathfrak{p}\ (k,\tau)} \mathfrak{p} & \text{otherwise} \end{cases}$$

where the intersection is taken over all prime ideals $\mathfrak{p}$ which are of type (k,τ). It is a local complete noetherian reduced flat $\mathcal{O}$-algebra of residue field $\mathbb{F}$.

Thanks to Breuil-Mézard's result which we will recall in the section 5.3, after inverting p the condition of being of type (k,τ) is Zariski closed, as Kisin showed in the Corollary in [Kis09, p. 642].

5.2 Semistable representations of dimension 2

In this section we will see how to solve the above deformation problem with $\tau = triv$ and $k = 2$. Hence, in this case we have $K = K_0 = \mathbb{Q}_p$ and we stress that p is odd, so that $k - 1 \leq p - 2$ always.

The idea is to consider every type of semistable 2-dimensional representation of G_p with coefficients in E and Hodge–Tate weights $(0,1)$ (this can be generalized for $1 \leq k \leq p$, k even, but it easier to describe the classification for $k = 2$). As shown in Example 3.14.9, there are four different types of such representations corresponding to four different types of weakly admissible filtered (ϕ, N)-modules, each of these parametrizes by a pair of values, and for different pair of values we get pairwise non-isomorphic modules: therefore, this pair characterizes exactly the weakly admissible module.

Thanks to p-adic Hodge theory and in particular Corollary 3.14.5, we can also say that such semistable representations are parametrized by this pair of values. Thus, we denote:

$V = V_{\text{st},2}(D(\mu_1,\mu_2)) = V(\mu_1,\mu_2)$ if V is crystalline reducible (split and non-split, case (1) and (2)), with $\mu_1,\mu_2 \in \overline{\mathbb{Z}_p}^{\times}$;

$V = V_{\text{st},2}(D(\mu,\nu)) = V(\mu,\nu)$ if V is crystalline irreducible (case (3)), with $\mu \in \overline{\mathbb{Z}_p}^{\times}$ and $\nu \in \mathfrak{m}_{\overline{\mathbb{Z}_p}}$;

$V = V_{\mathrm{st},2}(D(\mu, \mathfrak{L})) = V(\mu, \mathfrak{L})$ if V is semistable non-crystalline (case (4)), with $\mu \in \overline{\mathbb{Z}_p}^{\times}$ and $\mathfrak{L} \in \overline{\mathbb{Q}_p}$.

With this characterization, we can classify all the isomorphism classes of the residual representations they give:

Proposition 5.2.1. *Let V be a 2-dimensional semistable representation of G_p with coefficients in E and Hodge–Tate weights $(0,1)$, $T \subset V$ a G_p-stable $\mathcal{O}$-lattice and $\overline{T}$ its reduction modulo ϖ. We have:*

1. *if V is crystalline reducible, $V = V(\mu_1, \mu_2)$ then*

$$\overline{T} \simeq \begin{pmatrix} \omega\lambda(\bar{\mu}_2^{-1}) & * \\ 0 & \lambda(\bar{\mu}_1^{-1}) \end{pmatrix} \text{ with } * \text{ peu ramifié}$$

2. *if V is crystalline irreducible, $V = V(\mu, \nu)$ then*

$$\overline{T}|_{I_p} \simeq \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix} \text{ and } \det(\overline{T}) \simeq \omega\lambda(\bar{\mu}^{-1})$$

3. *if V is semistable non-crystalline, $V = V(\mu, \mathfrak{L})$ then:*

   (a) *if $val(\mathfrak{L}) < 1$ then*

$$\overline{T} \simeq \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \lambda(\bar{\mu}^{-1}) \text{ with } * \text{ peu ramifié}$$

   (b) *if $val(\mathfrak{L}) \geq 1$ then*

$$\overline{T} \simeq \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \lambda(\bar{\mu}^{-1}) \text{ with } * \text{ très ramifié or } \overline{T} \simeq \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \otimes \lambda(\bar{\mu}^{-1})$$

Remark 5.2.2. For the definition of peu and très ramifié extensions, see §1.5.

The proof of this Proposition is given in [BM02, Prop. 4.1.1, 4.2.1]: it is important because the authors give explicitly examples of strongly divisible modules for each case and these computations will be useful in the description of the universal deformation rings.

Let us explain two examples.

Consider $V = V(\mu_1, \mu_2)$ crystalline reducible: since it has a stable 1-dimensional representation and its Hodge–Tate weights are $(0,1)$, V is an extension of $\lambda(\mu_1^{-1})$ by $\chi\lambda(\mu_2^{-1})$, where $\lambda(\mu_i^{-1})\colon G_p \to \mathcal{O}^{\times}$ is the unramified character which sends the arithmetic Frobenius in μ_i^{-1}. Thus, clearly $\overline{T}$ is an extension of $\lambda(\bar{\mu}_1^{-1})$ by $\omega\lambda(\bar{\mu}_2^{-1})$. Further, the condition "peu ramifié" is deduced by [FL82, §9] and [Edi92, Prop. 8.2].

Now, consider $V = V(\mu, \mathfrak{L})$ semistable non-crystalline with $val(\mathfrak{L}) < 1$. In this case, we can see better what role the category $\mathfrak{M}^1$ and $\widetilde{\mathrm{MF}}^{f,1}_{\mathrm{tor}}$ play in the construction. First, up to twisting by an unramified character, we may assume $V \simeq V(1, \mathfrak{L})$. As above, V is an extension of 1 by χ, so $\overline{T}$ is an extension of 1 by ω. Let $\mathcal{D}(1, \mathfrak{L}) = S_{\mathcal{O}}[1/p] \otimes_E D(1, \mathfrak{L}) \simeq S_{\mathcal{O}}[1/p]e_1 \oplus S_{\mathcal{O}}[1/p]e_2$ with $(\mathrm{Fil}^1, \phi, N)$-structures described as in §4.3. For $val(\mathfrak{L}) < 1$, one can check that the $S_{\mathcal{O}}$-submodule $\mathcal{M}$ of $\mathcal{D}(1, \mathfrak{L})$ generated by $(pe_1, \mathfrak{L}e_2)$ (provided with the induced structure) is a strongly divisible module, and the object $\mathcal{M} \otimes \mathbb{F}$ lies in $\mathfrak{M}^1$ corresponding to the following element in $\mathrm{MF}^{f,1}_{\mathrm{tor}}$:

$$\left(M = \mathbb{F}f_1 + \mathbb{F}f_2,\ \mathrm{Fil}^1 M = \mathbb{F}f_1,\ \phi_0(f_2) = f_2,\ \phi_1(f_1) = f_1 + f_2\right).$$

Therefore, $T_{\mathrm{st},2}(\mathcal{M}) \otimes_{\mathcal{O}} \mathbb{F} \simeq \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}$ with $*$ nonzero peu ramifié.

5.3 Deformations of type $(2, \tau)$ in the peu ramifiée case

In this section, we will state a special case of a theorem of Breuil-Mézard in [BM02] and we will discuss what Kisin pointed out in [Kis08] in a footnote at page 514. As before, we recall that we are studying deformations of type $\tau = triv$ with Hodge–Tate weights $(0, 1)$, so with $k = 2$.

If $\bar{\rho}\colon \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \to \mathrm{GL}_2(\mathbb{F})$ is a continuous representation such that $\mathrm{End}_{\mathbb{F}[G_p]}(\bar{\rho}) = \mathbb{F}$, then, up to replacing $\mathbb{F}$ by a finite extension independent of $\bar{\rho}$, we can distinguish the following three cases, which correspond to a different isomorphism class of the ring $R(2, triv, \bar{\rho})$:

(1) $\bar{\rho}|_{I_p} \notin \left\{ \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix} \right\}$;

(2) $\bar{\rho}|_{I_p} \in \left\{ \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \text{ with } * \text{ très ramifié}, \begin{pmatrix} \omega_2 & 0 \\ 0 & \omega_2^p \end{pmatrix} \right\}$ or $\bar{\rho} \simeq \begin{pmatrix} \omega\lambda(\alpha) & * \\ 0 & \lambda(\beta) \end{pmatrix}$ with $\alpha, \beta \in \mathbb{F}^\times$, $\alpha \neq \beta$;

(3) $\bar{\rho} \simeq \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \lambda(\alpha)$ with $\alpha \in \mathbb{F}^\times$ and $*$ peu ramifié.

Now, we explain why we classify the residual representations in this way and we try to give an idea of the shape of the respective deformation ring.

All the residual representation in (1) must arise as a reduction modulo ϖ of a lattice in a 2-dimensional semistable representation with Hodge–Tate weights $(0, 1)$, so by Proposition 5.2.1, since our starting representation $\bar{\rho}$ is not inside that set, there is no deformation of type $(2, triv)$: in other words, $R(2, triv, \bar{\rho}) = 0$.

For the residual representations $\bar{\rho}$ as in point (2), Breuil–Mézard describe explicitly some strongly divisible modules $\mathcal{M}$ over the ring $\mathcal{O}[\![X]\!]$ ([BM02, §5.2.1 and Prop.5.2.4.1]), whose corresponding $\mathcal{O}[\![X]\!]$-lattices $T_{\mathrm{st},2}(\mathcal{M})$ are deformations of $\bar{\rho}$ to $\mathcal{O}[\![X]\!]$. So we have a continuous local $\mathcal{O}$-algebra homomorphism $f\colon R^{\mathrm{univ}} \to \mathcal{O}[\![X]\!]$ which is actually surjective (see point (ii) in the proof [BM02, Thm. 5.3.1]). Moreover, any semistable deformation ρ with Hodge–Tate weights $(0,1)$ of $\bar{\rho}$ to $\overline{\mathbb{Z}_p}$ is isomorphic (up to scalar extension) to the corresponding lattice of exactly one specialisation $\mathcal{M}_x := \mathcal{M} \otimes_{\mathcal{O}[\![X]\!]} \overline{\mathbb{Z}_p}$ along a unique morphism $x\colon \mathcal{O}[\![X]\!] \to \overline{\mathbb{Z}_p}$ ([BM02, Cor. 5.2.4.2.]), hence the morphism f factors through a surjective map $g\colon R(2, triv, \bar{\rho}) \twoheadrightarrow \mathcal{O}[\![X]\!]$ and induces a bijection between maximal spectra $\text{m-Spec}\,\mathcal{O}[\![X]\!][1/p] \simeq \text{m-Spec}\,R(2, triv, \bar{\rho})[1/p]$. Finally, by commutative algebra considerations, we conclude that in this case $R(2, triv, \bar{\rho}) \simeq \mathcal{O}[\![X]\!]$ ([BM02, Thm. 5.3.1, point (ii), second case]).

In the case (2), every residual representation comes from the reduction modulo ϖ of a lattice inside exactly one type in the classification of 2-dimensional semistable representation with Hodge–Tate $(0,1)$ (see Prop. 5.2.1). However, in point (3), we have that this equivalence class of $\bar{\rho}$ arises from two different cases: a crystalline non-split representation and a semistable non-crystalline one. In particular, in the last section we saw two examples which gives two equal residual representations when we put $\alpha = \beta$ in the crystalline case. Therefore, in the description of strongly divisible modules given by the authors in [BM02, §5.2.1], Breuil–Mézard write down two different modules over the same ring, say a crystalline module $\mathcal{M}_{\mathrm{cris}}$ over $\mathcal{O}[\![X]\!]$ and a semistable one $\mathcal{M}_{\mathrm{st}}$ over $\mathcal{O}[\![Y]\!]$ in order to distinguish the two cases. As in the case (2), the authors get two surjection maps $f_1\colon R^{\mathrm{univ}} \twoheadrightarrow \mathcal{O}[\![X]\!]$ and $f_2\colon R^{\mathrm{univ}} \twoheadrightarrow \mathcal{O}[\![Y]\!]$ which factors through the ring $R(2, triv, \bar{\rho})$. Thanks to [BM02, Cor. 5.2.4.2], every $x\colon R^{\mathrm{univ}} \to \overline{\mathbb{Z}_p}$ which corresponds to a semistable representation with Hodge–Tate weights $(0,1)$ factors through f_1 if it lies a crystalline non-split representation or through f_2 if it lies inside a semistable non-crystalline one. Therefore, we claim that the following equality holds:

$$\text{m-Spec}\,R^{\mathrm{univ}}[1/p] \cap (V(\ker f_1) \cup V(\ker f_2)) = \text{m-Spec}\,R(2, triv, \bar{\rho})[1/p]$$

In fact, pick an element x in the LHS, then consider the specialization over x of $\mathcal{M}_{\mathrm{cris}}$ if x lies in $V(\ker f_1)$ or of $\mathcal{M}_{\mathrm{st}}$ if x lies in $V(\ker f_2)$: it corresponds to a deformation of $\bar{\rho}$ to $\overline{\mathbb{Z}_p}$ with the desired properties, so x lies inside $\text{m-Spec}\,R(2, triv, \bar{\rho})$. Moreover, the LHS is closed inside the RHS and it is dense by definition of $R(2, triv, \bar{\rho})$, hence the equality.

The authors of [BM02] have asserted in Theorem 5.3.1 that the map $(f_1, f_2)\colon R(2, triv, \bar{\rho}) \to O[\![X]\!] \times O[\![Y]\!]$ induces an isomorphism after inverting p. Kisin has pointed out (see footnote in [Kis08, p. 514]) that this is not true as the two irreducible components intersect in the generic fibre.

Our results show that $R(2, triv, \bar{\rho}) \simeq \mathcal{O}[\![X, Y]\!]/(XY)$, so indeed the two irreducible components intersect in the generic fibre.

5.4 The strongly divisible module over $\mathcal{O}[\![X,Y]\!]/(XY)$

At the end of previous section, we said that we can still use the strongly divisible modules $\mathcal{M}_{\mathrm{cris}}$ and $\mathcal{M}_{\mathrm{st}}$ of Breuil and Mézard in a concrete fashion. In particular, we are able to construct such a strongly divisible module over $\mathcal{O}[\![X,Y]\!]/(XY)$ and describe the ring $R(2, triv, \bar\rho)$ for $\bar\rho$ as in the case (3) in the section 5.3.

We will show that the following module $\mathcal{M}$ endowed with the structures defined below is a strongly divisible module over $\mathcal{O}[\![X,Y]\!]/(XY)$.

$$\begin{aligned}
\mathcal{M} :=& \quad S_{\mathcal{O}[\![X,Y]\!]/(XY)} \cdot E_1 \oplus S_{\mathcal{O}[\![X,Y]\!]/(XY)} \cdot E_2 \\
\mathrm{Fil}^1\mathcal{M} :=& \quad (\mathrm{Fil}^1 S_{\mathcal{O}[\![X,Y]\!]/(XY)})\mathcal{M} + S_{\mathcal{O}[\![X,Y]\!]/(XY)} \left(E_1 + \frac{p([\alpha]^{-1}+X)}{[\alpha]^{-2} - p([\alpha]^{-1}+X)^2} E_2 \right) \\
\phi\colon& \quad E_1 \mapsto p([\alpha]^{-1}+X)E_1 \\
& \quad E_2 \mapsto \frac{[\alpha]^{-2}}{[\alpha]^{-1}+X} E_2 \\
N\colon& \quad E_1 \mapsto \frac{1}{[\alpha]^{-1}(1-p)} Y E_2 \\
& \quad E_2 \mapsto 0.
\end{aligned}$$

We claim that if we consider the reduction maps $\pi_1\colon \mathcal{M} \twoheadrightarrow \mathcal{M}/Y\mathcal{M}$ and $\pi_2\colon \mathcal{M} \twoheadrightarrow \mathcal{M}/X\mathcal{M}$ we get back, after a suitable change of basis, their starting modules $\mathcal{M}_{\mathrm{cris}}$ and $\mathcal{M}_{\mathrm{st}}$ as in [BM02, §5.2.1, point (ii)]. In the following, we define by R to be the ring $\mathcal{O}[\![X,Y]\!]/(XY)$ and by $\mu = \frac{p([\alpha]^{-1}+X)}{[\alpha]^{-2}-p([\alpha]^{-1}+X)^2}$ to be the coefficient of E_2 in the filtration submodule.

First, the image of π_1 is the following module:

$$\begin{aligned}
\mathcal{M}_1 :=& \quad S_{\mathcal{O}[\![X]\!]} \cdot E_1 \oplus S_{\mathcal{O}[\![X]\!]} \cdot E_2 \\
\mathrm{Fil}^1\mathcal{M}_1 :=& \quad (\mathrm{Fil}^1 S_{\mathcal{O}[\![X]\!]})\mathcal{M}_1 + S_{\mathcal{O}[\![X]\!]} \left(E_1 + \frac{p([\alpha]^{-1}+X)}{[\alpha]^{-2} - p([\alpha]^{-1}+X)^2} E_2 \right) \\
\phi\colon& \quad E_1 \mapsto p([\alpha]^{-1}+X)E_1 \\
& \quad E_2 \mapsto \frac{[\alpha]^{-2}}{[\alpha]^{-1}+X} E_2 \\
N\colon& \quad E_1 \mapsto 0 \\
& \quad E_2 \mapsto 0.
\end{aligned}$$

After changing basis with the transformation $e_1 := E_1 + \mu E_2$, $e_2 := E_2$ we obtain exactly the module $\mathcal{M}_{\mathrm{cris}}$.

Secondly, the image of π_2 is the following:

$$\begin{aligned}
\mathcal{M}_2 :=\quad & S_{\mathcal{O}[\![Y]\!]} \cdot E_1 \oplus S_{\mathcal{O}[\![Y]\!]} \cdot E_2 \\
\mathrm{Fil}^1\mathcal{M}_2 :=\quad & (\mathrm{Fil}^1 S_{\mathcal{O}[\![Y]\!]})\mathcal{M}_2 + S_{\mathcal{O}[\![Y]\!]}\left(E_1 + \frac{p}{[\alpha]^{-1}(1-p)}E_2\right) \\
\phi\colon\quad & E_1 \mapsto p[\alpha]^{-1}E_1 \\
& E_2 \mapsto [\alpha]^{-1}E_2 \\
N\colon\quad & E_1 \mapsto \frac{1}{[\alpha]^{-1}(1-p)}YE_2 \\
& E_2 \mapsto 0.
\end{aligned}$$

Then, rescaling just one of the two basis vector, for example $e_1 := E_1$ and $e_2 := 1/([\alpha]^{-1}(1-p))E_2$, we get the original semistable module $\mathcal{M}_{\mathrm{st}}$.

The check that $\mathcal{M}$ actually defines a strongly divisible module is an easy computation for most of properties 4.6.2 (recall that $k = 2$). In fact, except for the second property, it is a straightforward calculation:

(i) $\mathrm{Fil}^1\mathcal{M} \supseteq (\mathrm{Fil}^1 S_R)\mathcal{M}$: by construction.

(iii) $\phi(sx) = \phi(s)\phi(x)$ for any $s \in S_R$, $x \in \mathcal{M}$ by construction.

(iv) Since $\phi(\mathrm{Fil}^i S_R) \subset p^i S_R$ for $0 \le i \le p-1$, $\phi(E_1) = p([\alpha]^{-1} + X)E_1$ and μ is a divisible by p, we get:

$$\begin{aligned}
\phi(\mathrm{Fil}^1\mathcal{M}) &= \phi\big((\mathrm{Fil}^1 S_R)\mathcal{M} + S_R(E_1 + \mu E_2)\big) = \\
&= \phi(\mathrm{Fil}^1 S_R)\phi(\mathcal{M}) + \phi(S_R)\phi(E_1 + \mu E_2) \subseteq \\
&\subseteq pS_R\phi(\mathcal{M}) + pS_R(\cdots) \subseteq p\mathcal{M}.
\end{aligned}$$

Further, $\phi(\mathcal{M}) = S_R p([\alpha]^{-1} + X)E_1 + S_R[\alpha]^{-2}/([\alpha]^{-1} + X)E_2$. Since $[\alpha]^{-1} + X$ and $[\alpha]^{-2}/([\alpha]^{-1} + X)$ are invertible in R (and thus in S_R), we obtain that E_1 can be written as an element inside $S_R\phi(E_1 + \mu E_2)$, while E_2 inside $S_R\phi(\mathcal{M})$, thus $\phi(\mathrm{Fil}^1\mathcal{M})$ generates $p\mathcal{M}$ over S_R.

(v) $N(sx) = N(s)x + sN(x)$ for any $s \in S_R$, $x \in \mathcal{M}$ by construction.

(vi) We check the equality $N\phi = p\phi N$ on the basis; recall that in R we have $XY = 0$:

$$\begin{aligned}
N\phi(E_1) &= N(p([\alpha]^{-1} + X)E_1) = p([\alpha]^{-1} + X)N(E_1) = \\
&= p\frac{[\alpha]^{-1} + X}{[\alpha]^{-1}(1-p)}YE_2 = (XY = 0) = \\
&= p\frac{[\alpha]^{-1}}{[\alpha]^{-1}(1-p)}YE_2 = \frac{p}{(1-p)}YE_2
\end{aligned}$$

and

$$\phi N(E_1) = \phi\left(\frac{1}{[\alpha]^{-1}(1-p)}YE_2\right) = \frac{1}{[\alpha]^{-1}(1-p)} \cdot \frac{[\alpha]^{-2}}{[\alpha]^{-1}+X}YE_2 =$$

(since $XY = 0$, then $[\alpha]^{-1}Y = [\alpha]^{-1}Y + XY = ([\alpha]^{-1} + X)Y$)

$$= \frac{[\alpha]^{-1}+X}{(1-p)([\alpha]^{-1}+X)}YE_2 = \frac{1}{(1-p)}YE_2.$$

For E_2, the equality automatically holds because it is an eigenvector for ϕ and vanishes on N.

(vii) Since $N(\mathrm{Fil}^{i+1}S_R) \subset \mathrm{Fil}^i S_R$ for $0 \le i \le p-1$, we have:

$$\begin{aligned}(\mathrm{Fil}^1 S_R)N(\mathrm{Fil}^1\mathcal{M}) &= (\mathrm{Fil}^1 S_R)\big(N(\mathrm{Fil}^1 S_R)\mathcal{M} + (\mathrm{Fil}^1 S_R)N(\mathcal{M}) + \\ &\qquad + N(S_R)(E_1+\mu E_2) + S_R N(E_1+\mu E_2)\big) \\ &= (\mathrm{Fil}^1 S_R)\big(N(\mathrm{Fil}^1 S_R)\mathcal{M} + (\mathrm{Fil}^1 S_R)\big((Y/([\alpha]^{-1}(1-p))\,E_2\big) + \\ &\qquad + S_R(E_1+\mu E_2) + S_R(Y/([\alpha]^{-1}(1-p))\,E_2)\big) \\ &\subset (\mathrm{Fil}^1 S_R)\mathcal{M} + (\mathrm{Fil}^1 S_R)(E_1+\mu E_2) \ \subset\ \mathrm{Fil}^1\mathcal{M}.\end{aligned}$$

As we said before, in order to prove the second property $\mathrm{Fil}^1\mathcal{M} \cap I\mathcal{M} = I\mathrm{Fil}^1\mathcal{M}$ for any ideal $I \subset R$, we need some preliminary results. Of course, the inclusion $\mathrm{Fil}^1\mathcal{M} \cap I\mathcal{M} \supseteq I\mathrm{Fil}^1\mathcal{M}$ is trivial, so it is enough to show the other inclusion. In the following, we will use that this property holds for the modules $\mathcal{M}_{\mathrm{cris}}$ and $\mathcal{M}_{\mathrm{st}}$, as proved in [BM02, Thm. 5.2.4.1].

We start with an easy result.

Lemma 5.4.1. *The module $\mathcal{M}$ identifies as the fiber product of $\mathcal{M}_{\mathrm{cris}} = \mathcal{M}/Y\mathcal{M}$ and $\mathcal{M}_{\mathrm{st}} = \mathcal{M}/Y\mathcal{M}$ over $\mathcal{N} := \mathcal{M}/(X,Y)\mathcal{M}$.*

Proof. We begin showing that the natural map $R = \mathcal{O}[\![X,Y]\!]/(XY) \to \mathcal{O}[\![X]\!] \times_{\mathcal{O}} \mathcal{O}[\![Y]\!]$ defined by $f(X,Y) \mapsto (f(X,0), f(0,Y))$ is a isomorphism. In fact, the kernel of this map is exactly (XY) (which annihilates in R) and given a pair $(h_1(X), h_2(Y))$ in the target satisfying $c := h_1(0) = h_2(0)$, we can consider the element $h_1 + h_2 - c \in R$ which has as image exactly that pair. In other words, since $XY = 0$, every element in R can be written as a sum of one formal power series in X and one in Y having the same constant term.

This isomorphism induces a map $S_R \to S_{\mathcal{O}[\![X]\!]} \times_{S_{\mathcal{O}}} S_{\mathcal{O}[\![Y]\!]}$ defined by the projection on the coefficients. As above, it is clear surjective, while for the injectivity we see that the kernel is the intersection $XS_R \cap YS_R =$ (by the Artin–Rees' lemma) $= S_{XR} \cap S_{YR} = 0$.

Further, we can push over this isomorphism in the free module $\mathcal{M}$ considering the maps π_1 and π_2 given by, resp., reducing modulo Y and modulo X. But, as shown at the beginning, that reduced modules corresponds, resp., to $\mathcal{M}_{\text{cris}}$ and $\mathcal{M}_{\text{st}}$, so, denoting by $\mathcal{N} := \mathcal{M}/(X,Y)\mathcal{M}$, we have the following cartesian diagram:

$$\begin{tikzcd}
\mathcal{M} \arrow[dr, "\xi"] \arrow[drr, bend left, "\pi_1"] \arrow[ddr, bend right, "\pi_2"'] & & \\
& \mathcal{M}_{\text{cris}} \times_{\mathcal{N}} \mathcal{M}_{\text{st}} \arrow[r] \arrow[d] & \mathcal{M}_{\text{cris}} \arrow[d] \\
& \mathcal{M}_{\text{st}} \arrow[r] & \mathcal{N}
\end{tikzcd}$$

Finally, the map ξ is an isomorphism because $\mathcal{M}$ is a free module over S_R of rank 2 as the fibred module $\mathcal{M}_{\text{cris}} \times_{\mathcal{N}} \mathcal{M}_{\text{st}}$. $\square$

In particular, the module $\mathcal{M}$ inherits all the good property of the two modules $\mathcal{M}_{\text{cris}}$ and $\mathcal{M}_{\text{st}}$.

Now, we go back to our problem: $\text{Fil}^1\mathcal{M} \cap I\mathcal{M} \subseteq I\text{Fil}^1\mathcal{M}$ for any ideal $I \subset R$. By using the above identification and that $\mathcal{M}_{\text{cris}}$ and $\mathcal{M}_{\text{st}}$ satisfy this property ([BM02, Prop.5.2.4.1]), we need the following result.

Lemma 5.4.2. *With the above notation,* $\pi_1^{-1}(I\text{Fil}^1\mathcal{M}_{\text{cris}}) \cap \pi_2^{-1}(I\text{Fil}^1\mathcal{M}_{\text{st}}) = I\text{Fil}^1\mathcal{M}$.

Remark 5.4.3. Of course, with the notation $I\text{Fil}^1\mathcal{M}_{\text{cris}}$ we mean that the image of I inside $\mathcal{O}[\![X]\!]$ multiply $\mathcal{M}_{\text{cris}}$ and similarly with $I\text{Fil}^1\mathcal{M}_{\text{st}}$.

Proof. Since the image under π_1 and π_2 of $\text{Fil}^1\mathcal{M}$ is surjective onto, respectively, $\text{Fil}^1\mathcal{M}_{\text{cris}}$ and $\text{Fil}^1\mathcal{M}_{\text{st}}$, we have that $\pi_1(I\text{Fil}^1\mathcal{M}) = I\text{Fil}^1\mathcal{M}_{\text{cris}}$ and $\pi_2(I\text{Fil}^1\mathcal{M}) = I\text{Fil}^1\mathcal{M}_{\text{st}}$. Moreover, $\pi_1^{-1}(I\text{Fil}^1\mathcal{M}_{\text{cris}}) = I\text{Fil}^1\mathcal{M} + Y\mathcal{M}$ and similarly for the other one. Hence, we get:

$$\pi_1^{-1}(I\text{Fil}^1\mathcal{M}_{\text{cris}}) \cap \pi_2^{-1}(I\text{Fil}^1\mathcal{M}_{\text{st}}) = (I\text{Fil}^1\mathcal{M} + Y\mathcal{M}) \cap (I\text{Fil}^1\mathcal{M} + X\mathcal{M}) =: A.$$

Pick an element $a \in A$, so in particular $a \in I\text{Fil}^1\mathcal{M} + Y\mathcal{M}$, hence we may write $a = m_1 + Ym_2$ where $m_1 \in I\text{Fil}^1\mathcal{M}$ and $m_2 \in \mathcal{M}$. Thus, $Ym_2 = a - m_1 \in A$ because $I\text{Fil}^1\mathcal{M} \subseteq A$. Now, we look at its image under π_2: $Y\pi_2(m_2) \in I\text{Fil}^1\mathcal{M}_{\text{st}}$. Therefore:

$$\begin{aligned} Y\pi_2(m_2) \in I\text{Fil}^1\mathcal{M}_{\text{st}} \cap Y\mathcal{M}_{\text{st}} &= \text{Fil}^1\mathcal{M}_{\text{st}} \cap I\mathcal{M}_{\text{st}} \cap Y\mathcal{M}_{\text{st}} = \\ &= \text{Fil}^1\mathcal{M}_{\text{st}} \cap (I \cap (Y))\mathcal{M}_{\text{st}} = (I \cap (Y))\text{Fil}^1\mathcal{M}_{\text{st}} \end{aligned}$$

where the last equality is given by the result in [BM02, Prop. 5.2.4.1]. So, $Ym_2 \in (I \cap (Y))\text{Fil}^1\mathcal{M} + X\mathcal{M}$, but since the ideals (X) and (Y) intersect only trivially in S_R, we conclude that $Ym_2 \in (I \cap (Y))\text{Fil}^1\mathcal{M}$, i.e. $a \in I\text{Fil}^1\mathcal{M}$. $\square$

Therefore, we conclude with the following chain of inclusions:

$$\begin{aligned} \text{Fil}^1\mathcal{M} \cap I\mathcal{M} &\subseteq \pi_1^{-1}(\text{Fil}^1\mathcal{M}_{\text{cris}} \cap I\mathcal{M}_{\text{cris}}) \cap \pi_2^{-1}(\text{Fil}^1\mathcal{M}_{\text{st}} \cap I\mathcal{M}_{\text{st}}) = \\ &= \pi_1^{-1}(I\text{Fil}^1\mathcal{M}_{\text{cris}}) \cap \pi_2^{-1}(I\text{Fil}^1\mathcal{M}_{\text{st}}) = \\ &= I\text{Fil}^1\mathcal{M}. \end{aligned}$$

5.5 End of the proof

Now we know that $\mathcal{M}$ defines a strongly divisible module over $R = \mathcal{O}[\![X,Y]\!]/(XY)$. In the following, we will finally prove that the ring R is isomorphic to the universal deformation ring $R(2, triv, \bar{\rho})$, with $\bar{\rho} \simeq \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \lambda(\alpha)$ with $\alpha \in \overline{\mathbb{F}_p}^{\times}$ and $*$ peu ramifié, as in point (3).

Since $\mathcal{M}$ is a strongly divisible module, then the lattice $T_{\mathrm{st},2}(\mathcal{M})$ is a deformation of $\bar{\rho}$ to R, so Corollary 4.6.8 gives us a continuous map $f\colon R^{\mathrm{univ}} \to R$ of local complete $\mathcal{O}$-algebras.

Lemma 5.5.1. *The map $f\colon R^{\mathrm{univ}} \to R$ is surjective.*

Proof. We follows the same ideas given by Breuil and Mézard.

In order to prove the surjection, it sufficies to check it on tangent spaces, i.e., that so is the map $R^{\mathrm{univ}} \to R/(\pi, \mathfrak{m}_R^2) = \mathbb{F}[X,Y]/(X^2, XY, Y^2)$. But this is equivalent to say that $T_{\mathrm{st},2}(\mathcal{M}) \otimes_R \mathbb{F}[X,Y]/(X^2, XY, Y^2) \simeq T_{\mathrm{st},2}(\mathcal{M} \otimes_R \mathbb{F}[X,Y]/(X^2, XY, Y^2))$ cannot define over a strict $\mathbb{F}$-algebra of $\mathbb{F}[X,Y]/(X^2, XY, Y^2)$. Since $T_{\mathrm{st},2}$ is fully faithful (Corollary 4.6.8), this can be reformulated as saying that $\mathcal{M} \otimes_R \mathbb{F}[X,Y]/(X^2, XY, Y^2)$ does not come from an extension by scalars of a subobject inside $\mathfrak{M}^1$ equipped with an action of a such $\mathbb{F}$-subalgebra. But the variable Y is inside only in the value of $N(E_1)$, so Y must lie inside such a subalgebra. Moreover, if we consider $\mathcal{M} \otimes_R \mathbb{F}[X,Y]/(Y, (X^2, XY, Y^2)) = \mathbb{F}[X]/(X)^2$, i.e. we see all the other coefficients different by Y, we get the starting Breuil–Mézard's module $\mathcal{M}_{\mathrm{cris}}$, in which the image of ϕ depends on X without coming from a tensor with elements in $\mathbb{F}[X]/(X)^2$. $\square$

Proposition 5.5.2. *The map $f\colon R^{\mathrm{univ}} \to R$ induces a bijection*

$$\text{m-Spec}\, R[1/p] \xrightarrow{\simeq} \text{m-Spec}\, R(2, triv, \bar{\rho})[1/p].$$

Proof. First of all, we notice that both R^{univ} and R are quotient of $\mathcal{O}[\![X_1, \dots, X_n]\!]$ for some n where $\mathcal{O}$ is a complete DVR, thus by [GD66, §10.5.7] the rings $R^{\mathrm{univ}}[1/p]$ and $R[1/p]$ are Jacobson rings and the residue field at any maximal ideal of these rings are finite extensions of $E = \mathcal{O}[1/p]$. So, we can extend the local $\mathcal{O}$-algebra homomorphism $f\colon R^{\mathrm{univ}} \to R$ to a $\mathcal{O}$-algebra homomorphism $\tilde{f}\colon R^{\mathrm{univ}}[1/p] \to R[1/p]$ of Jacobson rings.

Since the map f is surjective by Lemma 5.5.1, the pre-image of a maximal ideal in $R[1/p]$ is again a maximal ideal in $R^{\mathrm{univ}}[1/p]$, so the map $\tilde{f}$ induces a well-defined injective map

$$f^{\sharp}\colon \text{m-Spec}\, R[1/p] \hookrightarrow \text{m-Spec}\, R^{\mathrm{univ}}[1/p]$$

sending $\mathfrak{n}$ to $\tilde{f}^{-1}(\mathfrak{n})$. Moreover, to a maximal ideal $\mathfrak{n}_x$ of $R[1/p]$ corresponds a specialization map $\tilde{x}\colon R[1/p] \twoheadrightarrow E'$ for some finite extension of E'/E, i.e., a map $x\colon R \twoheadrightarrow \mathcal{O}_{E'}$, while the maximal ideal $f^\sharp(\mathfrak{n})$ corresponds to the pre-composition $R^{\text{univ}} \xrightarrow{f} R \xrightarrow{x} \overline{\mathbb{Z}_p}$ of x by f. Now, we will show that $f^\sharp$ is a bijection on the subset m-Spec $R(2, triv)[1/p]$.

Now, we will show that $f^\sharp$ surjects onto the subset m-Spec $R(2, triv, \bar{\rho})[1/p]$ and thus, in particular, that the map f factors through a map $g\colon R(2, triv, \bar{\rho}) \to R$. Let $\mathfrak{m}_y \in$ m-Spec $R(2, triv, \bar{\rho})[1/p]$, i.e., $y\colon R^{\text{univ}} \to \overline{\mathbb{Z}_p}$ induces a deformation $\rho_y\colon G_p \to \mathrm{GL}_2(R^{\text{univ}}) \to \mathrm{GL}_2(\overline{\mathbb{Z}_p})$ of $\bar{\rho}$ to $\overline{\mathbb{Z}_p}$ of type $(2, triv)$. It is enough to find a ring homomorphism $x\colon R \to \overline{\mathbb{Z}_p}$ with the property that the deformation $\rho_{f\circ x}$ to $\overline{\mathbb{Z}_p}$ is equivalent to ρ_y.

By Proposition 4.6.10, we know that there is a strongly divisible module $\mathcal{M}_y$ such that $T_{\text{st},2}(\mathcal{M}_y) = \rho_y$, and by the fully faithfulness of $T_{\text{st},2}$ (Corollary 4.6.8) it is enough to find a strongly divisible module $\mathcal{M}_x$ over $\mathcal{O}$ isomorphic (up to scalar extension) to $\mathcal{M}_y$.

Thanks to the classification of 2-dimensional p-adic semistable representations as in § 5.2, the deformation ρ_y is inside either a crystalline non-split representation $V(\mu_1, \mu_2)$ or a semistable non-crystalline representation $V(\mu, \mathfrak{L})$ for a unique pair of values, resp., $(\mu_1, \mu_2) \in \overline{\mathbb{Z}_p}^\times \times \overline{\mathbb{Z}_p}^\times$ or $(\mu, \mathfrak{L}) \in \overline{\mathbb{Z}_p}^\times \times \overline{\mathbb{Q}_p}$; moreover, these representations are pairwise non-isomorphic. Then, one of these pairs characterized completely the deformation ρ_y.

Furthermore, the determinant of any deformation is fixed by the residual representation $\bar{\rho}$ and, in our case, it is equal modulo $\mathfrak{m}_{\overline{\mathbb{Z}_p}}$ to α^{-2}. Hence, the deformation ρ_y is uniquely described by one of the two values, and by equivalence between the two integral categories, so is the corresponding strongly divisible module $\mathcal{M}_y$. For this reason, Breuil and Mézard considered universal deformation rings with one variable because the $\mathcal{O}$-algebra morphisms $\mathcal{O}[\![X]\!] \to \overline{\mathbb{Z}_p}$ are uniquely determined by the image of the variable X. In other words, by the proof of Proposition 5.2.1, the strongly divisible module $\mathcal{M}_y$ is characterized by either the module $\mathcal{M}_{\text{cris}} \otimes_{\mathcal{O}[\![X]\!]} \overline{\mathbb{Z}_p}$ along an $\mathcal{O}$-algebra homomorphism $\mathcal{O}[\![X]\!] \to \overline{\mathbb{Z}_p}$ if the deformation lies inside the crystalline locus or the module $\mathcal{M}_{\text{st}} \otimes_{\mathcal{O}[\![Y]\!]} \overline{\mathbb{Z}_p}$ along a fixed $\mathcal{O}$-algebra homomorphism $\mathcal{O}[\![X]\!] \to \overline{\mathbb{Z}_p}$ if the deformation lies inside the semistable non-crystalline locus. Finally, with our 'universal' strongly divisible module $\mathcal{M}$, we can collect both information in a unique $\mathcal{O}$-algebra homomorphism $x\colon R \to \overline{\mathbb{Z}_p}$ and the module $\mathcal{M}_x := \mathcal{M} \otimes_R \overline{\mathbb{Z}_p}$ is isomorphic to $\mathcal{M}_y$. □

Finally, we are ready to conclude our work.

Theorem 5.5.3. *The morphism $f\colon R^{\text{univ}} \to R := \mathcal{O}[\![X,Y]\!]/(XY)$ induces an isomorphism $g\colon R(2, triv, \bar{\rho}) \to R$ of $\mathcal{O}$-algebras.*

Proof. As shown in the proof of Proposition 5.5.2, the map f induces a map g through $R(2, triv, \bar{\rho})$. By Lemma 5.5.1, we can say further that g is surjective. Summarizing, we have the following commutative diagram:

$$\begin{tikzcd} R(2, triv, \bar{\rho})[1/p] \arrow[r, "\tilde{g}"] & R[1/p] \\ R(2, triv, \bar{\rho}) \arrow[u, hook] \arrow[r, two heads, "g"] & R \arrow[u, hook] \end{tikzcd}$$

Hence, in order to show the injectivity of g it is enough to show the isomorphism $R(2, triv, \bar{\rho})[1/p] \simeq R[1/p]$. Since the rings are p-torsion free, the map $\tilde{g}$ is surjective too, so it remains the injective part.

So, pick an element $r \in R(2, triv, \bar{\rho})[1/p]$ satisfying $\tilde{g}(r) = 0$. Since trivially we have $0 \in \bigcap \mathfrak{n}$ where the intersection is taken among the $\mathfrak{n} \in$ m-Spec $R[1/p]$, the bijection m-Spec $R(2, triv, \bar{\rho})[1/p] \simeq$ m-Spec $R[1/p]$ proved in Proposition 5.5.2 implies that $r \in \bigcap \mathfrak{m}$, where this intersection is among $\mathfrak{m} \in$ m-Spec $R(2, triv, \bar{\rho})[1/p]$. Note that, in a Jacobson ring, the Jacobson radical is equal to the nilradical: since, by definition, $R(2, triv, \bar{\rho})$ is reduced, we conclude that $r = 0$.

Therefore, the morphism $g\colon R(2, triv, \bar{\rho}) \to R$ is an isomorphism of local $\mathcal{O}$-algebras, as desired. □

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