Monodromy representations associated to a continuous map

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Abstract

The aim of this thesis is to give a geometrical meaning to the induced monodromy representation. More precisely, given $f : X \to Y$ a continuous map, the associated functor $f : \Pi_1(X) \to \Pi_1(Y)$ induces a functor $\mathrm{Rep}_k(\Pi_1(X)) \overset{\mathrm{ind}_f}{\to} \mathrm{Rep}_k(\Pi_1(Y))$ of the corresponding categories of representations. We will define a functor $f_*^{LCSH} : \mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$ from the category of locally constant sheaves on X to that of locally constant sheaves on Y in a way that the monodromy representation $\mu_{f_*^{LCSH}\mathscr{F}}$ is given by $\mathrm{ind}_f(\mu_{\mathscr{F}})$, where $\mu_{\mathscr{F}}$ denotes the monodromy representation of a locally constant sheaf $\mathscr{F}$ on X.

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Introduction

Given $H \leq G$ an inclusion of groups, classical constructions in representation theory are the induced and restricted representations. On one hand, the restricted representation allows us to define a representation of H given one of G. On the other hand, the induced representation defines in a natural way a representation of G given a representation of H. This construction can be generalized for any morphism $\phi : H \to G$. Even further, there is a parallel construction in the case of groupoids: using the right Kan extension of the restricted representation we may construct the induced representation.

A well-known result in algebraic topology is the equivalence between the representations of the fundamental groupoid $\Pi_1(X)$ of a topological space X (called *monodromy representations*) and the category $\mathrm{LCSH}(k_X)$ of locally constant sheaves over X. This equivalence, given by the monodromy functor, gives us a geometric interpretation of the monodromy representations, since locally constant sheaves of sets over X have a corresponding interpretation as coverings of X.

Given a continuous map $f : X \to Y$, it induces a functor $f : \Pi_1(X) \to \Pi_1(Y)$. Hence we can consider the associated induced representation. Our aim is to give a geometrical meaning to this construction. In other words, we want to construct a functor $\mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$ which corresponds to the induced representation at the level of monodromy representations. If f is a Serre fibration, this functor is nothing but the direct image functor f_*. However, in general f_* need not preserve locally constant sheaves, so we will factor f as a homotopy equivalence and a Serre fibration. Using such factorization we will be able to define the desired functor.

In Section 1, we will recall some classical results in category theory, mainly Kan extensions and the basics of representation of groupoids.

In Section 2, we will review the theory of locally constant sheaves on topological spaces. In particular, how homotopy equivalences between topological spaces imply equivalences of the categories of locally constant sheaves. Then we prove that if $f : X \to Y$ is a Serre fibration the direct image functor f_* preserves locally constant sheaves. Finally, we prove that any continuous map f defines a functor $f_*^{LCSH} : \mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$.

In Section 3, we will study the monodromy representations of a topological space. In particular, the equivalence between $\mathrm{LCSH}(k_X)$ and the category $\mathrm{Rep}_k(\Pi_1(X))$ of monodromy representations of a locally arcwise connected and semi-locally simply connected topological space. We will use this correspondence to give a geometrical meaning to the restricted and induced representation at the level of monodromy representations. More precisely, we will prove that for a continuous map $f : X \to Y$, the restricted representation corresponds to the inverse image sheaf, and if f is a Serre fibration the induced representation corresponds to the direct image sheaf. Finally, we will show that for any continuous map $f : X \to Y$, the functor f_*^{LCSH} corresponds to the induced representation, namely that the following diagram quasi-commutes

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(X)) & \xleftarrow[\mathrm{ind}_f]{} & \mathrm{Rep}_k(\Pi_1(X)) \\ {\scriptstyle \mu_Y}\uparrow & & {\scriptstyle \mu_X}\uparrow \\ \mathrm{LCSH}(k_Y) & \xleftarrow[f_*^{\mathrm{LCSH}}]{} & \mathrm{LCSH}(k_X) \end{array}$$

1 Preliminary Notions

In this section the fundamental notions for the rest of the thesis will be laid. Mainly, the Kan extensions of functors and the representations of fundamental groupoids will be introduced.

1.1 Notation

k will denote a commutative ring unless stated otherwise. We will denote by $\mathrm{Mod}(k_X)$ the category of sheaves of k-modules over a topological space X.
The category of functors between $\mathcal{C}$ and $\mathcal{D}$ will be denoted by $\mathrm{Fct}(\mathcal{C},\mathcal{D})$.
If G is a group, one can consider it as the category with only one element $\{*\}$, where $\mathrm{Hom}(*,*) = G$, and we will denote it with the same notation if it is clear by the context.

1.2 Kan extension of functors

The contents of this section are classical, and here we follow [1].
A fundamental tool that we are going to use is the Kan extension of functors, which is a way to construct in an abstract manner left and right adjoins of a functor between categories of functors.
Let $\mathcal{C}$, $\mathcal{I}$ and $\mathcal{J}$ be three categories, and let $\varphi : \mathcal{J} \to \mathcal{I}$ be a functor. We define a functor

$$\varphi_* : \mathrm{Fct}(\mathcal{I},\mathcal{C}) \to \mathrm{Fct}(\mathcal{J},\mathcal{C})$$

as follows

$$\varphi_*(\alpha) := \alpha \circ \varphi \quad \text{for } \alpha \in \mathrm{Fct}(\mathcal{I},\mathcal{C})$$

Definition 1.1. (i) If the functor φ_* admits a left adjoint, we denote it by $\varphi^\dagger$

(ii) If the functor φ_* admits a right adjoint, we denote it by $\varphi^\ddagger$

Theorem 1.2. *Let $\varphi : \mathcal{J} \to \mathcal{I}$ be a functor with $\mathcal{J}$ small.*

(i) Assume that $\mathcal{C}$ admits small inductive limits. Then $\varphi^\dagger : \mathrm{Fct}(\mathcal{J},\mathcal{C}) \to \mathrm{Fct}(\mathcal{I},\mathcal{C})$ exists and for any $\beta \in \mathrm{Fct}(\mathcal{J},\mathcal{C})$ we have

$$\varphi^\dagger\beta(i) \simeq \varinjlim_{(\varphi(j)\to i)\in\mathcal{J}_i} \beta(j) \quad \text{for } i \in \mathcal{I}$$

(ii) Assume that $\mathcal{C}$ admits small inductive limits. Then $\varphi^{\ddagger} : \mathrm{Fct}(\mathcal{J},\mathcal{C}) \to \mathrm{Fct}(\mathcal{I},\mathcal{C})$ exists and for any $\beta \in \mathrm{Fct}(\mathcal{J},\mathcal{C})$ we have

$$\varphi^{\ddagger}\beta(i) \simeq \varprojlim_{(i\to\varphi(j))\in\mathcal{J}^i} \beta(j) \quad \textit{for } i \in I$$

Remark 1.3. Note that, since $(\varphi \circ \psi)_* = \psi_* \circ \varphi_*$, it follows that $(\varphi \circ \psi)^{\dagger} \simeq \varphi^{\dagger} \circ \psi^{\ddagger}$ and $(\varphi \circ \psi)^{\ddagger} \simeq \varphi^{\ddagger} \circ \psi^{\ddagger}$.

1.3 Representation of Grupoids

In this section most of the results follow [2], [3] and [4].
Recall that a groupoid is a category in which every morphism is an isomorphism.
Note that any groupid is equivalent to a disjoint union of groups. More precisely, if we choose a representative $d_i \in \mathrm{Ob}(\mathcal{G})$ for every isomorphic class of objects of $\mathcal{G}$ and set $G_i = \mathrm{Aut}(d_i)$, we get the inclusion functor $i : \bigsqcup_i G_i \to \mathcal{G}$, where $\bigsqcup_i G_i$ denotes the disjoint union category, with objects $\{(\{*\},i) | i \in I\}$ and morphisms

$$Hom_{\bigsqcup_i G_i}((\{*\},i),(\{*\},j)) = \begin{cases} G_i & \text{if } i = j \\ \emptyset & \text{else} \end{cases}$$

Then i is essentially surjective and it is clear that it is fully faithful. Hence, $\mathcal{G} \simeq \bigsqcup_i G_i$

Definition 1.4. Let $\mathcal{G}$ be a groupoid. A $\mathcal{G}$-representation over k is a functor from $\mathcal{G}$ to $\mathrm{Mod}(k)$. We define the category of $\mathcal{G}$-representations over k as $\mathrm{Rep}_k(\mathcal{G}) := \mathrm{Fct}(\mathcal{G}, \mathrm{Mod}(k))$.

If G is a group, considered as a groupoid, we recover the classical definition of representation of groups.
Note that since $\mathrm{Mod}(k)$ admits small injective and projective limits, so does $\mathrm{Rep}_k(G)$. This allows us to use Theorem 1.2.

Definition 1.5. (i) Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a functor of groupoids and set

$$\begin{aligned} \mathrm{res}_{\varphi} = \varphi_* : \mathrm{Rep}_k(\mathcal{G}) &\to \mathrm{Rep}_k(\mathcal{H}) \\ F &\mapsto F \circ \varphi \end{aligned}$$

(ii) We define $\mathrm{ind}_{\varphi} : Rep_k(\mathcal{H}) \to \mathrm{Rep}_k(\mathcal{G})$ as the right Kan extension of res_{φ}.

(iii) We will define $\mathrm{coind}_{\varphi} : \mathrm{Rep}_k(\mathcal{H}) \to \mathrm{Rep}_k(\mathcal{G})$ as the left Kan extension of res_{φ}

Proposition 1.6. *If $\varphi : \mathcal{H} \to \mathcal{G}$ is an equivalence of groupoids then* $\mathrm{res}_\varphi : \mathrm{Rep}_k(\mathcal{G}) \to \mathrm{Rep}_k(\mathcal{H})$ *is an equivalence of categories with* $\mathrm{ind}_\varphi \simeq \mathrm{coind}_\varphi$ *as quasi-inverse.*

Proof. Let ψ be a quasi-inverse for φ. Then, we have that $(\psi \circ \varphi)_* = \varphi_* \circ \psi_*$. Since $\psi \circ \varphi \simeq id_{\mathcal{G}}$, we have that $\varphi_* \circ \psi_* \simeq (id_{\mathcal{G}})_* = id_{\mathrm{Rep}_k(\mathcal{G})}$. Similarly, we have that $\psi_* \circ \varphi_* \simeq id_{\mathrm{Rep}_k(\mathcal{H})}$. Hence ψ_* is a quasi-inverse of φ_*. Since a quasi-inverse is both a right and a left adjoint, by unicity, up to isomorphism, of adjoints, it follows that $\psi \simeq \mathrm{ind}_\varphi \simeq \mathrm{coind}_\varphi$. □

We would like to give an explicit construction to the functors ind_φ and coind_φ, this will be realized by using groupoid algebras.

Definition 1.7. Let $\mathcal{G}$ be a groupoid. The groupoid algebra $k[\mathcal{G}]$ has an underlying k-module with one generator e_g for each morphism of $\mathcal{G}$. The algebra structure is given by

$$e_g \cdot e_h = \begin{cases} e_{gh} & \text{if g and h are composable} \\ 0 & \text{otherwise} \end{cases}$$

Remark 1.8. Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a morphism of groupoids. Then it induces an algebra morphism

$$\tilde{\varphi} : k[\mathcal{H}] \to k[\mathcal{G}]$$
$$\sum k_i e_h \mapsto \sum k_i e_{\varphi(h)}$$

Note that $\tilde{\varphi}$ is an isomorphism if φ is an equivalence.

Proposition 1.9. *If If $\mathcal{G}$ has a finite number of isomorphism classes of objects, there is an equivalence*

$$\delta : \mathrm{Rep}_k(\mathcal{G}) \xrightarrow{\sim} \mathrm{Mod}(k[\mathcal{G}]) \tag{1}$$

Proof. If $\mathcal{G}$ has a finite number of isomorphism classes of objects, it means that $\mathcal{G} \simeq G_1 \sqcup \ldots \sqcup G_n := \tilde{\mathcal{G}}$, where $\mathrm{Ob}(\tilde{\mathcal{G}}) = \{c_1, \ldots, c_n\}$. Hence by Proposition 1.6 we get an equivalence $\mathrm{Rep}_k(\mathcal{G}) \simeq \mathrm{Rep}_k(\tilde{\mathcal{G}})$ via the restriction functor. So we only need to prove that $\mathrm{Rep}_k(\tilde{\mathcal{G}}) \simeq \mathrm{Mod}(k[\tilde{\mathcal{G}}])$.
Let $F \in \mathrm{Rep}_k(\tilde{\mathcal{G}})$. It induces a $k[\tilde{\mathcal{G}}]$ module, M_F as follows:

$$M_F := \bigoplus_{i=1}^{n} F(c_i)$$

where the $k[\tilde{\mathcal{G}}]$ structure is given by $e_g \cdot (m_1, \dots, m_n) = (0, \dots, 0, F(g) \cdot m_i, 0 \dots 0)$, since $F(g) \in \mathrm{Aut}(F(c_i))$ if $g \in G_i = \mathrm{Aut}_{\tilde{\mathcal{G}}}(c_i)$.
On the other hand $M \in \mathrm{Mod}(k[\tilde{\mathcal{G}}])$ induces a representation F_M by setting $F_M(c_i) = e_{id_{c_i}} M$ and $F_M(g) = e_g \cdot -$. Clearly both equivalences are inverse to one another, and hence the result follows. □

Proposition 1.10. *Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a functor of groupoids having finite number of isomorphism classes of objects. Let W be the $k[\mathcal{G}]$-module associated to a k-representation F of $\mathcal{G}$. Then $res_\varphi(F) \in \mathrm{Rep}_k(\mathcal{H})$ corresponds to the restriction of scalars of W onto $k[\mathcal{H}]$.*

Proof. Since φ induces a morphism $\tilde{\varphi}$ of groupoid algebras we need to check that the following diagram quasi-commutes

$$\begin{array}{ccc} \mathrm{Rep}_k(\mathcal{G}) & \xrightarrow[\sim]{\delta} & \mathrm{Mod}(k[\mathcal{G}]) \\ \downarrow{\scriptstyle \mathrm{res}_\varphi} & & \downarrow{\scriptstyle \tilde{\varphi}^*} \\ \mathrm{Rep}_k(\mathcal{H}) & \xrightarrow[\sim]{\delta} & \mathrm{Mod}(k[\mathcal{H}]) \end{array}$$

where $\tilde{\varphi}^*$ denotes the restriction of scalars. Let $F \in \mathrm{Rep}_k(\mathcal{G})$, then

$$\delta \circ \mathrm{res}_\varphi(F) = \bigoplus_j e_{id_{\varphi(d_j)}} F(\varphi(d_j)).$$

On the other hand,

$$\tilde{\varphi}^* \circ \delta(F) = \tilde{\varphi}^*(\bigoplus_i e_{id_i} F(\varphi(c_j))) =: \tilde{\varphi}^*(M).$$

Take $e_{id_{d_j}}$, then $e_{id_{d_j}} \cdot M = e_{id_{\varphi(d_j)}} M = F(\varphi(d_j)))$. Hence, since $\tilde{\varphi}^*(M)$ can be decomposed as the direct sum $\bigoplus_j e_{id_{d_j}} \cdot M$ the result follows. □

Proposition 1.11. *Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a functor of groupoids having finite number of isomorphism classes of objects, and let V be the $k[\mathcal{H}]$-module associated to a k-representation of $\mathcal{H}$. Then* $\mathrm{ind}_\varphi(V)$ *can be realized as*

$$\mathrm{ind}_\varphi(V) \simeq \mathrm{Hom}_{k[\mathcal{H}]}(k[\mathcal{G}], V)$$

and $\mathrm{coind}_\varphi(V)$ *can be realized as*

$$\mathrm{coind}_\varphi(V) \simeq k[\mathcal{G}] \otimes_{k[\mathcal{H}]} V$$

Proof. Since ind_φ is the right adjoint of res_φ, by Proposition 1.10 and the unicity up to isomorphism of the right adjoint, we have that $\mathrm{ind}_\varphi(V) \simeq \mathrm{Hom}_{k[\mathcal{H}]}(k[\mathcal{G}],V)$, since $\mathrm{Hom}_{k[\mathcal{H}]}(k[\mathcal{G}],V)$ is the right adjoint of the extension of scalars in the category of modules.
Similarly, coind_φ is the left adjoint of res_φ, hence since the left adjoint of restriction of scalars is the extension of scalars, by the same argument as for the induction the results follows. □

Proposition 1.12. *Let $\varphi : \mathcal{H} \to \mathcal{G}$ be a functor of groupoids, where $\mathcal{G},\mathcal{H}$ are equivalent to finite groupoids. Then* $\mathrm{ind}_\varphi \simeq \mathrm{coind}_\varphi$.

Proof. By hypothesis, $\mathcal{G} \simeq G_1 \sqcup \ldots \sqcup G_n$ and $\mathcal{H} \simeq H_1 \sqcup \ldots \sqcup H_m$. Note that we can consider the induced representation on $\mathcal{G}$ one group at a time, so we can assume that $\mathcal{G} \simeq G$, a finite group G. For the case of finite groups we have that the induced and the coinduced representation coincide, since we have an isomorphism

$$\begin{aligned} \mathrm{Hom}_{k[H]}(k[G],V) &\to k[G] \otimes_{k[H]} V \\ f &\mapsto \sum_{g \in P} e_{g^{-1}} \otimes_{k[H]} f(g) \end{aligned}$$

where P is a set of distinct representatives of the right $\varphi(H)$-cosets in G.
Let V be a representation of $\mathcal{H}$. Then we have that

$$\mathrm{ind}_\varphi(V) \simeq \mathrm{Hom}_{k[\mathcal{H}]}(k[G], \bigoplus_i e_{id_{d_i}} F(d_i)) \simeq \bigoplus_i \mathrm{Hom}_{k[\mathcal{H}]}(k[G], F(d_i)),$$

since $k[G]$ is finitely generated. We have that $\mathrm{Hom}_{k[\mathcal{H}]}(k[G],F(d_i)) = \mathrm{ind}_\varphi(F(c_i))$, hence ind_φ can be decomposed as the direct sum of induced representations of finite groups. Hence, we have

$$\begin{aligned} \mathrm{ind}_\varphi(V) &\simeq \bigoplus_i \mathrm{ind}_\varphi(F(c_i)) \simeq \bigoplus_i \mathrm{coind}_\varphi(F(c_i)) \simeq \bigoplus_i k[\mathcal{G}] \otimes_{k[\mathcal{H}]} F(c_i) \simeq k[\mathcal{G}] \otimes_{k[\mathcal{H}]} V \\ &\simeq \mathrm{coind}_\varphi(V) \end{aligned}$$

and the result follows. □

2 Locally Constant Sheaves

In this section we will study locally constant sheaves and their behavior under continuous maps. More precisely, we will prove that if f is a homotopy equivalence, then $f^{-1} : \mathrm{LCSH}(k_Y) \to \mathrm{LCSH}(k_X)$ is an equivalence. and that if f is a Serre fibration, then $f_* : \mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$ is well defined.
When it comes to Serre fibrations, we will prove that any f can be decomposed as the composition of a homotopy equivalence and a Serre fibration.
In this section we will follow [5], [6], [7] and [8].

2.1 Locally Constant Sheaves

Definition 2.1. Let X be a topological space.

(i) We call a sheaf $\mathscr{F}$ on X constant if it is the sheaffification of a constant presheaf on X

(ii) A sheaf $\mathscr{F}$ on X is locally constant if there exists an open cover $\{U_i\}_i$ of X such that $\mathscr{F}|_{U_i}$ is a constant sheaf on U_i

Remark 2.2. (i) We will denote the category of locally constant sheaves of k-modules over X by $\mathrm{LCSH}(k_X)$

(ii) We will denote by M_X the constant sheaf with stalk M

Now we want to prove that every locally constant sheaf on the interval is a constant sheaf. Firstly, we need a couple of lemmas.

Lemma 2.3. *Let $M,N \in \mathrm{Mod}(k)$. Then $(\mathrm{Hom}_k(M,N))_X \simeq \mathscr{H}om_{k_X}(M_X,N_X)$*

Lemma 2.4. *Let $X = U_1 \cup U_2$ be a covering of X by two open sets. Let $\mathscr{F}$ be a sheaf on X and assume that*

(i) $U_1 \cap U_2$ is connected and non empty;

(ii) $\mathscr{F}|_{U_i}$ is a constant sheaf.

Then $\mathscr{F}$ is a constant sheaf.

Proof. Since $\mathscr{F}|_{U_i}$ is constant, there exists M_i such that $(M_i)_X|_{U_i} \simeq \mathscr{F}|_{U_i}$. Since $U_1 \cap U_2 \neq \emptyset$ and is connected, we have that $M_1 \simeq M_2$, hence we may assume that $M_1 = M_2 = M$. We can define an isomorphism $\theta_{ij} = \theta_1 \circ \theta_2^{-1} : M_X|_{U_1\cap U_2} \xrightarrow{\simeq} M_X|_{U_1\cap U_2}$. By Lemma 2.3 $\mathscr{H}om_{k_X}(M_X,M_X)(U_1 \cap U_2) \simeq \mathrm{Hom}(M,M)$, hence $\theta_{12} \in \mathrm{Aut}(M,M)$. Using the same correspondence, we find that θ_{12} extends to an isomorphism $\theta : M_X \simeq M_X$ over X. Define $\alpha_i : F|_{U_i} \xrightarrow{\simeq} (M_X)|_{U_i}$ by $\alpha_1 := \theta_1$ and $\alpha_2 := \theta|_{U_2} \circ \theta_2$. Hence α_1 and α_2 glue together to an isomorphism $\alpha : F \xrightarrow{\simeq} M_X$. □

Proposition 2.5. *Let $\mathscr{F}$ be a locally constant sheaf on $[0,1]$. Then:*

(i) $\mathscr{F}$ is a constant sheaf.

(ii) If $t \in [0,1]$, the morphism $\Gamma(I;\mathscr{F}) \to F_t$ is an isomorphism.

(iii) In particular, if $\mathscr{F} = M_{[0,1]}$ for $M \in \mathrm{Mod}(k)$, then

$$M \cong F_0 \leftarrow \Gamma([0,1];M_{[0,1]}) \to F_1 \cong M$$

is the identity on M.

Proof. (i) follows from Lemma 2.4.
(ii) and (iii) follow immediately form the properties of constant sheaves. □

From here on, all the results are taken from [6], adapted from locally constant staks to locally constant sheaves.

Lemma 2.6. *(i) Let $\mathscr{F}$ be a constant sheaf on a locally connected space X. Then let $U \subset X$ be a connected open subset. Then for each $x \in X$, the natural map $\mathscr{F}(U) \to \mathscr{F}_x$ is an isomorphism*

(ii) Let X be a locally connected topological space and let $\mathscr{F}$ be a locally constant sheaf on X. Then every point $x \in X$ has a connected neighborhood V such that the natural map $\mathscr{F}(V) \to \mathscr{F}_x$ is isomorphism.

Proof. For (i), assume that $\mathscr{F} \cong M_X$, and let U be a connected open neighborhood of x then $\mathscr{F}|_U \simeq M_U$. Let $\{V\}$ be a system of neighborhoods of X such that each $V \subset U$. We might assume, moreover, that every V is connected. In this situation we have $M_U \simeq M_V$. Therefore, $\mathscr{F}(U) \simeq \mathscr{F}(V)$. Then it follows that $\mathscr{F}(U) \simeq \varinjlim_{V \ni x} \mathscr{F}(V) = \mathscr{F}_x$
(ii) follows immediately from (i) □

Lemma 2.7. *Let X, Y be topological spaces, let $f : X \to Y$ be a continuous maps If $\mathscr{F} \in \mathrm{LCSH}(k_X)$, then $f^{-1}\mathscr{F} \in \mathrm{LCSH}(k_Y)$. In particular, $f^{-1}M_X \simeq M_Y$ for any $M \in \mathrm{Mod}(k)$.*

Proof. Let $\{U_i\}_i$ be such that $\mathscr{F}|_{U_i}$ is constant. Then, $f^{-1}\mathscr{F}$ is constant over $f^{-1}(U_i)$, since

$$\varinjlim_{f(f^{-1}U_i)\subset V} \mathscr{F}|_V = \varinjlim_{U_i\subset V} \mathscr{F}|_V = \mathscr{F}|_{U_i},$$

which is a constant sheaf. $\square$

In particular, we get a functor

$$f^{-1} : \mathrm{LCSH}(k_Y) \to \mathrm{LCSH}(k_X)$$

2.2 Invariance by Homotopy

Proposition 2.8. *Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. Let $g := (id, f) : [0,1]\times X \to [0,1]\times Y$, and let $p : [0,1]\times X \to X$ and $q : [0,1]\times Y \to Y$ be the projections:*

$$\begin{array}{ccc} [0,1]\times X & \xrightarrow{g} & [0,1]\times Y \\ \big\downarrow p & & \big\downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

Let $\mathscr{G}$ be a locally constant sheaf on $[0,1]\times Y$. Then the natural morphism $f^{-1}q_*\mathscr{G} \to p_*g^{-1}\mathscr{G}$ is an isomorphism.

First we need the following result

Lemma 2.9. *In the notation above, every point $t \in [0,1]$ has a neighborhood $I \subset [0,1]$ such that $\mathscr{G}([0,1]\times Y) \to \mathscr{G}(I\times Y)$ is an isomorphism.*

Proof. There exists an open cover $\{I_i \times U_j\}$ of $[0,1]\times Y$ such that $\mathscr{G}$ is constant on each $I_i \times U_j$. By fixing j, we've got that $\mathscr{G}$ is locally constant over $[0,1]\times U_j$, and hence it is constant by Proposition 2.5. Therefore, $\mathscr{G}([0,1]\times U_j) \to \mathscr{G}(I\times U_j)$ is an isomorphism. $\square$

Proof of Proposition 2.8. Let $y \in Y$, $t \in [0,1]$. Let $\{U\}$ be a fundamental systems of neighborhoods of y. Due to Lemma 2.9, we can pick for each U an open set $I_U \subset [0,1]$ such that the restriction functor $\mathscr{G}([0,1]\times U) \to \mathscr{G}(I_U \times U)$ is an isomorphism. We might choose I_U such that $I_U \times U$ form a fundamental system of neighborhoods of (t,y). Then it follows that the restriction functor of stalks $(q_*\mathscr{G})_y \to \mathscr{G}_{(t,y)}$ is an isomorphism, since

$$(q_*\mathscr{G})_y = \varinjlim_{y\in U} q_*\mathscr{G}(U) = \varinjlim_{(t,y)\in[0,1]\times U} \mathscr{G}([0,1]\times U) \simeq \varinjlim_{(t,y)\in I_U\times U} \mathscr{G}(I_U\times U) = \mathscr{G}_{(t,x)}$$

Now let $x \in X$, then we have natural isomorphisms

$$(f^{-1}q_*\mathscr{G})_x \simeq (q_*\mathscr{G})_{f(x)} \simeq \mathscr{G}_{(t,f(x))} \simeq (g^{-1}\mathscr{G})_{(t,x)} \simeq (p_*g^{-1}\mathscr{G})_x.$$

hence the natural isomorphism is an isomorphism. □

Theorem 2.10. *Let X be a topological space, and let p denote the projection map* $[0,1]\times X \to X$. *Then* p_* *and* p^{-1} *are inverse equivalences between the categories of locally constant sheaves on X and the category of locally constant sheaves on* $[0,1]\times X$

Proof. Let $\mathscr{F} \in \mathrm{LCSH}(k_X)$, and let $\mathscr{G} \in \mathrm{LCSH}(k_{[0,1]\times X})$.
Now let $x \in X$ and $i_x : \{x\} \hookrightarrow X$ be the inclusion. Similarly let $j_x : [0,1] \simeq [0,1]\times x \hookrightarrow [0,1]\times X$, and let $q : [0,1] \simeq [0,1]\times x \to x$. By Proposition 2.8 the natural map $(p_*p^{-1}\mathscr{F})_x = i_x^{-1}p_*p^{-1}\mathscr{F} \to q_*j_x^{-1}p^{-1}\mathscr{F}$ is an isomorphism. On the other hand, $q_*j_x^{-1}p^{-1}\mathscr{F} = q_*q^{-1}i_x^{-1}\mathscr{F} = q_*q^{-1}(\mathscr{F}_x)$. The natural map $\mathscr{F}_x \to (p_*p^{-1}\mathscr{F})_x \simeq q_*q^{-1}(\mathscr{F}_x)$ coincides with the adjunction map $\mathscr{F}_x \to q_*q^{-1}(\mathscr{F}_x)$. Since $q^{-1}\mathscr{F}_x$ is constant, $\mathscr{F}_x \to q_*q^{-1}(\mathscr{F}_x)$ is an equivalence by propostion 2.6. Therefore, $\mathscr{F} \to p_*p^{-1}\mathscr{F}$ is an isomorphism of sheaves
As for $\mathscr{G}$, let $(t,x) \in [0,1]\times X$. Then, there is an equivalence $(p^{-1}p_*\mathscr{G})_{(t,x)} \simeq (p_*\mathscr{G})_x$. Proposition 2.8 gives us an equivalences $(p_*\mathscr{G})_x \simeq q_*j_x^{-1}\mathscr{G} = j_x^*\mathscr{G}([0,1])$. The locally constant sheaf $j_x^{-1}\mathscr{G}$ is constant over $[0,1]$, so $j_x^{-1}\mathscr{G}([0,1]) \simeq \mathscr{G}_{(t,x)}$ by Proposition 2.6. It follows that $p^{-1}p_* \to \mathscr{G}$ is an isomorphism of sheaves. □

Recall that a two topological spaces X and Y are homotopy equivalent if there exists $f : X \to Y$ and $g : Y \to X$ such that $f\circ g$ is homotopic to id_Y and $g\circ f$ is homotopic to id_X.

Corollary 2.11. *Let* $f : X \to Y$ *be a homotopy equivalence. Then the functor* $f^{-1} : \mathrm{LCSH}(k_Y) \to \mathrm{LCSH}(k_X)$ *is an equivalence of categories.*

Proof. For (i) let $g : Y \to X$ be an homotopy inverse to f. Let $h : [0,1]\times X \to X$ be a homotopy between $g\circ f$ and id_X. Let $p : [0,1]\times X \to X$ denote the projection, and for $t \in [0,1]$ let $i_t : X \to [0,1]\times X$ be such that $i_t(x) = (t,x)$. Then $i_t^{-1} \simeq p_*$ by Theorem 2.10. It follows that $i_0^{-1} \simeq i_1^{-1}$ and that $(h\circ i_0)^{-1} \simeq (h\circ i_1)^{-1}$. On the other hand, $h\circ i_0 = g\circ f$ and $h\circ i_1 = id_X$, so $f^{-1}\circ g^{-1} \simeq id_{\mathrm{LCSH}(k_X)}$. In a similar manner, we can construct an equivalence $g^{-1}\circ f^{-1} \simeq id_{\mathrm{LCSH}(k_Y)}$

□

Theorem 2.12. *Let X be a locally contractible topological space, and let* $\mathscr{F}$ *be a be a sheaf on X. Then, the following are equivalent:*

(i) The sheaf $\mathscr{F}$ is locally constant.

(ii) If U and V are two open subsets of X with $V \subset U$, and the inclusion map $V \hookrightarrow U$ is a homotopy equivalence, then the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is an isomorphism.

(iii) If U and V are two contractible open subsets of X, ans $V \subset U$, then $\mathscr{F}(U) \to \mathscr{F}(V)$ is an isomorphism.

(iv) There exists a collection $\{U_i\}_i$ of contractible open subsets of X such that each point $x \in X$ has a fundamental system of neighborhoods of the form $\{U_{i_j}\}_j$, and such that $\mathscr{F}(U_{i_k}) \to \mathscr{F}(U_{i_j})$ is an isomorphism whenever $U_{i_j} \subset U_{i_k}$.

Proof. To prove that (i) implies (ii) let U and V be such that the conditions in (ii) are satisfied. Then corollary 2.11 implies that $\mathscr{F}(U) \to \mathscr{F}(V)$ is an isomorphism, so (i)⇒(ii)
The fact that condition (ii) implies condition (iii) and that condition (iii) implies condition (iv) are obvious.
For seeing that (iv) implies (i) let $\mathscr{F}$ fulfill the conditions on (iv). We will show that $\mathscr{F}$ is locally constant. To do so, we need to show that its restriction to each of the U_i is constant. Each $x \in U_i$ has a fundamental system of neighborhoods, $\{U_{i_j}\}_j$ such that for each j $U_{i_j} \subset U_i$. Due to the hypothesis, $\mathscr{F}(U_i) \simeq \mathscr{F}(U_{i_j})$, so the map $(F)(U) \to (F)_x$ is also an isomorphism (same argument as in Lemma 2.6). Then due to Lemma 2.6 $\mathscr{F}|_{U_i} \simeq M_{U_i}$, where $M = \mathscr{F}_x$. □

2.3 Serre fibrations

In this section we will follow [7] and [8]. Set $I := [0,1]$

Definition 2.13. (i) A continuous map $f : X \to Y$ is said to have the homotopy lifting property with respect to A if, given a continuous map $g : A \times I \to Y$ and a continuous map $\tilde{g}_0 : A \to X$ lifting $g(\cdot,0)$, i.e. $f \circ \tilde{g}_0 = g(\cdot,0)$, then there exists a continuous map $\tilde{g} : A \times I \to X$ lifting $g(\cdot,t)$.

(ii) A Serre fibration is a continuous map $f : X \to Y$ that has the homotopy lifting property for all spaces A of the form I^n.

Example 2.14. Any covering space, more generally any locally trivial bundle, is a Serre fibration.

Proposition 2.15. *Let $\mathscr{F} \in \mathrm{LCSH}(k_X)$. If f is a Serre fibration, then $f_*\mathscr{F} \in \mathrm{LCSH}(k_Y)$.*

Proof. Let $V \subset U \subset X$ be open subsets of X. By Theorem 2.12 we need to show that the restriction map $f_*\mathscr{F}(U) \to f_*\mathscr{F}(V)$ is an isomorphism. Since f is a Serre fibration, the inclusion $f^{-1}(V) \hookrightarrow f^{-1}(U)$ is a homotopy equivalence and the proposition follows from the corollary 2.11. □

If f is a Serre fibration we get a functor

$$f_* : \mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$$

Recall that the compact-open topology defined on the space $\mathscr{C}(I,Y)$ of continuous maps $I \to Y$ is the topology induced by the sub-basis consisting of the sets $M(K,U) := \{f : K \to U | f$ continious, and $f(K) \subset U\}$ where $K \subset I$ is compact and $U \subset Y$ is open. In particular $\mathscr{C}(I,Y)$ is the space of paths on Y.

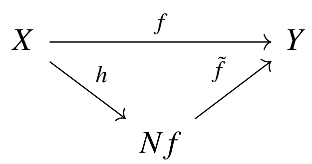
Proposition 2.16. *In the notation above:*

(i) The evaluation map $e : \mathscr{C}(I,Y) \times I \to Y$, $e(f,y) = f(y)$, is continuous.

(ii) A map $f : I \times Z \to Y$ is continuous if and only if the map $\hat{f} : X \to \mathscr{C}(I,Y)$, $\hat{f}(x)(y) = f(y,x)$, is continuous.

Proof. For (i) $(f,s) \in \mathscr{C}(I,Y) \times I$ and let $U \subset I$ be a neighborhood of $f(s)$. Since I is compact, continuity of f implies that there exists a compact neighborhood of s, $K \subset I$ such that $f(K) \subset U$. Then $M(K,U) \times K$ is a neighborhood of (f,s) in $\mathscr{C}(I,Y) \times I$ such that $e(M(K,U) \times K) \subset U$, hence e is continuous.
(ii) Suppose $f : I \times X \to Y$ is continuous. To show that $\hat{f}$ is continuous we need to show that for any $M(K,U) \subset \mathscr{C}(I,Y)$ set of the sub-basis, the set $\hat{f}^{-1}(M(K,U)) = \{x \in X | f(K,x) \subset U\}$ is open in X. Since $f^{-1}(U)$ is an open subset of the compact set $K \times \{x\}$, there exists $V \subset I$ and $W \subset X$ such that $K \times \{x\} \subset V \times W \subset f^{-1}(U)$. I follows that W is a neighborhood of z in $\hat{f}^{-1}(M(K,U))$ and hence $\hat{f}$ is continuous. □

Now we shall see that any continuous map $f : X \to Y$ can be decomposed as the composition of a homotopy equivalence and a fibration. This would mean that f factors as



where $\tilde{f}$ is a fibration and h is a homotopy equivalence and set $Nf := \{(x,\gamma) | x \in X$ and $\gamma : I \to Y$ such that $\gamma(0) = f(x)\}$ with the subspace topology induced from the inclusion $Nf \subset X \times \mathscr{C}(I,Y)$. Let γ_x^C denote the constant path on x.

Proposition 2.17. *Let $f : X \to Y$ be a continuous map. Then $f = \tilde{f} \circ h$, with a homotopy equivalence $h : X \to Nf$, $h(x) = (x, \gamma^c_{f(x)})$, and $\tilde{f} : Nf \to Y$, $\tilde{f}(a,\gamma) = \gamma(1)$ is a Serre fibration.*

Proof. $\tilde{f}$ is continuous since the evaluation map $\mathscr{C}(I,Y) \times I \to Y$, $(\gamma,t) \mapsto \gamma(t)$ is continuous by proposition 2.16 (i).
Now, we have to check that $\tilde{f}$ is a Serre fibration. Let $g(\cdot,t) : I^n \to Y$ be a homotopy and give a lift for $g(\cdot,0)$, $\tilde{g}(\cdot,0) : I^n \to Nf$, $\tilde{g}(x,0) = (h(x),\gamma_x)$ for $h : I^n \to X$ and $\gamma_x : I \to Y$. Now, we shall define the lift $\tilde{g}(x,t) = (h(x),\gamma_x * g_{[0,t]}(x))$, where $\gamma_x * g_{[0,t]}$ denotes the product of paths between γ_x and the path induced by $[0,t] \ni s \mapsto g(x,s)$. The composition is well defined, since $g(x,0) = \tilde{f} \circ \tilde{g}(x,0) = \gamma_x(1)$. Now, to check that $\tilde{g}(\cdot,t)$ is continuous we consider it as a map $I^n \times I \to Nf \subset X \times \mathscr{C}(I,Y)$ and use Proposition 2.16 (ii) from which we deduce that continuity of $I^n \times I \to X \times \mathscr{C}(I,Y)$ is equivalent to continuity for $I^n \times I \times I \to X \times Y$.
As for h being a homotopy equivalence, let $p : Nf \to X$ be the natural projection. Then, clearly $id_X = p \circ h$. On the other hand, $id_{Nf} \simeq h \circ p$ since we can define the continuous deformation $g : Nf \times I \to Nf$ of Nf onto $h(X)$ by

$$g(x,\gamma)(t) := (x,\gamma_t)$$

where $\gamma_t(s) = \gamma((1-t)s)$, hence g is indeed a deformation from Nf to X. □

Definition 2.18. Let $f : X \to Y$ and $p : Nf \to X$ be the natural projection. For $\mathscr{G} \in \mathrm{LCSH}(k_X)$ set

$$f_*^{LCSH}\mathscr{G} := \tilde{f}_* p^{-1}\mathscr{G} \tag{2}$$

where $\tilde{f}$ is as in Proposition 2.17.

Remark 2.19. Note that by Propositions 2.17, 2.11 and 2.15, we have that $f_*^{LCSH}\mathscr{G} \in \mathrm{LCSH}(k_Y)$ for any $\mathscr{G} \in \mathrm{LCSH}(k_X)$, hence we get a functor

$$f_*^{LCSH} : \mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$$

Remark 2.20. If f is a Serre fibration, then

$$f_* \simeq f_*^{LCSH}$$

since $X \to N_f$ is a fiber homotopy equivalence, hence by Theorem 2.10 the result follows.

3 Monodromy Representations

We will start this section by introducing the well known notions of fundamental groupoid of a topological space X and the monodromy functor from the category of locally constant sheaves over X to the representations of the fundamental groupoid. The first half of the section will be devoted to constructing a quasi-inverse for the monodromy functor in the case X is semi-locally simply connected. Given a continuous map $f : X \to Y$, we will use this equivalence to show that the restricted representation of the induced functor on the fundamental groupoids corresponds to the inverse image functor at the level of locally constant sheaves. On the other hand, if f is a Serre fibration, the induced representation corresponds to the direct image functor. Finally, we will use the results from the previous section to construct a general adjoint to the inverse image functor in the category of locally constant sheaves.
Here we follow [1], [5] and [9].

3.1 The fundamental grupoid

Definition 3.1. Let X be a topological space. We define $\Pi_1(X)$ as the category given by $\mathrm{Ob}(\Pi_1(X)) = X$ and whose morphisms are homotopy classes of paths between two points. Note that every morphism is an isomorphism, so it is a grupoid. We call this category the fundamental grupoid of X.

Remark 3.2. Note that if X is arcwise connected then for any $x_0 \in X$ the natural functor $\pi_1(X,x_0) \underset{i_{x_0}}{\overset{\simeq}{\to}} \Pi_1(X)$ is an equivalence. Indeed, since every object in $\Pi_1(X)$ is isomorphic to x_0, i_{x_0} is essentially surjective. Moreover it is clear that it is fully faithful. It follows that if X is any topological space, and $X = \bigsqcup_i U_i$ a decomposition on its arcwise connected components, then $\Pi_1(X) \simeq \bigsqcup_i \pi_1(U_i,x)$.

3.2 The monodromy functor and its quasi-inverse

Definition 3.3. The objects in the category $\mathrm{Rep}(\Pi_1(X)) = \mathrm{Fct}(\Pi_1(X), \mathrm{Mod}(k))$ are called monodromy representations of X.

Definition 3.4. We call Monodromy functor the functor

$$\mu : \mathrm{LCSH}(k_X) \to \mathrm{Rep}_k(\Pi_1(X))$$

defined by Let $\mathscr{F} \in \mathrm{LCSH}(k_X)$ then $\mu_{\mathscr{F}} : \Pi_1(X) \to \mathrm{Mod}(k)$ is as follows

- $\mu_{\mathscr{F}}(x) = \mathscr{F}_x$
- $\mu_{\mathscr{F}}([\gamma]) : \mathscr{F}_{\gamma(0)} \simeq (\gamma^{-1}\mathscr{F})_0 \overset{\simeq}{\leftarrow} \gamma^{-1}\mathscr{F}(I) \overset{\simeq}{\rightarrow} (\gamma^{-1}\mathscr{F})_1 \simeq \mathscr{F}_{\gamma(1)}$

Where the isomorphism is well defined since $\mathscr{F}$ locally constant implies $\gamma^{-1}\mathscr{F}$ constant, by Proposition 2.5.

Lemma 3.5. *The monodromy functor* μ *is faithful.*

Proof. Let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be two morphisms of locally constant sheaves. Assume that $\mu(\varphi) = \mu(\psi)$. This induces an equality on the level of stalks, $\varphi_x = \psi_x : F_x \to G_x$ for any $x \in X$. Therefore, $\varphi = \psi$ □

Recall that a topological space is *semi-locally simply connected* if for any $x \in X$ there exists a neighborhood U such that any loop in U is contractible in X, in other words, the induced group morphism $\pi_1(U,x) \to \pi_1(X,x)$ is trivial, i.e. we get the commutative triangle

$$\begin{array}{ccc} \pi_1(U,x) & \longrightarrow & \pi_1(X,x) \\ & \searrow \quad \nearrow & \\ & \{*\} & \end{array}$$

The main objective of this section is to prove the following theorem:

Theorem 3.6. *Let X be semi-locally simply connected topological space. Then the monodromy functor* μ *is an equivalence of categories.*

To prove the equivalence we shall construct a quasi-inverse ν of μ. We start by defining ν without any assumptions on the space X

Definition 3.7. Let $F : \Pi_1(X) \to \mathrm{Mod}(k)$ be a functor. Then for every $U \subset X$ open we define

$$\nu(F)(U) = \varprojlim_{\Pi_1(U)} F \circ i_U,$$

where $i_U : \Pi_1(U) \to \Pi_1(X)$ is the map induced by the inclusion $U \subset X$.

Remark 3.8. (i) Consider first the projection map

$$\pi : \bigsqcup_{x \in X} F(x) \to X.$$

Then $\nu(F)(U)$ identifies with the set of local sections of π which are compatible with the action of $\Pi_1(X)$, in other words

$$\nu(F)(U) \simeq \{f : U \to \bigsqcup_{x \in X} F(x) \mid \pi \circ f = id_U \text{ and } f(\gamma(1)) = F([\gamma])(f(\gamma(0)) \text{ for any path } \gamma \text{in } U\}$$

Hence we recover the definition in section in chapter 6 section F of the exercises from [9].

(ii) Let U be an arcwise connected open subset of X and let $x_0 \in U$. We get

$$\pi_1(U,x_0) \xrightarrow{i_{x_0} \simeq} \Pi_1(U) \xrightarrow{i_U} \Pi_1(X) \xrightarrow{F} \mathrm{Mod}(k)$$

and, we can write $\nu(F)(U) \simeq \varprojlim_{\pi_1(U,x_0)} F \circ i_U \circ i_{x_0} = F(x_0)^{\pi_1(U,x_0)}$, where $F(x_0)^{\pi_1(U,x_0)}$ denotes the subspace of invariant elements of $F(x_0)$ under the action of $\pi_1(U,x_0)$.

Lemma 3.9. *In the notation above,* $\nu(F) \in \mathrm{Mod}(k_X)$

Proof. First, we will show that $\nu(F)$ is a presheaf. Clearly $\nu(F)(U) \in \mathrm{Mod}(k)$, since for every $x \in \Pi_1(U)$, $F(x) \in \mathrm{Mod}(k)$. As for the restriction maps, let $V \subset U \subset X$ and let $i_{VU} : V \hookrightarrow U$ be the inclusion. Then there exists a morphism

$$\varprojlim(F \circ i_U) \overset{\mathrm{res}_{UV}}{\to} \varprojlim(F \circ i_U \circ i_{UV})$$

The commutativity of the restriction maps follows from the commutativity of the diagram with the inclusions

$$\begin{array}{ccc} \Pi_1(V) & \xrightarrow{i_{VU}} & \Pi_1(U) \\ {\scriptstyle i_{WV}}\uparrow & \nearrow {\scriptstyle i_{WU}} & \\ \Pi_1(W) & & \end{array}$$

Therefore $\nu(F)$ is a presheaf. Now, we will see that $\nu(F)$ satisfies both sheaf axioms. Let $U \subset X$ be open and let $\{U_i\}_i$ be an open cover of U.

(1) Let $s \in \nu(F)(U)$ such that $s|_{U_i} = 0$. Then, since s is a map, clearly $s = 0$.

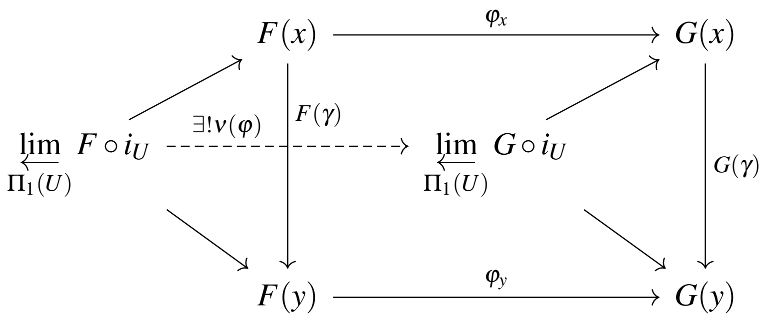
(2) Let $s_i \in \nu(F)(U_i)$ such that $s_i|_{U_{ij}} = s_j|_{U_{ij}}$. Then, we can define a map $s : U \to \bigsqcup F(x)$ by $s(x) = s_i(x)$ if $x \in U_i$. The compatibility conditions ensure that s is well defined. Moreover, we have that $\pi \circ s(x) = \pi \circ s_i(x) = x$, where i is such that $x \in U_i$. We only need to check the final compatibility regarding γ. Let $\gamma : I \to U$ be a continuous path in U. Then, since I is compact and γ continuous, $\gamma(I) \subset U$ is compact and we can get

a finite subcover $\{U_{i_j}\}_{j=1}^{n}$ such that $\gamma(I) \subset \bigcup_j U_{i_j}$. We can order the U_{i_j} in the order by which γ crosses them. Now let z_{j+1} be a point of $U_{i_j,i_{j+1}} \cap \gamma$. We can decompose the path in pieces as follows. Let γ_1 be the path that follows γ from $\gamma(0)$ to z_2 and let γ_n the path that follows γ from z_n to $\gamma(1)$. For $j = 2 \dots n-2$, let γ_j be the path that follows γ from x_j to x_{j+1}. Therefore, we have that $\gamma = \gamma_n \dots \gamma_1$ and each $\gamma_j \subset U_j$. Then, we have that

$$\begin{aligned} F([\gamma])(f(\gamma(0))) =& F([\gamma_n \dots \gamma_1])(f(\gamma_n \dots \gamma_1(0))) \\ =& F([\gamma_n]) \dots F([\gamma_1])(f(\gamma_n \dots \gamma_1(0))) \\ =& f(\gamma(1)) \end{aligned}$$

Therefore, $\nu(F)$ is a sheaf □

With this, we can define a functor $\nu : \mathrm{Rep}_k(\Pi_1(X)) \to \mathrm{Mod}(k_X)$, that sends $F \in \mathrm{Rep}_k(\Pi_1(X))$ to $\nu(F)$, and a morphism of functors $\varphi : F \Rightarrow G$ to the morphism of functors induced by



The composition and the compatibility with restrictions is preserved naturally from the uniqueness of $\nu(\varphi)$ stated above. Trivially $\nu(id_F) = id_{\nu(F)}$. Hence $\nu : \mathrm{Rep}_k(\Pi_1(X)) \to \mathrm{Mod}(k_X)$ is a functor.

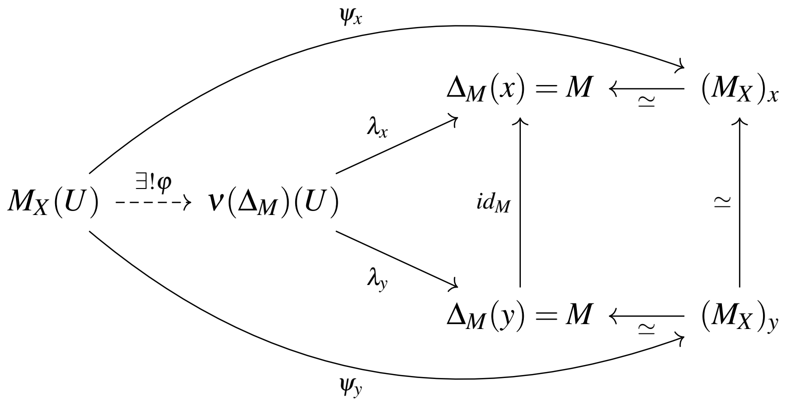
Example 3.10. Let $\Delta_M : \Pi_1(X) \to \mathrm{Mod}(k)$ be the constant functor that sends every $x \in X$ to $M \in \mathrm{Mod}(k)$ and every γ to the identity. Then

$$\nu(\Delta_M)(U) = \varprojlim_{\Pi_1(U)} \Delta_M \circ i_U = \varprojlim_{\Pi_1(U)} \Delta_M \simeq \varprojlim_{\pi_0(\Pi_1(U))} \Delta_M.$$

Here $\pi_0(\Pi_1(U))$ denotes the set of isomorphism classes of $\Pi_1(U)$, which is in one to one correspondence with the set $\pi_0(U)$ of the arc-wise connected components of U. Therefore,

$$\nu(\Delta_M)(U) \simeq \varprojlim_{\pi_0(U)} \Delta_M \simeq \mathrm{Hom}_{\mathbf{Set}}(\pi_0(U), M)$$

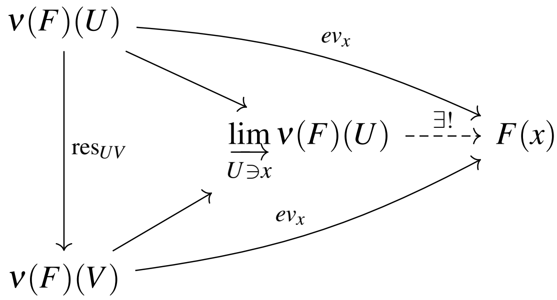
Hence we have a natural morphism from $M_X(U) := \{f : U \to M | f \text{ locally constant}\} \hookrightarrow \mathrm{Hom}_{\mathbf{Set}}(\pi_0(U), M)$.
Note that if U is arcwise connected, then we can construct an isomorphism



Note that since U is connected, then due to Lemma 2.6 ψ_x, and ψ_y are isomorphisms. Moreover, since U is arcwise connected $\nu(\Delta_M)(U) \simeq \mathrm{Hom}_{\mathbf{Set}}(\pi_0(U), M) = \mathrm{Hom}_{\mathbf{Set}}(\{*\}, M) \simeq M$, hence λ_x and λ_y are also isomorphisms. Therefore, due to the commutativity of the triangles, φ is an isomorphism.

Lemma 3.11. *For any $x \in X$ we have a morphism $\nu(F)_x \to F(x)$. Moreover, if X is locally arcwise connected, the morphism is an isomorphism.*

Proof. The first claim is due to definition of stalk:



so ev_x, the evaluation map of $s_x \in (\nu(F))_x$ at x is the desired map.
If X is locally arcwise connected, firstly we want to prove that ev_x is injective. Let $x \in U \subset X$ be open and $s_U \in \nu(F)(U)$. We will denote the projection of s_U onto the stalk as $[s_U]$. Suppose s_U such $ev_x[s_U] = s_U(x) = 0$. We want to prove that $[s_U] = 0$.
Since X is locally arcwise connected, there exists $V \ni x$, $V \subset U$ arcwise connected. Let $s_V \in \nu(F)(V)$ be such that $[s_V] = [s_U]$. We claim that $s_V = 0$, hence $[s_V] = 0$. Indeed, since V is arcwise connected, for any point $y \in V$ there exists a path γ from x to y hence,

$s_V(y) = F([\gamma])(s_V(x)) = F([\gamma])(0) = 0$.
To check the surjectivity of ev_x, for any $m \in F(x)$ we need to find a section $s \in \nu(F)(U)$ such that $s(x) = m$. Suppose that U is arcwise connected, and for $y \in U$ define $s(y) = F([\gamma])(m)$ for γ a path from x to y. Clearly $s \in \nu(F)(U)$, and hence ev_x is surjective. □

Example 3.12. Now we would like to construct an example where ev_x is not injective. Let X be the comb space, figure 1, and let $\Delta_M \in \mathrm{Rep}_k(\Pi_1(X))$ be the constant functor with value

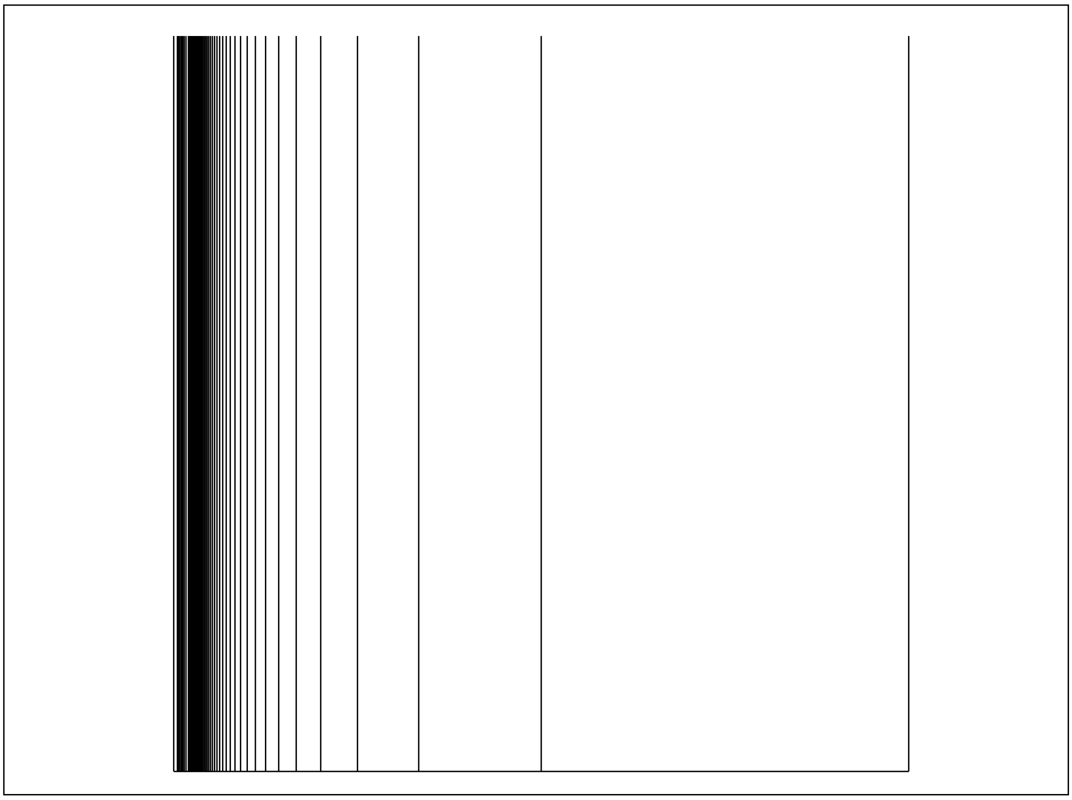


Fig. 1 Depiction of the comb space

M. We will label each tooth of the comb for its position, being the n-th comb the one over $\frac{1}{n}$ and the 0-th the one over 0. Consider $\{U_i\}_i$ a fundamental system of open neighborhoods of $(0,1)$ such that U_i contains infinitely many connected components of X. This can be done since $(1,0)$ is a limit point. We will call $\Lambda_i \subset \mathbb{N} \cup \{0\}$ the subset indexes of teeth with non empty intersection with U_i. Note that for all $i \in I$, $0 \in \Lambda_i$ and $|\Lambda_i| = |\mathbb{N}|$.

We have that

$$\nu(\Delta_M)_{(0,1)} \simeq \varprojlim_{U \ni (0,1)} \mathrm{Hom}_{\mathbf{Set}}(\pi_0(U), M) \simeq \varprojlim_{U_i} \mathrm{Hom}_{\mathbf{Set}}(\pi_0(U_i), M),$$

where the last equivalence follows from U_i being a fundamental system of neighborhoods. Since $|\pi_0(U_i)| = |\Lambda_i|$

$$\nu(\Delta_M)_{(0,1)} = \varprojlim_{U_i} \mathrm{Hom}_{\mathbf{Set}}(\Lambda_i, M)$$

Note that $\mathrm{Hom}_{\mathbf{Set}}(\Lambda_i, M)$ are nothing but sequences of elements of M, and $\varprojlim_{U_i} \mathrm{Hom}_{\mathbf{Set}}(\Lambda_i, M)$ sets that $s_i \sim s_j$ if there exists $k \in I$ such that $U_k \subset U_i, U_j$ and such that the series coincide on Λ_k.
Therefore, let $s_0, s_1 \in \nu(F)(U_i)$ for some i such that $s_0 := (0,0,0...)$ and $s_1 := (0,1,1...)$. For $i = 0,1$ $ev_{(0,1)}(s_i) = 0$, however, both of the sections are distinct on the stalk by construction.

Example 3.13. Example 3.12 gives us a case in which $\nu(\Delta_M)$ is not locally constant. If it were, there would exists $U \ni (0,1)$ such that $\nu(\Delta_M)|_U \simeq M_U$. In this case, For $V \subset U$, $V \ni (0,1)$ connected we would have $\nu(\Delta_M)(V) \simeq (M_X)_{(0,1)} = M$ by Lemma 2.6, which leads to contradiction.

Proposition 3.14. *If X is locally arcwise connected the following diagram quasi-commutes:*

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(X)) & \xleftarrow[\mu]{} & \mathrm{LCSH}(k_X) \\ & \searrow^{\nu} & \downarrow i \\ & & \mathrm{Mod}(k_x) \end{array}$$

More precisely, there exists an isomorphism of functors $i \overset{\theta}{\Rightarrow} \nu \circ \mu$.

Proof. To define θ, let $\mathscr{F}$ be a locally constant sheaf. Let $x, y \in U$ and let γ be a path from x to y. Then the following diagram holds:

$$\begin{array}{ccccc} & & \xrightarrow{\phi_x} & & \mu(\mathscr{F})(y) = \mathscr{F}_y \\ & & & \nearrow & \uparrow \mu(\mathscr{F})(\gamma) \\ \mathscr{F}(U) & \overset{\exists!\theta_{\mathscr{F}}(U)}{\dashrightarrow} & \varprojlim_{\Pi_1(U)} \mu(\mathscr{F}) \circ i_U & & \\ & & & \searrow & \\ & & \xrightarrow{\phi_y} & & \mu(\mathscr{F})(x) = \mathscr{F}_x \end{array}$$

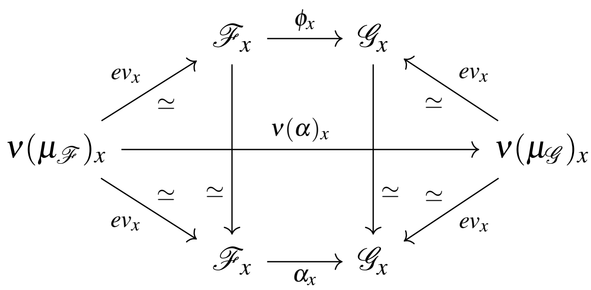
Therefore, we define $\theta_{\mathscr{F}} : \mathscr{F} \to \nu(\mu_{\mathscr{F}})$, and hence $\theta : i \to \nu \circ \mu$.
Now, let X be locally arcwise connected and take $\mathscr{F} \in \mathrm{LCSH}(k_X)$. Since X is locally arcwise connected, there exists an open cover $\{U_i\}_i$ of X of arcwise connected open subsets such that $\mathscr{F}|_{U_i} \simeq M_{U_i}$. Moreover, we have that $\mu_{\mathscr{F}}|_{U_i} \simeq \mu_{M_{U_i}} \simeq \Delta_M$, the constant functor. As U_i is arcwise connected, we have that $\nu(\Delta_M)|_{U_i} \simeq M_{U_i}$, by Example 3.10. Therefore, $\nu(\mu_{\mathscr{F}}|_{U_i}) \simeq M_{U_i}$. Thus, since they are locally isomorphic on an open cover of X, they are globally isomorphic. Hence $\nu \circ \mu \simeq i$ □

Corollary 3.15. *If X is locally arcwise connected μ is fully faithful.*

Proof. Faithfulness was already proven in Lemma 3.5.
Fullness follows from Proposition 3.14, since for $\alpha \in \mathrm{Hom}_{\mathrm{Rep}_k(\Pi_1(X))}(\mu_{\mathscr{F}}, \mu_{\mathscr{G}})$ we get the following diagram

$$\begin{array}{ccc} \nu(\mu_{\mathscr{F}}) & \xrightarrow{\nu(\alpha)} & \nu(\mu_{\mathscr{G}}) \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle\simeq} \\ \mathscr{F} & \overset{\phi}{\dashrightarrow} & \mathscr{G} \end{array}$$

where ϕ is induced and makes the diagram commute. Note that the vertical arrows on the level of stalks translate to $\nu(\mu_{\mathscr{F}})_x \to \mu_{\mathscr{F}}(x) = \mathscr{F}_x$, hence by the universal property they restrict to the the evaluation map ev_x. We claim that $\mu(\phi) = \alpha$. Indeed, let $x \in X$, then



where the vertical arrows are induced by the isomporhpism ev_x and hence they are the identity. All the squares commute by contruction. Hence $\mu(\phi)_x = \alpha_x$. □

Lemma 3.16. *If X is semi-locally simply connected, then $\nu(F) \in \mathrm{LCSH}(k_X)$. Hence we get a functor*

$$\nu : \mathrm{Rep}(\Pi_1(X)) \to \mathrm{LCSH}(k_X)$$

Proof. Since X is semi-locally simply connected, for $x_0 \in X$, there exists $U \ni x_0$ open connected subset of X such that any loop in U is contractible in X. Therefore, we have that

i_U factors as

$$\begin{array}{ccc} \Pi_1(U) & \xrightarrow{\quad i_U \quad} & \Pi_1(X), \\ & \searrow \qquad \nearrow & \\ & \{*\} & \end{array}$$

where $\{*\}$ denotes the category with only one morphism and the functor $\{*\} \to \Pi_1(X)$ sends $*$ to x_0. In this setting, $F \circ i_U$ is constant on U, hence $\nu(F)|_U \simeq \nu(F \circ i_U) \simeq \nu(\Delta_{F(x_0)})$ is the constant sheaf on U with stalk $F(x_0)$. Since we can cover X with such U's, $\nu(F)$ is locally constant. $\square$

Theorem 3.17. *If X is semi-locally simply connected and locally arcwise connected, then μ and ν are quasi inverse to each other.*

Proof. By Lemma 3.16 and Proposition 3.14 we deduce that $id_{\mathrm{LCSH}(k_X)} \simeq \nu \circ \mu$.
Now, we want to prove that $\mu \circ \nu \simeq id_{\mathrm{Rep}_k(\Pi_1(X))}$. In order to do this, we will explicitly construct such isomorphism.
Let $F \in \mathrm{Rep}(\Pi_1(X))$. Then, since X is locally arcwise connected, $\mu \circ \nu(F)(x) = \nu(F)_x \overset{ev_x}{\simeq} F(x)$.
We will build the data for an isomorphism of functors. Let $x, y \in X$ such that there exists a path $\gamma : I \to X$ from x to y. Consider the diagram

$$\begin{array}{ccccc} \nu(F)_x & & \xrightarrow{\mu\circ\nu(F)([\gamma])} & & \nu(F)_y \\ & \nwarrow & & \nearrow & \\ {\scriptstyle ev_x}\downarrow{\scriptstyle \simeq} & & \nu(F)(U) & & {\scriptstyle \simeq}\downarrow{\scriptstyle ev_y} \\ & \swarrow & & \searrow & \\ F(x) & & \xrightarrow{F([\gamma])} & & F(y) \end{array}$$

where all the diagonal arrows are the natural ones. The upper triangle commutes by definition of the monodromy functor, the lower by definition of ν and the side ones by Lemma 3.11. Hence the diagram commutes. Therefore, we get an isomorphism of functors $\mu \circ \nu \simeq id_{\mathrm{Rep}_k(\Pi_1(X))}$. $\square$

3.3 Monodormy representation associated to a Serre fibration

Let X and Y be topological spaces, and $f : X \to Y$ a continuous map. Then f induces a functor

$$\begin{aligned} f : \Pi_1(X) &\to \Pi_1(Y) \\ x &\mapsto f(x) \\ [\gamma] &\mapsto [f \circ \gamma] \end{aligned}$$

With the notation of Section 1.3 we get a functor

$$\mathrm{res}_f = f_* : \mathrm{Rep}_k(\Pi_1(X)) \to \mathrm{Rep}_k(\Pi_1(Y))$$

Proposition 3.18. *(i) The following diagram quasi-commutes.*

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xrightarrow{\mathrm{res}_f} & \mathrm{Rep}_k(\Pi_1(X)) \\ \mu_Y \uparrow & & \mu_X \uparrow \\ \mathrm{LCSH}(k_Y) & \xrightarrow{f^{-1}} & \mathrm{LCSH}(k_X) \end{array}$$

(ii) If X, Y are locally arcwise connected, the following diagram quasi-commutes

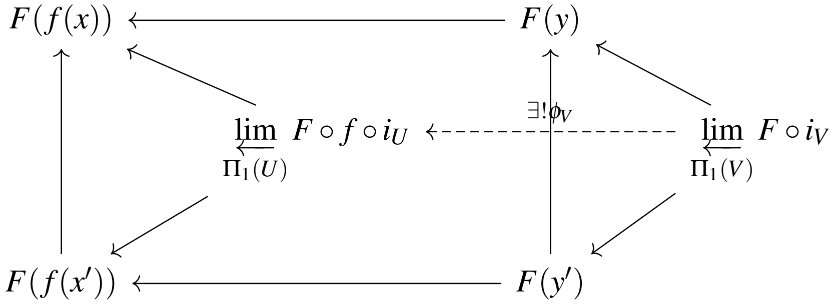
$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xrightarrow{\mathrm{res}_f} & \mathrm{Rep}_k(\Pi_1(X)) \\ \downarrow \nu_Y & & \downarrow \nu_X \\ \mathrm{Mod}(k_Y) & \xrightarrow{f^{-1}} & \mathrm{Mod}(k_X) \end{array}$$

Proof. For (i), let $\mathscr{F} \in \mathrm{LCSH}(k_Y)$. Then

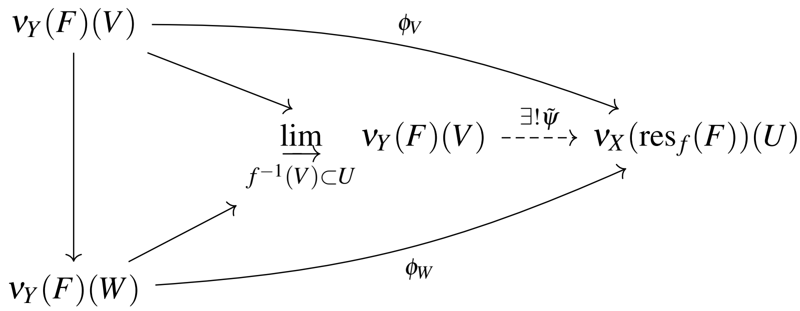
$$\mathrm{res}_f \circ \mu_Y(\mathscr{F}) = \left(\begin{aligned} \mathrm{res}_f \circ \mu_Y(\mathscr{F}) : \Pi_1(X) &\to \mathrm{Mod}(k) \\ x &\mapsto \mathscr{F}_{f(x)} \\ x \sim x' &\mapsto \mathscr{F}_{f(x)} \simeq \mathscr{F}_{f(y')} \end{aligned} \right)$$

$$\simeq \left(\begin{aligned} \mu_X(f^{-1}\mathscr{F}) : \Pi_1(X) &\to \mathrm{Mod}(k) \\ x &\mapsto (f^{-1}\mathscr{F})_x \\ x \sim x' &\mapsto (f^{-1}\mathscr{F})_x \simeq (f^{-1}\mathscr{F})_{x'} \end{aligned} \right) = \mu_X(f^{-1}\mathscr{F})$$

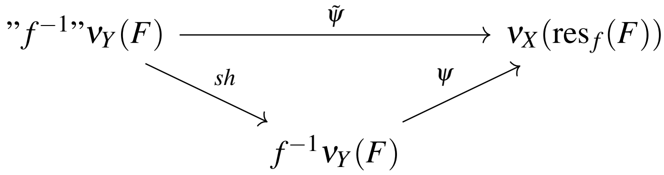
So the diagram quasi-commutes.
For (ii), let $F \in \mathrm{Rep}_k(\Pi_1(Y))$. Let $U \subset X$ and let $V \subset Y$ such that $f(U) \subset V$.



where $y, y' \in Y$ are such that $f(x) = y$ and $f(x') = y'$. Hence for any $V \subset Y$ such that $f(U) \subset V$ we can define a morphism $\phi_V : \nu_Y(F)(V) \to \nu_X(\mathrm{res}_f(F))(U)$



Hence we have a morphism of presheaves $\tilde{\psi} : "f^{-1}"\nu_Y(F) \to \nu_X(\mathrm{res}_f(F))$. Since $\nu_X(\mathrm{res}_f(F))$ is a sheaf, we know that $\tilde{\psi}$ factors as



where *sh* denotes the sheafification functor. Hence we have defined a morphism $\psi : f^{-1}\nu_Y(F)(U) \to \nu_X(\mathrm{res}_f(F))$. Under the assumption that X and Y are locally arcwise connected, we have that for any $x \in X$

$$(f^{-1}\nu_Y(F))_x \simeq \nu_Y(F)_{f(x)} \simeq F(f(x)) \simeq \mathrm{res}_f(F)(x) \simeq \nu_X(\mathrm{res}_f(F))_x.$$

Hence, ψ_x is an isomorphism for all x. $\square$

Recall that if f is a Serre fibration

$$f_* : \mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$$

Proposition 3.19. *If f is a Serre fibration. If X and Y are semi-locally simply connected ans locally arcwise connected, the following diagrams quasi-commute:*

(i)

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xleftarrow{\mathrm{ind}_f} & \mathrm{Rep}_k(\Pi_1(X)) \\ \mu_Y \uparrow & & \mu_X \uparrow \\ \mathrm{LCSH}(k_Y) & \xleftarrow{f_*} & \mathrm{LCSH}(k_X) \end{array}$$

(ii)

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xleftarrow{\mathrm{ind}_f} & \mathrm{Rep}_k(\Pi_1(X)) \\ \downarrow \nu_Y & & \downarrow \nu_X \\ \mathrm{LCSH}(k_Y) & \xleftarrow{f_*} & \mathrm{LCSH}(k_X) \end{array}$$

Proof. For (i), we have that $\mathrm{ind}_f \circ \mu_X$ is right adjoint to $\nu_X \circ \mathrm{res}_f$ and $\mu_Y \circ f_*$ is right adjoint to $f^{-1} \circ \nu_Y$. Let $\Gamma : f^{-1} \circ \nu_Y \xrightarrow{\sim} \nu_X \circ \mathrm{res}_f$ be the unique isomorphism induced from the commutativity from Proposition 3.18. Since $\mathrm{ind}_f \circ \mu_X \circ \Gamma$ is a right adjoint for $\mu_Y \circ f_*$, there exists a unique isomorphism $\mu_Y \circ f_* \to \mathrm{ind}_f \circ \mu_X \circ \Gamma$. On the other hand, we have an isomorphism $- \circ \Gamma^{-1} : \mathrm{ind}_f \circ \mu_X \circ \Gamma \to \nu_Y \circ \mathrm{ind}_f$, hence by composing both isomorphisms we have that $\nu_Y \circ \mathrm{ind}_f \simeq f_* \circ \nu_X$
(ii), follows from a parallel argument as the one in (i) □

If f is not a fibration, we have to use the decomposition of f studied in Section 2.3:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {}_{p}\nwarrow\!\searrow^{h} \qquad \nearrow_{\tilde{f}} & \\ & Nf & \end{array}$$

where $\tilde{f}$ is a Serre fibration, p is the natural projection and h is a homotopy inverse for p.
Recall that

$$f_*^{LCSH} := \tilde{f}_* \circ p^{-1} : \mathrm{LCSH}(k_X) \to \mathrm{LCSH}(k_Y)$$

Theorem 3.20. *Let X and Y be semi-locally simply connected and locally arcwise connected, and let $f : X \to Y$ be a continuous map. Then the following diagrams commute*

(i)

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xleftarrow{\mathrm{ind}_f} & \mathrm{Rep}_k(\Pi_1(X)) \\ {\scriptstyle \mu_Y}\uparrow & & {\scriptstyle \mu_X}\uparrow \\ \mathrm{LCSH}(k_Y) & \xleftarrow{f_*^{\mathrm{LCSH}}} & \mathrm{LCSH}(k_X) \end{array}$$

(ii)

$$\begin{array}{ccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xleftarrow{\mathrm{ind}_f} & \mathrm{Rep}_k(\Pi_1(X)) \\ \downarrow{\scriptstyle \nu_Y} & & \downarrow{\scriptstyle \nu_X} \\ \mathrm{LCSH}(k_Y) & \xleftarrow{f_*^{LCSH}} & \mathrm{LCSH}(k_X) \end{array}$$

Proof. (i) Consider the following diagrams:

$$\begin{array}{ccccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xleftarrow{\mathrm{ind}_{\tilde{f}}} & \mathrm{Rep}_k(\Pi_1(Nf)) & \xleftarrow{\mathrm{res}_p} & \mathrm{Rep}_k(\Pi_1(X)) \\ {\scriptstyle \mu_Y}\uparrow & & {\scriptstyle \mu_{Nf}}\uparrow & & {\scriptstyle \mu_X}\uparrow \\ \mathrm{LCSH}(k_Y) & \xleftarrow{\tilde{f}_*} & \mathrm{LCSH}(k_{Nf}) & \xleftarrow{p^{-1}} & \mathrm{LCSH}(k_X) \end{array}$$

By Propositions 3.18 and 3.19 we know that both squares commute. Moreover, since p is a homotopy equivalence, by Corollary 2.11 he have that p^{-1} is an equivalence of categories. Since both μ_X and μ_{N_f} are equivalences too, it follows that res_p is an equivalence. We need to show that $\mathrm{ind}_f = \mathrm{ind}_{\tilde{f}} \circ \mathrm{res}_p$, i.e. $\mathrm{ind}_h = \mathrm{res}_p$. This follows from the fact that $id_{\mathrm{LCSH}(k_X)} = (p \circ h)^{-1} = h^{-1} \circ p^{-1}$ and that $id_{\mathrm{LCSH}(k_{Nf})} \simeq (p \circ h)^{-1} = p^{-1} \circ h^{-1}$. Hence p^{-1} and h^{-1} are quasi inverse to each other, and due to the uniqueness of adjointness $\mathrm{res}_p \simeq \mathrm{ind}_h$. Since $\mathrm{ind}_f \simeq \mathrm{ind}_{\tilde{f}} \circ \mathrm{ind}_h$ the result follows.

(ii) Similarly we can decompose (ii) into the following commutative diagrams.

$$\begin{array}{ccccc} \mathrm{Rep}_k(\Pi_1(Y)) & \xleftarrow{\mathrm{ind}_{\tilde{f}}} & \mathrm{Rep}_k(\Pi_1(Nf)) & \xleftarrow{\mathrm{res}_p} & \mathrm{Rep}_k(\Pi_1(X)) \\ \downarrow{\scriptstyle \nu_Y} & & \downarrow{\scriptstyle \nu_{Nf}} & & \downarrow{\scriptstyle \nu_X} \\ \mathrm{LCSH}(k_Y) & \xleftarrow{\tilde{f}_*} & \mathrm{LCSH}(k_{Nf}) & \xleftarrow{p^{-1}} & \mathrm{LCSH}(k_X) \end{array}$$

□

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