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Brauer-Manin for Homogeneous Spaces

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Introduction

Let X be a smooth variety defined over a number filed k. A leading problem in arithmetic geometry is to determine whether or not X has any k-points. Due to the difficulty of this task, we are mainly concerned in two problems:

i) finding necessary conditions for $X(k)$ to be non-empty;

ii) understanding the "size" of $X(k)$.

Our approach relies on a "local-to-global" strategy, indeed we would like to study the diagonal embedding $X(k) \hookrightarrow \prod_{v \in \Omega_k} X(k_v)$, where k_v is the completion of k at the prime v. In particular a first refinement of the problems is the following:

i) *if* $\prod_{v \in \Omega_k} X(k_v)$ *is non-empty, can we deduce that* $X(k)$ *is non-empty?* (known as *Hasse principle*);

ii) *is the diagonal embedding dominant?* (known as *Weak approximation*, or *Strong approximation*, depending on the topology of $\prod_{v \in \Omega_k} X(k_v)$).

Positive examples are known:

i) the Hasse principle is true for smooth projective quadrics, Brauer-Severi varieties, smooth projective cubics in $\mathbb{P}^n_{\mathbb{Q}}$ with $n \geq 9$ and torsors under semisimple simply connected algebraic groups;

ii) k-rational varieties, smooth projective intersection of two quadrics with a k-point in $\mathbb{P}^n_k$ with $n \geq 6$, satisfy the weak approximation property.

But is not often the case. Manin, in the 1970, considered the *Brauer group* of X, $\mathrm{Br}(X)$, and for any $B \subset \mathrm{Br}(X)$ introduced the subset $X(k)^B$ of $\prod_{v \in \Omega_k} X(k_v)$ such that

$$X(k) \subset X(k)^{\mathrm{Br}(X)} \subset X(k)^B \subset \prod_{v \in \Omega_k} X(k_v),$$

where the inclusions hold thanks to the *Global Class Field Theory*. Of course if $X(k)^B = \emptyset$ implies that $X(k) = \emptyset$, conversely we say that *the Brauer-Manin obstruction to the Hasse Principle is the only one* if we have

$$X(k)^B \neq \emptyset \Rightarrow X(k) \neq \emptyset$$

for every variety X in a certain class. Analogously is not hard to define the *Brauer-Manin obstruction to weak and strong approximation* (see Section 3.1).

Manin's construction accounts for almost all known examples where the Hasse Principle fails, and seems having a nice behaviour whenever X admits an extra structure, for example when an algebraic group G acts on X (when the action is simply transitive, resp. transitive, we say that X is a *torsor*, resp. a *homogeneous space*, under G). Of course some conditions on G have to be imposed, in particular G has to be simply connected, in virtue of the following result (the proof can be found in [Rap12], page 9).

Theorem 0.0.1 (Minchev)**.** *Let X be an irreducible normal variety over a number field k such that $X(k) \neq \emptyset$. If there exists a non trivial connected unramified covering $f : Y \to X$ defined over an algebraic closure $\overline{k}$, then X does not have strong approximation off any finite set S of places of k.*

Moreover, since the case of torsors under abelian varieties is already discussed (see for example [Sko01], Theorem 6.2.3) we may assume G to be linear, in particular we are mainly concerned with *semisimple simply connected groups*. The first chapter summarises, for the most part, the results from the theory of linear algebraic groups over fields of characteristic zero we will need to succeed in this study.

The general strategy is to deduce the arithmetic properties on a variety with an action of G from the properties of G, of course in the following order:

G semisimple simply connected k-group $\Rightarrow$ k-torsors under G $\Rightarrow$ k-homogeneous spaces under G.

The first step was, as mentioned above, the proof of the Hasse principle for torsors under a semisimple simply connected group G (due to Kneser, Harder and Chernousov), which can be written as

$$H^1(k, G) \longrightarrow \prod_{v \in \Omega_{\mathbb{R}}} H^1(k, G) \text{ is bijective.}$$

But how far is a homogeneous space from being a torsor (at least from a "Hasse principle point of view")? This question can not be answered just in terms of the first group of Galois cohomology. Following [Bor93], the second chapter of the thesis introduces the formalism of the *nonabelian* H^2, which provides the right setting for a first answer (see Section 2.2).

Theorem 0.0.2. *Let X/k a homogeneous space of a semisimple simply connected algebraic group G/k with connected stabilizer $\overline{H}$. There exists a k-lien L, whose underling algebraic group is $\overline{H}$, and an element $\eta_X \in H^2(k, L)$ such that if η_X is neutral, then the Hasse principle (with respect to the real places) holds.*

The rest of the chapter is devoted to application of this result, in particular we prove an Hasse principle for the nonabelian H^2 and we describe many conditions on the connected stabilizer to have the Hasse principle, see Theorem 2.6.2.

These first results show that the case of homogeneous space is significantly harder than the case of torsors, and various conditions on the stabilizer have to be considered. After the unconditioned discussed above, Chapter 3 deals with the Brauer-Manin obstruction. Following [Bor96], we prove that for a homogeneous space of a connected linear group with connected stabilizer and for a homogeneous space of a simply connected group with abelian stabilizers, the Brauer-Manin obstructions explain the failure of the Hasse principle and of weak approximation (see Section 3.2).

The study of strong approximation and integral Hasse principle, which are closely related to each other (cf. Theorem 4.1.6), took fifteen years from the article [Bor96]. Indeed the analogues results in case the stabilizers are connected/abelian have been proven by Colliot-Thélène-Xu in [CTX09] and Borovoi-Demarche in [BD13]. For an arbitrary stabilizer this study remains a difficult task and only during the present year some results have been achieved. The last chapter presents Demarche's counterexample (presented in [Dem15]) to strong approximation with Brauer-Manin obstruction when the stabilizer is nilpotent. Namely we have the following (cf. Theorem 4.2.1).

Theorem 0.0.3. *Let G be a semisimple simply connected algebraic k-group, p be a prime number, H be a non commutative finite group of order p^n and set $X = G/H$. For any finite set $S_0 \subset \Omega_k$, if k contains the p^{n+1}th roots of unity, then the obstruction of Brauer Manin to strong approximation for X is not the only one.*

Moreover, a general result of [LX15] ensures the existence of an $\mathcal{O}_{k,S_0}$-model of X, for which the failure of the integral Hasse principle is not explained by Brauer-Manin; it remains an interesting question to give an explicit construction of such a model, maybe with the structure of homogeneous space, extending the one of X.

To complete the story we cite the main result of [LA15] about weak approximation: if that the Brauer-Manin obstruction is the only obstruction to weak approximation for homogeneous spaces of the form SL_n/H with H finite, then this is also the case for every homogeneous space G/H with connected G and arbitrary H. Apart from this, not much else is known.

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Notation

In general k will be a field of characteristic zero, $\overline{k}$ a fixed algebraic closure and Γ_k the absolute Galois group of k.

Let G/k be a linear algebraic group. In the following we will often consider the following k-groups, which deserve a name:

- G° for the connected component of $1_G \in G$;
- G' for the derived subgroup of G;
- $\pi_0(G)$ for the group of connected component of G, i.e. the quotient G/G° (it is a finite group);
- G^{u} for the unipotent radical of G (it is a unipotent characteristic subgroup);
- G^{red} for the quotient G°/G^{u} (it is a reductive group);
- G^{ss} for the derived subgroup of G^{red} (it is a semisimple group);
- G^{tor} for the quotient $G^{\mathrm{red}}/G^{\mathrm{ss}}$ (it is the biggest toric quotient of G);
- G^{ssu} for the kernel of the canonical projection $G^\circ \to G^{\mathrm{tor}}$ (it is an extension of G^{ss} by G^{u});
- G^{sc} for the universal covering of G^{ss} (it is a semisimple simply connected group);
- when G is reductive we define G^{ad} as the quotient of G by its center (it is a semisimple group with trivial center).

The properties in brackets will be explained and proven in the first chapter and used systematically in the subsequent chapters.

Let k be a field of characteristic zero, and X a k-variety, i.e. a geometrically irreducible algebraic variety X over a field k; in our discussion X will always be smooth. The following is standard notation:

- $\mathrm{Br}(X) := H^2(X_{et}, \mathbb{G}_m)$, the *(cohomological) Brauer group* of X;
- $\mathrm{Br}_0(X) := \mathrm{Im}(\mathrm{Br}(k) \to \mathrm{Br}(X))$;
- $\mathrm{Br}_1(X) := \mathrm{Ker}(\mathrm{Br}(X) \to \mathrm{Br}(X_{\overline{k}}))$;
- $\mathrm{Br}_a(X) := \mathrm{Br}_1(X)/\mathrm{Br}_0(X)$, the *arithmetic Brauer group* of X;
- $\mathrm{Br}_{nr}(X) := \mathrm{Br}_{nr}(k(X)/k) \subset \mathrm{Br}(X)$, the *unramified Brauer group* (it can be identified with the Brauer group of any smooth proper k-variety birational to X);
- If k is a number field and $S \subset \Omega_k$ a finite set, we define
$$Б_S(X) := \mathrm{Ker}\left(\mathrm{Br}_a(X) \to \prod_{v \notin S} \mathrm{Br}_a(X_{k_v})\right).$$
We set $Б_\omega(X) := \bigcup_S Б_S(X)$. If $S = \emptyset$ we write $Б(X)$ for $Б_S(X)$.

As in [Ser73], for a discrete, not necessary commutative, group A with an action of Γ_k, we write $H^1(k, A)$ for the pointed set of classes of continuous 1-cocycles, the so called *first group of nonabelian Galois cohomology*.

Chapter 1

Arithmetic of Linear Algebraic Groups

We present a short overview on the main results we are going to use in this work. Generically we follow [Mil11], [BT65], [PR94], over an algebraically closed field [MT11], for the more arithmetic part [Mil11], [Har06] and [San81].

Definition 1.0.4. An *algebraic group* over a field k is a group object in the category of schemes of finite type over k.

The building blocks of the theory of algebraic groups, in the sense that all others can be constructed by successive extensions, are the following:

- finite (constant and non constant);
- unipotent groups;
- semisimple algebraic groups;
- algebraic tori;
- abelian varieties, i.e. connected algebraic groups whose underlying scheme is projective.

From now on we will deal mainly with the affine case.

Definition 1.0.5. An algebraic k-group is a *linear algebraic group* if it satisfies one of the following equivalent properties:

- its underlying scheme is affine,
- it admits a closed embedding in GL_n for some $n \geq 0$.

Example. A constant group scheme is affine if and only if is finite.

Theorem 1.0.6 (Cartier)**.** *Every linear algebraic group over a field of characteristic zero is smooth.*

Proof. For the proof see [Mil11], Theorem 6.31. □

From now on we specialize our selves to the case of characteristic 0 (the case we are interested in). More generally many statements will remain true if k is perfect of positive characteristic and G is smooth. With *algebraic group* we will mean an algebraic group defined over a field k; some times we could omit to remark that the group is linear.

1.1 Homogeneous spaces over a field

Before starting our overview about the structure of algebraic groups we introduce the the notion of homogeneous space and we present the "quotient construction".

Definition 1.1.1. Let G/k be a linear algebraic group. Let X/k be a (non empty) variety equipped with a right action of G. We say that X is homogeneous (resp. principal homogeneous) under G if $G(\overline{k})$ acts transitively (resp. simply transitively) on $X(\overline{k})$.

Example. Let q be a non degenerate quadratic form. The projective variety given by the zero locus of q is a homogeneous space under the orthogonal group $O(q)$.

Following [Bor69], Chapter II, Section 6 we briefly discuss the "quotient construction". Before we recall some general facts of classical algebraic geometry.

Definition 1.1.2. Let $\pi : V \to W$ be a k-morphism of k-varieties. We say that π is a *quotient morphism* if it is surjective, open, and if $U \subset V$ is an open, then π induces an isomorphism between $k[\pi(U)]$ and the set of elements of $k[U]$ which are constant on the fibers of $\pi_{|U} : U \to \pi(U)$.

Quotient morphisms have the universal property the reader can expect: Let $\pi : V \to W$ be a quotient morphism, and $\alpha : V \to Z$ any morphism constant on the fibers of π, then there exists a unique map $\beta : W \to Z$ such that $\alpha = \beta \circ \pi$.

Lemma 1.1.3. *Let $\pi : V \to W$ be a surjective open morphism of irreducible varieties. If W is an integral variety, then π is a quotient morphism.*

Proof. For the proof see [Bor69], Chapter II, Lemma 6.2. □

We can now consider the case of the quotient of V by G.

Definition 1.1.4. Let G be a k-group acting on A k-variety V. An *orbit map* is a surjective morphism $\pi : V \to W$ of varieties such that the fibers of π are the orbits of G on V. A *k-quotient* of V by G is an orbit map $V \to W$ which is also a quotient morphism in the sense of the previous definition.

As above we have the same universal property with regard to every map $\alpha : V \to Z$ constant on the orbits of G. In particular the quotient, if it exists, it is unique. We write V/G for such an object.
About the existence, in our setting, we have a classical result.

Theorem 1.1.5. *Let G be an affine k-group and H be a closed k-subgroup of G. Then the quotient $\pi_H : G \to G/H$ exists over k, and G/H is a smooth quasi-projective k-variety. If H is normal in G then G/H is an affine k-group and π_H is a group map.*

Proof. For the proof see [Bor69], Chapter II, Theorem 6.8. □

Remark. Being representable remains a delicate issue; we state here two interesting results about quotients in higher dimensions.

- Let S be a locally noetherian scheme of dimension less or equal to 1, $G \to S$ a group scheme locally of finite type and $H \to S$ a flat subgroup of G. Then the *fppf*-sheaf G/H is represented by an S-scheme of finite type. For the proof we refer the reader to [Ana73], Theorem 4.c.
- The previous result can not be generalized in dimension bigger than 1. Let $S = \mathbb{A}^2_k$, and $G = (\mathbb{G}_{a,S})^2$, in [Ray70], Lemma X.14 is constructed a subgroup $N \subset G$ étale such that G/N is not representable by a scheme.

We are now ready to prove the following lemma, which will play a crucial role in Chapter 3 to treat a special reduction via the "fibration method", cf. [Bor96] Lemma 3.1.

Lemma 1.1.6. *Let G/k be a linear group, N a normal subgroup of G. Given a homogeneous space X/k under G, there exists a quotient $Y = X/N$ (i.e. a homogeneous space of G/N) and a smooth G-equivariant map $\varphi : X \to Y$ such that it is a quotient morphism in the sense of Definition 1.1.4. In particular it is surjective and its fibers are orbits of N.*

Proof. Let $\overline{H}$ the stabilizer of a $\overline{k}$-point of X, and consider the canonical quotient map $\pi_{\overline{H}} : G_{\overline{k}} \to G_{\overline{k}}/\overline{H} \cong X_{\overline{k}}$. Since N is normal, $\overline{H}_0 := \overline{H} \cdot N_{\overline{k}}$ is a subgroup of $G_{\overline{k}}$. In virtue of the universal property of the quotient maps, applied over $\overline{k}$, it make sense to consider the following natural commutative diagram:

$$\begin{tikzcd} G_{\overline{k}} \arrow[rr, "\pi_{\overline{H}_0}"] \arrow[dr, "\pi_{\overline{H}}"'] & & G_{\overline{k}}/\overline{H}_0 =: \overline{Y} \\ & G_{\overline{k}}/\overline{H} \cong X_{\overline{k}} \arrow[ur, "\overline{\varphi}"'] & \end{tikzcd}$$

.

Since $\pi_{\overline{H}}$ and $\pi_{\overline{H}_0}$ are smooth and surjective, the same holds for $\overline{\varphi}$; moreover the fibers of $\overline{\varphi}$ are the orbits of $N_{\overline{k}}$.

To conclude, applying Lemma 1.1.3 to show that is is a quotient map, it is enough to construct (Y, φ), a k-form of $(\overline{Y}, \overline{\varphi})$. To do this, notice that Γ_k acts continuously on $\overline{Y}$, via $\overline{\varphi}$, since it acts on $X_{\overline{k}}$. By Galois Descent this is enough to define, taking the fixed points of the action, a k-form (Y, φ). □

As in classical group theory we have the following characterization.

Lemma 1.1.7. *Let X/k be a homogeneous space under G. Then the following are equivalent:*

- *$X(k) \neq \emptyset$,*
- *X is isomorphic to the quotient variety G/H for some subgroup H of G.*

Remark. If G is a commutative group then every subgroup is normal. In particular every G-homogeneous space is a G-torsor.

We end this section introducing two important subgroups of a given group G.

Definition 1.1.8. Let G/k be a linear group, and H/k a subgroup. The functors *centralizer*, denoted with $C_G(H)$, and *normalizer*, denoted with $N_G(H)$, are representable by closed subgroups called the *centralizer of H in G* and the *normalizer of H in G*; in particular we define the *center* of G as $C_G(G)$. For the proof we refer the reader to [ABD+66] VIII.6.7.

1.2 Kernels and quotients

Theorem 1.2.1. *Let $f : G \to H$ a morphism of algebraic groups. Then it admits the following factorization*

$$G \twoheadrightarrow G/\operatorname{Ker} f \hookrightarrow H$$

where the first map is the projection to the quotient and the second one a closed immersion.

Proof. For the proof see [Gro62], Corollary 7.4. It make sense to consider the quotient $G/\operatorname{Ker} f$, thanks to Theorem 1.1.5. □

Another general result is the following.

Proposition 1.2.2. *Let $f : G \to H$ be a morphism of linear algebraic groups. Then $\operatorname{Im}(f) = f(G)$ is a Zariski closed subgroup of H.*

Proof. The Zariski-closure $\overline{\operatorname{Im}(f)}$ is a closed subgroup of H (exactly as for topological groups). To conclude it is enough to show that $\operatorname{Im}(f) = \overline{\operatorname{Im}(f)}$. By Chevalley's Theorem $\operatorname{Im}(f)$ is constructible, so it contains an open subset U of $\overline{\operatorname{Im}(f)}$. Since the set $\{f(g) \cdot U\}_{g \in G}$ is an open covering of $\operatorname{Im}(f)$, it follows that $\operatorname{Im}(f)$ is open and dense in $\overline{\operatorname{Im}(f)}$. By density, for any $h \in \overline{\operatorname{Im}(f)}$ the set $(h \cdot \operatorname{Im}(f)) \cap \operatorname{Im}(f)$ is not empty, hence h belongs to $\operatorname{Im}(f) \cdot (\operatorname{Im}(f))^{-1} \subset \operatorname{Im}(f)$. □

1.3 Connectedness

Definition 1.3.1. An algebraic group is *connected* if it underlying topological space is connected (for the Zariski topology).

Proposition 1.3.2. *Let G be a linear algebraic group.*

a) The irreducible components of G are pairwise disjoint, so they are the connected components of G.

b) The irreducible component G° containing 1_G is a closed normal subgroup of finite index in G.

c) Any closed subgroup H of G of finite index contains G°.

Proof. a) Let X, Y be two irreducible components of G with non trivial intersection. We can assume $1_G \in X \cap Y$. The image of $X \times Y$ via the multiplication $m : G \times G \to G$ is irreducible and contains X and Y and so, by maximality, we have $X = m(X \times Y) = Y$.

b) G° and $(G^\circ)^{-1}$ are irreducible components of G containing 1_G, hence they must coincide. Analogously $m(G^\circ \times G^\circ) \subset G^\circ$, therefore it is a subgroup of G. It is normal since $g^{-1}G^\circ g$ is irreducible and contains 1_G.

Since an affine algebraic variety has only finitely many maximal irreducible subsets it is enough to notice that every irreducible components is a translation of G°.

c) Since $|G : H^\circ| = |G : H| \cdot |H : H^\circ|$ is finite and $H^\circ \leq G^\circ \leq G$, being G° and H° connected, we have

$$G^\circ = \bigsqcup gH^\circ = H^0.$$

□

Remark. Notice that Theorem 1.0.6 implies the point *a*) of the previous proposition.

Remark. A connected linear algebraic group can not have proper closed subgroup of finite index. In particular, for a finite group G we always have $G^\circ = 1$.

Definition 1.3.3. The *derived subgroup* G' of an algebraic group G is the intersection of all the subgroups containing the scheme theoretical image of the map $G \times G \to G$, given by the association $(x, y) \mapsto [x, y] := xyx^{-1}y^{-1}$.

Proposition 1.3.4. *Let G be a connected linear group. Then G', the derived group of G, is a closed and connected subgroup of G.*

Proof. Consider the maps $f_i : G^{2i} \to G$ given by $(x_1, \ldots, x_i, y_1, \ldots, y_i) \mapsto [x_1, y_1] \ldots [x_i, y_i]$ for any $i \geq 0$. Each image $\mathrm{Im}(f_i)$ is closed, by Proposition 1.2.2, and connected, since it is the image of a connected; hence irreducible subgroup. Thus $\mathrm{Im}(f_1) \subset \cdots \subset \mathrm{Im}(f_i) \subset \ldots$ is a chain of connected subgroups, and G' is their union. The chain stabilizes, since it is bounded by $\dim(G)$, hence G' is a closed and connected subgroup of G. □

Proposition 1.3.5. *Consider an exact sequence of linear algebraic group*

$$1 \to A \to B \to C \to 1.$$

If A and C are connected so does B; conversely, if B is connected, so also is C.

Proof. Consider the exact sequence defining the group of connected components of B:

$$1 \to B^\circ \to B \xrightarrow{\pi} \pi_0(B) \to 1.$$

Assume A and C to be connected. Since A lies in the kernel of $B \xrightarrow{\pi} \pi_0(B)$, then, by the universal property of quotients, we have the factorization $\pi : B \to B/A \cong C \to \pi_0(B)$; but C is connected, hence $\mathrm{Im}(\pi) = 1$. It follows that the inclusion $B^\circ \to B$ is an isomorphism.

The other implication is clear since C is the image of B via the projection map. □

1.4 Unipotent groups

Theorem 1.4.1 (Characterizations of unipotent groups)**.** *Let G/k a linear algebraic group and k of characteristic zero. The following are equivalent.*

a) G is unipotent;

b) G admits a closed embedding in $\mathbb{U}_n$ for some $n \geq 0$;

c) G admits a finite composition series such that each successive quotient is isomorphic to $\mathbb{G}_a$;

d) every element of $G(\overline{k})$ is unipotent, i.e. every $u \in G(\overline{k})$ satisfies $(u-1)^n = 0$;

e) the underlying k-scheme of G is isomorphic to $\mathbb{A}^n_k$ for some $n \geq 0$.

Proof. For the proof see [Poo16], Theorem 5.4.8. □

Corollary 1.4.2. *Unipotent groups, over a field of characteristic zero, are connected.*

Proof. It is clear from the above characterization, using e). □

Corollary 1.4.3. *Subgroups, quotients, and extensions of unipotent groups are unipotent.*

Proof. For subgroups is clear from the above characterization, using b). For quotients it follows from c). More details can be found in [Mil11], Corollary 15.7. □

It is not hard to prove the following.

Proposition 1.4.4. *Let k a field of characteristic zero. If G is a commutative unipotent group over k, then $G \cong (\mathbb{G}_a)^n$, as group schemes, for some $n \geq 0$.*

A general fact is the following. For a more standard and elementary approach see for example [PR94], section 2.2.

Proposition 1.4.5. *Let k a field of characteristic zero. For any unipotent k-group G we have $H^1(k, G) = 0$. If G is commutative unipotent then $H^i(k, G) = 0$ for any $i \geq 1$.*

Proof. First of all notice that $H^i(k, \mathbb{G}_a) = 0$ for any $i > 0$. This is clear since $\mathbb{G}_a$ is a quasi-coherent module over k, hence its cohomology can be computed on the Zariski site; fields have trivial Zariski-cohomological dimension since are irreducible topological spaces. We conclude by induction on dimension of G: By Theorem 1.4.1 we can write an exact sequence

$$1 \to H \cong \mathbb{G}_a \to G \to G/H \to 1$$

which yields an exact sequence of pointed sets

$$H^1(k, H) \to H^1(k, G) \to H^1(k, G/H)$$

Since the term on the left is zero by what is proved above, and the right one also by induction, we have that $H^1(k, G)$ is trivial. For the second statement the same argument applies to the cohomological exact sequence of above for any $i \geq 1$ (alternatively one can use Proposition 1.4.4). □

These two results explain why, for our purpose, the unipotent part of an algebraic group plays no role and lead us to the definition of the next section.

Remark. The proof of Proposition 1.4.5 shows also that any $\mathbb{G}_{a,k}$-torsor over an affine k-variety is trivial, since, over an affine scheme X, the functor $\Gamma(X, -) : \mathrm{Qcoh}(X) \to \mathrm{Ab}$ is exact (and provides the same cohomology as the derived functor of $\Gamma(X, -) : \mathrm{Sh}(X) \to \mathrm{Ab}$, when restricted to the quasi-coherent modules).

We end this section with some useful results.

Definition 1.4.6. An algebraic group G/k is called *nilpotent* if it admits a central series of finite length, i.e. consider

$$\gamma_1 = G, \ \gamma_2 = [G, G], \ldots, \gamma_n = [\gamma_{n-1}, G], \ \ldots \tag{1.4.0.1}$$

then there exists k such that $\gamma_k = 1$.

By point c) of Theorem 1.4.1 it is clear that unipotent groups are nilpotent.

Proposition 1.4.7. *Let G be an algebraic non trivial unipotent k group. Then the center of G is non trivial.*

Proof. Since G is nilpotent, we can consider the central series 1.4.0.1. Chose n such that $\gamma_n = 1$ and $\gamma_{n-1} \neq 1$. γ_n is trivial if and only if for all $x \in \gamma_{n-1}$ and for all $g \in G$, $x^{-1}g^{-1}xg = 1$, or, equivalently $xg = gx$. The last condition means that γ_{n-1} is a subgroup of $Z(G)$. □

1.5 Reductive groups

Definition 1.5.1. A linear algebraic group G is *solvable* if there exists a finite sequence of algebraic groups

$$1 = G_n \subset G_{n-1} \subset \cdots \subset G_0 = G$$

such that each G_{i+1} is normal in G_i and G_i/G_{i+1} is commutative.

Example. Subgroups, quotients, and extensions of solvable algebraic groups are solvable.

To give sense to the next definitions we need a preliminary result. In this section k will always denote a field of characteristic zero.

Proposition 1.5.2. *Let G/k be an algebraic group.*

a) There exists a largest connected normal solvable subgroup of G, denoted RG and called the radical *of G.*

b) There exists a largest connected normal unipotent subgroup, denoted R_uG and called the unipotent radical *of G.*

The idea is to consider the product of all connected normal solvable (or unipotent) subgroups of G. This amounts to prove the following lemma.

Lemma 1.5.3. *Let N and H be algebraic subgroups of G with N normal. If H and N are solvable (resp. unipotent, resp. connected), then HN is solvable (resp. unipotent, resp. connected).*

Proof. It is enough to consider the following exact sequence

$$1 \to N \to HN \to HN/N \cong H/H \cap N \to 1$$

where the last isomorphism can be checked at the level of group functor thanks to the construction of the quotients. As remarked above, if H is solvable, so also is its quotient $H/H \cap N$; this concludes since HN is an extension of solvable groups.

The same argument works also for unipotent groups (thanks to Corollary 1.4.3,) and for connected (thanks to Proposition 1.3.5). □

Definition 1.5.4. A *semisimple* group is a connected linear algebraic group whose geometric radical, $R(G_{\overline{k}})$, is trivial. A *reductive* group is a connected linear algebraic group whose geometric unipotent radical is trivial.

Remark. Semisimple groups, in particular, are reductive.

Another useful construction is the following.

Theorem 1.5.5. *Let G be a connected solvable group, then there exists a unique connected normal algebraic subgroup G_u of G such that*

- *G_u is unipotent,*
- *G/G_u is of multiplicative type (in the sense of Definition 1.7.1).*

Taking G_u commutes with base change of the field.

Proof. See [Mil11], Theorem 16.33. □

Proposition 1.5.6. *Let G an algebraic group, radicals commute with base change of the base field. Moreover $R_uG = (RG)_u$.*

Proof. The result follows by uniqueness and maximality of the objects involved and the usual Galois Descent's theorem. □

Immediately we have.

Corollary 1.5.7. *Let G be a connected algebraic group (over a perfect field k).*

- *G is semisimple if and only if $RG = 1$.*
- *G is reductive if and only if $R_uG = 1$.*

Example. Let G be connected algebraic group, the quotient group G/RG is semisimple, and G/R_uG is reductive.

1.6 Algebraic tori

Definition 1.6.1. A *split torus* is an algebraic group isomorphic to a finite product of copies of $\mathbb{G}_m$. A *torus* is a k-algebraic group such that $T_{\overline{k}}$ is a split torus.

Remark. If k is an algebraically closed field, then all tori are split.

Let G be an algebraic group over k. A k-torus $T \subset G$ is said to be *maximal* if $T_{\overline{k}}$ is maximal in $G_{\overline{k}}$, i.e. $T_{\overline{k}}$ is not properly contained in any other torus.

Theorem 1.6.2 (Existence of maximal k-tori)**.** *Let G be a smooth connected affine group over an arbitrary field k. Then G contains a maximal k-torus T.*

Proof. For the proof, in this generality, see [Con10] Theorem 1.1. If k is assumed to be perfect easier arguments are possible. □

Theorem 1.6.3 (Conjugacy Theorem for maximal tori)**.** *Let $\overline{G}$ be a connected group over an algebraically closed field $\overline{k}$. If T and T_0 are maximal tori in $\overline{G}$, then $T_0 = g^{-1}Tg$ for some $g \in \overline{G}(\overline{k})$.*

Proof. For the proof see [Spr98], Theorem 6.3.5. We can deduce this result from Theorem 1.9.2... □

1.6.1 The variety of maximal tori, weak approximation

Let G be a reductive k-group, $T \subset G$ a maximal k-torus and $N = N_G(T)$ its normalizer. Thanks to Theorem 1.6.3, the map $g^{-1}Tg \mapsto gN$ gives a bijection between the maximal tori of G and the points of the *k-variety of maximal tori*, $\mathcal{T} := G/N$. Notice that the k-points of $\mathcal{T}$ correspond to the maximal k-tori of G; in particular it does not depend on the choice of the k-torus T.

Theorem 1.6.4. *In characteristic zero, $\mathcal{T}$ is a rational smooth k-rational variety, i.e. biregularly isomorphic to an affine space.*

Proof. It is smooth since it is an homogeneous space under G, smooth group (see Section 1.1). The rationality is established in [PR94], Theorem 2.18, for a sketch of proof see for example [Gil07b], Theorem 2.3. □

It is a classical result that if X is a irreducible, smooth k-rational variety, then it satisfies weak approximation. We just sketch the argument: The k-rationality of X implies the existence of a biregular k-isomorphism between a Zariski open subset U of $\mathbb{A}^{\dim(X)}$ and a Zariski open subset W of X. $\mathbb{A}^{\dim(X)}$ has the weak approximation property since it is a product of $\mathbb{A}^1$ which does (a classical result of elementary number theory), and since the v-adic topology on the k_v-points of $\mathbb{A}^{\dim(X)}$ is stronger than the Zariski topology also U has the weak approximation property; by biregularity the same holds for W, i.e. $W(k)$ is dense in $\prod_{v\in\Omega_k} W(k_v)$. To conclude it is enough to prove that $W(k_v)$ is dense in the k_v-points of X, but, since X is smooth, this can be done thanks to the Implicit Function Theorem, see for example [PR94], Lemma 3.2.

Weak approximation for $\mathcal{T}$, which is satisfied by the above discussion, can be expressed in a nice way.

Corollary 1.6.5. *Let S be any finite subset of Ω_k. Given $T(v)$ a maximal k_v-torus of G for all $v \in S$, there exists a maximal k-torus T of G which, for any $v \in S$, is conjugate to T_v via a k_v-point of G.*

Proof. For any $v \in S$, let x_v be the point in $\mathcal{T}$ corresponding to $T(v)$. Every torus in the conjugacy class $\{g^{-1}T(v)g\}_{g\in G(k_v)}$ corresponds exactly to a point of the orbit $U_v = G(k_v)\cdot x_v$. If U_v is open in $\mathcal{T}$, then, by the weak approximation on $\mathcal{T}$, there exists $x \in \mathcal{T}(k) \cap \prod_{v\in S} U_v$, and it corresponds to the torus T we were looking for. We conclude since we can apply [PR94], Proposition 3.3, Corollary 2. on U_v. □

Remark. Notice that weak approximation for $\mathcal{T}$ holds independently of the fact that this property holds or not on G. This will remain an isolated phenomenon: in general we try to deduce arithmetic properties on an homogeneous space from properties of the group acting on it.

1.7 Groups of multiplicative type

Definition 1.7.1. A *group of multiplicative type* over k is a commutative linear k-group which is an extension of a finite group by a torus.

Recall that the *module of characters* of G is the abelian group $\widehat{G} = \mathrm{Hom}(G_{\overline{k}}, \mathbb{G}_m)$, equipped with the action of the Galois group Γ_k (induced by the action of Γ_k on $\mathbb{G}_m$), it will be also denoted with $X^*(G)$. Analogously the *module of co-characters* will be denoted with $X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G_{\overline{k}})$.

We state two classical results about groups of multiplicative type.

Theorem 1.7.2 (Rosenlicht's Theorem)**.** *Let X, Y be geometrically irreducible k-varieties, then $\overline{k}[X]^*/\overline{k}^*$ is a free abelian group of finite rank and there is an isomorphism of groups*

$$\overline{k}[X \times_k Y]^*/\overline{k}^* \cong \overline{k}[X]^*/\overline{k}^* \times \overline{k}[Y]^*/\overline{k}^*.$$

If G is a connected linear group we have

$$k[G]^*/k^* \cong \widehat{G}(k).$$

Proof. For the proof see [Vos98] I, 3.4; cf. also [San81], Lemma 6.5. $\square$

Theorem 1.7.3. *Let G be a k-group. The association $G \rightsquigarrow \hat{G}$ gives an equivalence of category between the category of k-groups of multiplicative type and the category of discrete Γ_k-modules of finite type. Moreover a sequence of groups of multiplicative type is exact if and only if the dual of Γ_k-modules of characters is exact.*

The following result will be useful in many occasion, in particular applied to the inner action to show that some subgroups of a connected linear group are central.

Theorem 1.7.4 (Rigidity)**.** *Let G be a connected (linear) k-group G, and $N \subset G$ a normal k-subgroup of multiplicative type. Then N is central in G.*

We prove the result thanks to two lemmas.

Lemma 1.7.5. *If G is connected and N a finite normal subgroup, then N is central.*

Proof. For any $a \in N$ consider the map $G \to G$ given by $x \mapsto x^{-1}ax$. Notice that the image of the map is contained in N (since N is normal), contains 1_G (taking $x = 1_G$), contains just one point (the image of a connected is connected but N is finite). Hence a commutes with any element of G. $\square$

Lemma 1.7.6. *Let M be a finitely generated $\mathbb{Z}$-module. The only map $M \to M$ that fits in the commutative diagrams*

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\pi"] & M \arrow[d, "\pi"] \\ M/nM \arrow[r, "\mathrm{Id}"] & M/nM \end{tikzcd}$$

for all $n \in \mathbb{N}$, is the identity map.

Proof. We may assume M to be torsion free. Let $\alpha : M \to M$ be a map fitting in the diagrams, we can write it as a matrix $(a_{i,j})$, with $a_{i,j} \in \mathbb{Z}$. By assumption we have that $a_{i,j} = 0 \mod m$ for all $i \neq j$, for all $m \in N$, i.e. $a_{i,j} = 0$, analogously for $i = j$ we have $a_{i,i} = 1$. The result is proven. $\square$

Thanks to the equivalence of categories displayed by Theorem 1.7.3, we have that the only endomorphism of a group of multiplicative type whose restriction to the n-torsion is the identity map for any $n \in \mathbb{N}$, is the identity.

Proof of Theorem 1.7.4. Thanks to the first lemma the n-torsion of N is central in G for any $n \in \mathbb{N}$. The union of all the n-torsion point of N is still central, and so does their Zariski-closure. In virtue of the second lemma, this is enough to conclude. $\square$

Remark. Notice that the groups of multiplicative type are not closed by extensions. For example let $\sigma : \mathbb{G}_m \to \mathbb{G}_m$ be the automorphism given by $x \mapsto x^{-1}$, $\mathbb{G}_m \rtimes \langle \sigma \rangle$ is not even commutative.

Another property for a torus we will consider is the following.

Definition 1.7.7. A k-torus T is called *anisotropic* if $X^*(T)^{\Gamma_k}$ is trivial.

Proposition 1.7.8. *Every torus admits a largest split sub-torus T_s and a largest anisotropic sub-torus T_a. Moreover $T_s \cap T_a$ is finite and $T = T_s \cdot T_a$ (over k).*

Proof. For the proof see [Mil11], Corollary 14.26. □

1.8 More about semisimple groups

We can characterise this properties in terms of commutative subgroups (if k is not perfect, also in the smooth case, just one implication holds) as follows.

Proposition 1.8.1. *Let G be a connected algebraic group. The following hold.*

a) G is semisimple if and only if every connected normal commutative subgroup is trivial.

b) G is reductive if and only if every connected normal commutative subgroup is a torus.

Proof. We prove just the easy implication, for the converse we refer the reader to [Mil11], Proposition 17.7. In a) , if G is semisimple and $H \subset G$ connected normal, then $H_{\overline{k}} = 1$, hence $H = 1$. In b), If G is reductive and $H \subset G$ is as above, then $H_{\overline{k}} \subset RG_{\overline{k}}$, which has no unipotent subgroups. □

Theorem 1.8.2. *If G is reductive then the derived group G' of G is semisimple, the connected center $Z(G)^\circ$ of G is a torus, and $Z(G) \cap G'$ is the (finite) center of G'; moreover*

$$G = Z(G)^\circ \cdot G'.$$

Proof. As usual it is enough to prove the result over the algebraic closure of k. Since RG is connected and $(RG)_u = 0$, then Theorem 1.5.5 ensure that RG is a torus. By Theorem 1.7.4 is contained in $Z(G)^\circ$ and by maximality of RG we conclude that $Z(G)^\circ = RG$ is a torus.

Notice that $Z(G)^\circ \cdot G'$ is a normal subgroup of G and the quotient $G/Z(G)^\circ \cdot G'$ is trivial: It is commutative (being a quotient of G/G') and semisimple (being a quotient of $G/Z(G)^\circ = G/RG = G^{\text{ss}}$, since G is reductive). Hence the decomposition $G = Z(G)^\circ \cdot G'$ is proven.

Consider the exact sequence (using the isomorphism as in the proof of Lemma 1.5.3)

$$1 \to Z(G)^\circ \cap G' \to G' \to G/RG \to 1,$$

since $Z(G)^\circ \cap G'$ is finite (for this computation we refer the reader to [Mil11], the proof of Theorem 17.20), G' must be semisimple. Of course $Z(G) \cap G'$ is contained in $Z(G')$, but we have also $Z(G') \subset Z(G)$, since $G = Z(G)^\circ \cdot G'$ and $Z(G)^\circ$ is commutative; hence $Z(G) \cap G'$ is the center of G'. The result is proven. □

We can summarise the properties proven until now in the following proposition.

Proposition 1.8.3. *Let G be a reductive $\overline{k}$-group. The following are equivalent.*

- *G is semisimple;*
- *there is no normal subtorus $\mathbb{G}_m \subset G$;*
- *the center of G is a finite group scheme;*
- *the derived subgroup of G is equal to G;*
- *G is character free;*
- *any invertible function on G is constant.*

Proof. We have already proven that, for a semisimple group G, $G = G^{\text{ss}}$. It follows that $\widehat{G} = 1$; Rosenlicht's Theorem implies that $k[G]^* \cong k^*$. The previous proposition shows that, if G is semisimple, $G = G'$. For missing equivalences we refer the reader to [CT07], Proposition 4.17. □

1.9 Borel and parabolic subgroups

Definition 1.9.1. A *Borel subgroup* of a linear algebraic group G is a subgroup B such that $B_{\overline{k}}$ is a maximal connected solvable subgroup of $G_{\overline{k}}$.

The main goal of this section is to prove the following results.

Theorem 1.9.2. *Let G a reductive group over k.*

a) The quotients by a Borel subgroups produce a projective variety.

b) Any two Borel subgroups of G are conjugate by an element of $G(\overline{k})$.

Remark. It is not possible to obtain an analogue of Theorem 1.6.2 for Borel subgroups. Indeed *quasi-split groups* over a field are reductive groups with a Borel subgroup defined over the field. In [Ser73], III-2.2 Theorem 1 is proven that, for a perfect field k, each linear k-group contains a Borel subgroup defined over k if and only if $H^1(k, G) = 0$ for all connected linear group L. In particular this implies that k has cohomological dimension smaller or equal to 1, hence it is possible to find a non quasi-split group, for example, over any number field.

Theorem 1.9.3. *For any Borel subgroup B of G, $G = \cup_{g\in G(\overline{k})} g^{-1}Bg$.*

Proof. For the proof see [Mil11], V. Theorem 3.24. □

Remark. Bruhat decomposition.

In the case $k = \overline{k}$ a generalization of Borel subgroups is also useful.

Definition 1.9.4. Let $\overline{G}$ be a group over $\overline{k}$. A subgroup P of $\overline{G}$ is *parabolic* if the quotient variety $\overline{G}/P$ is projective.

Theorem 1.9.5. *Let $\overline{G}$ be a connected algebraic $\overline{k}$-group and P a subgroup. Then P is parabolic if and only if it contains a Borel subgroup.*

Proof. For the proof see [Mil11], V-Theorem 3.27. The only missing ingredient to prove this result is the Borel fixed point theorem. □

1.10 Split reductive groups

Definition 1.10.1. A reductive group is *split* if it contains a split maximal torus.

Remark. Having a torus that is maximal among the split tori, i.e. a maximal split torus, is a different property.

Example. Any reductive group over an algebraically closed field, thanks to Remark 1.6.

Example. GL_n is split reductive (over any field), with maximal split torus $\mathbb{D}_n$. SL_n is a split semisimple group, with split maximal torus given by the diagonal matrices of determinant 1.

1.10.1 Kneser-Tits conjecure

In this section we briefly describe a result we will use in what follows. For more about this topic we refer the reader to [Gil07a] and [Tit77].

For a (connected) reductive k-group G we write $G^+(k)$ for the normal subgroup of $G(k)$ generated by $U(k)$, where U runs trough the k-subgroups of G isomorphic to the additive group $\mathbb{G}_a$. The quotient

$$W(G, k) := G(k)/G^+(k)$$

is called the *Whitehead group* of the group G/k. The original conjecture of Kneser-Tits asserts that $W(G, k) = 1$ for any simply connected group G such that $G(k)/Z(G(k))$ is simple.

In the exposé of Tits, [Tit77], are presented many evidences of the conjecture (and also counterexamples, due to Platonov); in particular we are interested in the following.

Theorem 1.10.2. *Let G be a split reductive group on a field of characteristic 0, then $W(G, k) = 1$.*

Proof. For the proof see [Tit77], Theorem 1.1.2. □

Remark. Notice that any k-group G with trivial Whitehead group has the following interesting property: It is "$\mathbb{A}^1_k$-connected", in the sense that for any $x, y \in G(k)$ there are algebraic maps

$$h_1, \ldots, h_m : \mathbb{A}^1_k \to G$$

such that $h_1(0) = x$, $h_i(1) = h_{i+1}(0)$ for $i = 1, \ldots, m-1$ and $h_m(1) = y$.

1.11 Root Data from algebraic group

1.11.1 Lie algebra associated and the adjoint representation

Denote with $k[\epsilon]$ the ring of dual numbers and denote $\pi : k[\epsilon] \to k$ the map given by $\epsilon \mapsto 0$, and $i : k \to k[\epsilon]$ the canonical inclusion.

Definition 1.11.1. Let G an algebraic group, its *Lie algebra* is $\mathrm{Lie}(G) = \mathrm{Ker}(G(k[\epsilon]) \to G(k))$.

The following proposition is just a corollary of the general theory of algebraic derivations.

Proposition 1.11.2. *Let I be the kernel of $\epsilon : \mathcal{O}(G) \to k$. Then* $\mathrm{Lie}(G)$ *is isomorphic to $Hom_k(I/I^2, k)$, in particular it is a k-vector space.*

Example. $\mathrm{Lie}(\mathbb{G}_a) = k$ and $\mathrm{Lie}(\mathrm{GL}_n) = \mathfrak{gl}_n$, the Lie algebra of $n \times n$-matrices over k with the Lie bracket given by $[X, Y] = XY - YX$; $\mathrm{Lie}(\mathrm{SL}_n) = \mathfrak{sl}_n := \{A \in \mathfrak{gl}_n \mid \mathrm{tr}(A) = 0\}$

Given an algebraic group G we have a canonical morphism of algebraic group

$$Ad : G \to \mathtt{Aut}(\mathrm{Lie}(G))$$

given by

$$x \mapsto Ad(x)$$

where $Ad(x)$ is the automorphism defined by $y \mapsto i(x)yi(x)^{-1}$. We call the map Ad the *adjoint representation.*

1.11.2 Root Data

Regarding the root datum we follow [Spr79], section 1 and 2.

Definition 1.11.3. A *root datum* is a quadruple $\Psi = (X, R, X^\vee, R^\vee)$ where: X and $X^\vee$ are free $\mathbb{Z}$-modules of finite rank in duality by a pairing $X \times X^\vee \to Z$ denoted by $\langle , \rangle$, R and $R^\vee$ are finite subsets of X and $X^\vee$ in bijection $\alpha \leftrightarrow \alpha^\vee$. For $\alpha \in R$ define an endomorphism s_α of X, respectively $s_{\alpha^\vee}$ of $X^\vee$, by

$$s_\alpha : x \mapsto x - \langle x, \alpha^\vee \rangle \alpha, \quad s_{\alpha^\vee} : y \mapsto y - \langle \alpha, y \rangle \alpha^\vee.$$

And impose the following axioms:

(RD1) For all $\alpha \in R$, $\langle \alpha, \alpha^\vee \rangle = 2$;

(RD2) For all $\alpha \in R$, $s_\alpha(R) \subset R$ and $s_{\alpha^\vee}(R^\vee) \subset R^\vee$.

Remark. From the first axiom we have that $s_\alpha(\alpha) = -\alpha$, $s_\alpha^2 = 1$ and $s_\alpha(x) = x$ if $\langle x, \alpha^\vee \rangle = 0$. Hence s_α should be thought as an abstract reflection in the hyperplane orthogonal to $\alpha^\vee$.

The main goal of this section is to associate to each pair (G, T) made of a reductive group G and a split maximal torus T, a root datum $\Psi(G, T)$. If we fix also a Borel subgroup containing T, we can obtain something more: a *based root datum.*

Let (G, T) be a split reductive group, and consider the adjoint representation

$$Ad : G \to \mathrm{GL}(\mathrm{Lie}(G)).$$

Recall the standard fact that semisimple element (i.e. diagonalizable over $\overline{k}$) are simultaneously diagonalizable if and only if they commute (see, for example, [Con14], Theorem 5.1). Since the image of T via Ad is made by semisimple commutative element an action on $\mathrm{Lie}(G)$ be can consider the decomposition

$$\mathrm{Lie}(G) = \mathfrak{g}_0 \oplus \bigoplus_\chi \mathfrak{g}_\chi,$$

where $\mathfrak{g}_0$ is the subspace on which T acts trivially and $\mathfrak{g}_\chi$ is the subspace with the action given by a non trivial character χ. The non zero χ appearing in this decomposition are called *roots* of (G,T), denoted with R; it is a finite subset of $X^*(T)$. This association determines the root datum $\Psi(G,T)$ attached to (G,T).

The importance of the root data is due to the following important result.

Theorem 1.11.4 (Chevalley Classification Theorem)**.** *Two semisimple linear algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum there exists a semisimple algebraic group which realizes it.*

Proof. For the proof see [Spr79], Theorem 2.9 *i*). □

1.11.3 Isogeny, More Root Data

Definition 1.11.5. A surjective homomorphism $G \to H$ of algebraic k-groups with finite kernel is called an *isogeny*. Notice that, by Theorem 1.7.4, we have that (in characteristic zero) every isogeny is central.

In the category of split reductive groups we define a *central isogeny* of (G,T) to (G',T') as an isogeny $G \to G'$ such that $\mathrm{Im}(T) = T'$. In terms of root data the following definition make sense: Let $\Psi = (X,R)$, $\Psi' = (X',R')$ be two root data, a homomorphism $f : X' \to X$ is called an *isogeny* of Ψ' in Ψ if

- f is injective and $\mathrm{Im}\, f$ has finite index in X;
- f induces a bijection of R and R', and its transpose a bijection between $R^\vee$ and $R'^\vee$.

Proposition 1.11.6. *If ϕ is a central isogeny of (G,T) onto (G',T'), then $\phi^* : \chi^*(T) \to \chi^*(T')$ is an isogeny of $\Psi(G',T')$ into $\Psi(G,T)$. Moreover, given two isognenies ϕ, ϕ' as before, if $\phi^* = \phi'^*$ then they differ by conjugation by an element of t, i.e. there exists $t \in T$ with $\phi' = \phi \circ \mathrm{Int}(t)$.*

Proof. For the proof see [Spr79], Proposition 2.5 and Theorem 2.9 *ii*). □

Corollary 1.11.7. *There is a split exact sequence*

$$1 \to \mathtt{Int}(G) \to \mathtt{Aut}(G) \to \mathtt{Aut}(\phi_0(G)) \to 1,$$

where $\phi_0(G)$ is the based rood datum associated to a triple (G,B,T), with G a connected reductive algebraic group.

Proof. See [Spr79], Proposition 2.13, Corollary 2.14. □

1.12 Important properties of semisimple simply connected algebraic groups

Definition 1.12.1. Let G be semisimple k-group, we say that G is *semisimple simply connected* if there is no connected finite étale Galois cover of $G_{\overline{k}}$. An arbitrary (connected) k-group G is called *simply connected* if it is an extension of a semisimple simply connected group by an unipotent. In particular a reductive simply connected group must be semisimple.

The properties of G, a semisimple simply connected algebraic group defined over a field k of characteristic zero, we will use more often are the following:

- $k[G]^* \cong k^*$;
- $\mathrm{Pic}(G) = 0$;
- $\mathrm{Br}(G) \cong \mathrm{Br}(k)$.

The first property is clear and holds for any semisimple group, as proven in Proposition 1.8.3. The last two require more work, we devote this section to their proofs.

Remark. Suppose G is a k-group such that $k[G]^* = k^*$ and $\mathrm{Pic}(G) = 0$. Of course those assumptions are satisfied by any unipotent group, but if we suppose G to be reductive then G must be a semisimple simply connected algebraic group. The proof of this fact will be clear using Proposition 1.8.3 and 1.12.5.

1.12.1 The existence of a simply connected covering

The results discussed in this section are useful in many situation.

Theorem 1.12.2. *Let G be a semisimple k-group. There exists a uniquely defined isogeny $G^{sc} \to G$, where G^{sc}/k is a semisimple simply connected k-group. The kernel is a finite k-group of multiplicative type whose character group is the Γ_k-module given by* $\mathrm{Pic}(G_{\overline{k}})$.

Proof. For the proof see [CT07], Proposition 4.22. Over $\overline{k}$ see also [FI73], Section 4. □

Studying arithmetic properties of a group G, it is quite common to encounter "covering" of a group G, i.e. short exact sequences

$$1 \to B \to \widetilde{G} \to G \to 1$$

with different properties on B (eg. finite or a torus) and on $\widetilde{G}$ (eg. semisimple simply connected, a product of a quasi trivial torus and a semisimple simply connected group). The key role of such coverings is to deduce arithmetic properties on G proving the same properties on $\widetilde{G}$, taking in account the subgroup B. In particular in Chapter 3 we will need the following.

Theorem 1.12.3. *Let G be a reductive k-group. There exists a central extension of k-groups*

$$1 \to Z \to G_1 \to G \to 1$$

such that G_1^{ss} is simply connected and N is a torus.

Proof. For the proof of this result, originally proved by Langlands, we refer to [MS82], Proposition 3.1. □

If the group G is not reductive, we can refine the previous result as follows.

Theorem 1.12.4. *Let G be a connected k-group. There exists an extension*

$$1 \to Z \to G_0 \to G \to 1$$

where $(G_0)^{ssu}$ is simply connected and Z is a torus.

Proof. Thanks to the Levi decomposition (see [PR94], Theorem 2.3), we can write G as the semidirect product $G^{\mathrm{u}} \rtimes G^{\mathrm{red}}$. Consider the exact sequence produced by the previous result:

$$1 \to Z \to G_1 \to G^{\mathrm{red}} \to 1,$$

it is enough to define G_0 as the semidirect product $G^{\mathrm{u}} \rtimes G_1$. □

1.12.2 The Picard Group of a semisimple simply connected algebraic group

The main goal of this section is to prove that $\mathrm{Pic}(G)$ is trivial when G is a semisimple simply connected algebraic group defined over a number field k.

Proposition 1.12.5. *For a semisimple $\overline{k}$-group $\overline{G}$, the following properties are equivalent.*

- *$\overline{G}$ is semisimple simply connected;*
- *There is no non-trivial isogeny $\overline{G}_1 \to \overline{G}$;*
- $\mathrm{Pic}(\overline{G}) = 0$.

Proof. For the proof see [CT07], Proposition 4.19. See also Lemma 6.9, (iii) of [San81]. □

To show that the Picard group vanishes already over k, we recall the exact sequence of [San81], Lemma 6.9 (i); where $\mathrm{Br}_a(G)$ is defined as $\mathrm{Ker}(\mathrm{Br}(G) \to \mathrm{Br}(G_{\overline{k}}))/\mathrm{Br}(k)$.

Lemma 1.12.6. *Let G/k be a connected linear group. The following sequence is exact:*

$$0 \to H^1(k, \widehat{G}) \to \mathrm{Pic}(G) \to \mathrm{Pic}(G_{\overline{k}})^{\Gamma_k} \to H^2(k, \widehat{G}) \to \mathrm{Br}_a(G) \to H^1(k, \mathrm{Pic}(G_{\overline{k}})) \to H^3(k, \widehat{G}).$$

Proof. The sequence displayed is nothing else, using Rosenlicht's Lemma, than the quotient of the low degree exact sequence of the Hochshild-Serre spectral sequence associated to the Galois covering $G_{\overline{k}} \to G$. $\square$

Since G is semisimple, Proposition 1.8.3 implies that $\widehat{G} = 1$. In particular the exact sequence displayed above shows that $\mathrm{Pic}(G)$ and $\mathrm{Pic}(G_{\overline{k}})^{\Gamma_k} = 0$ are canonically isomorphic. The result is proven.

Example. We prove that $\mathrm{SL}_{n,k}$ is semisimple simply connected. Thanks to Proposition 1.8.3 and 1.12.5, it is enough to prove that it is a semisimple group with trivial Picard group.

Notice that $\mathrm{SL}_{n,k}$ is reductive, since it is a normal subgroup of $\mathrm{GL}_{n,k}$ which is a reductive group, and semisimple, since its center is $\mu_{n,k}$, a finite group scheme. As a k-variety $\mathrm{GL}_{n,k}$ is isomorphic to $\mathrm{SL}_{n,k} \times \mathbb{G}_m$, in particular we have $0 = \mathrm{Pic}(\mathrm{GL}_{n,k}) \cong \mathrm{Pic}(\mathrm{SL}_{n,k}) \oplus \mathrm{Pic}(\mathbb{G}_m)$, where the first equality holds since $\mathrm{Pic}(\mathbb{A}_k^{n^2})$ surjects onto $\mathrm{Pic}(\mathrm{GL}_{n,k})$, and the second using [San81], Lemma 6.6. It follows that $\mathrm{Pic}(\mathrm{SL}_{n,k}) = 0$.

1.12.3 The Brauer Group of a semisimple simply connected algebraic group

The main goal of this section is to prove that $\mathrm{Br}(G)$ is canonically isomorphic to $\mathrm{Br}(k)$ when G is a semisimple simply connected group and k a number field. The proof is taken from [Gil09], see Theorem 4.3.

Theorem 1.12.7 (Bhatt)**.** *Let $\overline{k}$ be an algebraically closed field of characteristic zero. Let $X/\overline{k}$ be an integral, quasi compact, regular scheme with function field $E = \overline{k}(X)$. Let $\xi \in X(E)$ be the generic point.*

Assume there exists a rational point $x \in X(\overline{k})$ and morphisms

$$h_1, \dots, h_m : \mathbb{A}_E^1 \to X$$

such that $h_1(0) = x$, $h_i(1) = h_{i+1}(0)$ for $i = 1, \dots, m-1$ and $h_m(1) = \xi$. Then $\mathrm{Br}(X) = 0$.

Before proving the theorem we recall two useful facts; for more details about the proposition see [Mil80], Example 2.22.

Theorem 1.12.8. *Let E be a field of characteristic zero.* $\mathrm{Br}(\mathbb{A}_E^n)$ *is isomorphic to* $\mathrm{Br}(E)$ *via the structural map $\mathbb{A}_E^n \to E$.*

Proof. For the proof see [AG60], Theorem 7.5. $\square$

Proposition 1.12.9. *Let k be any field. Let X/k be an irreducible, quasi compact, regular scheme and $j : \xi \to X$ the inclusion of the generic point. Then we have an exact sequence of étale X-sheaves*

$$0 \to \mathbb{G}_m \to j_*\mathbb{G}_m \to \bigoplus_{x \in X^{(1)}} (i_x)_*\mathbb{Z} \to 0.$$

The short exact sequence displayed above, combined with the Leray spectral sequence, implies the exactness of the following

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(E).$$

Moreover we notice that the map $\mathrm{Br}(X) \to \mathrm{Br}(E)$ is the one induced in cohomology by $\xi \to X$.

Proof of Bhatt. Thanks to the discussion above, it is enough to prove that the map induced by the inclusion of the generic point $\mathrm{Br}(X) \to \mathrm{Br}(E)$ is zero. We denote with $s_0, s_1 : \overline{k} \to \mathbb{A}_{\overline{k}}^1$ the rational points 0 and 1, analogously for $s_{0,E}$, $s_{1,E} : E \to \mathbb{A}_E^1$.

Claim 1. Let $f : \mathbb{A}_E^1 \to X$ be such that $s_{0,E}^* \circ f^* : \mathrm{Br}(X) \to \mathrm{Br}(E)$ is the zero map, then also $s_{1,E}^* \circ f^*$ does. The maps $s_{0,E}^*$, $s_{1,E}^*$ are two sections of $\pi_E^* : \mathrm{Br}(E) \to \mathrm{Br}(\mathbb{A}_E^1)$; since π_E^* is an isomorphism (thanks to Theorem 1.12.8) we have $s_{0,E}^* = s_{1,E}^*$.

Claim 2. Let $x \in X(k) \subset X(E)$ and $f : \mathbb{A}_E^1 \to X$ a map such that $f(0) = x$, then $s_{0,E}^* \circ f^* : \mathrm{Br}(X) \to \mathrm{Br}(E)$ is trivial. Notice that the composition $f \circ s_{0,E} : E \to \mathbb{A}_E^1 \to X$, thanks to the assumption $f(0) = x$, factorized through k, hence $s_{0,E}^* \circ f^*$ factorizes through $\mathrm{Br}(\overline{k}) = 0$.

We end by induction applying Claim 1 and 2 to get that $s_{1,E}^* \circ h_m^* : \mathrm{Br}(X) \to \mathrm{Br}(E)$ is zero, where $h_m \circ s_{1,E}$ is the inclusion of the generic point in X. The theorem is proven. $\square$

Thanks to the Kneser-Tits Conjecture for split groups (see the Remark in Section 1.10.1) we have that $\mathrm{Br}(G_{\overline{k}}) = 0$. To deduce our initial claim it is enough to prove that $\mathrm{Br}_a(G) = 0$. To do this it is enough to consider the exact sequence of Lemma 1.12.6: Since G is semisimple simply connected we have $\mathrm{Pic}(G_{\overline{k}}) = 0$ (see the previous section) and $\widehat{G} = 1$, it follows that $\mathrm{Br}_a(G) = 0$. The result is proven.

Corollary 1.12.10. *Let G be a semisimple simply connected group, and H a k-subgroup of G. Write $X := G/H$ for the corresponding homogeneous space and $\pi : G \to X$ for the projection map.The kernel of the map $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(G)$ is canonically isomorphic to $\mathrm{Br}(X)/\mathrm{Br}(k)$.*

Proof. Denote with $\pi_X : X \to k$ and $\pi_G : G \to k$ the structural maps of X and G. The above theorem states that $\pi_G^* : \mathrm{Br}(G) \to \mathrm{Br}(k)$ is an isomorphism; by functoriality it makes the following diagram commutative:

$$\begin{tikzcd} \mathrm{Br}(k) \arrow[r, "\mathrm{Id}"] \arrow[d, "\pi_X^*"] & \mathrm{Br}(k) \arrow[d, "\pi_G^*"] \\ \mathrm{Br}(X) \arrow[r, "\pi^*"] & \mathrm{Br}(G) \end{tikzcd}$$

To conclude it is enough to show that the kernel of $(\pi_G^*)^{-1} \circ \pi^* : \mathrm{Br}(X) \to \mathrm{Br}(k)$ is isomorphic to $\mathrm{Br}(X)/\mathrm{Br}(k)$; but this is true by abstract non sense and the commutativity of the previous diagram. □

1.13 More about the center

Proposition 1.13.1. *Let G be a linear algebraic group, and G^{sc} the special covering of G'. The composition map $\rho : G^{sc} \to G' \to G$ induces an isomorphism*

$$G^{sc}/Z(G^{sc}) \cong G/Z(G).$$

Proof. As usual, thanks to the Levi decomposition we may assume G to be reductive. Let μ be the kernel of $G^{sc} \to G'$, since μ is finite Proposition 1.7.4 ensures that $\mu = \mu \cap Z(G^{\mathrm{sc}})$. Moreover we have the following morphism of short exact sequence induced by the natural inclusions:

$$\begin{tikzcd} 1 \arrow[r] & \mu = \mu \cap Z(G^{\mathrm{sc}}) \arrow[r] \arrow[d, "\mathrm{Id}"] & Z(G^{\mathrm{sc}}) \arrow[r] \arrow[d] & Z(G') \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mu \arrow[r] & G^{\mathrm{sc}} \arrow[r] & G' \arrow[r] & 1 \end{tikzcd},$$

where the first sequence is right exact again by the Rigidity property. This proves the result when G is semisimple, and it is enough to conclude in virtue of Theorem 1.8.2. □

Proposition 1.13.2. *Let G be a connected linear algebraic group and G' its derived subgroup. The following is exact*

$$1 \to Z(G') \to Z(G) \to G^{tor} \to 1.$$

Proof. It follows from the proof of Theorem 1.8.2, after a reduction to $G = G^{\mathrm{red}}$. □

Lemma 1.13.3. *Let G/k be a connected algebraic group, Z its center. Then the center of G/Z is unipotent.*

Proof. For the proof see [ABD+66], Exposé XVII, Lemma 7.3.2. □

Proposition 1.13.4. *Let G be a reductive $\overline{k}$-group. The quotient of G by its center is a semisimple group which is centerless.*

Proof. For the proof see [CT07], Proposition 4.21. □

1.14 Some vanishing results

In this section we resume some vanishing results for the second Tate-Shafarevich group with coefficient in a torus. Important tools will be the Tate-Nakayama and Poitou-Tate dualities.

Let k be a number field and T a k-torus. As usual we set

$$Ш^2(k,T) = \mathrm{Ker}\left(H^2(k,T) \to \prod_{v\in\Omega_k} H^2(k_v,T)\right)$$

Analogously we define $Ш^1(k,T)$ and $Ш^1(k,\widehat{T})$. Moreover we define also $Ш^2_\omega(\hat{T}) := \{\alpha \in H^2(k,\hat{T}), \alpha_v = 0 \in H^2(k_v,\hat{T})$ for almost all $v \in \Omega_k\}$.

We state and prove Lemma 6.7 of [Bor93]. The proof is obtained from Lemma 1.9 of [San81] and Lemma 3.4.1 of [Bor92].

Lemma 1.14.1. *Let T and k as above. Assume at least one of the following holds:*

i) T is a quasi-trivial k-torus (i.e. a product of tori of the form $R_{K/k}\mathbb{G}_m$ for K/k finite);

ii) There exists $v_0 \in \Omega_k$ such that $T_{k_{v_0}}$ is k_{v_0}-anisotropic (see Definition 1.7.7);

iii) T is split over a finite cyclic extension K/k;

iv) T is one dimensional.

Then $Ш^2(k,T)$ vanishes.

Proof. The exactness of Brauer-Hasse-Noether implies that $Ш^2(k,\mathbb{G}_m) = 0$, this proves i), since

$$H^2(K,\mathbb{G}_m) \cong H^2(k,R_{K/k}\mathbb{G}_m)$$

by the exactness of the Weil restriction functor.

The strategy to prove ii) and iii) is the same:

Claim 1. $Ш^2(k,T) \cong Ш^2(K/k,T)$ where K/k is a finite galois extension that splits T.

Fact 2. $Ш^2(K/k,T)$ is the cokernel of the map $H^1(K/k,T(\mathbb{A}_K)) \to H^1(K/k,T(\mathbb{A}_K)/T(K))$, which can be identified (thanks to Tate-Nakayama duality) to the cokernel of

$$\varsigma : \amalg_{v\in\Omega_k} H^{-1}(\mathrm{Gal}(K_w,k_v),X_*(T_K)) \to H^{-1}(\mathrm{Gal}(K/k),X_*(T_K));$$

where w is a place of K over v and H^{-1} denotes the homological group H^0_T modified *à la Tate*.

Claim 3. ς is surjective.

To prove Claim 1. notice that $H^1(K,\widehat{T}) = 0$ since $\widehat{T}$ is a permutation Γ_k-module. Thanks to the infilation-restriction exact sequence we deduce that $Ш^1(K/k,\widehat{T}) \cong Ш^1(k,\widehat{(T)})$. The duality provided by Poitou-Tate for tori proves the claim.

To prove Claim 3. we distinguish the two cases we are dealing with. In ii) we have that

$$H^0(\mathrm{Gal}(K_{w_0},k_{v_0}),X_*(T)) = 0,$$

and in particular the norm map, at the place v_0, is the zero map. It follows that the component of ς at the place v_0 is a quotient of the identity map, by construction of the Tate cohomology group H^{-1}, hence surjective. In iii), in the whole reasoning, we can choose K to be the cyclic extension that splits T. There exists $v \in \Omega_k$ such that $\mathrm{Gal}(K_w,k_v)$ is canonically isomorphic to $\mathrm{Gal}(K,k)$, by applying Čebotarev Density Theorem to a generator of the cyclic group $\mathrm{Gal}(K,k)$. Hence ς contains a copy of the identity map.

Since the k-forms of $\mathbb{G}_m$ are parametrized by $H^1(k,\mathtt{Aut}(\mathbb{G}_m))$, and $\mathtt{Aut}(\mathbb{G}_m)$ has two elements, we have just two possibilities: or T is isomorphic to $\mathbb{G}_m$ or it splits over a quadratic field extension of k, hence iv) implies iii). □

Being a quasi-trivial k-torus is equivalent to say that $\widehat{T}$ is an induced Γ_k-module. In particular we have also two more results.

Lemma 1.14.2. *Let T be a quasi-trivial k-torus. Then $H^1(k, \widehat{T}) = Ш^2_\omega(k, \widehat{T}) = 0$.*

Proof. The statement about $H^1(k, \widehat{T})$ follows from Shapiro's Lemma (cf. [Har12], Theorem 1.20), which ensures that induced modules are acyclic. For the second part notice that if $T = \mathbb{G}_m$, the results follows since $Ш^2_\omega(\mathbb{Z}) = 0$, by Čebotarev as in [Har12], Proposition 9.2. This is enough to conclude apply again Shapiro's and to reduce the case of an arbitrary quasi-trivial torus to $\mathbb{G}_m$. □

Proposition 1.14.3. *Let T be a quasi-trivial k-torus. Then* $\mathrm{Pic}(T) = 0$*.*

Proof. By the exact sequence 1.12.6 we have that, for any torus T, $\mathrm{Pic}(T)$ is isomorphic to $H^1(k, \widehat{T})$, which is zero since T is quasi trivial (as proven above). □

1.14.1 Tate-Nakayama duality

Theorem 1.14.4. *Let k be a non-archimedean local field of characteristic 0. Let T be a group of multiplicative type. The cup product defines a duality between the discrete group $H^2(k, T)$ and the compact group $H^0(k, X^*(T))^\wedge$, where the upper script means the completion relative to the topology of subgroups of finite index. In particular $H^2(k, T) = 0$ if and only if $X^*(T)^{\Gamma_k} = 0$, where $X^*(T)$ denotes the group of characters of T.*

Proof. For the proof see [Mil06], 0.2 Corollary 2.4 □

More about Tate-Nakayama duality for tori can be found in [Ono63], in particular the second section is devoted to prove the Fact we used during the proof of Lemma 1.14.1.

1.15 Important exact sequences

Theorem 1.15.1. *Let G/k be a semisimple simply connected group over a local field of characteristic zero or a number field and let Z be the center of G. The connecting map $\delta_G : H^1(k, G^{ad}) \to H^2(k, Z)$, induced by the exact sequence $1 \to Z \to G \to G^{ad} \to 1$, is surjective.*

Before proving the Theorem we need a lemma for the local case.

Lemma 1.15.2. *Let G be as above, and k a local field (real or non-archimedean), there exists a maximal k-torus T of G such that $H^2(k, T) = 1$.*

Proof. If $k = \mathbb{R}$ this is Lemma 6.18 of [PR94], if k is non-archimedean Theorem 6.21 of [PR94] produce an anisotropic maximal k-torus T which, thanks to Tate-Nakayama duality, satisfies the condition $H^2(k, T) = 1$. □

Proof of the Theorem. Local case. Let T be the maximal torus of the Lemma, and denote T' the image of T in G^{ad}, it is a subgroup of G^{ad} thanks to Proposition 1.2.2. Since Z is contained in T we have a commutative diagram wit exact rows:

$$\begin{tikzcd}
1 \arrow[r] & Z \arrow[r, "a"] \arrow[d, "\mathrm{Id}"] & T \arrow[r] \arrow[d] & T' \arrow[r] \arrow[d, "i"] & 1 \\
1 \arrow[r] & Z \arrow[r] & G \arrow[r] & G^{\mathrm{ad}} \arrow[r] & 1
\end{tikzcd}.$$

By functoriality of cohomology, we obtain the following commutative diagram:

$$\begin{tikzcd}
H^1(k, T') \arrow[r, "\delta_T"] \arrow[d, "i_*"] & H^2(k, Z) \arrow[r, "a_*"] \arrow[d, "\mathrm{Id}"] & H^2(k, T) = 1 \\
H^1(k, G^{\mathrm{ad}}) \arrow[r, "\delta_G"] & H^2(k, Z) &
\end{tikzcd}$$

It follows that $\delta_G \circ i_*$ is an epimorphism, and so δ_G does. □

Global case. Fix an element $\eta \in H^2(k, Z)$. We start showing that there exists a finite set of places $S = S_\eta$, containing an archimedean place, such that the restrictions $\eta_v \in H^2(k_v, Z)$ are trivial for all $v \in \Omega_k - S$.

We choose a finite extension K/k that $\eta \in \mathrm{Im}(H^2(K/k, Z) \xrightarrow{\inf} H^2(k, Z))$, for example the fixed field of the subgroup of Γ_k acting trivially on the cocycle representing η. By functoriality η_v belongs to $\mathrm{Im}(H^2(K_w/k_v, Z) \xrightarrow{\inf} H^2(k_v, Z))$ where w is a place of K over v. Up to a finite number of primes, we can suppose K_w/k_v to be unramified, hence $H^2(K_w/k_v, Z) = 1$ (since the maximal unramified extension of k_v is a C_1 field, see [Har12] Thm 5.1, and the fact that Z is finite) and the claim is proven.

For any place $v \in S$ we choose a k_v-torus T_v as in Lemma 1.15.2. By Weak Approximation, as explained in Section 1.6.1, there exists a k-torus T in G such that, as k_v-torus, it is isomorphic to T_v for all $v \in S$. As above set T' for the image of T in G^{ad} and consider the following "local to global" diagram chasing with exact rows:

$$\begin{array}{ccccc} H^1(k, T') & \xrightarrow{\quad\delta_T\quad} & H^2(k, Z) & \xrightarrow{\quad a_*\quad} & H^2(k, T) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{v \in \Omega_k} H^1(k_v, T') & \longrightarrow & \prod_{v \in \Omega_k} H^2(k_v, Z) & \longrightarrow & \prod_{v \in \Omega_k} H^2(k_v, T) \end{array}$$

Since S contains a real place, by the proof of Lemma 1.15.2, one of the T_vs appearing in $\prod_{v \in \Omega_k} H^2(k_v, T)$ is anisotropic, hence Proposition 1.14.1 *ii*) ensures that the vertical map $H^2(k, T) \to \prod_{v \in \Omega_k} H^2(k_v, T)$ is injective.

As in the local case it is enough to show that our $\eta \in H^2(k, Z)$ belongs to $\mathrm{Ker}(a_*)$ or, equivalently that it goes to zero in $\prod_{v \in \Omega_k} H^2(k_v, T)$. But this is true since outside S it is already zero in $\prod_{v \in \Omega_k} H^2(k_v, Z)$, and in S we have imposed the whole $H^2(k_v, T)$ to be zero. The theorem is proven. $\square$

$\square$

Proposition 1.15.3. *Let k a number field and G/k be a connected group. Then the natural map $H^1(k, G) \to \prod_{v \in \Omega_\infty} H^1(k_v, G)$ is surjective.*

Proof. For the proof see [PR94], Proposition 6.17. The proof consists in two steps: a reduction to the case of tori, and the proof for tori. We sketch a simple argument to show the result when $G = T$ a k-torus, using the dualities of the previous section. The cokernel of $H^1(k, G) \to \prod_{v \in \Omega_\infty} H^1(k_v, G)$ can be identified, as in the proof of Lemma 1.14.1, with $Ш^2_{\Omega_\infty}(k, T)$. The result follows since $\Gamma_{\mathbb{R}}$ is cyclic. $\square$

We conclude this section with an important and hard fact.

Theorem 1.15.4. *Let G be a semisimple simply connected group over a non archimedean local field k. Then $H^1(k, G) = 1$.*

Proof. This is Theorem 6.4 of [PR94], its proof occupies the sections 6.6 (for classical groups) and 6.7 (for Exceptional groups). A uniform proof, based on Bruhat-Tits theory, is also possible. $\square$

1.16 Extension of algebraic groups

With Ext^r we will denote the *Yoneda Ext*. For much more about this we refer to [Mil06] or the notes [Har06].

Proposition 1.16.1. *Let k be a field of characteristic zero. Let N be a finite k-group scheme. For any $r \geq 0$, there exists a canonical isomorphism*

$$\mathrm{Ext}^r_k(N, \mathbb{G}_m) \cong \mathrm{Ext}^r_{\Gamma_k}(N(\overline{k}), \overline{k}^*).$$

Remark. From now on, in this situation, we will confuse the two extension groups.

Proof. For $r = 0$ the two groups are clearly the same. Thanks the duality between $\overline{k}$-groups and groups of multiplicative we have that $\mathrm{Ext}^s_{\overline{k}}(F_{\overline{k}}, \mathbb{G}_m) = \mathrm{Ext}^s(\mathbb{Z}, \widehat{F}) = 0$, for any s. From the degeneracy of the spectral sequence displayed above we have

$$\mathrm{Ext}^r_k(N, \mathbb{G}_m) \cong H^r(k, \mathrm{Hom}_{\mathbb{Z}}(N(\overline{k}), \overline{k}^*)) \cong \mathrm{Ext}^r_{\Gamma_k}(N(\overline{k}), \overline{k}^*),$$

where the last isomorphism holds thanks to the composition low of derived functors, just noticing that $\mathrm{Hom}_{\mathbb{Z}}(-, \overline{k}^*)$ is exact. $\square$

1.16.1 Central extensions Hochschild cohomology

From now on $\mathrm{Ext}^c_k(H, \mathbb{G}_m)$ will denote the abelian group of isomorphism classes of central extensions of k-algebraic groups of H by $\mathbb{G}_m$.

Remark. If H is a connected linear algebraic group, then any extension is central, and it is possible to prove that $\mathrm{Ext}^c_k(H, \mathbb{G}_m)$ is isomorphic to $\mathrm{Pic}(H)$, see [CT08] Corollary 5.7.

For the definition of Hochschild cohomology we refer the reader to [DG80] II.3 Section 1. We just collect here two results we will use later.

Proposition 1.16.2. *Let H be a constant k-algebraic group. There exists a canonical isomorphism of abelian groups $H^2(H, k^*) \cong H^2_0(H, \mathbb{G}_m)$, where the second group is the group of Hochschild cohomology of the trivial action of k-group H on the k-group $\mathbb{G}_m$.*

Proposition 1.16.3. *Let M be a G-module, then $\mathrm{Ext}^c_k(G, M) \cong H^2_0(G, M)$.*

Proof. For the proof see [DG80] II.3 Proposition 2.3 $\square$

1.16.2 Brauer group and central extensions

Thanks to the previous results, we are now ready to state and prove an isomorphism which will play a crucial role in the proof of Theorem 4.2.1 (cf. [Dem15], Proposition 1.1).

Let k be a field of characteristic zero and G be a semisimple simply connected k-group. Let H be a constant finite k-subgroup of G, write $X := G/H$ for the corresponding homogeneous space and $\pi : G \to X$ for the projection map. Write $\mathrm{Br}(X, G)$ for the kernel of the map $\pi^* : \mathrm{Br}(X) \to \mathrm{Br}(G)$ (cf. Corollary 1.12.10).

Theorem 1.16.4. *There exists a canonical isomorphism*

$$\Delta_X : \mathrm{Ext}^c_k(H, \mathbb{G}_m) \xrightarrow{\cong} \mathrm{Br}(X, G) \cong \mathrm{Br}(X)/\mathrm{Br}(k).$$

Moreover Δ_X is functorial in H in the following sense: for any $Z \subset H$, set $Y := G/K$ for the corresponding homogeneous space and $f : Y \to X$ the canonical morphism; the natural diagram

$$\begin{tikzcd}
\mathrm{Ext}^c_k(H, \mathbb{G}_m) \arrow[r, "\Delta_X"] \arrow[d] & \mathrm{Br}(X)/\mathrm{Br}(k) \arrow[d, "f_*"] \\
\mathrm{Ext}^c_k(Z, \mathbb{G}_m) \arrow[r, "\Delta_Y"] & \mathrm{Br}(Y)/\mathrm{Br}(k)
\end{tikzcd}$$

is commutative.

Proof. Notice that $\pi : G \to X$ is an X-torsor under the group H. Since H is a constant group scheme we can consider (see for example [Mil13], Theorem 14.9) the Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(H, H^q(G, \mathbb{G}_m)) \Rightarrow H^{p+q}(X, \mathbb{G}_m).$$

Since $H^0(G, \mathbb{G}_m) = k[G]^* = k^*$, the last part of the low degree exact sequence becomes

$$H^0(k, \mathrm{Pic}(G)) \to H^2(H, k^*) \to \mathrm{Ker}(\mathrm{Br}(X) \xrightarrow{\pi^*} \mathrm{Br}(G)) \to H^1(H, \mathrm{Pic}(G)),$$

where the action of H on k^* is trivial by construction. As explained in the beginning of Section 1.12, we have $\mathrm{Pic}(G) = 0$; it follows that $H^2(H, k^*)$ is canonically isomorphic to $\mathrm{Br}(X, G)$.

We have $H^2(H, k^*) \cong H^2_0(H, \mathbb{G}_m) \cong \mathrm{Ext}^c_k(H, \mathbb{G}_m)$ where the first is an isomorphism by Proposition 1.16.2, and the second by Proposition 1.16.3. Hence we have constructed a canonical isomorphism $\Delta_X : \mathrm{Ext}^c_k(H, \mathbb{G}_m) \to \mathrm{Br}(X, G)$. Corollary 1.12.10 concludes the proof. $\square$

1.17 More about homogeneous spaces

1.17.1 Symmetric spaces

In this section we present an interesting example of homogeneous space.

Definition 1.17.1. Let G/k an algebraic group. A *symmetric spaces* for G is an homogeneous space X/k such that the stabilizer $\overline{H}$ is the group of invariants of an involution of $G_{\overline{k}}$, i.e. of a $\sigma \in \mathtt{Aut}(G_{\overline{k}})$ such that $\sigma^2 = \mathrm{Id}$.

The reason why we are interested in symmetric space under semisimple simply connected algebraic groups is due to the following theorem.

Definition 1.17.2. Let $\overline{G}$ a linear algebraic $\overline{k}$-group. An automorphism $\sigma : \overline{G} \to \overline{G}$ is *semisimple* if there exists an embedding $\overline{G} \hookrightarrow \overline{G}_0$ such that σ is realized by conjugation by some semisimple element $s \in \overline{G}_0$.

Theorem 1.17.3 (A basic theorem on connectedness)**.** *Let $\overline{G}$ be a semisimple, simply connected $\overline{k}$-group. If σ a semisimple automorphism, then the group of fixed points $\overline{G}_\sigma$ is connected and reductive.*

Proof. For the proof see [Ste68], Theorem 8.1. □

This result applies for symmetric spaces $(G, \overline{H})$. Since G is semisimple, the group $\mathtt{Out}(G)$ is finite, hence every automorphism can be realized as conjugation in a larger group and we can diagonalize every involution (k has characteristic different from 2). This shows that the stabilizer $\overline{H}$ is connected.

Chapter 2

Abelianization of the second nonabelian Galois Cohomology

In this chapter we present the main results proven by Borovoi in [Bor93]. Borovoi's results are also discussed in [Sko01] (briefly) and in the master thesis [LA10] (with many details). Moreover the results we present here, on nonabelian Galois cohomology, have been recently extended to a general base scheme, see [GA12].

2.1 Lien and the second nonabelian Galois Cohomology

Let k be a field of characteristic zero, and fix an algebraic closure $\overline{k}$. For a $\sigma \in \Gamma_k$ we denote with $\sigma^* : \mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(\overline{k})$, the morphism induced by $\sigma^{-1} : \overline{k} \to \overline{k}$. Let $\overline{p} : \overline{G} \to \overline{k}$ an algebraic $\overline{k}$-group. We denote by $\mathtt{SAut}(\overline{G}) \subset \mathtt{Aut}(\overline{G}/k)$ the subgroup of *semialgebraic* (or semilinear) k-automorphisms of $\overline{G}$. Those are the elements $s \in \mathtt{Aut}(\overline{G}/k)$ such that $\overline{p} \circ \varphi = (\sigma^*) \circ \overline{p}$, for some $\sigma \in \Gamma_k$. Such a σ is unique, it is enough to consider the restriction to $\overline{k}$ of the automorphism of the ring of regular function $\overline{k}[\overline{G}]$ induced by s^{-1}. The map $q : \mathtt{SAut}(\overline{G}) \to \Gamma_k$ sending s to σ is an homomorphism. An element $s \in \mathtt{SAut}(\overline{G})$ such that $q(s) = \sigma$ is called σ-*semialgebraic*.

A semialgebraic automorphism $s \in \mathtt{SAut}(\overline{G})$ is an algebraic automorphism of $\overline{G}$ (i.e an automorphism of $\overline{G}$ as an algebraic group over $\overline{k}$) if and only if $q(s) = 1$; if $q(s) \neq 1$ then s is not a morphism of $\overline{k}$-varieties. The group of algebraic automorphism of $\overline{G}$ will be denoted with $\mathtt{Aut}(\overline{G})$. For any $x \in \overline{G}(\overline{k})$ we define $\mathrm{int}(x)$ for the inner automorphism of $\overline{G}$ given by $y \mapsto xyx^{-1}$, and we write $\mathrm{Int}(\overline{G})$ for the *group of inner automorphisms*, i.e. the subgroup of $\mathtt{Aut}(\overline{G})$ given by $\mathrm{int}(x)$ for $x \in G(\overline{k})$. We can identify $\mathrm{Int}(\overline{G})$ with the quotient of $\overline{G}(\overline{k})$ by the $\overline{k}$-points of the center. Since it is a normal subgroup in $\mathtt{SAut}(\overline{G})$, it make sense to define $\mathtt{Out}(\overline{G}) := \mathtt{Aut}(\overline{G})/\mathrm{Int}(\overline{G})$ and $\mathtt{SOut}(\overline{G}) := \mathtt{SAut}(\overline{G})/\mathrm{Int}(\overline{G})$.

We have the following exact sequences:

$$1 \to \mathtt{Aut}(\overline{G}) \to \mathtt{SAut}(\overline{G}) \xrightarrow{q} \Gamma_k \tag{2.1.0.1}$$

$$1 \to \mathtt{Out}(\overline{G}) \to \mathtt{SOut}(\overline{G}) \xrightarrow{q} \Gamma_k, \tag{2.1.0.2}$$

where the second is obtained from the first by taking the quotient by $\mathrm{Int}(\overline{G})$.

We will consider the group $\mathtt{SAut}(\overline{G})$ equipped with the weak topology with respect to the family of evaluation maps $\mathrm{ev}_x : \mathtt{SAut}(\overline{G}) \to \overline{G}(\overline{k}), \varphi \mapsto \varphi(x)$, where $\overline{G}(\overline{k})$ has the discrete topology (see also [FSS98]). In other words, this is the coarsest group topology on $\mathtt{SAut}(\overline{G})$ such that the stabilizers of geometric points of $\overline{G}$ are open.

Definition 2.1.1. A k-*lien* (k-band, k-kernel) is a pair $L = (\overline{G}, \kappa)$ where $\overline{G}$ is a $\overline{k}$-group and $\kappa : \Gamma_k \to \mathtt{SOut}(\overline{G})$ is a group homomorphism satisfying

- κ is a splitting of 2.1.0.2, i.e. $q \circ \kappa = \mathrm{Id}_{\Gamma_k}$;
- κ can be lifted to a continuous map $f : \Gamma_k \to \mathtt{SAut}(\overline{G})$.

If the map $f : \Gamma_k \to \mathtt{SAut}(\overline{G})$ is also an homomorphism we say that L is *representable*.

For a k-lien $L = (\overline{G}, \kappa)$ we define the second Galois cohomology set $H^2(k, L) = H^2(k, \overline{G}, \kappa)$ trying to emulate the description in terms of cocycles of the abelian H^2.

Definition 2.1.2. A 2-cocycle is a pair (f, u) of continuous maps $f : \Gamma_k \to \mathtt{SAut}(\overline{G})$, $u : \Gamma_k \times \Gamma_k \to \overline{G}(\overline{k})$ such that for any $\sigma, \tau, \rho \in \Gamma_k$ the following formulas hold:

$$\mathrm{int}(u_{\sigma,\tau}) \circ f_\sigma \circ f_\tau = f_{\sigma\tau}, \tag{2.1.0.3}$$

$$u_{\sigma,\tau\rho} \cdot (f_\sigma(u_{\tau,\rho})) = u_{\sigma\tau,\rho} u_{\sigma,\tau}, \tag{2.1.0.4}$$

$$f_\sigma \mod \mathrm{Int}\,\overline{G} = \kappa(\sigma). \tag{2.1.0.5}$$

The set of 2-cocycles is denoted with $Z^2(k, L)$. A cocycle of the form $(f, 1)$ is called a *neutral cocycle*.

Remark. Notice that when a cocycle is of the form $(f, 1)$, then the equation of 2.1.0.3 says that $f : \Gamma_k \to \mathtt{SAut}(\overline{G})$ is a continuous homomorphism.

The group $C(k, \overline{G})$ of continuous maps $c : \Gamma \to \overline{G}(\overline{k})$ acts on $Z^2(k, L)$ acts on the left via the formula $c \cdot (f, u) = (c \cdot f, c \cdot u)$ where

$$(c \cdot f)_\sigma = \mathrm{int}(c_\sigma) \circ f_\sigma, \tag{2.1.0.6}$$

$$(c \cdot u)_{\sigma,\tau} = c_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot f_\sigma(c_\tau)^{-1} \cdot c_\sigma^{-1}. \tag{2.1.0.7}$$

By direct computation one shows that the action is well defined.

Definition 2.1.3. The quotient set $H^2(k, L) = Z^2(k, L)/C(k, \overline{G})$ is called the *second Galois cohomology set of k with coefficient in L*. A *neutral cohomology class* in $H^2(k, L)$ is the class of a neutral cocycle.

From now on we will denote with Cl the projection map $Z^2(k, L) \to H^2(k, L)$, $(f, u) \mapsto \mathrm{Cl}(f, u)$.

Remark. In the abelian case this definition agrees with the usual one.

Remark. A possible more natural way to define our H^2 is to follow the algebraic interpretation of the group $H^2(G, A)$ for a G-module A. Namely, this group parametrizes the equivalence classes of extensions of the form $0 \to A \to E \to G \to 0$ (cf. [Wei94], Classification Theorem 6.6.3). This path will lead us to the following equivalent description of $H^2(k, L)$. Consider an extensions of topological groups

$$1 \to \overline{G}(\overline{k}) \to E \xrightarrow{q} \Gamma_k \to 1. \tag{2.1.0.8}$$

We do not require that the arrows to be continuous but we require that 2.1.0.8 is *locally split*, in the sense that there exists a finite field extension K/k such that the induced map $E \cap q^{-1}(\Gamma_K) \xrightarrow{q_K} \Gamma_K$ admits a continuous homomorphic section. Two such extensions are called equivalent if there is a topological isomorphism of their middle terms commuting with the identities on the boundary terms. For the proof that the two definition agree see [LA10], Proposition 2.6.3.

In [FSS98] the condition of being locally split for the extensions is replaced by the equivalent conditions on the maps of 2.1.0.8, namely the first map has to be *strict*, i.e. the (discrete) topology on $\overline{G}(\overline{k})$ must be induced by the topology on E, and the second must be open onto its image (for this equivalence see [HS02], Appendix A: Extensions of Topological Groups).

Lemma 2.1.4 (Galois Descent). *Let $\overline{G}$ a $\overline{k}$-algebraic group. The continuous homomorphic sections of $q : \mathtt{SAut}(\overline{G}) \to \Gamma_k$ are in natural bijective correspondence with the k-forms of $\overline{G}$.*

Proof. If $\overline{G} = G \times \overline{k}$, the map $f_G : \Gamma_k \to \mathtt{SAut}(\overline{G})$ given by $\sigma \mapsto 1 \times \sigma^*$ is an homomorphic section of $q : \mathtt{SAut}(\overline{G}) \to \Gamma_k$, and it is continuous because the preimage of the stabilizer of $\overline{g} \in \overline{G}(\overline{k})$ is the open subgroup of the absolute Galois group, defined by the the field of definition of $\overline{g}$ (which is a finite extension of k).

Conversely for a $\overline{k}$-group $\overline{G}$ any continuous homomorphic sections $f : \Gamma_k \to \mathtt{SAut}(\overline{G})$ induces an action of Γ_k on $\overline{G}(\overline{k})$ such that the stabilizers are open (as above). We obtain a k-form G of $\overline{G}$ defining $G = \mathrm{Spec}(\overline{k}[\overline{G}]^{\Gamma_k})$. To check that G is actually a k-variety and that $G_{\overline{k}}$ is isomorphic to $\overline{G}$ see [Ser75], Ch V-20, Prop. 12 and [BS63], 2.12. ∎

Remark. Consider $L = (\overline{G}, \kappa)$ a k-lien with $\overline{G}$ abelian then κ is trivial, since $\mathtt{Aut}(\overline{G}) = \mathtt{Out}(\overline{G})$. By Galois Descent we always obtain a k-form G of $\overline{G}$.

In general the set $H^2(k,L)$ does not necessarily have neutral elements, may contain more than one neutral element, or can be empty. However for a k-group G, the class of $(f_G, 1)$, as constructed above, is a canonical neutral element (cf. Construction 2.3.1).

Construction 2.1.5. Let $L = (\overline{G}, \kappa)$ be a k-lien, and $\overline{N}$ be a normal $\overline{k}$-subgroup of $\overline{G}$, denote with $\overline{G}'$ the quotient group of $\overline{G}$ by $\overline{N}$ and $\pi : \overline{G} \to \overline{G}'$ the projection to the quotient. If $\overline{N}$ is fixed by all the semialgebraic automorphisms of $\overline{G}$, e.g. $\overline{N} = Z(\overline{G})$, then it makes sense to consider the homomorphism $\pi : \mathtt{SAut}(\overline{G}) \to \mathtt{SAut}(\overline{G}')$ and the homomorphism $\kappa' = \pi \circ \kappa : \Gamma_k \to \mathtt{SOut}(\overline{G}')$. We have constructed the *quotient k-lien* $L' = (\overline{G}', \kappa')$. Moreover we can define a canonical map

$$\pi_* : H^2(k,L) \to H^2(k,L'),\ \mathrm{Cl}(f,u) \mapsto \mathrm{Cl}(\pi \circ f, \pi \circ u)$$

which restricts to the set of neutral elements.

Two examples of this construction we will often use are the following.

- If $\overline{N}$ contains the derived subgroup of $\overline{G}$ then $\overline{G}'$ is abelian, and so $\kappa' : \Gamma_k \to \mathtt{SOut}(\overline{G}')$ is a continuous homomorphism $\Gamma_k \to \mathtt{SAut}(\overline{G}')$. By Galois descent it determines a k-form G' of $\overline{G}'$, and we obtain a map $\pi_* : H^2(k,L) \to H^2(k,G')$, where $H^2(k,G')$ is the usual Galois cohomology group for abelian Γ_k-modules.

- Let $L = (\overline{G}, \kappa)$ be a connected k-lien and consider $t : \overline{G} \to \overline{G}^{\mathrm{tor}}$, where $\overline{G}^{\mathrm{tor}}$ is the $\overline{k}$-torus $\overline{G}^{\mathrm{red}}/\overline{G}^{\mathrm{ss}}$. By construction the subgroup $\mathrm{Ker}(t)$ contains the derived subgroup of $\overline{G}$, and it is fixed by the elements of $\mathtt{SAut}(\overline{G})$. As in the previous case we obtain a k-form G^{tor} of $\overline{G}^{\mathrm{tor}}$ and a canonical map $t_* : H^2(k,L) \to H^2(k,G^{\mathrm{tor}})$. This map will pay a crucial role in the study of neutral elements.

2.1.1 The action of $H^2(k,Z)$

Let $L = (\overline{G}, \kappa)$ be a k-lien, and $\overline{Z}$ be the center of $\overline{G}$. We can consider the restriction homomorphism $\mathtt{SAut}(\overline{G}) \to \mathtt{SAut}(\overline{Z})$ which, since $\overline{Z}$ is abelian, factorizes through $\mathtt{SOut}(\overline{G})$. The composition $\Gamma_k \xrightarrow{\kappa} \mathtt{SOut}(\overline{G}) \to \mathtt{SAut}(\overline{Z})$, by Galois descent, determines a k-form Z of $\overline{Z}$, which will be called *the center of L*. Notice that, in the case $L = (G_{\overline{k}}, \kappa_G)$ for a k-group G, this notion agrees with the usual definition of center.

For any k-lien $L = (\overline{G}, \kappa)$ the abelian group $H^2(k,Z)$ acts on the set $H^2(k,L)$ by the formula $\mathrm{Cl}(\varphi) + \mathrm{Cl}(f,u) = \mathrm{Cl}(f, \varphi u)$ where $\varphi \in Z^2(k,Z)$, $(f,u) \in Z^2(k,L)$. The action is well defined since Z is commutative: for any $c \in C(k,\overline{G})$, we have $\varphi(cu) = c(\varphi u)$; the same reasoning applies to $c' \in C(k,Z)$.

Lemma 2.1.6. *Let L and Z be as above. If $H^2(k,L) \neq \emptyset$, then the action of $H^2(k,Z)$ on $H^2(k,L)$ is simply transitive.*

Proof. A detailed proof can be found in [LA10], Proposition 2.6.1. Here we just notice that, give two elements $\mathrm{Cl}(f,u), \mathrm{Cl}(f',u') \in H^2(k,L)$ we can always suppose $f = f'$. By 2.1.0.5 we know that $f_\sigma = f'_\sigma \mod \mathrm{Int}\,\overline{G}$ for all $\sigma \in \Gamma_k$; lifting the association $c_\sigma = f_\sigma (f'_\sigma)^{-1} \in \mathrm{Int}\,\overline{G}$ to $\overline{G}$, we obtain a continuous map $c : \Gamma_k \to \overline{G}(\overline{k})$ such that $cf' = f$. □

2.1.2 An exact sequence

Let G be a discrete Γ_k group, and B a subgroup of G (not necessary normal). The set of continuous maps $\Gamma_k \to G$ satisfying ${}^{\sigma}B = a_\sigma^{-1} B a_\sigma$ and $a_\sigma {}^{\sigma}a_\tau a_{\sigma\tau}^{-1} \in B$, denoted with $Z^1(\Gamma_k, G, B)$, is called *the set of relative 1-cocycle*. Given an element $a \in Z^1(\Gamma_k, G, B)$ the map

$$\kappa_a : \Gamma_k \to \mathtt{SOut}(B), \sigma \mapsto \left(b \mapsto a_\sigma {}^{\sigma}b a_\sigma^{-1} \mod \mathrm{Int}(B)\right)$$

makes (B, κ_a) a k-lien. We define $H^2(k, B\mathrm{rel}G)$ as the following set

$$\Big(\coprod_{a \in Z^1(\Gamma_k, G, B)} H^2(k, (B, \kappa_a)) \Big) / N$$

where N is the normalizer of B. For more details see [LA10].

Proposition 2.1.7. *Let G be a k group, and $N \subset G$ be a normal k-subgroup fixed by all the semialgebraic automorphisms of G. There exists a an exact sequence of pointed set*

$$1 \to H^0(k,N) \to H^0(k,G) \to H^0(k,G/N) \to H^1(k,N) \to$$
$$\to H^1(k,G) \to H^1(k,G/N) \to H^2(k,N\mathit{rel}G) \to H^2(k,L)$$

where $L = (G_{\overline{k}}, \kappa_G)$ is the trivial lien structure given by G on $G_{\overline{k}}$, as in Lemma 2.1.4.

Proof. For the proof see [LA10], Corollary 2.9.1.2 (see also Proposition 2.9.1 if N is not normal in G). □

2.2 The Springer class of a homogeneous space

To a homogeneous space X/k of an algebraic group G/k, we want to associate a lien $L = (\overline{H}, \kappa)$, where $\overline{H}$ is the stabilizer of some point $\overline{x}_0 \in X(\overline{k})$ (not necessarily defined on k), and a cohomology class $\eta_X \in H^2(k,L)$.

Construction 2.2.1 (the k-lien $L = (\overline{H}, \kappa)$)**.** For $\sigma \in \Gamma_k$ we write ${}^\sigma\overline{x}_0 = \overline{x}_0 \cdot g_\sigma$ where $g_\sigma \in G(\overline{k})$ in such a way that the map $\sigma \mapsto g_\sigma$ is continuous (up to multiply g_σ by an element of $\overline{H}(\overline{k})$). Consider $f_\sigma = \mathrm{int}(h_\sigma) \circ \sigma_* \in \mathtt{SAut}(G_{\overline{k}})$, it can be restricted to an element of $\mathtt{SAut}(\overline{H})$. Then the map $f : \Gamma_k \to \mathtt{SAut}(\overline{H})$ is continuous, the composition

$$\kappa : \Gamma \xrightarrow{f} \mathtt{SAut}(\overline{H}) \to \mathtt{SOut}(\overline{H})$$

is an homomorphism, and $L = (\overline{H}, \kappa)$ is a k-lien.

Construction 2.2.2 (the class $\eta_X \in H^2(k,L)$)**.** For any $\sigma \in \Gamma_k$ consider g_σ as above, and set

$$x \in G_{\overline{k}} \quad \nu_\sigma(x) = g_\sigma \cdot {}^\sigma x, \quad \nu_\sigma = l_{g_\sigma} \circ \sigma_*$$

where $l_{g_\sigma} : G_{\overline{k}} \to G_{\overline{k}}$ is the left translation by g_σ. Then ν_σ is a σ-semialgebraic $G_{\overline{k}}$ equivariant automorphism of $G_{\overline{k}}$, compatible with σ_*. Consider the following automorphism of $G_{\overline{k}}$:

$$\lambda_{\sigma,\tau} = \nu_{\sigma\tau} \circ \nu_\tau^{-1} \circ \nu_\sigma^{-1}$$

Notice that ν_σ^{-1} is the map sending $x \in G_{\overline{k}}$ to ${}^{\sigma^{-1}}(g_\sigma)^{-1} \cdot {}^{\sigma^{-1}}x$, and so, by direct computation, we have

$$\lambda_{\sigma,\tau}(x) = u_{\sigma,\tau} \cdot x$$

where $u_{\sigma,\tau} = g_{\sigma\tau} \cdot {}^\sigma g_\tau^{-1} \cdot g_\sigma^{-1} \in \overline{H}(\overline{k})$.

Let f be the map constructed above, the reader can check that $(f,u) \in Z^2(k,L)$ where $L = (\overline{H}, \kappa)$ as above. We set $\eta_X = \mathrm{Cl}(f,u)$ to be the *Springer Class of a homogeneous space X.*

Remark. The Springer Class has a nice description also as extension. Namely it is defined by

$$1 \to \overline{H} \to E_X \to \Gamma_k \to 1$$

where E_X is the subgroup of $G(\overline{k}) \rtimes \Gamma_k$ made of the elements $\overline{g}\sigma$ such that $\overline{g} \cdot \overline{x}_0 = \sigma(\overline{x}_0)$; cf. [FSS98] (5.1).

The two constructions show the existence of a $\overline{k}$-torsor $\overline{Y}$ under G and a G-equivariant map $\overline{\alpha} : Y \to X$ defined over $\overline{k}$, namely $\overline{Y} = G_{\overline{k}}$, and $\overline{\alpha} : \overline{Y} \to X_{\overline{k}}$ given by $g \mapsto \overline{x}_0 \cdot g$. We can interpret the class $\eta_X \in H^2(k,L)$ as an obstruction to define the pair $(\overline{Y}, \overline{\alpha})$ over k.

Proposition 2.2.3. *Let X/k be a (right) homogeneous space of an algebraic group G/k. The following hold.*

a) If $X(k) \neq \emptyset$ then η_X is neutral.

b) The class η_X is neutral if and only there exists a k-form (Y, α) of $(\overline{Y}, \overline{\alpha})$.

Proof. a) In this case the stabilizers $\overline{H}$ is defined over k, the element g_σ of Construction 2.2.1 can be chosen to be the identity, hence the $u_{\sigma,\tau}$ we have exhibited above is trivial.

b) If η_X is trivial we obtain the relation $1 = u_{\sigma,\tau} = g_{\sigma\tau} \cdot {}^\sigma g_\tau^{-1} \cdot g_\sigma^{-1}$ for a certain family (g_σ); this means that $\nu : \Gamma_k \to \mathtt{SAut}(G_{\overline{k}})$ is a (continuous) homomorphism. Via Galois descent it define a k form of the pair $(\overline{Y}, \overline{\alpha})$, hence we obtain a pair (Y, α) as wanted. Viceversa, given a k-form (Y, α) we can take ν_σ to be σ_*. As in the previous point, this concludes. □

Statement *b)* suggests that, if the class η_X is neutral, we could deduce the Hasse principle for homogeneous spaces from the Hasse principle for principal homogeneous spaces. This remark, combined with the Kneser, Harder and Chernousov's Hasse Principle, produces the following result.

Theorem 2.2.4. *Let X/k a homogeneous space of a semisimple simply connected algebraic group G/k with connected stabilizers. If the class η_X is neutral, then the Hasse principle (with respect to the real places) holds, i.e. if $X(k_v) \neq \emptyset$ for all archimedean places v of k, then $X(k) \neq \emptyset$.*

Proof. Since η_X is neutral we can consider $Y \to X$ as in Proposition 2.2.3. Moreover there exits a connected algebraic group H defined over k such that the isomorphism class of Y in $H^1(k, G)$ is defined up to twist by an element of $H^1(k, H)$. To produce such an H consider $\mathtt{Aut}_{X,G}(Y)$, the group of G-automorphism of $Y \to X$; over $\overline{k}$ it is isomorphic to the stabilizer of a $\overline{k}$-point of X, and inherits a non trivial action of Γ_k (since G, Y and X are defined over k), we define the group scheme H choosing a k-form of the above group. It is connected since it does on $\overline{k}$, by assumption: k-scheme is geometrically connected if and only if its base change to $\overline{k}$ is connected.

Choose some points $P_v \in X(k_v)$ for all $v \in \Omega_\infty$, and consider the classes of the fibers $[Y_{P_v}] \in H^1(k_v, H)$. Since the map

$$H^1(k, H) \longrightarrow \prod_{v \in \Omega_\infty} H^1(k_v, H)$$

is surjective for any connected k-group H (see Prop. 1.15.3), we can consider a k-torsor under H whose class restricts to $[Y_{P_v}]$ for all $v \in \Omega_\infty$. Twisting Y by this torsor, we force $Y(k_v) \neq \emptyset$ for all $v \in \Omega_\infty$, the result of Kneser, Harder and Chernousov for torsors under semisimple simply connected groups ensure that $Y(k) \neq \emptyset$; in particular $X(k) \neq \emptyset$. □

The rest of the chapter will be devoted to give a precise description of the neutral elements, in order to have sufficient condition to apply the previous theorem in many concrete situations.

2.3 Neutral cohomology classes

Let $\overline{G}$ be $\overline{k}$-group. To any k-form G of $\overline{G}$ we want to associate a neutral cohomology class $n(G)$, the *neutral cohomology class defined by the form G of $\overline{G}$.*

Construction 2.3.1. As in the proof of Galois Descent, to the k-form G we can associate a continuous homomorphism $f = f_G : \Gamma_k \to \mathtt{SAut}(\overline{G})$. Define $\kappa_G : \Gamma_k \to \mathtt{SOut}(\overline{G})$ via the association $\sigma \mapsto f_\sigma \mod \operatorname{Int}(\overline{G})$, therefore it makes sense to consider the neutral cohomology class $n(G) := \operatorname{Cl}(f_G, 1) \in H^2(k, (\overline{G}, \kappa_G)) =: H^2(k, G)$.

Remark. Let $\overline{G}, G$ be as in Construction 2.3.1, and set $Z = Z(G)$. Consider a cocycle $\psi \in Z^1(k, G/Z)$; the twisted group $G' = {}_\psi G$ (see [Ser73], I 5.3), determines, via Galois Descent, a continuous homomorphism $f' = f_{G'} : \Gamma_k \to \mathtt{SAut}(\overline{G})$ satisfying the relation $f'_\sigma = \psi_\sigma f_\sigma$. Thanks to the relations (due to the equality $(G/Z)(\overline{k}) = \operatorname{Int}(\overline{G})$)

$$f'_\sigma \mod \operatorname{Int}(\overline{G}) = f_\sigma \mod \operatorname{Int}(\overline{G}) = \kappa_G(\sigma),$$

the neutral cohomology class $n(G') = \operatorname{Cl}(f', 1)$, defined by $G' = {}_\psi G$ lies in $H^2(k, (\overline{G}, \kappa_G))$.

In particular, if the cocycle ψ is obtained as the projection of a cocycle with values in G, say $\varphi \in Z^1(k, G)$, we have $(f_{G'}, 1) = \varphi \cdot (f_G, 1)$; in particular in this case $n(G) = n(G')$.

The following lemma allows us to study the neutral classes $\eta \in H^2(k, L)$ for any k-lien $L = (\overline{G}, \kappa)$, in the easier case when L is a k-lien associated to a k-form of $\overline{G}$.

Lemma 2.3.2. *Let $L = (\overline{G}, \kappa)$ be a k-lien and $\eta \in H^2(k, L)$ a neutral class. There exists a k-from G of $\overline{G}$ such that $\eta = n(G)$. Moreover such a k-form is uniquely determined up to twisting by a cocycle $\varphi \in Z^1(k, G)$.*

Proof. For some homomorphism $f : \Gamma_k \to \mathtt{SAut}(\overline{G})$, we can write $\eta = \operatorname{Cl}(f, 1)$; by Galois Descent it corresponds to a k-form G of $\overline{G}$. By Construction 2.3.1, we have $\eta = n(G)$. To conclude it is enough to notice, thanks to the above Remark, that the relation $(f', 1) = \varphi \cdot (f, 1)$ for some $\varphi \in C(k, \overline{G})$ can hold if and only if $\varphi \in Z^1(k, G)$. This follows from the equality

$$1 = \varphi \cdot 1 = \varphi_{\sigma\tau} f_\sigma(\varphi_\tau)^{-1} \varphi_\sigma^{-1}.$$

□

The main goal of this section is to characterize neutral elements as follow.

Proposition 2.3.3. *Let G be a k-group, and Z its center. An element $\eta \in H^2(k,G)$ is neutral if and only if $\eta - n(G)$, in the sense of Lemma 2.1.6, belongs to the image of $\delta_G : H^1(k,G/Z) \to H^2(k,Z)$.*

With δ_G we mean the connecting map. We briefly recall its construction (cf. [Ser73], I-5.6): given a cocycle $\psi \in Z^1(k,G/Z)$, choose a continuous lifting of ψ in $C^1(k,G)$, namely $\sigma \mapsto \tilde{\psi}_\sigma \in G$, and define $\lambda_{\sigma,\tau} = \tilde{\psi}_\sigma {}^\sigma\tilde{\psi}_\tau \tilde{\psi}_{\sigma\tau}^{-1}$. One can check that λ is a 2-cocycle with values in Z, and that this association is well defined; it make sense to define $\delta_G(\mathrm{Cl}(\tilde{\psi})) = \mathrm{Cl}(\lambda)$.

Before proving the characterization of neutral elements we need two lemmas.

Lemma 2.3.4. *An element $\eta \in H^2(k,G)$ is neutral if and only if $\eta = n({}_\psi G)$ for some $\psi \in Z^1(k,G/Z)$.*

Proof. If $\eta = n({}_\psi G)$, then it is neutral by the above Remark. For the converse we write $n(G) = \mathrm{Cl}(f,1)$, $\eta = \mathrm{Cl}(f',1)$; as in the proof of Lemma 2.1.6, we have $f'_\sigma = \psi_\sigma f_\sigma$ for some $\psi \in C(k,G)$. By the equation of 2.1.0.5 we have $\psi_\sigma \in \mathrm{Int}(G_{\overline{k}}) = (G/Z)(\overline{k})$. To conclude it is enough to show that ψ is a cocycle. This follows from 2.1.0.3, since $\psi_{\sigma\tau} = \psi_\sigma f_\sigma \psi_\tau f_\sigma^{-1} = \psi_\sigma \cdot {}^\sigma\psi_\tau$. We have $\eta = \mathrm{Cl}(\psi f,1) = n({}_\psi G)$, hence the result is proven. □

Lemma 2.3.5. *Let G be a k-group, Z its center and $\psi \in Z^1(k,G/Z)$. Then $n({}_\psi G) - n(G) = \delta_G(\mathrm{Cl}(\psi))$.*

Proof. As in the previous lemma, we write $f, f' : \Gamma_k \to \mathtt{SAut}(G_{\overline{k}})$ for the homomorphisms defined by G and ${}_\psi G$, with the relation $f' = \psi f$.

Now we follow the construction displayed above: Let $\tilde{\psi} \in C^1(k,G)$ be a continuous lifting of ψ. By the relations of 2.1.0.6 and 2.1.0.7 we can write $(\psi f,1) = \tilde{\psi}\cdot(f,\lambda)$, where $\lambda_{\sigma,\tau} = \tilde{\psi}_{\sigma\tau}^{-1}\cdot\tilde{\psi}_\sigma\cdot f_\sigma(\tilde{\psi}_\tau)$. Since λ takes value in the center (because it is defined in terms of the lifting of a cocycle with values in G/Z), we can rewrite $\lambda_{\sigma,\tau} = \tilde{\psi}_\sigma f_\sigma(\tilde{\psi}_\tau)\tilde{\psi}_{\sigma\tau}^{-1}$. By definition we have $\mathrm{Cl}(\lambda) = \delta_G(\mathrm{Cl}(\psi))$. Our construction proves the following equalities:

$$\mathrm{Cl}(\varphi f,1) = \mathrm{Cl}(f,\lambda) = \mathrm{Cl}(\lambda) + \mathrm{Cl}(f,1) = \delta_G(\mathrm{Cl}(\psi)) + \mathrm{Cl}(f,1).$$

It follows that $n({}_\psi G) - n(G) = \mathrm{Cl}(\varphi f,1) - \mathrm{Cl}(f,1) = \delta_G(\mathrm{Cl}(\psi))$; which was the claim. □

Proof of Proposition 2.3.3. Let $\eta \in H^2(k,L)$ be a neutral element and write $\eta = n({}_\psi G)$, as in Lemma 2.3.4. By lemma 2.3.5 we have that $n({}_\psi G) - n(G)$ belongs to the image of δ_G.

Viceversa, let $\eta \in H^2(k,L)$ be any element such that $\eta - n(G) \in \mathrm{Im}(\delta_G)$, i.e. $\eta = \delta_G(\mathrm{Cl}(\psi)) + n(G)$ for some $\psi \in Z^1(k,G/Z)$. Lemma 2.3.5 allows us to write $n(G) = n({}_\psi G) - \delta_G(\mathrm{Cl}(\psi))$. Combining the two equations we get $\eta = n({}_\psi G)$, hence η is neutral. □

2.3.1 Reductive case

Let $\overline{G}$ be a connected reductive k-group and G_0 a k-form of $\overline{G}$ (it exists by Chevalley, see [Spr98], Theorem 9.6.2 and Theorem 16.3.3). In the same spirit of Section 1.11.3, we fix $T \subset G_0$ a split maximal torus and B a Borel subgroup containing T. To the triple (G_0,B,T) is associated the based root datum $\Psi(G_0,B,T)$. As in Proposition 1.11.6, we have an exact sequence

$$1 \to G_0^{\mathrm{ad}}(\overline{k}) \to \mathtt{Aut}(G_{0\overline{k}}) \to \mathtt{Aut}(\Psi) \to 1$$

To any roots α of Ψ we obtain a splitting $\mathtt{Aut}(\Psi) \to \mathtt{Aut}_k(G_0) \subset \mathtt{Aut}(G_{0\overline{k}})$ of the sequence displayed above (cf. [BT65] 2.3), and such a splitting is Γ_k-equivariant since Γ_k acts trivially on Ψ. Moreover, thanks to this exact sequence, we conclude that $\mathtt{Out}(\overline{G}) \cong \mathtt{Aut}(\Psi)$, Γ_k-equivariantly.

Lemma 2.3.6. *Let $\overline{G}$ be a connected reductive group. The set of k-lien $L = (\overline{G},\kappa)$, with $\overline{G}$ given, is canonically in bijection with the set of continuous homomorphisms $\Gamma_k \to \mathtt{Aut}(\Psi)$.*

Proof. The k-form G_0, obtained as above, defines a splitting of $\mathtt{SAut}(\overline{G}) \to \Gamma_k$ in virtue of Galois descent; in particular we can write $\mathtt{SOut}(\overline{G})$ as semidirect product of $\mathtt{Out}(\overline{G})$ and Γ_k. As discussed above Γ_k acts trivially on $\mathtt{Aut}(\Psi) \cong \mathtt{Out}(\overline{G})$, hence the semidirect product can be rewritten as

$$\mathtt{Aut}(\Psi) \times \Gamma_k \cong \mathtt{SOut}(\overline{G}). \tag{2.3.1.1}$$

Now the data of a k-lien structure corresponds to a continuous morphism $\Gamma_k \to \mathtt{SOut}(\overline{G})$, which splits 2.1.0.2, which, composed with the projection on the first factor, can be seen as a map $\Gamma_k \to \mathtt{Aut}(\Psi)$. Viceversa, given $f : \Gamma_k \to \mathtt{Aut}(\Psi)$, a k-lien structure is given by $(f,1) : \mathtt{Aut}(\Psi)\times\Gamma_k \cong \mathtt{Out}(\overline{G}) \to \Gamma_k$. □

We have developed all the machinery to prove the main result of the section.

Proposition 2.3.7. *Let $L = (\overline{G}, \kappa)$ be a connected reductive k-lien. The set of neutral cohomology classes in $H^2(k, L)$ is non empty.*

Proof. The previous lemmas associates to $L = (\overline{G}, \kappa)$ a morphism $\mu : \Gamma_k \to \mathtt{Aut}(\Psi)$. Consider the map ψ : spl $\circ \mu$, where spl : $\mathtt{Aut}(\Psi) \to \mathtt{Aut}(\overline{G})$ is a splitting associated to a certain root of Ψ. Since, as remarked above, Γ_k acts trivially on the image of spl we have that ψ belongs to $Z^1(k, \mathtt{Aut}(G_0))$, just because it is a homomorphism. We can consider $n({}_\psi G_0) \in H^2(k, L)$ (since $f_{{}_\psi G_0} = \kappa \mod \operatorname{Int}(\overline{G})$). Clearly it is a neutral element; the result is proven. $\square$

2.3.2 Non-reductive case

Let $L = (\overline{G}, \kappa)$ be a k-lien, not necessarily connected. The unipotent radical of $\overline{G}^\circ$, denoted with $\overline{G}^{\mathrm{u}}$, is a normal subgroup invariant under all elements of $\mathtt{SAut}(\overline{G})$. By the Quotient Construction 2.1.5 we can consider the k-lien $L^{\mathrm{red}} = (\overline{G}^{\mathrm{red}}, \kappa^{\mathrm{red}})$ and a canonical map $r : H^2(k, L) \to H^2(k, L^{\mathrm{red}})$ associated to the group $\overline{G}^{\mathrm{red}} = \overline{G}/\overline{G}^{\mathrm{u}}$. From the construction we know that the map r send neutral elements to neutral elements; viceversa we have the following result.

Proposition 2.3.8. *Let $L = (\overline{G}, \kappa)$ be k-lien. An element $\eta \in H^2(k, L)$ is neutral if and only if $r(\eta) \in H^2(k, L^{red})$ is neutral.*

Proof. If $\overline{G}^{\mathrm{u}} = 1$ the assertion is clear. We argue by induction. Consider the center $\overline{Z}$ of $\overline{G}^{\mathrm{u}}$, in virtue of Proposition 1.4.7 we have $\dim \overline{Z} > 0$. Since $\overline{Z}$ satisfies the assumptions of the Quotient Construction 2.1.5, we can consider the k-lien $L' = (\overline{G}', \kappa')$ and a canonical map $v : H^2(k, L) \to H^2(k, L')$ associated to $\overline{G}' = \overline{G}/\overline{Z}$. Since $\dim(\overline{G}'^{\mathrm{u}}) < \dim(\overline{G}^{\mathrm{u}})$, we may assume the result holds for L'.

Let $\eta = \operatorname{Cl}(f, u) \in H^2(k, L)$ be such that $r(\eta)$ is neutral. By the commutativity of the diagram

$$\begin{array}{ccc} H^2(k, L) & \xrightarrow{r} & H^2(k, L^{\mathrm{red}}) \\ \big\downarrow{\scriptstyle v} & & \big\downarrow \\ H^2(k, L') & \xrightarrow{r'} & H^2(k, L'^{\mathrm{red}}) \end{array}$$

and the previous remark, we know that $v(\eta) = \operatorname{Cl}(f', u')$ is neutral, i.e. $u' = 1$ and f' is an homomorphism (where f' and u' are the applications obtained form f and u taking the quotient, upto changing (f, u) with a suitable cohomologous cocycle). By restriction of the target of u to $\overline{Z}$ and via the map $\mathtt{SAut}(\overline{G}) \to \mathtt{SAut}(\overline{Z})$, we can interpret (f, u) as an element of $H^2(k, \overline{Z}, \kappa)$. Since $\overline{Z}$ is abelian, as in the Remark to Galois Descent, we have always a k-form Z of $\overline{Z}$ and, by Proposition 1.4.5 (which can by applied by Proposition 1.4.3) we have $H^2(k, \overline{Z}, \kappa) = H^2(k, Z) = 0$. It follows that $\operatorname{Cl}(f, u) = 1$ in $H^2(k, Z)$, i.e. there exists $c \in C^1(\Gamma_k, Z(\overline{k}))$ such that

$$c_{\sigma\tau} u_{\sigma,\tau} f_\sigma(c_\tau)^{-1} c_\sigma^{-1} = 1.$$

This means that $c \cdot (f, u)$ is a neutral cocycle, hence $\eta = \operatorname{Cl}(f, u) \in H^2(k, L)$ is neutral. The proposition is proven. $\square$

Corollary 2.3.9. *Let $(\overline{U}, \kappa)$ be a unipotent k-lien. Then any element $\eta \in H^2(k, (\overline{U}, \kappa))$ is neutral.*

Proof. By Proposition 1.4.2 we have $\overline{U}^{\mathrm{red}} = 1$, and so we can apply the previous result. $\square$

In the same spirit of Theorem 2.2.4, we can now prove the following; cf. [Bor96], Lemma 3.2.

Proposition 2.3.10. *Let U be a unipotent k-group and X/k a homogeneous space of U. Then $X(k) \neq \emptyset$.*

Proof. Let $\overline{H}$ be the stabilizers of a $\overline{k}$ point of X, since it is a subgroup of $U_{\overline{k}}$ it is unipotent (thanks to Proposition 1.4.3). The class $\eta_X \in H^2(k, (\overline{H}, \kappa))$ is neutral in virtue of the previous corollary. The result follows since any principal homogeneous space under a unipotent group has a k-point, as proven in Proposition 1.4.5. $\square$

We conclude this section presenting a different proof of Lemma 1.13 of [San81]. As usual we specialize our selves to the case of zero characteristic, but the proof works for every field (except for the statement about the unipotent radical, where the field must be assumed perfect).

Corollary 2.3.11. *Let G/K be a linear group, k a field of characteristic zero. If U is a k-subgroup, unipotent, invariant and k-solvable, then the canonical application $\pi : H^1(k, G) \to H^1(k, G/U)$ is a bijection. In particular this is the case when U is the unipotent radical of G.*

Proof. Since $H^1(k, U)$ is trivial and $H^2(k, (\overline{U}, \kappa))$ is made of neutral elements. Thanks to Proposition 2.1.7, we conclude. $\square$

2.4 Abelianization

A general principle, called *Whitehead principle* in [Toe02], for non abelian cohomology is the following: *Non-abelian cohomology is controlled by non-abelian cohomology in degree one (i.e. torsors theory) and usual abelian cohomology.* In our case this means that the H^1 and the H^2 of a connected (reductive) non commutative group can be understood in terms of the abelian groups H^1 and H^2 with values in a complex of abelian groups.

In this section, for a connected k-lien L, we define an abelian group $H^2_{ab}(k, L)$ and an *abelianization map* $\mathrm{ab} : H^2(k, L) \to H^2_{ab}(k, L)$ which will provide necessary and sufficient conditions for an element $\eta \in H^2(k, L)$ to be neutral (see Theorem 2.4.3).

Construction 2.4.1 ($H^2_{ab}(k, L)$)**.** Let $L = (\overline{G}, \kappa)$ be a reductive k-lien, and consider $\overline{G}^{\mathrm{ss}}$, $\overline{G}^{\mathrm{sc}}$ (cf. the notation in the introduction). We write $\overline{Z}$, $\overline{Z}^{\mathrm{ss}}$, $\overline{Z}^{\mathrm{sc}}$ for the center of $\overline{G}$, $\overline{G}^{\mathrm{ss}}$, $\overline{G}^{\mathrm{sc}}$, respectively, and Z, Z^{ss}, Z^{sc} for the center (in the sense of Section 2.1.1) of the k-forms G, G^{ss} and G^{sc}. The map $\overline{\rho} : \overline{G}^{\mathrm{sc}} \to \overline{G}$ induces an homomorphism $\rho : Z^{\mathbf{sc}} \to Z$ defined over k.

We define $H^2_{ab}(k, L) = \mathbb{H}^2(k, (Z^{(\mathrm{sc})} \xrightarrow{\rho} Z)[1])$, i.e. the Galois hypercohomology group of k with coefficient in the complex $(Z^{(\mathrm{sc})} \to Z)[1]$.

2.4.1 The abelianization map

Consider the short exact sequence of (abelian) complexes

$$1 \to (1 \to Z)[1] \xrightarrow{j} (Z^{(\mathrm{sc})} \xrightarrow{\rho} Z)[1] \to (Z^{\mathrm{sc}} \to 1)[1] \to 1.$$

This gives rise to the following exact sequence of abelian groups:

$$H^2(k, Z^{(\mathrm{sc})}) \xrightarrow{\rho_*} H^2(k, Z) \xrightarrow{j_*} H^2_{\mathrm{ab}}(k, L) \to H^3(k, Z^{(\mathrm{sc})}). \tag{2.4.1.1}$$

Thanks to Lemma 2.1.6, the map $j_* : H^2(k, Z) \to H^2_{\mathrm{ab}}(k, L)$ suffices to define an "association" $H^2(k, L) \to H^2_{\mathrm{ab}}(k, L)$; but to obtain a well defined map we need a Lemma.

Lemma 2.4.2. *Let $L = (\overline{G}, \kappa)$ be a connected reductive k-lien. Let $\eta, \eta' \in H^2(k, L)$ be neutral elements. Then $j_*(\eta - \eta') = 0$.*

Proof. Since the homomorphism $\rho : G^{\mathrm{sc}} \to G$ induces an isomorphism $G^{\mathrm{sc}}/Z^{(\mathrm{sc})} \cong G^{\mathrm{ad}}$ (as proven in Section 1.13), it makes sense to consider the following commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & Z^{(\mathrm{sc})} \arrow[r] \arrow[d, "\rho"] & G^{\mathrm{sc}} \arrow[r] \arrow[d, "\rho"] & G^{\mathrm{ad}} \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\
1 \arrow[r] & Z \arrow[r] & G \arrow[r] & G^{\mathrm{ad}} \arrow[r] & 1
\end{tikzcd}.$$

By functoriality we have the following commutative diagram:

$$\begin{tikzcd}
H^1(k, G^{\mathrm{ad}}) \arrow[r, "\delta_{G^{\mathrm{sc}}}"] \arrow[d, "\mathrm{Id}"] & H^2(k, Z^{(\mathrm{sc})}) \arrow[d, "\rho_*"] \\
H^1(k, G^{\mathrm{ad}}) \arrow[r, "\delta_G"] & H^2(k, Z)
\end{tikzcd}$$

Thanks to Lemma 2.3.2, we can apply Proposition 2.3.3 to write $\eta - \eta' \in \mathrm{Im}(\delta_G : H^1(k, G^{\mathrm{ad}}) \to H^2(k, Z))$; but $\mathrm{Im}(\delta_G) = \mathrm{Im}(\rho_*)$, which coincides with $\mathrm{Ker}(j_*)$ by the exactness of 2.4.1.1, i.e. $\eta - \eta' \in \mathrm{Ker}(j_*)$. $\square$

Let L be as in Lemma 2.4.2, the abelianization map $\mathrm{ab} : H^2(k, L) \to H^2_{\mathrm{ab}}(k, L)$ is given by the following association: for any $\eta \in H^2(k, L)$, choose $\eta' \in H^2(k, L)$ a neutral element (it is possible thanks to 2.3.7), and set

$$\mathrm{ab}(\eta) = j_*(\eta - \eta') \in H^2_{ab}(k, L)$$

The map is well defined since if $\eta'' \in H^2(k, L)$ is another neutral element we have just proven that $j_*(\eta' - \eta'') = 0$. Notice that if $\eta \in H^2(k, L)$ is a neutral element one can choose $\eta' = \eta$ and so $\mathrm{ab}(\eta) = 0$.

For a general $L = (\overline{G}, \kappa)$ a connected k-lien (non necessarily reductive) we define $H^2_{\mathrm{ab}}(k, L)$ as $H^2_{\mathrm{ab}}(k, L^{\mathrm{red}})$ and the abelianization map as the composition

$$\mathrm{ab} : H^2(k, L) \xrightarrow{r} H^2(k, L^{\mathrm{red}}) \xrightarrow{\mathrm{ab}} H^2_{\mathrm{ab}}(k, L^{\mathrm{red}}) = H^2_{ab}(k, L)$$

Also in the general case, thanks to Proposition 2.3.8, the abelianization map takes neutral cohomology classes to 0. For the opposite implication we have the following.

Theorem 2.4.3. *Let k be a p-adic field or a number field. Let $L = (\overline{G}, \kappa)$ be a connected k-lien. An element $\eta \in H^2(k, L)$ is neutral if and only if $\mathrm{ab}(\eta) = 0$ in $H^2_{ab}(k, L)$.*

Corollary 2.4.4. *If, in addition to Theorem 2.4.3, $\overline{G}$ is semisimple simply connected, then every element of $H^2(k, L)$ is neutral.*

Proof of Corollary 2.4.4. Since $\overline{G}$ is semisimple simply connected we have $Z^{(\mathrm{sc})} = Z$ and the complex $(Z^{(\mathrm{sc})} \to Z)[1]$ becomes $Z \xrightarrow{\mathrm{Id}} Z$. In particular we have $H^2_{\mathrm{ab}}(k, L) = 0$. Theorem 2.4.3 implies that every element is neutral. □

Proof of Theorem 2.4.3. By Proposition 2.3.8, we may assume L to be reduced. Let $\eta \in \mathrm{Ker(ab)}$. Thanks to Lemma 2.3.2 and Proposition 2.3.7 there exists a neutral class in $H^2(k, L)$ of the form $n(G)$ for some k-form G of $\overline{G}$. By assumption we have $j_*(\eta - n(G)) = 0$ and, thanks to the exactness of 2.4.1.1, we can write $\eta - n(G) = \rho_*(\chi)$ for some $\chi \in H^2(k, Z^{(\mathrm{sc})})$. Via Theorem 1.15.1, we can chose an element $\xi \in H^1(k, G^{\mathrm{ad}})$ such that $\delta_{G^{\mathrm{sc}}}(\xi) = \chi$. As explained in the proof of Lemma 2.4.2 we have $\delta_G = \rho_* \circ \delta_{G^{\mathrm{sc}}}$, in particular $\eta - n(G)$ belongs to the image of δ_G. Proposition 2.3.3 grantees that $\eta \in H^2(k, L)$ is neutral. □

2.5 Hasse principle for nonabelian H^2

Let $L = (\overline{G}, \kappa)$ be a connected k-lien where k is a local field of characteristic 0 or a number field. Consider the biggest $\overline{k}$-quotient torus of $\overline{G}$, $\overline{G}^{\mathrm{tor}} = \overline{G}^{\mathrm{red}}/\overline{G}^{\mathrm{ss}}$ and the canonical epimorphism $t : \overline{G} \to \overline{G}^{\mathrm{tor}}$. By construction $\overline{G}^{\mathrm{ss}}$ is a subgroup of $\mathrm{Ker}\, t$, and so any semialgebraic automorphism fixes $\mathrm{Ker}\, t$. As in the quotient construction we can consider a k-form G^{tor} of $\overline{G}^{\mathrm{tor}}$ and a canonical map $t_* : H^2(k, L) \to H^2(k, G^{\mathrm{tor}})$.

Let $Z, Z^{(\mathrm{sc})}, Z^{(\mathrm{ss})}$ be as in the previous section. Thanks to the discussion in Section 1.13 we have the following exact sequence of abelian groups

$$1 \to Z^{(\mathrm{ss})} \to Z \to G^{\mathrm{tor}} \to 1$$

which implies the exactness, in the category of abelian complexes, of

$$1 \to (Z^{(\mathrm{sc})} \to Z^{(\mathrm{ss})}) \to (Z^{(\mathrm{sc})} \to Z) \to (1 \to G^{\mathrm{tor}}) \to 1$$

We can consider the induced exact sequence in hypercohomology

$$H^1(k, G^{\mathrm{tor}}) \to H^3(k, \mathrm{Ker}\, \rho) \to H^2_{\mathrm{ab}}(k, L) \xrightarrow{t_{\mathrm{ab}}} H^2(k, G^{\mathrm{tor}}) \tag{2.5.0.2}$$

Lemma 2.5.1. *The composition $H^2(k, L) \xrightarrow{\mathrm{ab}} H^2_{ab}(k, L) \xrightarrow{t_{\mathrm{ab}}} H^2(k, G^{tor})$ is the canonical map $t_* : H^2(k, L) \to H^2(k, G^{tor})$.*

Proof. We can suppose $L = L^{\mathrm{red}}$. Identifying $H^2(k, L)$ with $H^2(k, Z)$, the map ab is nothing else than j_*. More details are provided in [LA10], Lemma 5.1.4. □

In particular, thanks to the easy part of Theorem 2.4.3, neutral elements of $H^2(k, L)$ are killed by t_*. For the opposite implication we discuss separately the local and the global case.

Proposition 2.5.2. *Let k be a p-adic field. Let $L = (\overline{G}, \kappa)$ be a connected k-lien. An element $\eta \in H^2(k, L)$ is neutral if and only if $t_*(\eta) = 0$.*

Proof. Since p-adic fields have strict cohomological dimension equal to 2 (see [Ser73], II-5.3, Prop. 15 or [Har12], Theorem 7.15) and $\operatorname{Ker}\rho$ is finite (it is contained in $Z(\overline{G}^{(\mathrm{sc})})$, by Theorem 1.7.4), the exact sequence 2.5.0.2 implies that t_{ab} is injective.

We conclude by Lemma 2.5.1 and Theorem 2.4.3. □

Theorem 2.5.3. *Let $L = (\overline{G}, \kappa)$ and k be as in the previous proposition. Assume that one of the following holds:*

i) G^{tor} is k-anisotropic (see Definition 1.7.7);

ii) $\overline{G}^{tor} = 1$;

iii) $\overline{G}$ is semisimple.

Then any element $\eta \in H^2(k, L)$ is neutral.

Proof. Clearly $iii) \Rightarrow ii) \Rightarrow i)$. If G^{tor} is k-anisotropic, by Tate-Nakayama Duality (see Theorem 1.14.4), the canonical map $t_* : H^2(k, L) \to H^2(k, G^{\mathrm{tor}}) = 0$ is the zero map. Proposition 2.5.2 ends the proof. □

Corollary 2.5.4. *Let k be a p-adic field, and let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an exact sequence of k-group. If G_1 is connected and $G_1^{tor} = 1$, then the map $H^1(k, G_2) \to H^1(k, G_3)$ is surjective.

Proof. The previous theorem ensure that every element in $H^2(k, G)$ is neutral. In virtue of Proposition 2.1.7, this concludes as in Corollary 2.3.11. □

The main theorem which will allow us to deduce the Hasse principle in many situations, via Theorem 2.2.4, is the following.

Theorem 2.5.5. *Let $L = (\overline{G}, \kappa)$ be a connected k-lien over a number field k. Consider an element $\eta \in H^2(k, L)$, the following are equivalent:*

a) η is neutral;

b) the restriction of η to $H^2(k_v, L)$ is trivial for all the archimedean places v of k and $t_*(\eta) = 0$ in $H^2(k, G^{tor})$.

Proof. Assume η to be neutral (in particular $H^2(k, L)$ is not empty). Via the commutativity of the following diagram

$$\begin{tikzcd}
H^2(k, L) \arrow[r, "\mathrm{ab}"] \arrow[d] & H^2_{\mathrm{ab}}(k, L) \arrow[d] \\
H^2(k_v, L) \arrow[r, "\mathrm{ab}"] & H^2_{\mathrm{ab}}(k_v, L)
\end{tikzcd}$$

and Theorem 2.4.3 we have that the condition on the restrictions is satisfied. Moreover $t_*(\eta) = 0$ thanks to the exact sequence 2.5.0.2 and the compatibility between t_{ab} and t_* explained by Lemma 2.5.1.

Viceversa consider the following "local-to-global" diagram chasing:

$$\begin{tikzcd}
H^1(k, G^{\mathrm{tor}}) \arrow[r] \arrow[d] & H^3(k, \operatorname{Ker}\rho) \arrow[r] \arrow[d] & H^2_{\mathrm{ab}}(k, L) \arrow[r, "t_{\mathrm{ab}}"] \arrow[d] & H^2(k, G^{\mathrm{tor}}) \arrow[d] \\
\prod_{v\in\Omega_\infty} H^1(k_v, G^{\mathrm{tor}}) \arrow[r] & \prod_{v\in\Omega_\infty} H^3(k_v, \operatorname{Ker}\rho) \arrow[r] & \prod_{v\in\Omega_\infty} H^2_{\mathrm{ab}}(k_v, L) \arrow[r] & \prod_{v\in\Omega_\infty} H^2(k_v, G^{\mathrm{tor}})
\end{tikzcd}$$

Notice that the rows are exact by 2.5.0.2, the first column map is surjective (by Proposition 1.15.3) and the second column map is an isomorphism (as explained in Proposition 2.5.2 $\operatorname{Ker}\rho$ is finite, so it follows from Poitou-Tate as in [Mil06], Theorem 4.10 (c)). Under the assumptions of $b)$, we can choose $x \in H^3(k, \operatorname{Ker}\rho)$, a lift of $\mathrm{ab}(\eta)$. Locally it uniquely corresponds to an element $\tilde{x} \in \prod_{v\in\Omega_\infty} H^3(k_v, \operatorname{Ker}\rho)$ which is mapped to zero in $\prod_{v\in\Omega_\infty} H^2_{\mathrm{ab}}(k_v, L)$, so it can be lifted to an element $x' \in \prod_{v\in\Omega_\infty} H^1(k_v, G^{\mathrm{tor}})$, and x' can be lifted to $x'' \in H^1(k, G^{\mathrm{tor}})$. Since x'' is mapped to x and x to $\mathrm{ab}(\eta)$, it must be $\mathrm{ab}(\eta) = 0$. The theorem is proven. □

The above theorem tells us that, to drop the assumption about the map t_*, it is enough to add some assumptions to ensure $Ш^2(k, G^{\mathrm{tor}}) = 0$. In particular we have the following.

Corollary 2.5.6 (a Hasse Principle for neutral elements). *Let $L = (\overline{G}, \kappa)$ and k be as in Theorem 2.5.5. Assume at least one of the following holds:*

i) $Ш^2(k, G^{tor}) = 0$;

ii) the k torus G^{tor} satisfies the assumptions of Lemma 1.14.1;

iii) $\overline{G}^{tor} = 1$;

iv) $\overline{G}$ is semisimple.

Then an element $\eta \in H^2(k, L)$ is neutral if and only if its restrictions to $H^2(k_v, L)$ are neutral for all $v \in \Omega_\infty$.

In particular, if k is a totally imaginary number field and $L = (\overline{G}, \kappa)$ a connected k-lien satisfying one of the assumptions $i) - iv)$, then any element of $H^2(k, L)$ is neutral.

Moreover we obtain a "local to global" version of Corollary 2.5.4.

Corollary 2.5.7. *Let k be a number field, and let*

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

be an exact sequence of k-group where G_1 is connected and $G_1^{tor} = 1$. If $\xi \in H^1(k, G_3)$ is such that $(\xi_v) \in \operatorname{Im}(\prod_{v\in\Omega_k} H^1(k_v, G_2) \to \prod_{v\in\Omega_k} H^1(k_v, G_3))$, then $\xi \in \operatorname{Im}(H^1(k, G_2) \to H^1(k, G_3))$.

2.6 Hasse principle and homogeneous spaces

Theorem 2.6.1 (local case). *Let k be a non archimedean local field of characteristic zero. Let G be a semisimple simply connected k-group and X an homogeneous space with connected stabilizers $\overline{H}$. Assume one of the following holds:*

i) the k-torus H^{tor} is k-anisotropic;

ii) $\overline{H}^{tor} = 1$;

iii) $\overline{H}$ is semisimple.

Then X has a k-point.

Proof. The Springer Class η_X, defined in Section 2.2, is neutral by Theorem 2.5.3. By Proposition 2.2.3 there exists a k-torsor $Y \to X$ under G; to conclude it is enough to show that Y has a k-point, and this is true since $H^1(k, G) = 0$ (see Theorem 1.15.4). □

Theorem 2.6.2 (global case). *Let k be a number field. Let G be a semisimple simply connected k-group and X an homogeneous space with connected stabilizers $\overline{H}$. Assume one of the following holds:*

i) $Ш^2(k, H^{tor}) = 0$;

ii) the k-torus H^{tor} is quasi-trivial;

iii) H^{tor} is k_{v_0} anisotropic for some place v_0;

iv) H^{tor} splits over a finite cyclic extension of k;

v) $\dim \overline{H}^{tor} \leq 1$*;*

vi) $\overline{H}^{tor} = 1$*;*

vii) $\overline{H}$ *is semisimple.*

Then the Hasse principle holds for X*.*

Proof. In virtue of Theorem 2.2.4 it is enough to show that the Springer Class η_X is neutral. To do this we apply Corollary 2.5.6. Since $X(k_v) \neq \emptyset$ there exists a k_v-principal homogeneous space $Y \to X$ under G, namely $Y = G$ and the map that sends g to $p_v \cdot g$ for some $p_v \in X(k_v)$ for all v, Proposition 2.2.3 implies that η_X is locally trivial. The result is proven. □

Adding some assumptions to control the infinite places of k, we can combine Theorem 2.6.2 with Theorem 2.6.1 to obtain the Hasse Principle with respect to the infinite places. For example in the following.

Corollary 2.6.3. *In the cases vi) and vii) of Theorem 2.6.2. If* $X(k_v) \neq \emptyset$ *for all* $v \in \Omega_\infty$*, then* $X(k) \neq \emptyset$*.*

Corollary 2.6.4. *If* X *is a projective variety, then the Hasse Principle holds for* X*.*

Proof. By definition the group $\overline{H}$ is a parabolic subgroup of $\overline{G}$. By Lemma 4.3.3 of [Har68], we have that H^{tor} is a quasi-trivial torus. The result is implied by Theorem 2.6.2 *ii*). □

Corollary 2.6.5. *Suppose that* X *is a symmetric homogeneous space of an absolutely simple simply connected* k*-group* G*. Then the Hasse principle holds for* X*.*

Proof. Thanks to the discussion in Section 1.17.1, the stabilizers of a point $\overline{H}$ is connected. From the classification of involutions of simple Lie algebras, as in [Hel78], Chapter 10, Section 5, page 514 we have that $\dim \overline{H}^{\mathrm{tor}} \leq 1$. In particular we can apply Theorem 2.6.2 *v*). □

Theorem 2.6.2 *vii*) can be generalized without assuming G to be semisimple, cf. [Bor96] Proposition 3.4.

Corollary 2.6.6. *Let* k *be a number field. Let* G *be simply connected linear* k*-group and* X *an homogeneous space with connected stabilizer* $\overline{H} = \overline{H}^{ssu}$ *(see the Notation at the beginning). If* $X(k_v) \neq \emptyset$ *for all* $v \in \Omega_\infty$*, then* $X(k) \neq \emptyset$*.*

Proof of Corollary 2.6.6. It is enough to reduce the general case to the case G semisimple simply connected, in this case the assumption $\overline{H} = \overline{H}^{\mathrm{ssu}}$ implies that $\overline{H}^{\mathrm{tor}} = 1$. Thanks to Lemma 1.1.6 there exists the quotient $Y = X/G^{\mathrm{u}}$, which is an homogeneous space under $G^{\mathrm{red}} = G/G^{\mathrm{u}}$, and a map $\varphi : X \to Y$. If $X(k_v) \neq \emptyset$ for all $v \in \Omega_\infty$, then the same is true for Y. Since G^{red} is a semisimple simply connected algebraic group, we can apply Corollary 2.6.3 (we are in case *vi*)) to ensure that there exists $y \in Y(k) \neq \emptyset$. The fiber of φ in y defines an homogeneous space under the action of G^{u}; by Proposition 2.3.10 it has a k-point, which, in particular, is also a k-point of X. □

Remark. The argument used to prove of Corollary 2.6.6 is a first application of the so called "fibration method", which will be used intensively in Chapter 3.

Chapter 3

The Brauer-Manin obstructions for homogeneous spaces with connected or abelian stabilizers

In this chapter we present the main results proven by Borovoi in [Bor96]. As stated in the introduction, the main goal is to prove that for a homogeneous space of a connected linear group with connected stabilizer and for a homogeneous space of a simply connected group with abelian stabilizers, the Brauer-Manin obstructions explain the failure of the Hasse principle and of weak approximation; see Theorem 3.2.4.

In this chapter k will denote a number field, G a connected linear k-group, X/k a homogeneous space under G, and $\overline{H}$ the stabilizer of a point $\overline{x}_0 \in X(\overline{k})$. If $S \subset \Omega_k$ is a finite set we write $k_S = \prod_{v \in S} k_v$, and $k_\infty = \prod_{v \in \Omega_\infty} k_v$.

3.1 Brauer-Manin obstructions

We collect here a brief overview about the Brauer-Manin obstruction, introducing the main notion we are going to discuss in this and in the following chapter. During this section let k be a number field and X be a k-variety, i.e. a separated scheme of finite type over k.

For any place $v \in \Omega_k$, we can endow $X(k_v)$ with the topology as v-adic space, and $X(\mathbb{A}_k)$ with the *adelic topology*, cf. [Con11].

Definition 3.1.1 (Weak approximation)**.** We say that X *satisfies weak approximation* if the diagonal map $X(k) \to \prod_{v \in \Omega_k} X(k_v)$ is dominant, in the right hand side equipped with the product topology.

Let S be a finite set of places of k. We denote with $\mathcal{O}_{k,S}$ the ring of *S-integers*, i.e. the set of $x \in k$ such that $v(x) \geq 0$ for all $v \in \Omega_k - S$. If $S \neq \emptyset$, $\mathcal{O}_{k,S}$ is a Dedekind domain which is not a field, and if $S \subset T$ are finite set of places then $\mathcal{O}_{k,S}$ is contained in $\mathcal{O}_{k,T}$. We denote with $\mathbb{A}_k^S$ the ring of *adeles outside* S, i.e. the ring obtained projecting $\mathbb{A}_k$ to $\prod_{v \in \Omega_k - S} k_v$ endowed with the adelic topology. We write $\pi^S : X(\mathbb{A}_k) \to X(\mathbb{A}_k^S)$ for the map induced by the projection $\pi : \mathbb{A}_k \to \mathbb{A}_k^S$.

Definition 3.1.2 (Strong approximation)**.** Assume $X(k) \neq \emptyset$. We say that X has *the strong approximation property off a finite set of places* $S \subset \Omega_k$ if the image of the diagonal map $X(k) \to X(\mathbb{A}_k^S)$ is dense.

We now define an important pairing, introduced for the first time by Manin, which will be fundamental to describe an obstruction to the existence of rational points. The main ingredient will be the main exact sequence of Class Field Theory. For more details we refer to [Poo16], [Mil13] and [Sko01]. A quick introduction is also offered in [CT15].

Theorem 3.1.3. *Let k be a number field, Ω_k its set of places. Then there is the following exact sequence*

$$0 \to \mathrm{Br}(k) \to \bigoplus_{v \in \Omega_k} \mathrm{Br}(k_v) \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0. \tag{3.1.0.1}$$

Proof. For the proof of this classical result of Global Class field theory see [SN86]. □

Let X/k a variety and $A \in \operatorname{Br}(X)$. If L is a k-algebra and $x \in X(L)$ then, by functoriality of $\operatorname{Br}(-)$, it induces a homomorphism

$$\operatorname{Br}(X) \to \operatorname{Br}(L), A \mapsto A(x).$$

Let X/k be a smooth and geometrically integral variety over a number field k. We are interested in the *Brauer-Manin pairing*:

$$\langle \ , \ \rangle : \operatorname{Br}(X) \times X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z},$$

which is defined by the following rule

$$(A, (P_v)) \mapsto \sum_{v \in \Omega_k} \operatorname{inv}_v(A(P_v)). \tag{3.1.0.2}$$

Where $A(P_v)$ makes sense thanks to the previous definition and inv_v are the *local invariant* maps appearing in the exact sequence of Theorem 3.1.0.1. The pairing is well defined, basically because the Brauer group of any $\mathcal{O}_v$ is zero.

Notice that, by the exact sequence 3.1.0.1, our pairing is trivial on $\operatorname{Br}_0(X)$, and so it can be defined also as a pairing from $\operatorname{Br}(X)/\operatorname{Br}_0(X)$.

Definition 3.1.4. We define $X(\mathbb{A}_k)^{\operatorname{Br}(X)}$ as the right kernel of the Brauer-Manin pairing, i.e. as the subset of $X(\mathbb{A}_k)$ orthogonal to all elements of $\operatorname{Br}(X)$.

Some fundamental properties are the following:

- The Brauer-Manin pairing is locally constant in the adelic topology, see Corollary 8.2.11 of [Poo16];
- $X(k) \subseteq X(\mathbb{A}_k)^{\operatorname{Br}(X)} \subseteq X(\mathbb{A}_k)$, where the first inclusion follows from the commutativity of the diagram

$$\begin{array}{ccc} X(k) & \hookrightarrow & X(\mathbb{A}_k) \\ \downarrow & & \downarrow \\ \operatorname{Br}(k) & \longrightarrow & \bigoplus_{v \in \Omega_k} \operatorname{Br}(k_v) \end{array}$$

  and the exact sequence 3.1.0.1.
- In particular the closure of the diagonal image of $X(k)$ via the diagonal embedding in the adelic points is contained in $X(\mathbb{A}_k)^{\operatorname{Br}(X)}$.

Remark (Functoriality). Let $f : X \to Y$ a k-morphism of smooth geometrically integral k-variety. Give $A \in \operatorname{Br}(X)$ and (P_v) an adelic point of Y, we have

$$\sum_{v \in \Omega_k} \operatorname{inv}_v(f^*A(P_v)) = \sum_{v \in \Omega_k} \operatorname{inv}_v(A(f(P_v)))$$

It follows

$$Y(\mathbb{A}_k)^{\operatorname{Br}(Y)} = \emptyset \Rightarrow X(\mathbb{A}_k)^{\operatorname{Br}(X)} = \emptyset.$$

Now, if the set $X(\mathbb{A}_k)^{\operatorname{Br}(X)}$ is empty of course the variety will not have rational point. As promised in the title of the section we introduce the link between the approximation properties and the Brauer-Manin pairing.

Definition 3.1.5. Let $\mathcal{C}$ be a class of varieties. We say that *the only obstruction to Hasse principle is given by the Brauer-Manin obstruction* if $X(\mathbb{A}_k)^{\operatorname{Br}(X)} \neq \emptyset$ implies $X(k) \neq \emptyset$ for every variety X in $\mathcal{C}$.

If X is not proper, the Brauer-Manin pairing between $\prod_v X(k_v)$ and $\operatorname{Br}(X)$ is not well defined. To get rid of this it is enough to consider the Brauer group of a smooth compactification of X, denoted with $\operatorname{Br}_{nr}(X)$; cf. Notation.

Definition 3.1.6. Given a variety X, we say that *the only obstruction to weak approximation is given by the Brauer-Manin obstruction* if $X(k)$ is dense in $(\prod_v X(k_v))^{\operatorname{Br}_{nr}(X)}$.

Definition 3.1.7. We say that X *satisfies the strong approximation property with Brauer-Manin obstruction off* S if $X(k)$, via the diagonal map, is dense in $\pi^S(X(\mathbb{A}_k)^{\mathrm{Br}(X)})$ with the adelic topology.

With $\overline{X(k)}^S$ we denote the adelic-closure of $X(k)$ in $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ in the sense of the previous definition.

Remark. Other variants of the Brauer-Manin pairing will also be discussed. In particular it is possible to consider Б(X), $\mathrm{Br}_a(X)$, ...

3.1.1 Notation

Following [Bor96] we represent the Brauer-Manin obstruction to the Hasse principle for X as an element $m_H(X) \in Б(X)^D = \mathrm{Hom}(Б(X), \mathbb{Q}/\mathbb{Z})$, and the Brauer-Manin obstruction to weak approximation off S as a map $m_{W,S}(X) : X(k_S) \to (Б_S(X)/Б(X))^D$.

Assume that $\prod_{v\in\Omega_k} X(k_v) \neq \emptyset$. Recall that the pairing

$$\langle\ ,\ \rangle : Б_\omega(X) \times \prod_{v\in\Omega_k} X(k_v) \to \mathbb{Q}/\mathbb{Z},$$

is defined over the classical Brauer Manin pairing choosing a lift $\tilde{b} \in \mathrm{Br}_1(X)$ of any $b \in Б_\omega(X)$. The result is a continuous well defined pairing (it does not depend on the choice of $\tilde{b}$ thanks to exact sequence 3.1.0.1); cf. [San81], Lemma 6.2.

Let $b \in Б(X) \subset Б_\omega(X)$ and $\tilde{x_v} \in \mathrm{Br}_1(X)$ be a representative of b. Since the localization of $\tilde{b}$ in $\mathrm{Br}_1(X_{k_v})$ lies in $\mathrm{Br}_0(X_{k_v})$, $\tilde{x_v} \in \mathrm{Br}(k)$ is independent of $x_v \in X(k_v)$; in particular $\langle b, (x_v)\rangle$ does not depend on the point (x_v). Thus we can consider $m_H(X) \in Б(X)^D$ defined as the map $b \mapsto \langle b, (x_v)\rangle$. Notice that $X(k) \neq \emptyset$ implies $m_H(X) = 0$.

Assume that $x \in X(k) \neq \emptyset$, and $S \subset \Omega_k$ a finite set. Recall that the pairing

$$\langle\ ,\ \rangle_S : Б_S(X) \times X(k_S) \to \mathbb{Q}/\mathbb{Z}, \langle b, (x_v)_{v\in S}\rangle \mapsto \sum_{v\in S} \mathrm{inv}_v(\tilde{b}(x_v) - \mathrm{inv}_v(\tilde{b}(x)))$$

is continuous and, thanks to exact sequence 3.1.0.1, it does not depend on the choice of the rational point. Moreover, if $b \in Б(X)$, then $\langle b, x_s\rangle_S = 0$ for any $x_S \in X(k_S)$; thus we can consider

$$Б_S(X)/Б(X) \times X(k_S) \to \mathbb{Q}/\mathbb{Z},$$

equivalently a continuous map $m_{W,S}(X) : X(k_S) \to (Б_S(X)/Б(X))^D$. Notice that if X satisfies weak approximation off S, then $m_{W,S}$ is zero.

In Section 3.4 we will use the functoriality of the two obstruction defined above as follows: Let X be such that $\prod_{v\in\Omega_k} X(k_v) \neq \emptyset$, and consider a morphism of k-varieties $\pi : X \to Y$. We have

- $\pi_*(m_H(X)) = m_H(Y)$, where $\pi_* : Б(X)^D \to Б(Y)^D$ is the map induced by π;
- if $X(k) \neq \emptyset$ the following is commutative

$$\begin{tikzcd} X(k_S) \arrow[r, "m_{W,S}(X)"] \arrow[d, "\pi"] & (Б_S(X)/Б(X))^D \arrow[d, "\pi_*"] \\ Y(k_S) \arrow[r, "m_{W,S}(Y)"] & (Б_S(Y)/Б(Y))^D \end{tikzcd}.$$

3.2 Statements of the main results

As usual, over $\overline{k}$, we write $X_{\overline{k}} = G_{\overline{k}}/\overline{H}$ (we are not assuming $X(k) \neq \emptyset$), and consider the following conditions:

$$G^{\mathrm{ssu}} \text{ is simply connected,} \tag{3.2.0.1}$$

$$\overline{H}^{\mathrm{mult}} := \overline{H}/\overline{H}^{\mathrm{ssu}} \text{ is abelian, hence of multiplicative type.} \tag{3.2.0.2}$$

Theorem 3.2.1 (The only obstruction to the Hasse principle)**.** *Let X be a homogeneous space. Consider the following assumption:*

- *G satisfies 3.2.0.1, and the stabilizer $\overline{H}$ of a $\overline{k}$-point of X satisfies 3.2.0.2.*

Under this condition the Brauer-Manin obstruction attached to $\text{Б}(X)$ is the only obstruction to the Hasse principle for X, i.e. if $X(\mathbb{A}_k)^{\text{Б}(X)} \neq \emptyset$, then $X(k) \neq \emptyset$.

For the next two statements we assume $X(k) \neq \emptyset$, and we write $X = G/H$ for some connected k-subgroup H.

Theorem 3.2.2 (The only obstruction to weak approximation). *Let $X = G/H$ be a homogeneous space. Consider the following assumption:*

- *G satisfies 3.2.0.1, and the stabilizer H of a k-point of X satisfies 3.2.0.2.*

Under this condition the Brauer-Manin obstruction attached to $\text{Б}_S(X)/\text{Б}(X)$ is the only obstruction to the weak approximation property for X, where $S \subset \Omega_k$ is a finite set.

Actually we will prove something stronger.

Theorem 3.2.3. *Under the assumptions of Theorem 3.2.2, if $x_S \in X(k_S)$ and $m_{W,S}(X)(x_S) = 0$, then x_S belongs to the closure of $X(k)$ in $\pi^S(X(\mathbb{A}_k))$ with respect to the product topology.*

The following result ensure that we can apply the previous results in the case we were interested in.

Theorem 3.2.4. *Assume one of the following holds:*

1. *G is a connected k-group and $\overline{H}$ connected;*
2. *G satisfies 3.2.0.1 and $\overline{H}$ is abelian (in particular it satisfies 3.2.0.2).*

Then the assumptions of Theorems 3.2.1, 3.2.2, 3.2.3 are satisfied.

We have already introduced all the technical prerequisites (thanks to Theorem 1.12.4) to prove the last result. Notice that only the first case requires an argument.

Proof of Theorem 3.2.4. In the first case we can consider X as a homogeneous space under another k-group G_0, with connected stabilizer $\overline{H}_0 \subset G_{0,\overline{k}}$ and such that G_0^{ssu} is simply connected. Namely it is enough to consider the object $G_0 \to G$ produced by Theorem 1.12.4; to check that the stabilizer $\overline{H}_0$ is connected consider the following exact sequence:

$$1 \to Z_{\overline{k}} \to \overline{H}_0 \to \overline{H} \to 1,$$

and conclude applying Proposition 1.3.5.

Now G_0 satisfies 3.2.0.1 and $\overline{H}_0/\overline{H}_0^{\text{ssu}} = \overline{H}_0^{\text{tor}}$ is clearly abelian. It follow that we can apply the three theorems we were considering to X seen as homogeneous space under G_0. The result is proven. ☐

3.3 Preliminary results

The strategy to prove the results stated in the previous section is to reduce them to known facts. In this reduction we will see at work the main theorems of Borovoi, proven in Chapter 2; but they are not enough. In particular the following plays a key role.

Theorem 3.3.1 (The case of a torus). *Let X/k be a homogeneous space of G, where G is a k-torus. The following hold:*

i) the assertion of Theorem 3.2.1 is true;

ii) the assertion of Theorem 3.2.3 is true;

iii) $X(k)$ is dense in $X(k_\infty)$.

Proof. This result was originally proved by Voskresenskii, see also [San81], Section 8. The first assertion can be also found in [Sko01], Theorem 6.2.1: it basically follows from the comparison of the Brauer-Manin pairing with the one given by Poitou-Tate, and the fact that the second one is not degenerate. The implication *ii)* $\Rightarrow$ *iii)*, due to Serre, can be found in [San81] Corollary 3.5. ☐

In Chapter 2 we have discussed just the Hasse principle, some times analogous results hold also for the weak approximation property. For example the analogous of Corollary 2.6.6 holds.

Theorem 3.3.2. *Let G be simply connected k-group and $X = G/H$ an homogeneous space with connected stabilizer $H = H^{ssu}$. Then $X(k)$ is dense in $X(k_S)$ for any finite set $S \subset \Omega_k$.*

Proof. This is one of the main result of the paper [Bor90]. In particular see Theorem 1.1, Theorem 1.4 and Corollary 1.7 of [Bor90]. □

It is worth noticing that we have also a version of Proposition 2.3.10 for weak approximation.

Proposition 3.3.3. *Let U be a unipotent k-group and X/k a homogeneous space of U. Then $X(k)$ is dense in $X(k_S)$, for any finite $S \subset \Omega_k$.*

Proof. Notice that $U(k)$ is dense in $U(k_S)$ since U, as k-variety, is isomorphic to $\mathbb{A}^n_k$, thanks to Theorem 1.4.1 *e*). To deduce the property for X from the approximation property for U, it is enough to show that $U(k_S)$ acts transitively on $X(k_S)$. Let H be the stabilizer of a k-point of X (which exists in virtue of Proposition 2.3.10), it is a subgroup of U, in particular it is unipotent. We have that $X(k_S) = U(k_S)/H(k_S)$ using the fact that $H^1(K, H) = 1$ for any field extension K/k, cf. the argument of [Ser73], I-Proposition 36. □

3.4 Proof of the main results

To prove the previous theorems we reduce the situation to a simpler case. From now on let G and X be as in Theorems 3.2.1 and 3.2.3.

Theorem 3.4.1. *It is enough to prove Theorem 3.2.1 and Theorem 3.2.3 under the following extra assumption:*

(†) *The homomorphism $\overline{H}^{mult} \to G^{tor}_{\overline{k}}$ induced by the inclusion $\overline{H} \subset G_{\overline{k}}$ is injective.*

The proof of Theorem 3.4.1 is based on the construction of an object $Y \to X$ satisfying (†). The construction of Y is not trivial: we will define a torsor $\overline{\pi} : \overline{Y} \to X_{\overline{k}}$ under $T_{\overline{k}}$, where T is a quasi trivial k-torus and we will discuss the existence of a k-form of the pair $(\overline{Y}, \overline{\pi})$. The argument we will use to show the existence of a k-form is close to the one of Section 2.2.

Construction 3.4.2 (A k-form H^{m} of $\overline{H}^{\mathrm{mult}}$)**.** Let $\sigma \in \Gamma_k$, as in Construction 2.2.1, we write ${}^\sigma\overline{x}_0 = \overline{x}_0 \cdot g_\sigma$ for a fixed family $g_\sigma \in G(\overline{k})$. For any $h \in \overline{H}(\overline{k})$ the following relation, by construction, holds:

$$\overline{x}_0 \cdot g_\sigma \cdot {}^\sigma h = {}^\sigma\overline{x}_0 \cdot {}^\sigma h = {}^\sigma(\overline{x}_0 \cdot h) = \overline{x}_0 \cdot g_\sigma.$$

In particular it make sense to consider the map

$$\nu_\sigma : \overline{H} \to \overline{H}, \quad h \mapsto g_\sigma \cdot {}^\sigma h \cdot g_\sigma^{-1},$$

which is a σ-semialgebraic automorphism in the sense of Section 2.1; by functoriality we can consider also $\nu^{\mathrm{m}}_\sigma \in \mathtt{SAut}(\overline{H}^{\mathrm{mult}})$. Thanks to the assumption 3.2.0.2 we have that ν^{m}_σ does not depend on the initial choice of the g_σs: Indeed, if we choose others $g'_\sigma = hg_\sigma$, we have $\nu'_\sigma = \mathrm{Int}(h) \circ \nu_\sigma$, but the inner automorphism becomes trivial on $\overline{H}^{\mathrm{mult}}$. The usual relation

$$\nu^{\mathrm{m}}_{\sigma\tau} = \nu^{\mathrm{m}}_\sigma \circ {}^\tau\nu^{\mathrm{m}}_\sigma$$

defines a k-form H^{m} of $\overline{H}^{\mathrm{mult}}$, thanks to Galois Descent.

To conclude we notice that the k-form H^{m} depends only on G and X. Indeed if $\overline{H}_1$ is the stabilizers of another point $\overline{x}_1 \in X(\overline{k})$, we can consider the isomorphism $\mu_g : \overline{H} \to \overline{H}_1$ given the conjugation by some $g \in G(\overline{k})$ such that $\overline{x}_1 = \overline{x}_0 \cdot g$. As in the previous reasoning we have that induced isomorphism $\mu^{\mathrm{m}}_g : H^{\mathrm{m}} \to H^{\mathrm{m}}_1$ does not depend on the choice of such a $g \in G(\overline{k})$.

Construction 3.4.3 (The torsor $\overline{\pi} : \overline{Y} \to X_{\overline{k}}$)**.** We choose a closed embedding $H^{\mathrm{m}} \hookrightarrow T$, where T is a quasi trivial k-torus. Define F as the k-product $G \times T$; we can now define a $\overline{k}$-homogeneous space $\overline{Y} := F_{\overline{k}}/\overline{H}$ under $F_{\overline{k}}$ and a $F_{\overline{k}}$-equivariant map

$$\overline{\pi} : \overline{Y} \to X_{\overline{k}}, \quad \overline{H} \cdot (g, t) \mapsto \overline{H} \cdot g.$$

The importance of this construction is enclosed in the following properties:

- $\overline{\pi} : \overline{Y} \to X_{\overline{k}}$ is a torsor under $T_{\overline{k}}$;

- the map $\overline{H}^{\mathrm{mult}} \to F_{\overline{k}}^{\mathrm{tor}}$ is injective, since H^{m} is a subgroup of T; cf. with the property (†);

- if there exists $x \in X(k)$ we have a k-form (Y, π) of $(\overline{Y}, \overline{\pi})$, just by considering the stabilizer of x.

The following Lemma is the technical heart of Theorem 3.4.1.

Lemma 3.4.4. *Suppose that $X(k_v) \neq \emptyset$ for any $v \in \Omega_k$. Then there exists a k-form (Y, π) of the pair $(\overline{Y}, \overline{\pi})$.*

To prove this lemma we construct a class $\eta \in H^2(k, T)$ which represents the obstruction of the existence of such a k-form, emulating the proof of Proposition 2.2.3.

Proof of Lemma 3.4.4. Step 1: The class $\eta \in H^2(k, T)$. The homomorphism $\sigma_* : X_{\overline{k}} \to X_{\overline{k}}$, induced by a $\sigma \in \Gamma_k$, can be lifted to a σ-semialgebraic automorphism $a_\sigma : \overline{Y} \to \overline{Y}$ as follows: Let $(g_\sigma \in G(\overline{k}))_\sigma$ be the family of Construction 3.4.2. Recall that $\overline{Y} = F_{\overline{k}}/\overline{H} = G_{\overline{k}} \times T_{\overline{k}}/\overline{H}$, and set

$$a_\sigma : \overline{Y} \to \overline{Y}, \quad \overline{H} \cdot (x, t) \mapsto \overline{H} \cdot (g_\sigma \cdot {}^\sigma x, {}^\sigma t).$$

Notice that, thanks to the relation ${}^\sigma\overline{x}_0 = \overline{x}_0 \cdot g_\sigma$, we have that $\sigma_* : G_{\overline{k}}/\overline{H} \to G_{\overline{k}}/\overline{H}$ is defined by the association $H \cdot x \mapsto \overline{H} \cdot (g_\sigma \cdot {}^\sigma x)$; in particular we have the commutativity $\overline{\pi} \circ a_\sigma = \sigma_* \circ \overline{\pi}$.

By Construction 3.4.3, we have that $\mathtt{Aut}_{F_{\overline{k}}, X_{\overline{k}}}(\overline{Y}) = T(\overline{k})$, cf. also the proof of Theorem 2.2.4. Therefore there exists a unique family $(d_{\sigma,\tau} \in T(\overline{k}))_{\sigma,\tau}$ such that the following relation holds

$$a_\sigma \cdot {}^\sigma a_\tau = d_{\sigma,\tau} a_{\sigma\tau}.$$

By direct computation we see that $(d_{\sigma,\tau})$ belongs to $Z^2(\Gamma_k, T(\overline{k}))$, and we define $\eta = \mathrm{Cl}(d_{\sigma,\tau}) \in H^2(k, T)$. Notice that η is independent of the choice of the family (a_σ), since inner automorphisms become trivial on T, as in Construction 3.4.2. □

Step 2: η is trivial. By assumption we have that X has a k_v point for any $v \in \Omega_k$, in particular, by the last property of Construction 3.4.3, we obtain a k_v-form of the $\overline{k}_v$-pair $(\overline{Y}, \overline{\pi})$ for any place v. It follows that, in the previous construction, the $a_\sigma : Y_{\overline{k}_v} \to Y_{\overline{k}_v}$ defying $\eta_v \in H^2(k_v, T)$ can be defined by the association $a_\sigma = \sigma_*$ since, as remarked above, η_v is independent of the (a_σ)s. In particular the relation $a_{\sigma\tau} = a_\sigma \cdot {}^\sigma a_\tau$ implies that $d_{\sigma,\tau} = 1$, i.e. η_v is trivial in $H^2(k_v, T)$ for any $v \in \Omega$. In other words we have proven $\eta \in \text{Ш}^2(k, T) = 0$, where the last equality holds since T is a quasi-trivial k-torus, as in Lemma 1.14.1 *i*). □

Step 3: η is "the obstruction". Since $\eta \in H^2(k, T)$ is trivial we can choose a family $(a_\sigma)_{\sigma \in \Gamma_k}$ of semialgebraic automorphism of $\overline{Y}$, satisfying $a_{\sigma\tau} = a_\sigma \cdot {}^\sigma a_\tau$; in particular, as usual, it determines a k-form (Y, π) of $(\overline{Y}, \overline{\pi})$. □

The Lemma is finally proven. □

Proof of Theorem 3.4.1. Consider the T-torsor $\pi : Y \to X$, which is defined over k thanks to Lemma 3.4.4, and notice that Y satisfies the assumption (†). We first prove that if X satisfies the assumptions of Theorems 3.2.1 and 3.2.3, then also Y does; supposing the Theorems are true for Y, we will conclude they hold for X.

Claim: $\text{Б}_S(X) \cong \text{Б}_S(Y)$. Since $\pi : Y \to X$ is a T-torsors, the following is exact, cf. (6.10.3) of Proposition 6.10 of [San81]:

$$\mathrm{Pic}(T) = 0 \to \mathrm{Br}_1(X) \xrightarrow{\pi_*} \mathrm{Br}_1(Y) \to \mathrm{Br}_a(T),$$

where the equality $\mathrm{Pic}(T) = 0$ holds since T is a quasi-trivial k-torus (see Proposition 1.14.3). Since $\pi : Y \to X$ is a map of k-schemes, we have that π_* is compatible with the natural maps $\mathrm{Br}(k) \to \mathrm{Br}(Y)$ and $\mathrm{Br}(k) \to \mathrm{Br}(Y)$. In particular, taking the cokernels, we get also the following exact sequence:

$$0 \to \mathrm{Br}_a(X) \xrightarrow{\pi_*} \mathrm{Br}_a(Y) \to \mathrm{Br}_a(T).$$

Since the same sequence remains exact replacing X with X_{k_v} for any place v outside a finite set of places S, we obtain, taking the kernels, the following exact sequence:

$$0 \to \text{Б}_S(X) \to \text{Б}_S(Y) \to \text{Б}_S(T).$$

Thanks to the natural exact sequence of Lemma 1.12.6, using that the geometric Picard group of a torus is trivial, we have that $\text{Б}_\omega(T)$ is isomorphic to $\text{Ш}^2_\omega(k,\widehat{T})$, which is zero as proven in Lemma 1.14.2. In particular we have that $\text{Б}_S(T)=0$ for any $S\subset\Omega_k$ finite, hence the canonical map $\text{Б}_S(X)\to\text{Б}_S(Y)$ is an isomorphism. Notice that, for $S=\emptyset$, we have also $\text{Б}(X)\cong\text{Б}(Y)$. □

Reduction for Theorem 3.2.1. First of all notice that, since $H^1(k_v,T)=0$ and $X(k_v)\neq\emptyset$ for any v, it follows that also $Y(k_v)\neq\emptyset$ for any $v\in\Omega_k$. Thanks to the previous claim $\pi_*:\text{Б}(Y)^D\to\text{Б}(X)^D$ is an isomorphism, by functoriality of the Brauer-Manin pairing, we have that the obstruction given by $\text{Б}(X)$ is the only obstruction to the Hasse principle also for Y. In particular if there exists $y\in Y(k)$, then $\pi(y)$ is a k-point of X. □

Reduction for Theorem 3.2.3. As in the previous reduction we have that $Y(k)\neq\emptyset$, since is given an element of $X(k)$. As above we have that $\pi_*:(\text{Б}_S(Y)/\text{Б}(Y))^D\to(\text{Б}_S(X)/\text{Б}(X))^D$ is an isomorphism, hence the equality $\pi_*(m_{W,S}(Y)(y_S))=m_{W,S}(X)(x_S)=0$ implies that $m_{W,S}(Y)(x_S)=0$. By assumption we have that y_S belongs to the closure of $Y(k)$ in $\pi^S(Y(\mathbb{A}_k))$, hence the same holds for x_S, by compatibility with the map π. □

Theorem 3.4.1 is finally proven. □

The "naturality" of reduction method of Borovoi should be now clear. Recall that X is a homogeneous space under a connected k-group G, such that G^{ssu} is simply connected, with stabilizers $\overline{H}$. Thanks to Lemma 1.1.6 we define V as the quotient X/G^{ssu}, and consider the canonical smooth map

$$\varphi:X\to V,$$

where V/k is a homogeneous space under $G/G^{\text{ssu}}=G^{\text{tor}}$ (for homogeneous spaces under tori Theorem 3.3.1 can be applied) and the fibers are homogeneous spaces of the simply connected group G^{ssu}, with stabilizers $\overline{H}_0:=\overline{H}\cap G^{\text{ssu}}_{\overline{k}}$. Notice that if we had $\overline{H}_0=\overline{H}_0^{\text{ssu}}$ or, in other words, $\overline{H}_0^{\text{mult}}=1$, then we could try to apply Corollary 2.6.6, for the Hasse principle, and Theorem 3.3.2, for weak approximation. See also the argument of Corollary 2.6.6.

We are now ready to ultimate the proof of the main results; cf. [Bor96] Proposition 3.5.

Proof of Theorem 3.2.1. Since $X(k_v)\neq\emptyset$ for any $v\in\Omega_k$ and by functoriality of the Brauer-Manin pairing, we have that V satisfies the assumptions of Theorem 3.3.1. It follows, by point *i*) of Theorem 3.3.1, that there exists $y\in V(k)\neq\emptyset$. Since $V(k)$ is dense in $V(k_\infty)$ (by Theorem 3.3.1 *iii*)) and $\varphi(X(k_\infty))$ is open in $V(k_\infty)$ (since φ is smooth, and applying the Implicit Function Theorem), we can choose $y\in V(k)\cap\varphi(X(k_\infty))$.

Consider the fiber $X_y=\varphi^{-1}(y)$. It is a homogeneous space under G^{ssu}, with stabilizer $\overline{H}_0:=\overline{H}\cap G^{\text{ssu}}_{\overline{k}}$, such that, by our choice of y, $X_y(k_\infty)\neq\emptyset$. In virtue of Theorem 3.4.1 we may assume that (†) holds, since G is connected we have the following injective morphism of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \overline{H}^{\text{ssu}} \arrow[r] \arrow[d] & \overline{H} \arrow[r] \arrow[d] & \overline{H}^{\text{mult}} \arrow[r] \arrow[d, "\text{injective by (†)}"] & 1 \\
1 \arrow[r] & G^{\text{ssu}}_{\overline{k}} \arrow[r] & G_{\overline{k}} \arrow[r] & G^{\text{tor}}_{\overline{k}} \arrow[r] & 1
\end{tikzcd}.$$

By the uniqueness of the Kernels we have $\overline{H}_0=\overline{H}^{\text{ssu}}$, in particular $\overline{H}_0$ is connected and $\overline{H}_0^{\text{mult}}=1$. Corollary 2.6.6 ensures that $X_y(k)\neq\emptyset$, in particular X has a k-point. The result is proven. □

Recall that, in the next statement, $X(k)\neq\emptyset$, and S is a finite set of places of k. The next argument is close to the one above, the only difference is how we will deal with the k_∞-points.

Proof of Theorem 3.2.3. Let $x_S\in X(k_S)$, and assume that $m_{W,S}(X)(x_S)$ is zero ; we first notice that we may assume that S contains all the real places of k. Indeed denote with Σ the union of S and the real places, and let $x_\Sigma\in X(k_\Sigma)$ be a point which restricts to x_S along the projection $X(k_\Sigma)\to X(k_S)$, it is enough to show that x_Σ belongs to $\overline{X(k)}^\Sigma$. To do this we prove that, for any open neighbourhood $\mathcal{U}_X$ of x_Σ in $\overline{X(k)}^\Sigma$, $\mathcal{U}_X\cap X(k)\neq\emptyset$.

Consider $y_\Sigma=\varphi(y_\Sigma)\in V(k_\Sigma)$, and notice that its projection on $V(k_S)$ is $y_S=\varphi(x_S)\in V(k_S)$, by functoriality. Since $m_{W,S}(V)(y_S)=\pi_*(m_{W,S}(X)(x_S))=0$, Theorem 3.3.1 *ii*) implies that y_S belongs to $\overline{V(k)}^S$ and, by [San81] (3.3.3), that $y_\Sigma\in\overline{V(k)}^\Sigma$, since the two Σ and S differ just by the real places.

Consider the open subset $\mathcal{U}_V := \varphi(\mathcal{U}_X)$ of $V(k_\Sigma)$ and notice that, by the previous step, $y_\Sigma \in Y(k) \cap \mathcal{U}_V$. As above the fiber X_{y_Σ} is a homogeneous space under G^{ssu} whose stabilizer H_0 has the property $H_0 = H_0^{\text{ssu}}$. By construction we have the $\emptyset \neq X_{y_\Sigma}(k_\Sigma) \subset X(k_\infty)$, hence we can apply Theorem 3.3.2 *ii*) and *iii*) to ensure that X_{y_Σ} has a k-point and that $X_{y_\Sigma}(k)$ is dense in $X_{y_\Sigma}(k_\Sigma)$.

Since $\mathcal{U}_X \cap X_{y_\Sigma}(k_\Sigma)$ is a non empty open subset in $X_{y_\Sigma}(k_\Sigma)$, it contains a k-point x. Clearly x belongs to $\mathcal{U}_X \cap X(k)$; our initial claim is proven. □

Chapter 4

Integral Brauer-Manin Obstruction for homogeneous spaces with finite nilpotent stabilizers

In this chapter we present the main result proven by Demarche in [Dem15] and a general theorem of [LX15]. In this chapter k will denote a number field.

4.1 Strong Approximation with Brauer Manin obstruction

Let k be a number field, we write Ω_k for the set of places of k, Ω_∞ for the subset of infinite places and Ω_f for the finite places. If $v \in \Omega_k$ we denote with k_v the completion of k in v, and with $\mathcal{O}_v$ its ring of integers (following the convention that $\mathcal{O}_v = k_v$ if $v \in \Omega_\infty$).

Let S be a finite set of places of k. We denote with $\mathcal{O}_{k,S}$ the ring of *S-integers*, i.e. the set of $x \in k$ such that $v(x) \geq 0$ for all $v \in \Omega_k - S$. If $S \neq \emptyset$, $\mathcal{O}_{k,S}$ is a Dedekind domain which is not a field, and if $S \subset T$ are finite set of places then $\mathcal{O}_{k,S}$ is contained in $\mathcal{O}_{k,T}$. We denote with $\mathbb{A}_k^S$ the ring of *adeles outside* S, i.e. the ring obtained projecting $\mathbb{A}_k$ to $\prod_{v\in\Omega_k - S} k_v$ endowed with the adelic topology. We write $\pi^S : X(\mathbb{A}_k) \to X(\mathbb{A}_k^S)$ for the map induced by the projection $\pi : \mathbb{A}_k \to \mathbb{A}_k^S$.

Let X be a k-variety, i.e. a separated scheme of finite type over k.

Definition 4.1.1. We say that X *satisfies the strong approximation property with Brauer-Manin obstruction off* S if $X(k)$, via the diagonal map, is dense in $\pi^S(X(\mathbb{A}_k)^{\mathrm{Br}(X)})$ with the adelic topology.

With $\overline{X(k)}^S$ we denote the adelic-closure of $X(k)$ in $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ in the sense of the previous definition.

4.1.1 Integral points

We introduce here also the notion of *integral points.*

Definition 4.1.2. Let X be a k-variety, we say that a separate scheme $\mathcal{X}$ finite over $\mathcal{O}_k$ is a *model* of X if $X \simeq \mathcal{X}_\eta$, with η the generic point of $\mathcal{O}_k$.

Usually we will suppose the model to be integral. The following are two classical facts.

Proposition 4.1.3. *Any two models of X are isomorphic out a finite number of places. Moreover if the variety is reduced (resp. irreducible, resp. proper) any model is reduced (resp. irreducible, resp proper) out a finite number of places.*

Theorem 4.1.4 (Existence of models)**.** *Let A be a Dedekind domain with field of fraction k. For any variety X over k there exists a model $\mathcal{X}$ over A. Moreover if X is affine or proper or projective then $\mathcal{X}$ can be chosen with the same property.*

Definition 4.1.5. For any finite place v, the canonical map

$$\mathcal{X}(\mathcal{O}_v) \to X(k_v),$$

is an inclusion, as a formal consequence of the *valutative criterion of separatedness*. The elements in the image of this map are called *integral points*. Notice that the integral points may depend on the choice of a model of X.

4.1.2 Integral model and strong approximation

This section is devoted to the proof of the following; cf. [LX15], Theorem 2.10.

Theorem 4.1.6. *Let k be a global field, X be a k-variety and $S \neq \emptyset$ a finite set of places. The following are equivalent*

a) X satisfies the strong approximation property with Brauer-Manin obstruction off S;

b) the only obstruction to the existence of integral points for every integral model $\mathcal{X}$ of X over $\mathcal{O}_{k,S}$ is the Brauer-Manin obstruction.

The main ingredient in the proof will be the following result.

Proposition 4.1.7. *Let X and S be as in Theorem 4.1.6. Given an open subset $U \subset X(\mathbb{A}_k^S)$ and a point $z \in U$, there exists a model $\mathcal{X}$ of X over $\mathcal{O}_{k,S}$ such that*

$$z \in \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \subset U.$$

Proof of Proposition 4.1.7. For the proof see [LX15], Corollary 2.9. □

Remark. In particular the subsets $\prod_{v\notin S} \mathcal{X}(\mathcal{O}_v) \subset X(\mathbb{A}_k^S)$ form a basis of the topological space $X(\mathbb{A}_k^S)$, where $\mathcal{X}$ runs through the $\mathcal{O}_{k,S}$-integral models of X. Moreover, if X is a smooth algebraic k-group it is possible to show that the sets $\prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$ are a fundamental system of neighbourhood of $1 \in X(\mathbb{A}_k^S)$, where $\mathcal{X}$ runs through the smooth group scheme models of X over $\mathcal{O}_{k,S}$, see the second part of Remark 2.6 in [LX15].

First of all we need an easy lemma.

Lemma 4.1.8. *Let $\mathcal{X}$ be an integral model of X over $\mathcal{O}_{k,S}$. Then the following natural diagrams with injective maps*

$$\begin{tikzcd} \mathcal{X}(\mathcal{O}_{k,S}) \arrow[r, hook] \arrow[d, hook] & \mathcal{X}(\mathbb{A}_{k,S}) \arrow[d, hook] \\ X(k) \arrow[r, hook] & \prod_{v\in\Omega_k} X(k_v) \end{tikzcd}$$

allows to identify $\mathcal{X}(\mathcal{O}_{k,S})$ with $X(k) \cap \mathcal{X}(\mathbb{A}_{k,S})$.

Proof. First of all notice that all the maps are injective thanks to the valutative criterion of separatedness and the fact that the canonical map

$$\mathcal{X}(\mathbb{A}_{k,S}) \to \prod_{v\in S} X(k_v) \times \prod_{v\notin S} \mathcal{X}(\mathcal{O}_v)$$

is bijective (see for example [Con11], Theorem 3.6).

Let $\sigma \in X(k)$ such that, when seen as an element of $X(k_v)$, lies in $\mathcal{X}(\mathcal{O}_v)$ for any $v \notin S$. To show that $\sigma \in \mathcal{X}(\mathcal{O}_{k,S})$ it is enough to see that, for any $v \notin S$, σ induces a map $U_v \to \mathcal{X}$, for some open neighbourhood U_v of $v \in \mathrm{Spec}(\mathcal{O}_{k,S})$, and glue these morphisms together. □

Proof of Theorem 4.1.6. a) ⇒ b). Let $\mathcal{X}$ be an integral model of X over $\mathcal{O}_{k,S}$ such that $\mathcal{X}(\mathbb{A}_{k,S})^{\mathrm{Br}(X)}$ is not empty. Since the the Brauer-Manin pairing is continuous we have that the $\mathcal{X}(\mathbb{A}_{k,S})^{\mathrm{Br}(X)}$ is open in $X(\mathbb{A}_k)^{\mathrm{Br}(X)}$, with regard to the adelic topology; hence, by Lemma 4.1.8 and the fact that π^S is open, we can write

$$\mathcal{X}(\mathcal{O}_{k,S}) = X(k) \cap \pi^S(\mathcal{X}(\mathbb{A}_{k,S})^{\mathrm{Br}(X)}) \neq \emptyset$$

by the assumption on X. An integral point is produced and the result is proven. □

b) $\Rightarrow$ *a)*. Let U be any open subset of $X(\mathbb{A}_k^S)$, with regard to the adelic topology, such that there exists $z \in U \cap \pi^S(X(\mathbb{A}_k)^{\mathrm{Br}(X)})$. By Proposition 4.1.7 we have an integral model $\mathcal{X}$ over $\mathcal{O}_{k,S}$ such that

$$z \in \prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \subset U.$$

In particular $\mathcal{X}(\mathbb{A}_{k,S})^{\mathrm{Br}(X)}$ is not empty, hence, by assumption, $\mathcal{X}(\mathcal{O}_{k,S}) \neq \emptyset$. Since the image of the natural map $\pi : \mathcal{X}(\mathcal{O}_{k,S}) \subset X(k) \to \mathcal{X}(A_k^S)$ lies in $\prod_{v \notin S} \mathcal{X}(\mathcal{O}_v) \subset U$, it must be, as in the proof of Lemma 4.1.8, $U \cap X(k) \neq \emptyset$ in $X(\mathbb{A}_k^S)$. The theorem is proven since U contains a rational point of X. $\square$

$\square$

4.2 Brauer-Manin obstruction to strong approximation

Let G be a semisimple simply connected algebraic k-group, H be a finite constant nilpotent k-subgroup of G and consider the homogeneous space $X := G/H$. The following results proves that, under suitable assumptions, *the obstruction of Brauer Manin to strong approximation* for X is not the only one.

Theorem 4.2.1. *Let p be a prime number and H be a non commutative finite group of order p^n. For any finite set $S_0 \subset \Omega_k$, if k contains the p^{n+1}th roots of unity, then there exists an adelic point $x \in X(\mathbb{A}_k)^{\mathrm{Br}(X)}$ such that $\pi^{S_0} \notin \overline{X(k)}^{S_0}$.*

Remark. Of course H has to be assumed non commutative, if not, this result would contradict what explained in the introduction. Thanks to this assumption n must be bigger than 2, and so, by the fact that k contains the p^{n+1}th roots of unity, it has no real places (also in the case $p = 2$).

The only reason why we choose H to be a p-group is, as every nilpotent group, the existence of a central cyclic non trivial subgroup Z contained in the derived subgroup of H.

The importance of such an Z is enclosed in the following lemma.

Lemma 4.2.2. *Under the assumption of Theorem 4.2.1, the canonical morphism* $\mathrm{Ext}_k^c(H, \mathbb{G}_m) \to \mathrm{Ext}_k^c(Z, \mathbb{G}_m)$ *is the zero map.*

Proof of Theorem 4.2.1. Step 1: a local to global diagram. Set $Y := G/Z$, $X := G/H$ and $f : Y \to X$ for the natural map. The induced map $f^* : \mathrm{Br}(X)/\mathrm{Br}(k) \to \mathrm{Br}(Y)/\mathrm{Br}(k)$ is the zero map, since, via the isomorphism established by Theorem 1.16.4 (and its functoriality), it corresponds to the map of Lemma 4.2.2.

To use this map we have to check that the Brauer Manin pairing $X(\mathbb{A}_k) \times \mathrm{Br}(X)/\mathrm{Br}(k) \to \mathbb{Q}/\mathbb{Z}$ can be slightly modified to obtain a pairing $P^1(k, H) \times \mathrm{Br}(X)/\mathrm{Br}(k) \to \mathbb{Q}/\mathbb{Z}$ (analogously for Y, with $P^1(k, Z)$ in place of $Y(\mathbb{A}_k)$), where $P^1(k, H)$ denotes the restricted product of the $H^1(k_v, H)$ with regard to the subsets $H^1(\mathcal{O}_v, H)$. The new pairing would produce a map $P^1(k, H) \xrightarrow{\partial_H} (\mathrm{Br}(X)/\mathrm{Br}(k))^D$ (analogously ∂_H for Y).

To do this notice that exact sequence defining X (as Γ_v-modules, for any $v \in \Omega_k$) gives us an exact sequence $X(k_v) \to H^1(k_v, H) \to H^1(k_v, G)$. Since v is not real, as remarked before, Theorem 1.15.4 implies that the map $X(k_v) \to H^1(k_v, H)$ is surjective. In particular we can lift any elements of $P^1(k, H)$ to an adelic point of X. This is enough to define the Brauer-Manin pairing as we wanted, because any two lifts differ by an element lying in the left kernel of the usual pairing, namely an adelic point of G.

The natural exact sequence

$$1 \to Z \to H \to H' := H/Z \to 1$$

induces the following "local-to-global" diagram with exact rows, where the first two 1s follow from the fact that Z and H are constant groups:

$$\begin{tikzcd}
1 \arrow[r] & H^1(k, Z) \arrow[r] \arrow[d] & H^1(k, H) \arrow[r] \arrow[d] & H^1(k, H') \arrow[r] \arrow[d] & H^2(k, Z) \arrow[d] \\
1 \arrow[r] & P^1(k, Z) \arrow[r] \arrow[d, "\partial_Z"] & P^1(k, H) \arrow[r] \arrow[d, "\partial_H"] & P^1(k, H') \arrow[r] & P^2(k, Z) \\
 & (\mathrm{Br}(Y)/\mathrm{Br}(k))^D \arrow[r, "(f^* = 0)^D"] & (\mathrm{Br}(X)/\mathrm{Br}(k))^D & &
\end{tikzcd}$$

Notice that the lower square is commutative thanks to the functoriality of our Brauer-Manin pairing (deduced from the functoriality of the usual pairing), and the map between the Brauer is zero by what explained above. □

Step 2: a contradiction. Heading for a contradiction suppose there exists a finite set of places S_0 such that $\pi^{S_0}(X(\mathbb{A}_k)^{\mathrm{Br}}) \subset \overline{X(k)}^{S_0}$. We call this assumption the *approximation hypothesis.*

Given an element $(z_v) \in P^1(k, Z)$, by the commutativity of the lower square, we have that $\partial_H(h_v) = 0$, where (h_v) is just (z_v) seen as an element of $P^1(k, H)$ via the inclusion. Since $\partial_H(h_v)$ is the map $\langle (h_v), - \rangle \operatorname{Br}(X)/\operatorname{Br}(k) \to \mathbb{Q}/\mathbb{Z}$, there exists an adelic point $h_v \in X(\mathbb{A}_k)$ orthogonal to Br. By the approximation hypothesis we have that, for all finite set of places $S \supset S_0$ there exists $h^S \in H^1(\mathcal{O}_{k,S}, H)$ such that $h^S_v = h_v$ for all $v \in S - S_0$. Write h'^S for the image of h^S in $H^1(\mathcal{O}_{k,S}, H')$.

Claim: $h'^S = 1$ for S big enough. Notice that the canonical map $H^1(\mathcal{O}_{k,S_0}, H') \to H^1(\mathcal{O}_{k,S}, H')$ is injective, since the elements in the two sets can be interpreted as classes of torsors. In the same fashion we have also that $H^1(\mathcal{O}_{k,S}, H') \hookrightarrow H^1(k, H')$, this can be also deduced by the inflation-restriction exact sequence since $H^1(\mathcal{O}_{k,S}, H') = H^1(G_S, H')$. Since $(h_v) \in P^1(k, H)$ lies in $P^1(k, Z)$, we have that $h'^S_v = 1$ for all $v \in S - S_0$; as in Example A.1 it follows that $h'^S \in H^1(\mathcal{O}_{k,S_0}, H')$. Since H' is finite, the pointed set $H^1(\mathcal{O}_{k,S_0}, H')$ is finite (same proof as in the abelian constant case, cf [Ser73]), and we can write its elements as $\{a_1, \dots a_m\}$ for some m. For all $i = 1, \dots, m$ choose a place $v_i \notin S_0$ such that $a_{i|v_i} \neq 1$ (if it is not possible set $v_i = \emptyset$). Consider now the finite set of places $S' := S_0 \cup \{v_i\}_{i=1}^m$, then the image of every element $h^{S'} \in H^1(\mathcal{O}_{k,S'}, H)$ obtained by the approximation hypothesis lies in $H^1(\mathcal{O}_{k,S_0}, H')$ (applying Harder's Lemma as before, thanks to our choice of S'). This argument proves that $h'^{S'}_v = 1$ for all place $v \notin S'$, in particular $h'^{S'} \in Ш^1_\omega(k, H')$, but, by Čebotarev, we have that $Ш^1_\omega(k, H') = 0$ (since H' is finite we can apply Proposition 9.2 of [Har12]). The claim is proven, and "big enough" means "containing S'". Moreover, as remarked before, $h'^S = 1$ also in $H^1(k, H')$, hence there exists $z^S \in H^1(k, Z)$ such that $z^S_v = z_v$ for all $v \in S - S_0$.

Recall that, by the nine terms exact sequence of Poitou-Tate, we have the exactness of $H^1(k, Z) \to P^1(k, Z) \to H^1(k, \widehat{Z})^D$ (about Poitou-Tate duality we refer the reader to [Mil06], Theorem 4.10). Since $H^1(k, \widehat{Z})$ is not finite, while $Ш^1_\omega(k, \widehat{Z})$ is finite, we can choose $(z_v) \in P^1(k, Z)$ not orthogonal to $H^1(k, \widehat{Z})$ (i.e. it is not zero in $H^1(k, \widehat{Z})^D$) such that $z_v = 0$ for all $v \in S_0$. Such an element, can not come from $H^1(k, Z)$, by Poitou-Tate, but this is the case by what the previous claim implies. The contradiction is established and the theorem proven. □

□

Remark. Thanks to Theorem 4.1.6, Theorem 4.2.1 implies the existence of $\mathcal{X}$, an $\mathcal{O}_{k,S_0}$-model of X, for which the failure of the integral Hasse principle is not explained by Brauer-Manin. At the moment we do not have an explicit description of such a model; it would be interesting to find a model with the structure of homogeneous space, extending the one of X.

4.2.1 Proof of Lemma 4.2.2

Consider (the class of) an extension $1 \to \mathbb{G}_m \to H_1 \to H \to 1 \in \operatorname{Ext}^c_k(H, \mathbb{G}_m)$, we want to show that its image $1 \to \mathbb{G}_m \to Z_1 \to Z \to 1$ in $\operatorname{Ext}^c_k(H, \mathbb{G}_m)$ is trivial. To do this we will perform a diagram chasing of the the following:

$$\begin{tikzcd}
\operatorname{Ext}^c_k(H,\mathbb{G}_m) \arrow[r, "Z \to H"] & \operatorname{Ext}^c_k(Z,\mathbb{G}_m) \\
\operatorname{Ext}^c_k(H,\mu_{p^n}) \arrow[u, "\mu_{p^n} \to \mathbb{G}_m"] \arrow[r, "Z \to H"] \arrow[d, "\mu_{p^n} \to \mu_{p^{n+1}}"] & \operatorname{Ext}^c_k(Z,\mu_{p^n}) \arrow[u, "\mu_{p^n} \to \mathbb{G}_m"] \arrow[d, "\mu_{p^n} \to \mu_{p^{n+1}}"] \\
\operatorname{Ext}^c_k(H,\mu_{p^{n+1}}) \arrow[r, "Z \to H"] & \operatorname{Ext}^c_k(Z,\mu_{p^{n+1}}) \arrow[uu, bend right, "\mu_{p^{n+1}} \to \mathbb{G}_m"']
\end{tikzcd}.$$

The two main results of Section 1.16.1 imply that $\operatorname{Ext}^c_k(H, \mathbb{G}_m)$ is isomorphic to $H^2(H, k^*)$, in particular it is a group of $|H| = p^n$-torsion (see Corollary 1.26 of [Har12]). Since the natural complex

$$\operatorname{Ext}^c_k(H,\mu_{p^n}) \xrightarrow{\mu_{p^n} \to \mathbb{G}_m} \operatorname{Ext}^c_k(H,\mathbb{G}_m) \xrightarrow{[p^n]=0} \operatorname{Ext}^c_k(H,\mathbb{G}_m)$$

is exact (see for example [ABD+66], Exposé XVII, Proposition A.2.1), we deduce that the map

$$\mathrm{Ext}^c_k(H, \mu_{p^n}) \to \mathrm{Ext}^c_k(H, \mathbb{G}_m)$$

is surjective.

By the surjectivity displayed above we can choose a lift, in $\mathrm{Ext}^c_k(H, \mu_{p^n})$, of the extension given by H_1, namely $1 \to \mu_{p^n} \to H_2 \to H \to 1$. Write $1 \to \mu_{p^n} \to Z_2 \to Z \to 1$ for the image of the extension given by H_2 in $\mathrm{Ext}^c_k(Z, \mu_{p^n})$, and consider its image $1 \to \mu_{p^{n+1}} \to Z_3 \to Z \to 1$ in $\mathrm{Ext}^c_k(Z, \mu_{p^{n+1}})$.

Claim: the extension given by Z_3 is k-split. The claim proves the result thanks to the commutativity of the big diagram, since the extension given by Z_1 is $\mathrm{Ext}^c_k(Z, \mathbb{G}_m)$ is the image of the the extension $1 \to \mu_{p^{n+1}} \to Z_3 \to Z \to 1$ along the (map induced by the) inclusion $\mu_{p^{n+1}} \to \mathbb{G}_m$.

Proof of the claim. Over $\overline{k}$ the extension given by Z_3 is trivial, since Z has cardinality equal to p and the extension is obtained pulling-back along the map $\mu_{p^n} \to \mu_{p^{n+1}}$ given by $x \mapsto x^p$. To conclude we prove that the action of Γ_k on Z_3 is trivial. To do this consider the image of the extension given by H_2 in $\mathrm{Ext}^c_k(H, \mu_{p^{n+1}})$, namely $1 \to \mu_{p^{n+1}} \xrightarrow{a} H_3 \to H \to 1$. The maps $\mu_{p^n+1} \xrightarrow{a} H_3$, $Z_2 \to H_2 \to H_3$ produce a (unique) map $Z_3 \to H_3$ which fits in the following injective morphism of short exact sequences:

$$\begin{tikzcd}
1 \arrow[r] & \mu_{p^{n+1}} \arrow[r] \arrow[d, "\mathrm{Id}"] & Z_3 \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mu_{p^{n+1}} \arrow[r, "a"] & H_3 \arrow[r] & H \arrow[r] & 1
\end{tikzcd}.$$

Notice that, by abstract non sense, the right square is a pull-back diagram.

For all $g, h \in H_3(\overline{k})$ and for all $\gamma \in \Gamma_k$ we have ${}^{\gamma}[g,h] = {}^{\gamma}(ghg^{-1}h^{-1}) = [{}^{\gamma}g, {}^{\gamma}h]$; since the action of Γ_k on H is trivial, the elements ${}^{\gamma}g$ and g differs by element in the center of H_3 (the extension of H_3 is central), hence we can write ${}^{\gamma}[g,h] = [g,h]$. In particular the action of Γ_k is trivial on $D(H_3)$.

Since the subgroup $Z_3(\overline{k})$ of $H_3(\overline{k})$ is contained, by construction of Z and the fact that the right square is a pull-back, in the subgroup generated by $D(H_3(\overline{k}))$ and $\mu_{p^{n+1}}(\overline{k}) \subset k^*$, since is $\overline{k}$-split, we have that the action of Γ_k on $Z_3(\overline{k})$ is trivial. Hence the extension is trivial not just as abstract groups but also as Γ_k-modules. $\square$

Appendix A

The one dimensional case

A.1 Harder's Lemma

Definition A.1.1. Let X be a scheme, $G \to X$ an fppf group scheme (not necessary commutative) over X, an X-*torsor* under G is a X-scheme $Y \to X$, with a G-action such that there exists a fppf covering $\{U_i \to X\}$ that trivialized Y, in the sense that $Y \times U_i \simeq G \times U_i$ compatible with the action of G.

Remark. Let X be scheme and G an group scheme over X which is affine. Then any *fppf* torsor on X under G is representable. This, and more, can be found in [Mil80] III Theorem 4.3.

Lemma A.1.2. *Let X be a scheme, $G \to X$ an affine, smooth group scheme. Given a Zariski open covering $X = U \cup V$, the following natural diagram is a pull-back:*

$$\begin{tikzcd} H^1(X,G) \arrow[r] \arrow[d] & H^1(U,G) \arrow[d] \\ H^1(V,G) \arrow[r] & H^1(U \cap V, G) \end{tikzcd}.$$

Proof. In virtue of the previous remark we can interpret the element in the H^1 as classes of torsors; at this point the diagram is nothing else than the gluing property of torsors in the intersection. ☐

Proposition A.1.3. *Let G be a group as in Lemma A.1.2 Let A be a DVR, K its fraction field. Write $\widehat{A}$, $\widehat{K}$ for the completions of A and K. The following natural diagram is a pull-back:*

$$\begin{tikzcd} H^1(A,G) \arrow[r] \arrow[d] & H^1(\widehat{A},G) \arrow[d] \\ H^1(K,G) \arrow[r] & H^1(\widehat{K}, G) \end{tikzcd}.$$

Proof. For the proof, which relays on a descent argument, see Proposition A.6 of [GP08]. ☐

We state here the so called Harder's lemma, in the more general version due to Gille and Pianzola, cf. [GP08], Corollary A.8.

Theorem A.1.4 (Harder's Lemma)**.** *Let X be a regular noetherian integral scheme of function field K. For $x \in X^{(1)}$, write $\widehat{\mathcal{O}}_x$ for the completion of the local ring $\mathcal{O}_{X,x}$ and $\widehat{K}_x$ for its fraction field. Let G be an affine, smooth group scheme. Let U be an open subset of X and $\gamma \in H^1(U,G)$. If, for any $x \in X^{(1)} - U$, $\gamma_{\widehat{K}_x} \in \mathrm{Im}(H^1(\widehat{\mathcal{O}}_x, G) \to H^1(\widehat{K}_x, G))$, then there exists an open subscheme $\widetilde{U} \subset X$ containing U such that the codimension of $X - \widetilde{U}$ is bigger or equal to 2, and γ belongs to the image of the natural map $H^1(\widetilde{U}, G) \to H^1(U,G)$.*

Proof. Write $X^{(1)} - U = \{x_1, \ldots, x_n\}$. By induction on n it is enough to prove the result when $X^{(1)} - U = x$. Thanks to Proposition A.1.3 there exists $\alpha \in H^1(\mathcal{O}_{X,x}, G)$ such that α and γ coincide in $U \times_X \mathcal{O}_{X,x} = K$. Thanks to Grothendieck-Margaux's result about the passage to the limit in nonabelian Čech cohomology (see [GP08], Proposition A.4 for the references), there exists an open neighbourhood

V_x of x in X and an $\widetilde{\alpha} \in H^1(V_x, G)$ such that $\widetilde{\alpha}$ and γ coincide at the generic point. The same argument implies that $\widetilde{\alpha}$ and γ coincide on a dense open subset W_0 of X.

Claim: there exists an open subset W of X such that $x \in U \cap W \subset W_0$. This concludes since, by Lemma A.1.2, we can glue the classes of γ and $\widetilde{\alpha}$ along $U \cap W$, otherwise stated $\gamma \in \mathrm{Im}(H^1(U \cup W, G) \to H^1(U, G))$. By construction x belongs to $U \cup W$, hence $\widetilde{U} := U \cup W$ is exactly as wanted.

To prove the claim consider the decomposition in irreducible component $X - W_0 = \overline{\{x\}} \cup Z_1 \cup \cdots \cup Z_m$, and notice that $x \notin Z_i$ since x has codimension one. In particular $W := (X - Z_1 \cup 2ts \cup Z_m) \cap V_x$ is an open subset of X as claimed. □

Example. We present an application of the previous theorem in the one dimensional case. Let k a number field and $S_0 \subset S \subset \Omega_k$ with S finite. Consider $X := \mathrm{Spec}(\mathcal{O}_{k,S_0})$, the open subset $U := \mathrm{Spec}(\mathcal{O}_{k,S}) \subset X$ and $\gamma \in H^1(U, G)$. If $\gamma_v \in H^1(\mathcal{O}_v, G)$ is trivial for any $v \in S - S_0$, then the $\widetilde{U}$ produced by the above theorem Theorem must be equal to X it self, since $\dim X = 1$, this means that

$$\gamma \in \mathrm{Im}(H^1(\mathcal{O}_{k,S_0}, G) \to H^1(\mathcal{O}_{k,S}, G)).$$

If the map $H^1(\mathcal{O}_{k,S_0}, G) \to H^1(\mathcal{O}_{k,S}, G)$ is injective (for example when G is finite), then the previous condition means exactly that γ lies in $H^1(\mathcal{O}_{k,S_0}, G)$.

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