Applications of Characteristic Classes and Milnor's Exotic Spheres

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1 Introduction

"In learning the sciences, examples are of more use than precepts"

This project represents a compressed introduction to fiber bundles and characteristic classes, with the aim of acquiring the minimal amount of machinery required to examine some of their copious and diverse applications to differentiable manifolds.

Very generally, characteristic classes assign collections of vector spaces parameterised by a space X to elements of the cohomology ring of X, in such a way that this assignment has certain reasonable properties. Before giving a less nebulous definition, let us talk about some questions on smooth manifolds where these classes can provide some potential insights.

Given a manifold, M, of dimension n an interesting question to ask is "what is the smallest k such that the manifold M can be immersed in $\mathbb{R}^{n+k}$"?. If the dimension of the manifold is n then by Whitney[18] such an immersion always exists when $k \geq n-1$. So we have an upper bound for our answer. On the other hand if the manifold is parallelizable (has a trivializable tangent bundle) by a theorem of Hirsch and Smale[5] we can immerse such a manifold into $\mathbb{R}^{n+1}$. The characteristic classes of the tangent bundle of a manifold measure, in some sense, how far this bundle is from being trivial, and so play a defining role in this question. An easy but useful obstruction to finding an embedding of a particular manifold in $\mathbb{R}^{n+k}$ will be given and the case of the real projective spaces will be treated in detail.

Certain characteristic classes can be defined only when particular structures are present on the tangent bundle of a manifold, important examples of such structure being an orientation, or complex or symplectic structures. The non-existence of these classes can then be used to provide evidence of the lack of existence of a certain structure on the manifold. A very easy illustration of this will be given by showing that for a sphere of dimension $4k$ cannot have the structure of a complex manifold.

Yet another general question in which these classes can help is the following "given a manifold M and a manifold N of dimension greater than M how many inequivalent ways can we embed M in N". This time the bundle of interest is the normal bundle, consisting of tangent vectors of N orthogonal to the tangent space of the embedded M in N. An investigation of a question of this type led to the following surprising discovery: there are at least 14 inequivalent smooth structures on the 7-sphere. A large part of this project will be devoted to this discovery of Milnor, which remains one of the most unexpected results in differential topology.

When Milnor first discovered the exotic differentiable structure of the 7-sphere it was generally assumed that any homeomorphism between smooth manifolds could be smoothed to give a diffeomorphism. Milnor's original aim was quite different[10]. He wished to study the topology of $2n$-dimensional manifolds which are $(n-1)$-connected. The choice of this type of manifold was a practical one: the homotopy class of such a manifold can be realised by attaching a $2n$-cell, e_{2n}, to a bouquet of n-spheres by idenfying the boundary ∂e_{2n} with $S^n \vee \cdots \vee S^n$ using an attaching map. A homotopy class of this type of structure depends on the homotopy class a the attaching map $f : \partial e_{2n} \to S^n \vee \cdots \vee S^n$. At the time homotopy group $\pi_{2n-1}(S^n \vee \cdots \vee S^n)$ was reasonably well understood when very few methods existed which would allow the calculation of general homotopy groups. It was possible, then, to work with the homotopy theory of such manifolds.

The "problem" arises when attempting to construct explicit examples of manifolds in the given homotopy class. An attempt to construct these manifolds for the relatively easy case $m = 1$ with a single sphere in the 'bouquet' led to Milnor's discovery of exotic differentiable structures.

To examine the properties of manifolds M_8 of the same homotopy type as an 8-cell attached to a 4-sphere we could look at the embedding of S^4 in the hypothetical M_8. In fact we look at a tubular neighbourhood of such a sphere or, equivalently, the normal disk bundle of S^4 in the manifold M_8. We can also ask the question in the other direction - given a an equivalence class of disk bundles over the 4-sphere, can a 8-cell can be attached to the boundary in such a way as to give a smooth, closed, orientable manifold? In order to do this the boundary of the disk bundle should, at least, be S^7. It is possible to find many disk bundles over S^4 with boundary S^7 and examples are constructed later. We can then attempt to attach an 8-cell to this boundary in such a way as to get an 8-manifold satisfying the above conditions. This is obviously possible in the when the boundary is the standard 7-sphere. However it can be shown that for some of the sphere bundles constructed later, this is impossible. Whence this bundle cannot have the differentiable structure of the standard 7-sphere.

Showing that the construction of such an M_8 is impossible requires computing the Pontrjagin numbers of this hypothetical manifold. Hizerbruch's signature theorem gives a formula for the second Pontrjagin class of the potential M_8 (and use of this theorem is why we have so many conditions on M_8). However for some of the given constructed sphere bundles this formula will give a fractional value for $p_2([M_8])$ which is impossible. The necessary conclusion is that such an M_8 cannot exist.

Conventions:

- a *map* is a continuous function between topological spaces. Sometimes the phrase continuous function or continuous map is used, to emphasise, or because I forgot this convention

- a *topologicalspace* is a Hausdorff and paracompact topological space
- Homology and cohomology are taken with integer coefficients, unless specified otherwise i.e. $H^n(X)$ means $H^n(X;\mathbb{Z})$.

2 Fiber Bundles

This is taken predominantly from[4, 13, 16]

2.1 The Category of Topological Fiber Bundles

The simplest example of a fiber bundle is the cartesian product of two topological spaces $E = B \times F$. This is called the trivial bundle with base space B and fiber F. Such an object is simply described with coordinates $E = \{(b, v) : b \in B, v \in F\}$

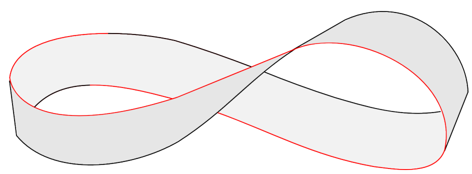

Figure 1: A failed attempt to embed the circle into the sphere by identifying elements (drawn in red) of neighbouring vector spaces

Examples of non-trivial fiber bundles occur when the fiber possesses a symmetry group which allows a "twisting" of the fibers as we travel along the base space. The fiber bundle looks locally like an ordinary cartesian product and "twisting" only becomes apparent when examining (potential) embeddings $B \hookrightarrow E$ of the base space in the total space. If the bundle is trivial it will always be possible to embed F copies of B into the total space. Indeed given any $v \in F$, there is an embedded copy of B given by $\{(b, v) : b \in B\}$. Once a "twist" is introduced however, it will not be possible to get such a collection of embeddings of B into the total space. In other words we no longer have the nice global coordinates (b, v) of the trivial bundle and it is impossible to "compare" neighboring fibers in a compatible way. A more technical definition is as follows:

Definition. *A fiber bundle consists of three topological spaces: E, the total space, B the base space and F the fibre with a continuous surjection $\pi : E \to B$ such that*

1. *B has a cover by open sets U_i on which $\pi^{-1}(U_i)$ is homeomorphic to the cartesian product $U_i \times F$*

2. *The homeomorphism $\phi_i : \pi^{-1}(U_i) \to U_i \times F$ is such that the following diagram commutes*

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[r, "\phi"] \arrow[d, "\pi"] & U_i \times F \arrow[dl] \\ U_i & \end{tikzcd}$$

The homeomorphisms ϕ_i are called the local trivialisations of the bundle. The bundle is said to be locally trivial over the open sets U_i and a cover of the base space for which the above homeomorphisms exist is called a trivialising open cover.

Let U_i , U_j be open sets on B so that the bundle is locally trivial over U_i and U_j and $U_i \cap U_j \neq \emptyset$. Then the local trivialisations ϕ_i, ϕ_j induce a homeomorphism on fibers over $b \in U_i \cap U_j$

$$\mathrm{g}_{ij} \coloneqq \phi_i \circ {\phi_j}^{-1}(b) : F \times \{b\} \to F \times \{b\}$$

These functions are called the transition functions of the fiber bundle. They take values in the topological group[1] $Homeo(F)$ and this is referred to as the structure group of the bundle.

Definition. *A fiber bundle morphism, defined for bundles with a common fiber F, is a pair of continuous funtions (f, g) between the base and total spaces of the bundle so that following diagram commutes*

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "\pi_1"] & E_2 \arrow[d, "\pi_2"] \\ B_1 \arrow[r, "f"] & B_2 \end{tikzcd}$$

i.e. g sends the fiber $\pi_1^{-1}(\{b\})$ of E_1 to the fiber $\pi_2^{-1}(\{f(b)\})$ of E_2.

A map between total spaces which is "fiberwise" (i.e. sends fibers to fibers) induces a map f on the base spaces for which the above diagram commutes. Thus the map f is completely determined by the map of total spaces g. If the bundles have a common base space B a "morphism of fibre bundles" is usually understood to have the identity on B as the induced map of base spaces i.e. we require $g(\pi_1^{-1}(\{b\})) \subseteq \pi_2^{-1}(\{b\})$. Naturally,

Definition. *An isomorphism of fiber bundles is a morphism of fiber bundles which has an inverse that is again a morphism of fiber bundles.*

[1] $Homeo(F)$ is a topological group whenever F is Hausdorff, locally connected and locally compact. In particular it is so when F is a topological manifold

In the statements and proofs that follow we will assume for simplicity that bundles over the same base space have the same trivialising open cover. We will see later that we can assume that this is always the case.

First a useful sufficient condition for a bundle map of two bundles over a common base space to be an isomorphism.

Lemma 2.1. *A morphism of fiber bundles $g : E \to E'$ over B is an isomorphism of fiber bundles if its restriction to each fiber $\pi^{-1}(\{b\})$ is a homeomorphism of fibers.*

Proof. Now assume that $g : E \to E'$ is a morphism of bundles which restricts to a homeomorphism of fibers on each fiber. g is clearly bijective. We need to check that the inverse morphism g^{-1} is continuous as a map from the total space E' to E. By composing with the trivialisations h_i,h'_i of the bundles E,E' we can see that the isomorphism of fibers induced by g induces an isomorphism of charts

$$\begin{aligned} U_i \times F &\to U_i \times F \\ (x, a) &\mapsto (x, \lambda_i(x)(a)) \end{aligned}$$

where $\lambda(x)$ is an element of the structure group depending continuously on $x \in B$. The inverse of this map is given by $(x, a) \mapsto (x, \lambda_i^{-1}(x)(a))$. But $\lambda_i(x)$ continuous implies $\lambda^{-1}(x)$ continuous (as $Homeo(F)$ is a topological group) and the composition $h_i^{-1} \circ (id, \lambda_i^{-1}) \circ h'_i(e') = g^{-1}(e')$ is therefore a continuous morphism from $\pi'^{-1}(U_i)$ to $\pi^{-1}(U_i)$. Thus g^{-1} is a bundle morphism as required. □

Corollary 2.2. *Two fiber bundles are isomorphic if there exists continuous functions $\lambda_i : V_i \to G$ so that the transition functions $g_{ij}, g'_{ij} : V_i \cap V_j \to G$ satisfy*

$$g'_{ij}(x) = \lambda_i(x) g_{ij}(x) \lambda_j^{-1}(x)$$

where $\lambda_i(x) : B \to Homeo(F)$ is continuous from the base space to the group of homeomorphisms of the fiber.

Proof. As above an isomorphism of fiber bundles induces a homeomorphism of fibers and in turn, upon composing with the trivialisation functions, homeomorphisms of fibers of the trivialisation of the bundle which will vary continuously with respect to a coordinate on the base space x. Looking at the commutative diagram

$$\begin{array}{ccc} h_i(\pi^{-1}\{x\}) & \xrightarrow{\lambda_i(x)} & h'_i(\pi^{-1}\{x\}) \\ \big\downarrow{\scriptstyle g_{ij}} & & \big\downarrow{\scriptstyle g'_{ij}} \\ h_j(\pi^{-1}\{x\}) & \xrightarrow{\lambda_j(x)} & h'_j(\pi^{-1}\{x\}) \end{array}$$

The result follows.

Conversely suppose we have $\lambda_i : V_i \to G$ so that the transition functions are related by $g'_{ij}(x) = \lambda_i(x) g_{ij}(x) \lambda_j^{-1}(x)$. Define the map $f_i : U_i \times F \to U_i \times F$ by $f_i(x, a) \to (x, \lambda_i^{-1}(x)(a))$. Define the morphism of fiber bundles $f : E \to E'$ locally by specifying $f(\pi^{-1}(U_i)) = h_i'^{-1} f_i h_i(\pi^{-1}(U_i))$. $f(\pi^{-1}(U_i))$ and $f(\pi^{-1}(U_j))$ agree on the bundle over the intersection of U_i and U_j and so do, in fact, give a morphism of fiber bundles. The resulting morphism clearly induces a homeomorphism of fibers and so is an isomorphism.

□

Two fiber bundles being isomorphic an equivalence relation and so one can speak of "equivalence classes of bundles over a base space". A "fiber bundle over B" is often understood to mean "an equivalence class of fiber bundles over B".

2.2 Structure group of Vector Bundles

Some important classes of fiber bundles are constrained to have a structure group smaller that $Homeo(F)$. Eliminating this flexibility often gives extra structure on our bundles. Important examples include (real) vector bundles. These are fiber bundles with individual fibers vector spaces over $\mathbb{R}$ and transition functions induced by the local trivialisations taking values in the general linear group $GL(n, \mathbb{R})$. Similarly a complex vector bundle of rank n has fibers which are vector spaces over $\mathbb{C}$ and structure group $GL(n, \mathbb{C})$.

A real vector bundle is said to be orientable if it is possible to choose a consistent orientation on the fibers i.e. a choice of orientation on each fiber so that the trivialisations $h_{ij} : U \times \mathbb{R}^n \to \pi^{-1}(U)$ is orientation preserving on each fiber (where $\mathbb{R}^n$ has the standard orientation). It is easy to see that we can make such a choice if and only if the transition functions have everywhere positive determinants so that the structure group takes values in $GL(n, \mathbb{R})^+$.

Given a complex vector bundle the underlying real bundle (given by simply "forgetting" the complex structure) is always orientable. This can be seen by looking directly at the transition functions for the bundle. For a complex vector bundle of rank n the transition functions of the underlying real bundle are found in $GL(n, \mathbb{R})^+$.

If the manifold M has a differentiable structure then it can be shown that M orientable if and only if its tangent bundle is orientable as a vector bundle.

If the base space B is a smooth manifold, then the total space E of a vector bundle can be given the structure of a smooth manifold. If the projection function is smooth we can call the vector bundle smooth. The next examples of fiber bundles should make it unsurprising that these particular bundles find wide applications in the field of differentiable topology.

2.3 Motivating Examples

2.3.1 Tangent and Normal Bundles

The tangent (or cotangent) bundle of a manifold is one of the most natural constructions of vector bundles over a manifold. This already provides examples and motivations for studying non-trivial fiber bundles. A classical theorem is that one cannot find an everywhere non-vanishing smooth vector field on S^2, so the tangent bundle on the 2-sphere cannot be trivial.

Another important example of a vector bundle is the concept of the normal bundle of a submanifold. Let $i : X \hookrightarrow M$ be an embedded submanifold. Consider tangent space of X at a point $x \in X$. This can be viewed as a subspace of the tangent space T_xM with inclusion given by

$$di_x : T_xX \to T_xM$$

Definition. *The normal space of X at x is defined as the quotient space $N_xX = {}^{T_xM}/_{T_xX}$. The normal vector bundle NX of X in M is defined as the union of the normal spaces over X*

$$NX = \{(x, v) : v \in N_xX\}$$

The tubular neighbourhood theorem states that the (open) normal disk bundle over $X \subset M$ is diffeomorphic to an open neighbourhood of X in M. Thus, by classifying the rank-n disk bundles over a space X we can examine its possible embeddings in an $dim(X) + n$ manifold.

2.3.2 Sphere Bundles and the Hopf Fibration

A sphere bundle is a fiber bundle with individual fibers n-spheres such that the transition functions induced by the local trivialisations take values in the group $O(n)$. A *section* of a vector bundle is a continuous function $s : B \to E$ with $\pi \circ s = Id$. A *metric* on a vector bundle $\pi : E \to B$ is a collection of inner products on the fibers the bundle that is continuous in the sense that if $s, t : B \to E$ are continuous sections then the the inner product on each fiber $\langle s(b), t(b)\rangle_b$ is a continuous function from B to $\mathbb{R}$. If the vector bundle has the structure of a smooth manifold we can also require a smoothly varying inner product.

We form the associated sphere bundle of a vector bundle by taking all vectors of norm one[2]. The associated (open) disk bundle of a vector bundle is composed of all vectors of norm less than one.

The famous Hopf fibration provides a (non-trivial) fibration of the 3-sphere over the 2-sphere in the following way:

A complex line through the origin in $\mathbb{C}^2$ is parameterised by (z, mz) where $m \in \mathbb{C} - 0$ is the (complex) slope of the line. Identify the 3-sphere as the set $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$. Each line will meet the 3-sphere in a circle $S^1 \subset \mathbb{C}^2$ which consists of points $\{(z, w) \in S^3 \mid \frac{w}{z} = m\}$ and each of these circles will be disjoint. Project each of these circles to an element of $\mathbb{C}P^1 \cong S^2$ which represents its slope and the set $(0, z)$ to the "point at infinity". Explicitly in homogenous coordinates

$$
\begin{aligned}
&p : S^3 \to \mathbb{C}P^1 \\
&p(\lambda, \mu) = [\lambda : \mu]
\end{aligned}
$$

and we get a fibration of S^3 over S^2 whose fibres are S^1. A completely analogous

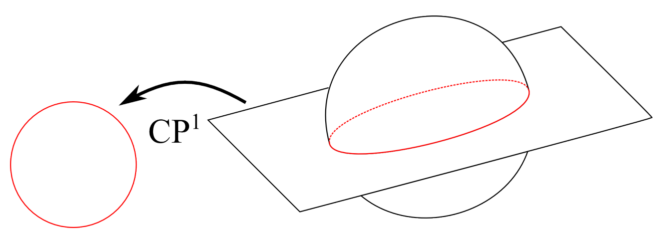

Figure 2: A graphical representation of the Hopf fibration with the circle in "S^3" being sent to the complex slope of the plane

construction works to find a fibration of the 7-sphere by 3-spheres. We can identify the 7-sphere as the set $S^7 = \{(q, r) \in \mathbb{H} \times \mathbb{H} \mid |q|^2 + |r|^2 = 1\}$. We can identify the 3-sphere with the set of quaternions of norm 1. A (quaternionic) line in $\mathbb{H}^2$ will intersect the 7-sphere along such a set of quaternions. Projecting a given 3-sphere of this form to a point representing its (quaternionic) slope is

[2]Every vector bundle (over a paracompact space) has a metric, this will be shown later

given by the mapping $(q,r) \to [q:r] \in \mathbb{H}P^1 \cong S^4$. Thus S^7 can be expressed as a fiber bundle over S^4 with fiber S^3.

This can be further extended to find a fibration of the 15-sphere over the 8-sphere by 7-spheres by using the structure of the octonions, though the details are slightly harder here. The division algebra structure of $\mathbb{C}, \mathbb{H}$ and $\mathbb{O}$ are used to construct these fibrations and by Adam's theorem sphere bundles (meaning fiber bundles with base space, fiber and total space all spheres) can only occur in these dimensions, and the rather less exciting case of the fibration of S^1 by 0-spheres.

2.4 The Pull Back of a Vector Bundle

Now that we are duly motivated, we give a construction in theory of vector bundles with particular importance to classifying the possible vector bundles over a given base space. Given a continuous function $f : B' \to B$ and a bundle $\pi : E \to B$ we create a bundle over a space B'.

Definition. *The pull back of the bundle $\pi : E \to B$ under a map $f : B' \to B$ has as total space the set*

$$f^*E = \{(b', e) \in B' \times E \mid f(b') = \pi(e)\} \subset B' \times E$$

equipped with the subspace topology and the projection function $\pi'(b', e) = b'$

This is clearly a vector bundle over B' with trivialising open cover $f^{-1}(U_i)$ where U_i is the trivialising cover for E and homeomorphisms to $f^{-1}(U_i) \times F$ induced by the corresponding trivialisations of $\pi : E \to B$. There is a bundle morphism

$$\begin{array}{ccc} f^*E & \xrightarrow{g} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

where g is the projection $f^*E \to E$ given by $g(b', e) = e$. Commutativity of the diagram is automatically satisfied. This leads to an alternative useful definition of the pull back of the bundle:

Lemma 2.3. *The pull back of the bundle $\pi : E \to B$ under the map $f : B' \to B$ is the unique vector bundle $f^*(E)$ over B such that there is a bundle morphism*

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ B' & \xrightarrow{f} & B \end{array}$$

so that $\tilde{f}$ restricts to an isomorphism on each fiber

Proof. g obviously restricts to a homeomorphism on each fiber. We need to show that any bundle satisfying this definition is equivalent to the one just described. Let E' be a vector bundle over B so that there is a bundle map $g : E' \to E$ which is fiberwise and restricts to a homeomorphism of fibers, on fibers. Then the bundle morphism sending each element v of a fiber of E' $(\pi'(v), g(v)) \in f^*(E)$ is a bundle morphism which restricts to homeomorphisms of fibers. Hence E' and $f^*(E)$ are equivalent bundles. □

In terms of the transition functions of the bundle we have the following:

Lemma 2.4. *The pull back bundle has as an open cover of the base space the set $f^{-1}(V_i)$ where V_i is the cover of B and as transition functions the pull back of the transition functions of B.*

Proof. This is immediate upon composing the trivialisations of the pull back bundle. □

Note that in this way a map $f : B' \to B$ induces a mapping $f^* : Vect(B) \to Vect(B')$ from the set $Vect(B)$ of equivalence classes of vector bundles over B to the set of vector bundles over B'. A key observation when categorising the vector bundles which can occur over a particular space B is that given two maps $f_1, f_2 : B \to A$ the pull back bundles of a vector bundle $E \to A$ are isomorphic if f_1 and f_2 are homotopic. The idea of the proof is to construct a vector bundle over $B \times [0, 1]$ which is $f_1^*(E)$ at $B \times \{0\}$ and $f_2^*(E)$ at $B \times \{1\}$. We can "untwist" the bundle over as we travel along (B, t) to show that these bundles are isomorphic. The proof is long but not difficult so we refer to [4], pg 20 for the details.

An immediate corollary is that homotopy equivalences of base spaces induce bijection between isomorphism classes of vector bundles over these bases. In particular for a contractible space there is a bijection $f^* : Vect(B) \xrightarrow{\sim} Vect(\{x\})$ for a point $x \in B$ and as the only possible vector bundle over a point is the trivial bundle, we conclude that any vector bundle over a contractible space is trivial.

To take this idea further we will now show how every vector bundle of an appropriate rank over a base space can be given as the pull back of a "universal bundle" with classes of vector bundles in correspondence with homotopy classes of maps between the base and the base space of the universal bundle. Of course, the problem of categorising vector bundles over a space is now given by the problem of classifying homotopy classes of maps from that space to a new complicated space which, in general, is still difficult.

2.5 Classifying Bundles: The Universal Bundle

Let M be a k-dimensional manifold embedded in $\mathbb{R}^n$. By the Whitney embedding theorem, such an embedding can always be realized as long as $n \geq 2k$. At each point $x \in M \subset \mathbb{R}^n$ the tangent space at x defines a k-dimensional subspace of the vector space $\mathbb{R}^n$ given by translating the space T_xM to the origin. This is a mapping from the manifold into the Grassmanian $G_k(\mathbb{R}^n)$, a manifold[3] whose points are the k-dimensional subspaces of $\mathbb{R}^n$.

Consider the following natural vector bundle[4] over $G_k(\mathbb{R}^n)$

$$E_k(\mathbb{R}^n) = \{(V, v) \in G_k(\mathbb{R}^n) \times \mathbb{R}^n \mid v \in V\}$$

i.e. the set of all vectors in the subspace $V \in G_k(\mathbb{R}^n)$. This is called the tautological bundle over $G_k(\mathbb{R}^n)$. The tangent bundle of M is clearly the pull back of this tautological bundle under the "generalised Gauss map"[5] sending each tangent space to the associated plane through the origin. With a little more work we can prove that in fact every rank-k real vector bundle over a base space B can be given as the pull back of the tautological bundle over a Grassmanian manifold.

Theorem 2.5. *Every rank-k bundle over a compact space X is the pullback of $E_k(\mathbb{R}^n)$ for some n, under a mapping $f : X \to G_k(\mathbb{R}^n)$. There is a bijection between homotopy classes of maps $[f] : X \to G_k$ and classes of isomorphic vector bundles over X.*

Proof. Let U_i be a open cover of the base space X such that the bundle is trivial over each U_i. Since X is compact we can find an open subcover so that the cardinality of this cover, m, is finite. Let $\lambda_i : U_i \to [0, 1]$ be a partition of unity subordinate to this open cover. Consider the trivialisations $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^k$ and the projection $p : U_i \times \mathbb{R}^k \to \mathbb{R}^k$. The composition of these functions is clearly a linear injection on each fiber. Let v be an element of the fiber above

[3] This is indeed a manifold cf Hatcher [4]

[4] This is indeed a vector bundle cf Hatcher [4]

[5] Named after the map of Gauss, who essentially defined this map for surfaces in $\mathbb{R}^3$

x i.e. and consider the function $h_i'(v) = \lambda_i(x)p(h_i(v))$. This induces a map

$$\begin{aligned} g : \pi^{-1}(\{x\}) &\to \mathbb{R}^k \oplus \cdots \oplus \mathbb{R}^k \cong \mathbb{R}^n \\ f(v) &= (\lambda_1(x)p(h_1(v)), \cdots, \lambda_m(x)p(h_m(v))) \end{aligned}$$

where $n = km$. This is clearly a linear mapping of the vector space $V = \pi^{-1}(\{x\})$ which defines a plane in $\mathbb{R}^n$, an element of $G_k(\mathbb{R}^n)$. Let $f(x)$ map $x \in X$ to the point in $G_k(\mathbb{R}^n)$ given by $g(\pi^{-1}(\{x\}))$. Then the pair (f, g) is a bundle map from $\pi' : E \to X$ to $\pi : E_k(R^n) \to G_k(\mathbb{R}^n)$ which is a linear injection (and so an isomorphism of vector spaces) on each fiber of E. Thus, E is isomorphic to the pull back of the bundle $E_k(R^n)$ under the map f.

□

This completes the proof when the bundle has a trivialising cover of finite cardinality. We can extend this to a paracompact base ($\Rightarrow$ countable trivialising cover) using the following.

Consider the inclusions $\mathbb{R}^n \subset \mathbb{R}^{n+1} \subset \cdots$ which lead to the inclusions $G_k(\mathbb{R}^n) \subset G_k(\mathbb{R}^{n+1}) \subset \cdots$. We take the direct limit and form the space

$$G_k = G_k(\mathbb{R}^\infty) = \bigcup_{n \in \mathbb{N}} G_k(\mathbb{R}^n)$$

equipped with the direct limit topology. Likewise we form the tautological bundle over G_k by taking

$$E_k = E_k(\mathbb{R}^\infty) = \bigcup_{n \in \mathbb{N}} G_k(\mathbb{R}^n)$$

It can be shown that E_k is indeed a vector bundle and by using an extension of the idea for compact manifolds we have that **every** vector bundle over a paracompact base, X, is the pullback of tautological bundle under a map $f : X \to G_k$. For this reason the space G_k is called the classifying space (of real vector bundles of rank k) and the bundle E_k is called the universal bundle.

For complex vector bundles the construction of a classifying space and universal bundle is completely analogous, first taking the Grassmanian $G_k(\mathbb{C}^n)$ whose points represent k-dimensional complex planes through the origin of $\mathbb{C}^n$, defining the tautological bundle similarly and then passing to the direct limit.

This theorem is useful as it allows us to make statements about properties of vector bundles by "pulling back" a property of the tautological bundle. In particular an immediate corollary is that every real vector bundle has a metric defined on fibers as follows: The fibers of $E_k(\mathbb{R}^n)$ are planes in $\mathbb{R}^n$. The vector bundle thus inherits a metric given by the restriction of the standard Euclidean metric of $\mathbb{R}^n$ to fibers of $E_k(\mathbb{R}^n)$. In turn every vector bundle is equipped with a metric which we get by pulling back the metric on $E_k(\mathbb{R}^n)$. Similarly every complex vector bundle has a Hermitian metric.

Unfortunately this theorem does not afford us the ability to classify bundles over X explicitly as finding all homotopy classes of maps $f : X \to G_k$ is too difficult in general. For this reason we now give another way of classifying vector bundles over spheres which reduces to the significantly easier[6] problem of calculating the homotopy groups $\pi_{k-1}(SO(n))$.

2.6 Classifying Bundles over Spheres

For the sphere there is another way of categorising vector bundles, which relies on the fact that an n-sphere can be covered by exactly two contractible open sets (in fact this property categorises topological n-spheres). First we reduce our very large structure group into a considerably smaller one

Lemma 2.6. *Let E be a rank-k real vector bundle with structure group $GL(k,\mathbb{R})$. Then E is equivalent to a bundle which has transition functions which take values in the subgroup $O(k) \subset GL(k,\mathbb{R})$. If E is orientable with structure group $GL(k,\mathbb{R})^+$ then E is equivalent to a bundle with transition functions in the subgroup $SO(k) \subset GL(k,\mathbb{R})^+$*

Proof. Let U_α be a trivialising open cover of the base space B. As the bundle is trivial over each of the U_α we can find k linearly independent sections, whose restrictions to each fibre form a basis of the vector space $\mathbb{R}^k$. Let $s_i^\alpha, 0 \leq i \leq k$ be such a collection of sections over a trivialising open subset U_α and $s_j^\beta, 0 \leq j \leq k$ such a collection of sections over a trivialising open subset U_β.

For all $x \in U_\alpha \cap U_\beta$ there is an induced metric on the fiber E_x. Therefore using the Gram-Schmidt orthogonalisation process we can find an orthonormal basis $t_i^\alpha(x)$ which is a linear combination of the $s_i^\alpha(x)$. Similarly we can construct an orthonormal basis $t_j^\beta(x)$ of E_x from the sections $s_j^\beta(x)$. Such a change of basis will be continuous as a function from the base space to $GL(n,\mathbb{R})$ and so produces an equivalent bundle. But the transition function at x is then from an orthonormal basis of $\mathbb{R}^k$ to an orthonormal basis of $\mathbb{R}^k$ and so takes values in $O(k)$. If the bundle is orientable, the bases can be ordered so that the transition functions preserve orientation and then will take values in $SO(k)$. So by a suitable choice of basis of the fibers of the bundle, it can be arranged that

[6]But in high dimensions still difficult

the structure group of a vector bundle is $O(k)$ and, if the bundle is orientable, in $SO(k)$. $\square$

We now put a vector bundle over a sphere in "normal form", where the bundle is trivial over two sets which cover the sphere so that the intersection is a band along the equator. The isomorphism classes of the vector bundle are the given by homotopy classes of maps from the equator to the structure group of the vector bundle. We are interested in the particular case of orientable real bundles, though similar constructions exist for non orientable real bundles and complex bundles.

Theorem 2.7. *Equivalence classes of rank-n orientable, real vector bundles $p : E \to S^k$ over the k-sphere are given by elements $\pi_{k-1}(SO(n))$*

Proof. Choose an open cover of S^k consisting of two open sets V_0, V_1 so that V_0, V_1 cover the upper and lower hemispheres of S^k and their intersection is a product $S^{k-1} \times (-\epsilon, \epsilon)$. Let (x, t) be coordinates on this intersection. Given a map $f : S^{k-1} \to SO(n)$ we form the vector bundle E_f over S^k associated to f as the quotient of the trivial bundles $V_{0,1} \times \mathbb{R}^n$ with $(x, t, v) \sim (x, t, f(x)(v))$ for $(x, t) \in V_0 \cap V_1$.

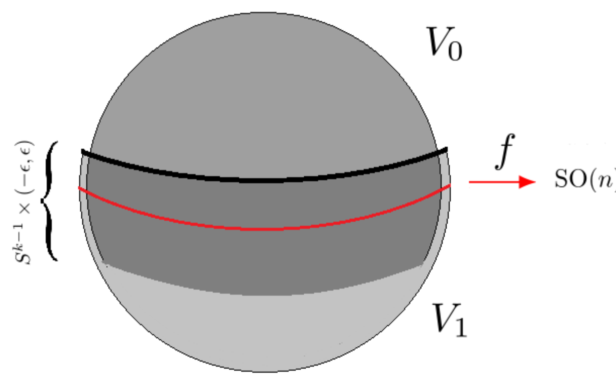

Figure 3: A sphere covered by two open sets

Conversely let $p : E \to B$ be a vector bundle over S^n. As V_0, V_1 are contractible, it is possible to find trivialisations of the vector bundle over these sets. We constuct the map $E \mapsto \pi_{k-1}(SO(n))$ by sending E to the homotopy class of the transition function $g_{01} : V_0 \cap V_1 \to SO(n)$ restricted to a sphere $(S^{k-1}, t_0) \subset V_0 \cap V_1$. A homotopy between any two such S^{k-1} leads to a homotopy the restrictions of g_{01} to these spheres, so this is a well defined operation. Now we wish to show that two bundles are equivalent iff they have the same associated homotopy class.

If E is a bundle and x_0 a chosen point on the equator we can always find a bundle equivalent to E with transition function $g_{01}(x_0) = e$, the identity in $SO(n)$ by sending the trivialisation of each fiber over V_1 with coordinates (b, v) to the homeomorphic image $(b, g_{01}(x_0)^{-1}v)$. If E, E' are equivalent bundles the trivialisations of E, E' over V_0, V_1 differ by maps $\lambda_{0,1} : V_{0,1} \to SO(n)$. Finally V_0, V_1 are contractible so we can find a retraction of each of these sets to x_0.

The transition functions of E, E' are related by $g'_{01}(x) = \lambda_0(x)g_{01}(x)\lambda_1(x)^{-1}$. But if $H^0(t,x)$ is a deformation retract of V_0 to x_0 and $H^1(t,x)$ deformation retract of V_1 to x_0, then $g(t,x) \mapsto \lambda_0(H^0(t,x))g_{01}(x)\lambda_1(H^1(t,x))^{-1}$ is a homotopy from g'_{01} to $\lambda_0(x_0)g_{01}\lambda_1(x_0)^{-1}$. Moreover as $SO(n)$ is path connected there is a path $\gamma(t)$ from $\lambda_0(x_0)$ to e and $\gamma(t)g_{01}\gamma(t)^{-1}$ is a homotopy from $\lambda_0(x_0)g_{01}(x)\lambda_1(x_0)^{-1}$ to $g_{01}(x)$. Therefore $g'_{01} \cong g_{01}$ as required.

Conversely let E, E' be vector bundles over S^k so that the restriction of the transition $(S^{k-1}, t_0) \subseteq V_0 \cap V_1$ are homotopic. Then $g'_{01}(x)g_{01}(x)^{-1}$ is homotopic to the constant map, so such a mapping can be extended over the whole disk D_0. Denote this extension by λ_0 and let λ_1 be the identity. We have continuous maps $\lambda_{0,1} : V_{0,1} \to (SO(n))$ with $g'_{01}(x) = \lambda_0(x)g_{01}(x)\lambda_1^{-1}$ and $g'_{10}(x) = \lambda_1(x)g_{01}(x)\lambda_0^{-1}$ so these bundles are equivalent, as required.

Thus the map $E \mapsto \pi_{n-1}(SO(n))$ is well defined with well defined inverse $\pi_{k-1}(SO(n)) \ni f \mapsto E_f$ and there is a bijection between vector bundles over S^k and elements of $\pi_{n-1}(SO(n))$.

3 Cohomology of Fiber Bundles

This is taken predominantly from [13, 3]

The cohomology of fiber bundles is difficult to calculate in general. For a special class of sphere bundle known as orientable sphere bundles, however, it is possible to use more elementary techniques. Let $p : E \to B$ be a rank n vector bundle. If E is a normed bundle we can form the sphere bundle S_E associated to E by taking the set of elements of E of norm one. If we know the cohomology of the base space we can calculate the cohomology of the associated sphere bundle using the long exact sequence

$$\cdots \to H^{i-n}(B) \to H^i(B) \to H^i(S_E) \to H^{i-n+1}(B) \cdots$$

known as the Gysin sequence. This is derived from the long exact sequence associated to the relative cohomology of the pair $H^*(E, E_0)$ where E_0 is the set of non-zero vectors in E using an isomorphism $H^i(B) \cong H^{i+n}(E, E_0)$ known as the Thom isomorphism.

3.1 Relative Cohomology of Pairs

Relative cohomology is dual to relative homology. The relative cohomology of a pair of topological spaces (X, A) is the cohomology of cochains which vanish on simplices in A. As with absolute cohomology if we take the direct sum of the relative cohomology groups $H^*(X, A)$ we have a graded ring where multiplication is now given by the relative cup product.

Many theorems in absolute cohomology have important generalisations for relative cohomology. The proof of the Thom isomorphism employs the following, which here are only stated. They play identical roles to their counterparts in absolute cohomology: The Künneth formula tells us the cohomology of a Cartesian product, given the cohomology of the pair and the Mayer-Vietoris sequence tells us how to construct cohomology classes on the union of topological spaces, given classes on the subspaces.

Theorem 3.1 (The Künneth Formula for Relative Cohomology). *Let $(X, A), (Y, B)$ be pairs of topological spaces and denote by p_1, p_2 the projections from $X \times Y$ onto the first and second factors respectively. Suppose $H^k(Y, B)$ is a free $\mathbb{Z}$-module for all k. Then the cross product homomorphism*

$$\begin{aligned} &H^*(X, A) \times H^*(Y, B) \to H^*(X \times Y, X \times B \cup A \times Y) \\ &(a, b) \mapsto p_1^*(a) \smile p_2^*(b) \end{aligned}$$

is an isomorphism of graded rings.

Theorem 3.2 (The Mayer Vietoris Sequence for Relative Cohomology)**.** *There exists a long exact sequence*

$$\cdots \to H^n(X_1 \cup X_2, A_1 \cup A_1) \to H^n(X_1, A_1) \oplus H^n(X_2, A_2)$$
$$\to H^n(X_1 \cap X_2, A_1 \cap A_2) \to H^{n+1}(X_1 \cup X_2, A_1 \cup A_1) \to \cdots$$

Where the first morphism is given as the pull back under the inclusion of subspaces and the second as the difference $i^* - j^*$ of the pullbacks of cohomology groups under the inclusions $i : X_1 \cap X_2 \hookrightarrow X_1$, $j : X_1 \cap X_2 \hookrightarrow X_2$

3.2 The Thom Isomorphism Theorem

In what follows a "choice of orientation" of a vector space is a choice of generator of the top relative cohomology group $H^n(V, V - 0)$. This is equivalent to the "usual" definition of orientation given by an ordering of basis elements of V[7]. Recall that an orientable bundle is one such that we can choose this generator in a way so that local trivialisations are orientation preserving on each fiber.

Theorem 3.3 (The Thom isomorphism theorem)**.** *Let $p : E \to B$ be an oriented real vector bundle of rank n. Denote by E_0 the set of non-zero vectors in E. There exists a unique class $u \in H^n(E, E_0; \mathbb{Z})$ such that for any fiber F the restriction $u|_{(F,F_0)} \in H^n(F, F_0; Z)$ is the class giving the orientation of F. Moreover $H^k(B) \cong H^{k+n}(E, E_0)$ and this isomorphism is given explicitly by $x \mapsto x \cup u$.*

The Thom isomorphism theorem is proved by first showing the case of the trivial bundle and then extending to general vector bundles using the Mayer Vietoris sequence. The class u is called the *Thom Class*

Lemma 3.4. *Let B be a space and $\mathbb{R}^n$ a real vector space with the canonical orientation. Then there exists a class $u \in H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$, unique up to a choice of orientation of $\mathbb{R}^n$ so that $u|_{(F,F_0)}$ is the orientation on the fiber F and $x \mapsto x \cup u$ is an isomorphism $H^k(B) \cong H^{k+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$*

Proof. By looking at the long exact sequence for relative cohomology $H^n(\mathbb{R}^n, \mathbb{R}^n_0) \cong \mathbb{Z}$ and $H^k(\mathbb{R}^n, \mathbb{R}^n_0)$ is trivial for all $k \neq n$. Therefore using the relative Künneth formula with the pairs $(B, \emptyset)$ and $(\mathbb{R}^n, \mathbb{R}^n_0)$ we have an isomorphism $H^k(B) \cong H^{k+n}(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$. Let 1 be the generating element of $H^0(B)$ and u be the pull back of the chosen generating element of $H^n(\mathbb{R}^n, \mathbb{R}^n_0)$ under the projection $(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0) \to (\mathbb{R}^n, \mathbb{R}^n_0)$. Then $1 \times u \in H^n(B \times \mathbb{R}^n, B \times \mathbb{R}^n_0)$ restricts to the orientation form on each fiber. The cup product $y \smile 1 \times u \mapsto y \times u$ is the isomorphism given by the Künneth formula. $\square$

[7] see appendix

We now need to show that if the vector bundle is orientable then there exists a unique class in $H^n(E, E_0)$ which restricts to the orientation form on each fiber. Then we need to show that cupping with this class gives the required isomorphism.

Proof. (of the Thom isomorphism theorem) Suppose that B is covered by two open sets U, V so that the vector bundle is trivial over U and V. Since E is orientable, we can give an orientation on each fiber of E so that the trivialisations are orientation preserving. By the lemma above there exist unique Thom classes $u_i \in H^n(U \times \mathbb{R}^n, U \times \mathbb{R}^n_0)$. As the trivialisations are orientation preserving these pull back to classes whose restriction to each fiber generate the cohomology $H^n(F, F_0)$. So we have the Thom isomorphism for the vector bundle restricted to U and V.

The restrictions of the class u_i to the bundle over $U \cap V$ restricts to an orientation form on each fiber, so we likewise have the Thom isomorphism theorem for $U \cap V$. The restriction of both classes u_i give the same orientation on each fiber as the trivialisations are orientation preserving. By the uniqueness of the choice of Thom class, these must agree on and so we have a well defined class u on E.

Let E_U be the restriction of the vector bundle to U, $E_\cap$ the restriction of E to $U \cap V$ and $\dot{E}$ the restriction to the set of non-zero vectors. By the relative Mayer-Vietoris sequence we have that

$$\cdots \to H^{n-1}(E_\cap, \dot{E}_\cap) \to H^n(E, \dot{E}) \xrightarrow{(i^*, j^*)} H^n(E_U, \dot{E}_U) \oplus H^n(E_V, \dot{E}_V)) \xrightarrow{\psi} H^n(E_\cap, \dot{E}_\cap) \to \cdots$$

is exact. By the Thom isomorphism theorem for $U \cap V$ we have that $H^{n-1}(E_\cap, \dot{E}_\cap) \cong H^{-1}(U \cap V) = 0$ and so (i^*, j^*) is injective. Thus the class u is unique.

We now need to prove that cupping with this form gives the Thom isomorphism. Consider the exact sequences

$$\begin{tikzcd}
H^n(E_U) \oplus H^n(E_V) \arrow[r] \arrow[d] & H^n(E_\cap) \arrow[r] \arrow[d] & H^{n+1}(E) \arrow[r] \arrow[d] & H^{n+1}(E_U) \oplus H^{n+1}(E_V) \arrow[r] \arrow[d] & H^{n+1}(E_\cap) \arrow[d] \\
H^n(E_U, \dot{E}_U) \oplus H^n(E_V, \dot{E}_V) \arrow[r] & H^n(E_\cap \dot{E}_\cap) \arrow[r] & H^{n+1}(E, \dot{E}) \arrow[r] & H^{n+1}(E_U, E_U) \oplus H^{n+1}(E_V, \dot{E}_V)) \arrow[r] & H^{n+1}(E_\cap, \dot{E}_\cap)
\end{tikzcd}$$

with isomorphisms from absolute to relative cohomology given on either side by the Thom isomorphism. Applying the five lemma we see that cupping with the form u gives an isomorphism from $H^n(E)$ to $H^n(E, \dot{E})$.

□

Here the result for two open sets is enough as spheres can be covered by two open sets. The result is easily extended to finite open sets and the result can be proved in the general case using a direct limit argument.

3.3 The Gysin Sequence

The bundle E_0 of non-zero vectors of E deformation retracts to S_E so we have the isomorphism $H^*(E_0) \cong H^*(S_E)$. Similarly $H^*(E) \cong H^*(B)$. Using these isomorphisms and the long exact sequence of cohomology groups of the pair (E, E_0) we get the Gysin sequence

$$\begin{tikzcd}
\cdots H^i(E,E_0) \arrow[r, "j"] & H^i(E) \arrow[r] & H^i(E_0) \arrow[r] & H^{i+1}(E,E_0)\cdots \\
\cdots H^{i-n}(B) \arrow[u, "\cup u"] \arrow[r, "\cup e"] & H^i(B) \arrow[u, "p^*"] \arrow[r] & H^i(S_E) \arrow[u, "\wr"] \arrow[r] & H^{i-n+1}(B)\cdots \arrow[u, "\cup u"]
\end{tikzcd}$$

for $i < k$, we have that $H^{i-n}(B) = 0$ and the Gysin breaks up into exact sequences $0 \to H^i(B) \to H^i(S_E) \to 0$ giving $H^i(B) \cong H^i(S_E)$. For higher cohomology groups the situation is more interesting and the relation between $H^i(B)$ and $H^i(S_E)$ depends on the Euler class, e which is defined as the image of the Thom class under the morphisms $(p^*)^{-1} \circ j(u)$.

If the bundle $S_E \to B$ has a section, we have $s \circ p = Id$ and the induced map on the cohomology of B is an isomorphism. Necessarily then $p^* : H^n(B) \to H^n(S_E)$ is injective. By the exactness of

$$H^0(B) \xrightarrow{\cup e} H^n(B) \xrightarrow{p^*} H^n(S_E)$$

then, the Euler class must be zero. Conversely, if the Euler class is non zero then there cannot be a nowhere vanishing section and in particular the bundle cannot be trivial.

3.4 The Thom Isomorphism Theorem with $\mathbb{Z}_2$ coefficients

Let us examine what "goes wrong" in the Thom isomorphism theorem when our bundle is non-orientable by looking at the classic case of the Mobius strip. The Mobius strip can be formed by taking two charts on the circle U_0, U_1 over which the bundles are trivial, twisting one and gluing together. We certainly have a Thom classes u_0, u_1 over the two trivial bundles but when we glue these together $u_0 = -u_1$ on one component of the intersection and so we cannot find a global well-defined form. However, if we take cohomology with $\mathbb{Z}_2$ coefficients, both forms evaluate to the unique non-trivial element in $H^1(F, F_0)$ and so these classes do agree $\mod 2$. Whence we have a statement of the Thom isomorphism theorem, valid for all vector bundles.

Theorem 3.5. *Let $p : E \to B$ be an real vector bundle. Denote by E_0 the set of non-zero vectors in E. There exists a unique class $u \in H^n(E, E_0; \mathbb{Z}_2)$ such that for any fiber F the restriction $u|_{(F,F_0)}$ is the unique non trivial class in $H^n(F, F_0; \mathbb{Z}_2)$. Moreover $H^k(B) \cong H^{k+n}(E, E_0)$ and this isomorphism is given explicitly by $x \mapsto x \cup u$.*

4 Characteristic Classes

Every vector bundle over a base space can be given as the pull back of a universal bundle of the appropriate rank. As pull back bundles under homotopy equivalent maps from the base space are equivalent, the set of vector bundles over a base space is given by the set homotopy equivalent classes of maps from the base space to the classifying space of rank n vector bundles. It is difficult to find all homotopy classes of these maps in general. However, we can at least look at the induced map on cohomology. Characteristic classes are the image of cohomology groups of the classifying space under this induced map. In particular, if the classifying map of the bundle induces a non-trivial map from the cohomology groups of the classifying space (i.e. there is a non-zero characteristic class associated to the bundle) the map to the classifying space the bundle is not null-homotopic and the bundle cannot be trivial.

A characteristic class of a vector bundle, then, is an element of the cohomology group of the base space. The associated classes can also be defined by an axiomatic definition - for instance, given a mapping $f : X \to B$ we obviously would ask that the pull back of the characteristic class be the characteristic class of the pull back bundle, a condition called naturality. In practise, the easiest way to calculate the classes of bundles is using the axiomatic definition. Historically (and more intuitively), non trivial characteristic classes indicated that one could not find an appropriate number of everywhere linearly independent, everywhere non-vanishing sections of the bundle which is obvious condition for triviality.

4.1 The Euler Class

The Euler class is the image of the Thom class under the canonical map from relative cohomology to absolute cohomology followed by restriction to the zero section. As seen in the last section, the Euler class vanishes if there exists a nowhere vanishing section of the vector bundle (though the converse is not true).

Lemma 4.1. *The Euler class is natural, i.e.* $f^*(e(E)) = e(f^*(E))$

Proof. By the definition of the pull back bundle there is a mapping from $f^*(E)$ to E which restricts to a linear isomorphism on each fiber. The pull-back of the Thom class on E under this mapping is a class which is non-zero on each fiber of $f^*(E)$. But this is necessarily the (unique) Thom class of $f^*(E)$. As the Euler class is just the Thom class restricted to the zero section we have that the Euler class of the pull-back bundle is the pull-back of the Euler class. $\square$

4.2 The Chern Classes

If we have a complex vector bundle with a Hermitian metric we can define the Chern classes. For an n-plane complex bundle there will be n of these classes with $c_i \in H^{2i}(B;\mathbb{Z})$. The non-vanishing of the i^{th} Chern class is an obstruction to finding $(n-i+1)$ linearly independent nowhere vanishing sections and Chern classes were initially defined in these terms. It can be difficult, however, to calculate the Chern classes when defined in this manner. The equivalent axiomatic definition is easier, in practise, to work with.

Definition. *Let $p : E \to B$ be a complex vector bundle. Chern classes are elements $c_i \in H^{2i}(B;\mathbb{Z})$ which satisfy the following properties. Let $c = 1 + c_1 + \cdots + c_n$. Then*

- *Naturality: $c_k(f^*E) = f^*(c_k(E))$ for $f : Y \to X$ for f a continuous function where f^*E is the pull-back bundle*
- *Stability: $c(E \oplus \epsilon) = c(E)$ for ϵ the trivial bundle*
- *Whitney sum formula: $c(E \oplus F) = c(E)c(F)$*
- *Normalization: c of the tautological line bundle on $\mathbb{C}P^1$ is a fixed generator of $H^2(\mathbb{C}P^1;\mathbb{Z})$*

Note that the product of total Chern classes given in the Whitney sum formula is the product in the graded cohomology ring $H^*(B)$. This is given by the cup product of elements and distributivity laws i.e $\gamma(\alpha+\beta) = \gamma \smile \alpha + \gamma \smile \beta$.

Theorem 4.2. *For a complex vector bundle over a paracompact space the Chern classes exist and are unique.*

Proof. An explicit construction of the Chern classes can be given in terms of the Euler class of affiliated vector bundles. These can be shown to satisfy the given axioms.

Let $p : E \to B$ be a complex vector bundle with a Hermitian metric. We can always find a Hermitian metric on E when the base is paracompact. Let E_{n-1} be the vector bundle which has as its base space, B_{n-1}, the set of non-zero vectors in E (denoted E_0 until now). An element $v \in B_{n-1}$ belongs to a fiber F of E. Define the fiber over v to be the set of vectors of F which are orthogonal to v with respect to the given Hermitian metric. This is a $(n-1)$-plane bundle over the base space B_{n-1}. Define E_{n-i} inductively by performing the same procedure on the new vector bundle.

As the real $2n$-plane bundle underlying a complex n-plane bundle is always orientable we can find an Euler class. Define the top Chern class c_n to be the Euler class of E. From the Gysin sequence we have that $(p^*)^{-1}$ is an isomorphism from $H^{2m}(B_{n-1})$ to $H^{2m}(B)$ whenever $m < n$. Define c_{n-1} to be the image of the Euler class of the bundle E_{n-1} under this isomorphism. Define c_{n-i} to be the image of the Euler class of E_{n-i} in $H^{2(n-i)}(B)$ under the successive isomorphisms $H^{2(n-i)}(B_k)\tilde{\to}H^{2(n-i)}(B_{k+1})$.

Proof of Naturality

The naturality of the top Chern class follows from the naturality of the Euler class of E. Let $f : B' \to B$ be a continuous mapping and consider the pull back back bundle $f^*(E)$. For the pull back bundle there is a commutative diagram

$$\begin{array}{ccc} f^*(E) & \xrightarrow{g} & E \\ \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

Denote by g_0 the restriction of g to non-zero vectors, which is a bundle map $(f^*(E))_0 \cong f^*(E_0) \to E_0$. If $f^*(E)$ has the pullback Hermitian metric and $v^\perp$ denotes subset of vectors in the fiber containing v which are perpendicular to $v \in (f^*(E))_0$ we have $g_0(v^\perp) = (g_0(v))^\perp$. Form the vector bundle E_{n-1} over the base $E_0 = B_{n-1}$ as detailed earlier. Likewise form the bundle $(f^*(E))_{n-1}$ over the base $(f^*(E))_0$. As $g_0(v^\perp) = (g_0(v))^\perp$ we have $(f^*(E))_{n-1} = g_0^*(E_{n-1})$. By this equivalence and the naturality of the Euler class on the bundle E_{n-1} then $f^*(c_{n-1}(E)) = c_{n-1}(f^*(E))$. The result on the other classes follows by induction.

Proof of Stability

Note that the bundle $E \oplus \epsilon$ with ϵ the trivial bundle has the obvious cross section $s(x) = (0, 1)$. Then the Euler class and so the top Chern class c_{n+1} is 0. Consider the pull back of the bundle $(E \oplus \epsilon)^\perp$ under the section s. The restriction of this bundle to image of the section s in $(E \oplus \epsilon)$ is clearly E itself over the (embedded) image of B in $(E \oplus \epsilon)$ and so pulls back to the bundle E over B.

Now consider the projection $p : (E \oplus \epsilon)_0 \to B$. Now by the definition of the Chern classes we have $c_i(E \oplus \epsilon) = (p^*)^{-1}(c_i((E \oplus \epsilon)^\perp))$. But $p \circ s = id$ and so we have

$$c_i(E \oplus \epsilon) = (p^*)^{-1}(c_i((E \oplus \epsilon)^\perp)) = s^*(c_i((E \oplus \epsilon)^\perp) = c_i(s^*(E \oplus \epsilon)^\perp)) = c_i(E) \tag{1}$$

as required.

Proof of Normalisation

This is seen in the next section in the calculation of the cohomology groups of complex projective space.

Whitney Sum Formula

This is proved by proving the corresponding statement for the universal bundle, the proof for a general bundle then follows. See [13] for details

□

As discussed a non-zero Euler class is an obstruction to finding an everywhere non-zero section of the bundle. The Euler class of the bundle $E^{\perp}$ being non-zero is an obstruction to finding a section of everywhere non-zero vectors perpendicular to vectors in the bundle E, so the relation between this construction and earlier constructions where the Chern classes measured an inability to find everywhere non-vanishing, linearly independent vector fields of the bundle is clear.

Let us define the conjugate bundle $\bar{E}$ associated to a complex vector bundle E as a complex vector bundle of the same dimension for which the action of $\mathbb{C}$ on each fiber is given by $(a + bi)v = av - ibv$. This is easily seen to be the vector bundle which has as transtion functions the conjugate of the transition functions of E.

We have the following

Lemma 4.3. *The Chern classes of the complex conjugate bundle, $\bar{E}$, are given by $c_i(\bar{E}) = (-1)^i c_i(E)$*

Proof. Let $\bar{E}$ denote the complex conjugate bundle. If $(v_1, \cdots, v_n)$ is the orientation of the complex bundle then $(v_1, iv_1, \cdots v_n, iv_n)$ is the orientation for the underlying real bundle. The orientation of the real bundle underlying the complex conjugate bundle is then $(v_1, -iv_1, \cdots v_n, -iv_n)$ and is so $(-1)^n$ times the orientation of the bundle E and so $e(\bar{E}) = (-1)^n e(E)$. As $(\bar{E})^{\perp} = \bar{E}^{\perp}$ the result follows. □

Corollary 4.4. *The odd Chern classes of a complexified bundle are 2-torsion elements*

Proof. For a complexified bundle $E \otimes \mathbb{C}$ we have $E \otimes \mathbb{C} \cong \overline{E \otimes \mathbb{C}}$ and the result follows directly from the above equation □

4.3 The Steifel-Whitney classes

We recall the version of the Thom isomorphism with coefficent ring $\mathbb{Z}_2$. We will now use this to form characteristic classes very similar to the Chern classes, which can be defined on real vector bundles. Define the “unoriented” Euler class of the bundle as the restriction of the Thom class $u \in H^n(E; \mathbb{Z}_2)$ to the zero section. The axioms definining the Steifel-Whitney classes should not be entirely surprising.

Definition. *Let $p : E \to B$ be a real vector bundle. Steifel-Whitney classes are elements $w_i \in H^i(B; \mathbb{Z}_2)$ which satisfy the following properties. Let $w = 1 + w_1 + \cdots + w_n$. Then*

- *Naturality: $w_k(f^*E) = f^*(w_k(E))$ for $f : Y \to X$ for f a continuous function where f^*E is the pull-back bundle*
- *Stability: $w(E \oplus \epsilon) = w(E)$ for ϵ the trivial bundle*
- *Whitney sum formula: $w(E \oplus F) = w(E)w(F)$*
- *Normalization: w of the tautological line bundle on $\mathbb{R}P^1$ is the unique non-zero element of $H^1(\mathbb{R}P^1; \mathbb{Z}_2)$*

The proof of the first three axioms is the same as that of the Chern classes, and writing $\mathbb{Z}_2$ everywhere. The proof of Normalization is given in the next section when calculating the cohomology ring of real projective space with $\mathbb{Z}_2$ coefficients.

4.4 The Pontrjagin Classes

It is not surprising that the Steifel-Whitney classes do not capture all information on the cohomology of a real vector bundle. We define another type of characterstic class, defined on real vector bundles. The Pontrjagin classes be described in terms of the Chern classes of the complexification of the bundle as follows: Let $p : E \to B$ be a real vector bundle. The i^{th} Pontrjagin class $p_i(E) \in H^{4i}(B)$ is given by $p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C})$ where $(E \otimes \mathbb{C})$ is the complexification of the bundle. The Ponrjagin classes satisfy

- $p_k(f^*E) = f^*(p_k(E))$
- $p(E \oplus \epsilon) = p(E)$ for ϵ the trivial bundle
- $p(E \oplus F) - p(E)p(F)$ is a 2-torsion element.

Proof. The naturality and stability of the Pontrjagin classes follows directly from the naturality of the Chern classes. For the Whitney sum formula we have that $(E \oplus F) \otimes \mathbb{C} = (E \otimes \mathbb{C}) \oplus (F \otimes \mathbb{C})$ and so for $0 \leq i \leq \frac{1}{2}(\text{rank}(E \oplus F))$

$$\begin{aligned} c(E \oplus F) &= \sum_i c_{2i-1}((E \otimes \mathbb{C}) \oplus (F \otimes \mathbb{C})) + \sum_i c_{2i}((E \otimes \mathbb{C}) \oplus (F \otimes \mathbb{C})) \\ &= \sum_i c_{2i-1}((E \otimes \mathbb{C}) \oplus (F \otimes \mathbb{C})) + \sum_i (-1)^i (p_i(E \oplus F)) \end{aligned}$$

and

$$\begin{aligned} c(E)c(F) &= (\sum_i c_{2i-1}(E) + \sum_i c_{2i}(E))(\sum_i c_{2i-1}(F) + \sum_i c_{2i}(F)) \\ &= (\sum_i c_{2i-1}(E) + \sum_i (-1)^i p_i(E))(\sum_i c_{2i-1}(F) + \sum_i (-1)^i p_i(F)) \end{aligned}$$

And so modding out elements of order 2

$$\begin{aligned} \sum_i (-1)^i (p_i(E \oplus F)) &= p(E)p(F) \mod 2 \\ p(E \oplus F) &= p(E)p(F) \mod 2 \end{aligned}$$

□

Lemma 4.5. *If E is a complex vector bundle then the complexification of the underlying real bundle $E \otimes \mathbb{C}$ is isomorphic over $\mathbb{C}$ to $E \oplus \bar{E}$*

Proof. We need to find a $\mathbb{C}$-linear vector space isomorphism on the fibers of the bundles.

The complexification of E as has its fibers the direct sum of two copies of the underlying real vector space $V \oplus V$ together with the almost complex structure $J(z, w) = (-w, z)$, where here z and w are considered pairs of real variables. The direct sum $E \oplus \bar{E}$ is pairs of complex variables with the action of $\mathbb{C}$ given by $\lambda(z, w) = (\lambda z, \bar{\lambda} w)$ for λ a complex constant. Consider the linear homomorphism $\phi : E \oplus \bar{E} \to E \otimes \mathbb{C}$ given by $\phi(z, w) \to (z + w, -iz + iw)$. Then for $z = x + iy$ we have $\phi(i(z, 0)) = \phi((iz, 0)) = (iz, z) = J(z, -iz) = J(\phi(z, 0))$ so ϕ is $\mathbb{C}$-linear in the first variable. Similarly $\phi(i(0, w)) = \phi((0, -iw)) = J(\phi(0, w))$. □

Lemma 4.6. *For any real vector bundle with a complex structure the Pontrjagin classes satisfy*

$$1 - p_1 + p_2 - \cdots \pm p_n = (1 - c_1 + \cdots \pm c_n)(1 + c_1 + c_2 \cdots + c_n)$$

Proof. $c(E \otimes \mathbb{C}) = c(E \oplus \bar{E}) = c(E)c(\bar{E})$ For i odd the associated Chern class is $c_i(E \oplus \bar{E}) = \sum_i (-1)^i (c_i(E) c_{k-i}(E)) = 0$. And so product $c(E \otimes \mathbb{C})$ is simply the sum of even Chern classes $c_{2i}(E \otimes \mathbb{C})$. By the definition of Pontrjagin classes this is $1 - p_1 + p_2 - \cdots \pm p_n$ □

Also we have an easy calculation which will be important later

Lemma 4.7. *The Pontrjagin classes of the tangent bundle of a sphere are trivial.*

Proof. The tangent space of the sphere at $\in S^n$ can be described by

$$T_xS = \{v \in \mathbb{R}^{n+1} : \langle x, v\rangle = 0\} \tag{2}$$

and the normal bundle at x as

$$N_xS = \{v \in R^{n+1} : v = \lambda x, \lambda \in \mathbb{R}\} \tag{3}$$

The normal bundle is trivial as we have the nowhere vanishing section (x, x). The Whitney sum of the tangent and normal bundles of a sphere is trivial and so, directly from the axioms for Pontrjagin classes, we have $p(TS^n) = p(TS^n \oplus \epsilon) = p(\epsilon^n) = p(\epsilon) = 1$ where ϵ^n is the n-fold Whitney sum of the trivial bundle. $\square$

The stability property of the Chern and Pontrjagin classes shows how they may "fail" to pick up non-triviality in a vector bundle. One of the simplest examples of this is S^5. The tangent bundle of the sphere will have all trivial Chern and Pontrjagin classes by above and the Euler class will also be zero, as it is for all spheres of odd dimension. The tangent bundle of S^5, however, is not isomorphic to the trivial bundle.

4.5 Characteristic Numbers

By evaluating combinations of characteristic classes of the tangent bundle on the fundamental class of the connected, orientable, closed manifold we can get some derived useful invariants.

Given an n dimensional complex manifold M with fundamental class $[M]$ and integers $i_1, \ldots, i_l$ such that $\sum \deg c_{i_j} = n$, the corresponding Chern number is defined $c_{i_1} \smile c_{i_2} \smile \cdots \smile c_{i_m}([M])$.

Similarly we can define the Steifel-Whitney numbers and, if the the dimension of the manifold is divisable by four, the Pontrjagin numbers. These are topological invariants of the manifold and are particularly important as they are complete invariants of a manifold's oriented bordism class i.e. there is an oriented bordism between two manifolds if and only if they have the same Steifel-Whitney and Pontrjagin numbers[8].

As the Pontrjagin numbers are a topological invariant we have an interesting corollory of a manifold possessing a non-zero Pontrjagin number. Reversing the orientation of manifold has no effect on the Pontrjagin classes (indeed we do not even need an orientation on M to define them), but reverses the sign of the fundamental class and so of a Pontrjagin number. Whence if a Pontrjagin number of M is non-zero there exists no orientation reversing homeomorphism of the manifold to itself.

4.6 Hizerbruch's Signature Theorem

Now we will discuss another topological invariant, defined for manifolds which are orientable, closed and of dimension $4k$, with k an integer.

First define the bilinear pairing

$$\begin{aligned} H^{2k}(M) \times H^{2k}(M) &\to H^{4k}(M) \\ (\alpha, \beta) &\mapsto \alpha \smile \beta \end{aligned}$$

There is explicit isomorphism between $H^{4k}(M)$ and $\mathbb{Z}$ given by evalating an element of the group on a chosen fundamental class $[M] \in H^{4k}(M)$. We can thus view the above pairing as defining a bilinear form from $H^{2k}(M)$ to $\mathbb{Z}$.

The basic property of the cup product

$$\alpha^p \smile \beta^q = (-1)^{pq}(\beta^q \smile \alpha^p)$$

ensures that this bilinear form is symmetric. The signature of the manifold is defined to be the signature of this quadratic form.

The signature of the manifold can be proved to be a topological invariant and, in fact, a bordism invariant. Moveover it is easily seen to be additive upon taking the disjoint union of manifolds and multiplicative upon taking Cartesian products. Thus, the signature defines an ring homomorphism from the oriented bordism ring to $\mathbb{Q}$ or equivalently an algebra homomorphism from the oriented bordism ring Ω tensored with $\mathbb{Q}$. We mentioned earlier that the Pontrjagin numbers and Stiefel Whitney numbers are a complete invariant for oriented

[8]For a basic definition of the oriented cobordism class and ring structure, see the appendix. Details can be found in [13]

bordism classes of manifolds. Tensoring the oriented bordism ring Ω with $\mathbb{Q}$ kills torsion, so it can be seen that the Pontrjagin classes determine completely the algebra homomorphisms $\Omega \otimes \mathbb{Q} \to \mathbb{Q}$. We can conclude that the signature of the manifold must be given by a linear combination of Pontrjagin numbers with coefficients in $\mathbb{Q}$.

The Hirzebruch Signature theorem gives explicit coefficients of this relation and gives the signature of the manifold as an explicit combination of the Pontrjagin numbers of the manifold. Verifiying the theorem is difficult, a proof is given in [6]. We will only need the result for the case of an 8-manifold, which is given explicitly by

$$\sigma(M) = \langle [M], \frac{1}{45}(7p_2(M) - p_1(M)^2 \rangle$$

One notable thing about this formula is that $\sigma(M)$ is necessarily an integer while the right hand side is an integer only for certain combinations of values of p_1 and p_2. This is what Milnor used to prove the existent of exotic differentiable structures on 7-spheres.

5 Applications: Projective Spaces and Complex Structures on Spheres

Having worked hard to gain the very useful tools of characteristic classes, we are now rewarded with a plethora of easily constructed, interesting results on differentiable manifolds. As an initial application we study projective space, manifolds ubiquitous in physics thanks to quantum mechanics, ubiquitous in algebraic geometry due to the very well behaved intersection theory of projective curves and ubiquitous in real life because we look at very far away things.

"Projective geometry is all geometry"

5.1 Cohomology Ring of Projective Space

The Gysin sequence provides an easy way to compute the cohomology of these important spaces

Theorem 5.1. *The cohomology groups of the projective space $\mathbb{C}P^n$ are given by the following.*

$$H^p(\mathbb{C}P^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & p \text{ even}, 0 \le p \le 2n \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Moreover the ring $H^*(\mathbb{C}P^n)$ is generated by the Euler class of the tautological bundle

Proof. Consider the bundle γ_0, the tautological bundle over $\mathbb{C}P^n$ less the zero section. The map $\mathbb{C}^{n+1} \ni v \mapsto (\mathbb{C}v, v)$ gives a homeomorphism between $\mathbb{C}^{n+1} - \{0\}$ and γ_0. Whence $H^i(\gamma_0) \cong \mathbb{Z}$ for $i = 0, 2n+1$ and zero otherwise[9]. Looking at the Gysin sequence

$$\cdots \to H^i(\gamma_0) \to H^i(\mathbb{C}P^n) \xrightarrow{\cup e} H^{i+2}(\mathbb{C}P^n) \to H^{i+1}(\gamma_0) \cdots$$

For $i = -1, -2$ the Gysin sequence gives the isomorphisms $H^0(\mathbb{C}P^n) \cong H^0(\gamma_0) \cong \mathbb{Z}$ and $H^1(\mathbb{C}P^n) \cong H^1(\gamma_0) = 0$. For all $0 \le i \le 2n$ cupping with the Euler class of the tautological bundle gives an isomorphism $H^i(\mathbb{C}P^n) \xrightarrow{\sim} H^{i+2}(\mathbb{C}P^n)$ and clearly all groups $H^i(\mathbb{C}P^n)$ are zero for $i \ge 2n$. Thus we have a complete description of the cohomology ring

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[e] \mod e^{n+1} \tag{5}$$

[9] Recall that the cohomology of the sphere is given by $H^0(S^n) \cong H^n(S^n) \cong \mathbb{Z}$ and $H^k(S^n) = 0$, $k \ne 0, n$, as can be found in any introduction to algebraic topology e.g. [3]. It is clear that $\mathbb{C}^{n+1} - \{0\}$ is homotopic to a $2n+1$ sphere and so these have identical cohomology groups

as required ☐

A completely analogous construction with the Gysin sequence of the tautological bundle over $\mathbb{R}P^n$, where now the "unoriented" Euler class lives in $H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ gives the result

Theorem 5.2. *The cohomology groups with $\mathbb{Z}_2$ coefficients of the projective space $\mathbb{R}$ are given by the following.*

$$H^p(\mathbb{R}P^n;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & 0 \leq p \leq n \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Moreover, if a generates $H^1(\mathbb{R}P^n;\mathbb{Z}_2)$ then a^j generates $H^j(\mathbb{R}P^n;\mathbb{Z}_2)$ for all $j \leq n$

So now we know something about the topology of these important spaces.

5.2 Immersions

The Whitney Embedding theorem was one of the first and remains one of the strongest and most famous results in immersion theory. Whitney proved that every n-dimnsional manifold can be immersed in $\mathbb{R}^m$ as long as $m \geq 2n-1$. In fact this sharp i.e. this is the lowest bound on m for which the result remains true for all manifolds, which can be seen by examining the immersions of real projective space. In this section we will get an obstruction to general immersions of a manifold given by the Steifel-Whitney classes of tangent bundle and prove the following

Theorem 5.3. *Let $n = 2^r, r \geq 1$. Then $\mathbb{R}P^n$ can be immersed in $\mathbb{R}^m$ if and only if $m \geq 2n-1$*

5.2.1 The Inverse Steifel-Whitney Classes

Let B be a space. We will construct a group stucture on a subring of the graded ring $H^*(B;\mathbb{Z}_2)$. Consider the set of all series

$$w = 1 + w_1 + w_2 + \cdots \in H^*(B;\mathbb{Z}_2) \tag{7}$$

which have leading term $1 \in H^0(B;\mathbb{Z}_2)$ and where $w_i \in H^i(B;\mathbb{Z}_2)$. We have the following

Lemma 5.4. *Every element $w \in H^*(B;\mathbb{Z}_2)$ with leading term 1 has a multiplicative inverse*

Proof. If $w = 1 + w_1 + w_2 + \cdots$ define w^{-1} by

$$w^{-1} = 1 + \sum_{n\geq 1}(w_1 + w_2 + \cdots)^n \tag{8}$$

We have

$$\begin{aligned}(w)(w^{-1}) &= (1 + (w_1 + w_2 + \cdots))(1 + \sum_n (w_1 + w_2 + \cdots)^n) \\ &= 1(1 + \sum_n (w_1 + w_2 + \cdots)^n) + (w_1 + w_2 + \cdots)(1 + \sum_n (w_1 + w_2 + \cdots)^n) \\ &= 1 + \sum_n (w_1 + w_2 + \cdots)^n + \sum_n (w_1 + w_2 + \cdots)^n = 1\end{aligned}$$

as all elements are of order 2. $\square$

5.2.2 Obstructions to Immersions

We can now show the non-existence of an potential immersion of a manifold into $\mathbb{R}^{n+k}$ by using the Steifel-Whitney classes and their inverses

Theorem 5.5. *The Whitney Duality Theorem Let M be an n-dimensional manifold which is immersed in $\mathbb{R}^{n+k}$. Then the Steifel-Whitney classes satisfy*

$$w(\nu) = w^{-1}(\tau) \tag{9}$$

where τ is the tangent bundle of the embedded M and ν the normal bundle

Proof. If M is immersed in $\mathbb{R}^{n+k}$ then the Whitney sum of the tangent bundle of M and the normal bundle of M is trivial.

Whence

$$w(\nu \oplus \tau) = 1 \Rightarrow w(\nu) = w^{-1}(\tau)$$

$\square$

This has the immediate corollary

Corollary 5.6. *Denote by $w_i^{-1}(\tau)$ the i^{th} homogeneous element*[10] *of $w^{-1}(\tau)$ Suppose $w_i^{-1}(\tau) \neq 0$ for some $i > k$. Then M cannot be immersed in $\mathbb{R}^{n+k}$*

[10] i.e. $w_i^{-1}(\tau) \in H^i(B;\mathbb{Z}_2)$

Proof. By the Whitney duality theorem $w_i(\nu) = w_i^{-1}(\tau)$, where ν is the normal bundle of the embedded M. But $w_i(\nu) = 0$ for all $i > k$. $\square$

5.2.3 Calculation of the Steifel-Whitney Classes

Now we need to find the Stiefel Whitney classes of the tangent bundle of $\mathbb{R}P^n$.

Lemma 5.7. *Let τ be the tangent bundle of $\mathbb{R}P^n$ and ϵ the trivial bundle. the Whitney sum $\tau \oplus \epsilon$ is the (n+1)-fold Whitney sum $\gamma \oplus \cdots \oplus \gamma$ of the tautological bundle of $\mathbb{R}P^n$*

Proof. We know that the direct sum of the tangent bundle of S^n with the normal bundle is isomorphic to the trivial bundle $S^n \times \mathbb{R}^n$. Quotient the bundle $T(S^n) \oplus N(S^n)$ by the relation $(x, v) \sim (-x, -v)$. The image of the tangent bundle of the sphere under this quotient is easily seen to give the tangent bundle of $\mathbb{R}P^n$. Nowhere vanishing section of the normal bundle, which consists of elements $\{(x, tx)|x \in S^n, t \in \mathbb{R}\}$ are quotiented by $(x, tx) \sim (-x, -x(t))$ and so give nowhere vanishing sections of the image given by $(\{\pm x\}, t)$. Whence, image of the normal bundle is the trivial bundle.

Now let us look at the image of $S^n \times \mathbb{R}^n$ under the above quotient. The above quotient of the bundle sends pairs $(x, t), (-x, -t)$ to the same equivalence class $[(x, t)]$. The trivial line bundle $S^n \times \mathbb{R}$ is isomorphic to the normal bundle under the isomorphism $(x, t) \mapsto (x, tx)$. But under the isomorphism from the trivial bundle to the normal bundle we have that $(-x, -t) \mapsto (-x, (-t)(-x)) = (-x, tx)$. Whence pairs of the form $(x, t), (-x, -t)$ are mapped to equivalence classes $(\{\pm x\}, tx)$. This is clearly isomorphic to the tautological bundle over $\mathbb{R}P^n$ and so we have our result.

$\square$

this has the immediate corollary

Corollary 5.8. *The total Steifel-Whitney class of the tangent bundle of $\mathbb{R}P^n$ is given by $w(\mathbb{R}P^n) = (1 + a)^{n+1}$ where a is the generator of $H^1(\mathbb{R}P^n)$*

Proof. As ϵ is the trivial bundle we have $w(\tau) = w(\tau \oplus \epsilon) = w(\gamma \oplus \cdots \oplus \gamma) = w(\gamma)w(\gamma) \cdots w(\gamma) = (1 + a)^{n+1}$ $\square$

Taking the binomial expansion for this series we find that

$$w_i(\mathbb{R}P^n) = \left(\binom{n+1}{i} \mod 2\right) a^i$$

Where as before a is the generating element of $H^1(\mathbb{R}P^n;\mathbb{Z}_2)$. The total Steifel-Whitney class of the tangent bundle of projective space is then given $w(\mathbb{R}P^n) = 1 + \sum_{i=1}^n w_i(\mathbb{R}P^n)$

Theorem 5.9. *Let $n = 2^r, r \geq 1$. Then $\mathbb{R}P^n$ can be immersed in $\mathbb{R}^m$ only if $m \geq 2n-1$*

Proof. Looking at the binomial coefficent $\binom{2^r+1}{i}$ mod 2 we can see this is non-zero for precisely $i = 1, 2^r$ and so the total Steifel-Whitney class of the tautological bundle is given by

$$w = 1 + a + a^n \tag{10}$$

As all cohomology groups $H^i(\mathbb{R}P^n)$ are zero for $i > n$ we have that $(a+a^n)^i = a^i$ for all $i \leq n$. Thus the total inverse Steifel-Whitney class is given by $w^{-1}(\mathbb{R}P^n) = 1 + a + a^2 \cdots a^n$ and by the Whitney Duality Theorem we have that $\mathbb{R}P^n$ cannot be embedded in $\mathbb{R}^m$ unless $m \geq 2n-1$. □

As mentioned before, this example shows that the Whitney theorem for immersions gives a sharp bound.

In constrast, now let $n = 2^r - 1$. We find that these $\mathbb{R}P^n$ are the only projective spaces with a chance of being "simple" in the following sense

Lemma 5.10. *$\mathbb{R}P^n$ has non-trivial Steifel-Whitney class if and only if $n+1$ is a power of 2. Hence $\mathbb{R}P^n$ cannot be parallelizable unless $n = 2^r - 1$.*

Proof. If $n = 2^r - 1$, so $n+1$ is a power of two then the coefficent of the Steifel-Whitney class $\binom{2^r}{i}$ mod 2 is always zero. So for $n = 2^r - 1$, $w(\mathbb{R}P^n) = 1$.

Conversely let $n + 1 = k2^r$ for k odd. Then $w(\mathbb{R}P^n)$ cannot be trivial. As the Steifel-Whitney classes have mod 2 coefficents we find

$$w(\mathbb{R}P^n) = (1+a)^{k2^r} = (1+a^{2^r})^k \tag{11}$$

But then $\binom{k}{1}$ mod $2 \neq 0$ and the Steifel-Whitney class is not trivial. The tangent bundle of $\mathbb{R}P^n$ cannot be trivial if it possesses a non-trivial characteristic class, so $\mathbb{R}P^n$ cannot be parallelizable unless $n+1$ is a power of two as required. □

Comment: Given a bilinear product operation

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \tag{12}$$

it is not difficult to prove (details in e.g. [13]) that such a product induces $n-1$ linearly independent nowhere vanishing sections of the tangent bundle of real projective $n-1$-space if such a product operation has no zero divisors. The above lemma then implies that bilinear product for $\mathbb{R}^n$ with no zero-divisors can only occur when n is a power of two. The cases $n = 1, 2, 4, 8$ of course refer to the division algebras of the reals, the complex numbers, the quaternions and the non-associative division algebra of the octonions respectively. There are no others, which can be proved using characteristic classes as a corollary of the Bott Periodicity theorem[1].

5.3 Obstructions to the existence of complex structures

As a final easy but impressive application of characteristic classes before moving on to the more laborious task of proving the existence of exotic differentiable structures on spheres, let us prove the following

Theorem 5.11. *The sphere S^{4k} cannot be a complex manifold*

First we need to calculate the Euler class of the tangent bundle of a sphere

Lemma 5.12. *The Euler class of an even dimensional sphere is twice a generator of the top homology group*

Proof. Consider the set $S^n \times S^n \setminus A$ where A is the anti-diagonal subset of the product space consisting of antipodal pairs $(x, -x)$. This is isomorphic to the tangent bundle of the sphere, with an explicit isomorphism given by sending the pair (x, y) to the element of the tangent plane of x which intersects the line through $(-x, y)$. All pairs (x, x) in the diagonal subset D are sent to the zero section of the tangent bundle, so we know that $S^n \times S^n \setminus (A, D)$ is isomorphic to the tangent bundle of the sphere less the zero section. Therefore, by excision we have that

$$\mathrm{H}^*(E, E_0) \cong H^*(S^n \times S^n \setminus A, S^n \times S^n \setminus (A, D)) \cong H^*(S^n \times S^n, S^n \times S^n \setminus D)$$

Now, $S^n \times S^n \setminus D$ homotopy equivalent to the subspace D. Indeed given a function

$$\begin{aligned} f : S^n \times S^n \setminus D &\to D \\ (x, y) &\mapsto (x, x) \end{aligned}$$

and a function

$$h : D \to S^n \times S^n \setminus D$$
$$(x, x) \mapsto (x, gx)$$

for some element of $g \in O(n)$ not equal to the identity, every set $(x_0, y) \subset S^n \times S^n \setminus D$ is a sphere minus a single point and so contractible to a point. Thus we can get a retraction of $S^n \times S^n \setminus D$ onto (x, gx) and so $S^n \times S^n \setminus D$ is homotopic to the subspace D. We have the isomorphisms

$$H^*(E, E_0) \cong H^*(S^n \times S^n, D)$$

From the long exact sequence of the cohmology of the pair $H^*(S^n \times S^n, D)$ we get the short exact sequence

$$0 \to H^n(S^n \times S^n, D) \to H^n(S^n \times S^n) \to H^n(D) \to 0$$

as $D \cong S^n$ implies $H^{n-1}(D) = 0$. A generator of the kernel of the map $H^n(S^n \times S^n) \to H^n(D)$ is of the form $\alpha - \beta$ where α, β are the pull backs of forms generating $H^n(S^n)$ under the canonical projection function $S^n \times S^n \to S^n$ chosen so that these restrict to the same generator of $H^n(D)$. Therefore $\alpha - \beta$ generates the cohomology group $H^n(S^n \times S^n, D)$ and corresponds to the Thom class.

Under the Thom isomorphism the Euler class is sent to the square of the Thom class. Now as α,β are the pull backs of the generating elements of the S^n we have that $\alpha^2 = \beta^2 = 0$ and so $(\alpha - \beta)^2 = -\beta\alpha - \alpha\beta$. If the dimension of the sphere is even we have that the cup product is symmetric and $(\alpha - \beta)^2 = -2\beta\alpha$ corressponds to twice a generating element of $H^n(E, E_0)$. Therefore $e(E)$ must be twice a generating element of $H^n(S^n)$. □

Now we prove that S^{4k} cannot have a complex structure.

Proof. For S^{4k} we know that $e \in H^{4k}(S^{4k})$ is twice a generating element of $H^{4k}(S^{4k})$. Whence if S^{4k} is a complex manifold we have for the top Chern class $c_{2k}(TS^{4k}) = e$.

Now by the definition of Pontrjagin classes we have

$$p_k(TS^{4k}) = \pm c_{2k}(TS^{4k} \otimes \mathbb{C}) = \pm c_{2k}(TS^{4k} \oplus \overline{TS^{4k}}) \tag{13}$$

But then by the Whitney sum formula and Lemma 4.7 we find that $p_k(TS^{4k}) = \pm c_{2k}(TS^{4k} \otimes \mathbb{C}) = \pm 2e \neq 0$ which is a contradiction, as the Pontrjagin classes for the tangent bundle of a sphere are trivial.

$\square$

Another corollory of Bott Periodicity further restricts the possible dimension of spheres which can have Chern classes defined on their tangent bundles. This can be used to prove that the only spheres which can possibly have a complex structure are S^2 and S^6. S^2 is of course the Riemann sphere. Whether or not S^6 possesses a complex structure remains an open question.

6 Construction of Milnor Spheres

6.1 The Structure of the Quaternions

This background material on quaternions and quaternionic projective space is taken predominantly from [14]. The material on exotic sphere is taken mostly from Milnor's original paper [9] and [17]

6.1.1 The Division Algebra $\mathbb{H}$

We will review some well known facts of the quaternions which will be used in the coming chapter.

The quaternions are a four dimensional vector space over $\mathbb{R}$ equipped with the usual sum and scalar multiplication such that the basis elements $1, i, j, k$ obey the relations $i^2 = j^2 = k^2 = ijk = -1$. Quaternion multiplication is then well defined by using the distributive law, giving $\mathbb{H}$ the structure of an algebra over $\mathbb{R}$. In fact this makes the quaternions a (non commutative) division algebra.

Given a quaternion $q = a + bi + cj + dk$ we define

- the conjugate q^* of q by $q^* = a - bi - cj - dk$
- the norm $\|q\|$ of q by $\|q\| = \sqrt{qq^*} = \sqrt{q^*q} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

The inverse of a quaternion is then given by $q^{-1} = \frac{q^*}{\|q\|^2}$.

Express $q = a + bi + cj + dk = a + \mathbf{v}$. The exponential of q is given by

$$exp(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!} = e^a \left(\cos \|\mathbf{v}\| + \frac{\mathbf{v}}{\|\mathbf{v}\|} \sin \|\mathbf{v}\| \right)$$

Analagous to the complex case we now have the polar form of q

$$q = \|q\| e^{\hat{n}\theta} = \|q\| \left(\cos(\theta) + \hat{n} \sin(\theta) \right)$$

exponentiating then becomes

$$q^\alpha = \|q\|^\alpha e^{\hat{n}\alpha\theta} = \|q\|^\alpha \left(\cos(\alpha\theta) + \hat{n} \sin(\alpha\theta) \right)$$

6.1.2 Quaternions and Rotations

The space of pure quaternions $\{h = a + bi + cj + dk \in \mathbb{H} | a = 0\}$ is identified with $\mathbb{R}^3$. The set of quaternions of norm one $q = \cos(\theta) + \hat{n}\sin(\theta)$ is identified naturally with $S^3 = \left\{x \in \mathbb{R}^4 : \|x\| = 1\right\}$. The operation of conjugation of a pure quaternion by a quaternion of norm one $h \to qhq^{-1}$ gives a rotation of angle 2θ about the axis $\hat{n} \in \mathbb{R}^3$. So S^3 identified with the unit quaternions provides a double cover of $SO(3)$ - the set of rotations of $\mathbb{R}^3$ - with q and $-q$ representing the same rotation.

We now wish to see how the unit quaternions act on $\mathbb{R}^4$. Consider an arbitrary rotation ϕ in $\mathbb{R}^4 \cong \mathbb{H}$. If $\phi(1) = q$ then $q^{-1}\phi$ is a rotation fixing 1 and so is a rotation of the pure quaternions. But then, as above, this rotation is given by $h \to \tilde{q}h\tilde{q}^{-1}$ for some unit quaternion $\tilde{q}$. The rotation ϕ is thus given by $h \to q\tilde{q}h\tilde{q}^{-1}$ and so there is an isomorphism $SO(4) \cong S^3 \times SO(3)$.

6.2 The tautological bundle on Quaternionic Projective Space

Quaternionic projective space is defined as the set of (quaternionic) one dimensional subspaces of $\mathbb{H} \times \mathbb{H}$. In other words it is the quotient space of $\mathbb{H} \times \mathbb{H}$ under the equivalence relation given by $(u_0, \cdots, u_n) \sim (\lambda u_0, \cdots, \lambda u_n)$. The tautological vector bundle over $\mathbb{H}P^1$ is defined

$$\gamma_{\mathbb{H}} = \{(x, v) \in \mathbb{H}P^1 \times \mathbb{H}^2 \text{ such that } [v] = x\}$$

The sphere bundle associated to the tautological vector bundle is in fact the 7-sphere. The fibers of the tautological vector bundle are quaternionic lines through the origin where the fibre over $[u_0 : u_1]$ is exactly the line $(q, \frac{u_1}{u_0}q)$ (as with the Hopf fibration we have a "point at infinity" at $[0 : u]$ and a line of "infinite slope"). Such a line will intersect the 7-sphere to give a set $\{(q, r) \in \mathbb{S}^7\ \frac{q}{r} = \frac{u_1}{u_0}\}$, which are exactly the elements of the tautological bundle of norm 1, so we get exactly the Hopf fibration of the 7-sphere discussed earlier.

Corollary 6.1. *The Euler class of the tautological vector bundle is a generator of $H^4(\mathbb{H}P^1)$. The total Chern class of the tautological bundle is $c[\gamma_{\mathbb{H}}] = 1 + e$. The total Pontrajagin class is $p[\gamma_{\mathbb{H}}] = 1 - 2e$*

Proof. Looking at the following portion of the Gysin sequence of the associated sphere bundle S_γ

$$H^3(S_\gamma) \to H^0(\mathbb{H}P^1) \xrightarrow{\cup e} H^4(\mathbb{H}P^1) \to H^4(S_\gamma) \cdots$$

The sphere bundle is homeomorphic to the 7-sphere so $H^k(S_\gamma) \cong 0$ for $k \neq 0, 1$. The base space is homeomorphic to the 4-sphere, with cohomology $H^0(\mathbb{H}P^1) \cong H^4(\mathbb{H}P^1) \cong \mathbb{Z}$. By the exactness of the Gysin sequence, then, cupping with the Euler class is an isomorphism sending $1 \in H^0(\mathbb{H}P^1)$ to a generating element of $H^4(\mathbb{H}P^1)$.

The first Chern class of the tautological bundle $c_1 \in H^2(\mathbb{H}P^1)$ is zero as $\mathbb{H}P^1$ has the cohomology of the 4-sphere. Since $\gamma_{\mathbb{H}}$ is a also a complex vector bundle of dimension 2 we have by Lemma 4.7

$$1 - p_1 + p_2 = (1 + c_2)(1 + c_2) = (1 + e)(1 + e) = 1 + 2e + e^2$$

but e^2 is zero as $H^8(\mathbb{H}P^1)$ is trivial.

6.3 Rank-4 real vector bundles over S^4

The discussion on quaternions and rotations gives a way to explicitly construct vector bundles over the 4-sphere. From Chapter 1, categorising rank-4 real vector bundles over S^4 is equivalent to classifying the homotopy classes of maps $S^3 \to SO(4)$, so first we catergorise these.

By above there is an isomorphism $SO(4) \cong S^3 \times SO(3)$. It is a well known fact that two maps from a connected, oriented manifold without boundary to a sphere are homotopic if and only if the degree of the mappings are equal (see [12], pg. 51). So classes of maps $\pi_3(S^3) = \mathbb{Z}$ are given by covering maps $f : S^3 \to S^3$ classified by $deg(f)$. Viewing S^3 as the space of unit quaternions and recalling the formula $q^\alpha = \|q\|^\alpha e^{\hat{n}\alpha\theta} = \|q\|^\alpha (\cos(\alpha\theta) + \hat{n}\sin(\alpha\theta))$ it is clear the map $q \to q^n$ is a covering map of degree n. So the composition

$$\begin{aligned} S^3 \to S^3 \times S^3 &\to S^3 \times SO(3) \to SO(4) \\ u \mapsto (u^a, u^b) &\mapsto (u^a, u^b q u^{-b}) \mapsto (q \mapsto u^{a+b} q u^{-b}) \end{aligned}$$

Defining the homomorphism

$$\begin{aligned} \phi_{hj} : S^3 &\to SO(4) \\ u &\mapsto (q \mapsto u^h q u^j) \end{aligned}$$

there is a group isomorphism

$$\begin{aligned} \mathbb{Z} \oplus \mathbb{Z} &\to \pi_3(SO(4)) \\ (h, j) &\to \phi_{hj} \end{aligned}$$

And so there is an isomorphism class of vector bundles over S^4, denoted E_{hj}, for each homotopy class of clutching functions ϕ_{hj}.

To construct these bundles explicitly we see that $\mathbb{H}P^1$ is diffeomorphic to the standard 4-sphere by the map

$$[u_0 : u_1] \mapsto \left(\frac{2\bar{u_1} u_0}{\|u_0\|^2 + \|u_1\|^2}, \frac{\|u_0\|^2 - \|u_1\|^2}{\|u_0\|^2 + \|u_1\|^2} \right)$$

The charts $U_i = \{[u_0 : u_1] \,|\, u_i \neq 0\}$ are both isomorphic to $\mathbb{H}$ with an explicit isomorphism given by $U_0 \ni [u_0 : u_1] \mapsto ({}^{u_1}/_{u_0})$, so a vector bundle over $\mathbb{H}P^1$ will be trivial over these charts. Form the vector bundle E_{hj} by taking trivial vector bundles (here, a real vector bundle of rank four or a quaternionic line bundle) over $\mathbb{H}$ and identifying

$$([1 : u], v) \sim \left([u^{-1} : 1], \frac{u^h v u^j}{\|u\|^{h+j}} \right) \tag{14}$$

and the homotopy class of the restriction of the map $v \mapsto \phi(v)$ to the equator $\|u\| = \|u\|^{-1} = 1$ gives an element of the homotopy group $\pi_3(SO(4))$.

□

Lemma 6.2. *The tautological bundle over $\mathbb{H}P^1$ is isomorphic to the vector bundle E_{01}.*

Proof. Vector bundles $E, \tilde{E}$ with transition functions $g_{ij}, \tilde{g}_{ij}$ are isomorphic if and only if we can find continuous mappings $\lambda_i, \lambda_j : V_i \cap V_j \to SO(4)$ so that the transition functions satisfy

$$\tilde{g}_{ij} = \lambda_i^{-1} g_{ij} \lambda_j$$

To get the transition functions of the tautological bundle over $\mathbb{H}P^1$, note that we have the following trivilisations of the bundle

$$h_0 : ([1 : t_0], (q, qt_0)) \to (t_0, q)$$
$$h_1 : ([t_1 : 1], (rt_1, r)) \to (t_1, r)$$

on U_0 and U_1 respectively. The transition functions from the chart U_0 to the chart U_1 is then given by

$$h_1 \circ h_0^{-1}(t_0, q) = h_1([1 : t_0], (q, qt_0)) = h_1([t_0^{-1} : 1], ((qt_0)t_0^{-1}, qt_0)) = (t_0^{-1}, qt_0)$$

so $g_{01}([1 : t_0])(q) = qt_0$. Similarly $g_{10}([1 : t_0])(q) = qt_0^{-1}$;

The transition functions of the bundle E_{01} is given by $\tilde{g}_{10}(u)(v) = \frac{vu}{\|u\|}$ which is just a continuous "resizing" of fibers. The functions $\lambda_1(u) = \|u\|$ and $\lambda_0(u) = 1$ satisfy $\tilde{g}_{10}(u) = \lambda_1(u)^{-1} g_{10}(u)\lambda_0(u)$ and $\tilde{g}_{01}(u) = \lambda_0(u)^{-1} g_{01}(u)\lambda_1(u)$ as required

□

6.4 Characteristic Classes of E_{hj}

Now the Euler and Pontrjagin classes of the bundle E_{hj} are calculated. These will be expressed using the Euler and Pontrjagin classes of the tautological bundle on $\mathbb{H}P^1$.

There is a continuous map from S^n to the wedge sum[11] $S^n \vee S^n$ collapsing the equator to a point. The homology and cohomology groups are given by $H_n(S^n \vee S^n) = H^n(S^n \vee S^n) \cong \mathbb{Z} \oplus \mathbb{Z}$. Let $\alpha_1 \in H^n(S^n \vee S^n)$ send the generator of the homology of the "first" sphere σ_1 to 1 and α_2 send the generator of the second sphere σ_2 to 1, i.e. $\alpha_i(\sigma_j) = \delta_{ij}$. The generator of $H_n(S^n)$ gets sent to the sum of generators $\sigma_1 + \sigma_2$ of $H^n(S^n \vee S^n)$ so that the pull back of $\alpha_1 + \alpha_2$ on the generator of $H_n(S^n)$ gives $f^*(h\alpha_1 + j\alpha_2)(\sigma) = (h\alpha_1 + j\alpha_2)(\sigma_1 + \sigma_2) = h\alpha_1(\sigma_1) + j\alpha_2(\sigma_2)$. Whence the induced pull back on the cohomology group sends the cohomology class corresponding to (h, j) to $h + j$. If h is a degree h mapping from the sphere to itself preserving the common point of the spheres then pulling back the generator α_2 by

[11] The wedge sum of spheres is quotient of the disjoint union of spheres obtained by identfying a single chosen point. See [3]

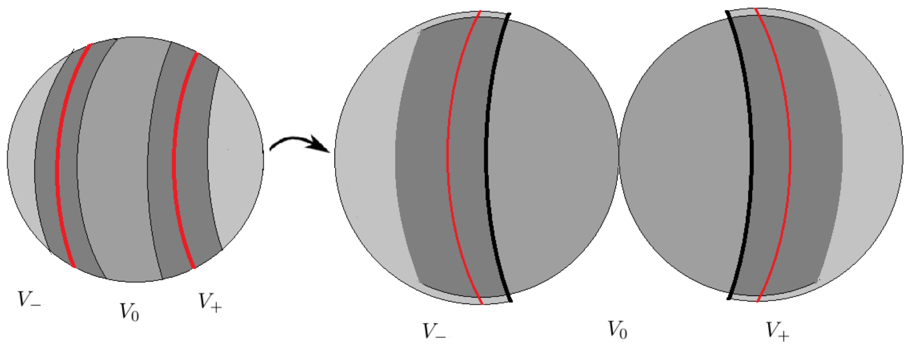

Figure 4: a mapping of sphere covered by three open sets to the wedge product of spheres

$$S^n \vee S^n \to S^n \vee S^n$$
$$(q, r) \to (h(q), r)$$

gives $h\alpha_1$. Likewise pulling back α_1 under the map $(q, r) \to (q, j(r))$ gives $j\alpha_2$.

Now consider the vector bundle on $S_4 \vee S_4$ with charts V_-, V_0, V_1 (shown below) which has clutching functions $u \to uv$ and $u \to vu$ on the equators of the first and second sphere. The pull back of this bundle under the mapping

$$S^4 \to S^4 \vee S^4 \to S^4 \vee S^4$$
$$h \to (q, r) \to (h(q), j(r))$$

This bundle is the pull back of such a bundle is equivalent to one given by extending the open sets V_- and V_+ and composing the tranisiton functions on the overlap. The homotopy class of the map from the equator to $SO(4)$ of this bundle is evidently $v \to u^h v u^j$. Consider the tautological bundle $\gamma_{\mathbb{H}} \cong E_{01}$ which we recall has the clutching function $v \to vu$. The conjugate of the tautolgical bundle is taken by conjugating each of the fibers. This is isomorphic to the bundle which has the conjugate clutching function $v \to u^{-1}v$[12]. Taking

[12] recall $q^{-1} = \frac{q^*}{\|q\|^2}$

the conjugate of each fiber reverses the underlying real orientation of the bundle, so $e(E_{-10}) = -e(E_{01})$. As reversing orientation preserves the Pontrjagin classes we find $p(E_{-10}) = p(E_{01})$. But since E_{hj} is given by then by the induced map on cohomology discussed above we have $e(E_{hj}) = e(-hE_{-10} + jE_{01}) = (h+j)e$ where e is the Euler class of the tautological bundle. Similarly $p_1(E_{hj}) = p_1(-hE_{-10}+jE_{01}) = (j-h)p_1$ where p_1 is the Pontrjagin class of the tautological bundle. As $p_1 = 2e$ we have $p_1(E_{hj}) = 2(j-h)e$

6.5 Milnor's Spheres

Taking the set of vectors of norm ≤ 1 in the previously constructed E_{hj} gives the associated disk bundle D_{hj}. The boundary of the fiber bundle is the associated sphere bundle B_{hj} consisting of vectors of norm one. The cohomology of B_{hj} can be deduced using the Gysin sequence and the cohomology of spheres. Looking at the portion

$$\cdots H^{i-4}(S_4) \xrightarrow{\cup e} H^i(S_4) \longrightarrow H^i(B_{hj}) \longrightarrow H^{i-4+1}(S_4) \cdots$$

there is an isomorphism and using $H^k(S^4) \cong 0,\ k \neq 0, 4$ we find $H^i(S_4) \cong H^i(B_{hj})$ for $i = 0, 1, 2, 5, 6$. For $i = 7$ the sequence gives an isomorphism between $H^7(B_{hj}) \cong H^4(S^4) \cong \mathbb{Z}$. As B_{hj} is of dimension seven the cohomology groups $H^i(B_{hj})$ are zero $i \geq 8$. For the cases $i = 3, 4$ the sequence

$$0 \longrightarrow H^3(B_{hj}) \longrightarrow H^0(S^4) \xrightarrow{\cup e} H^4(S_4) \longrightarrow H^4(B_{hj}) \longrightarrow 0$$

is exact so we find $H^4(B_{hj}) \cong \mathbb{Z}/(h+j)\mathbb{Z}$. If the Euler class is trivial (i.e. $h + j = 0$) then $H^3(B_{hj}) \cong \mathbb{Z}$. Otherwise $H^3(B_{hj}) = 0$. So for $h + j = 1$ the manifold B_{hj} has the same cohomology as the sphere S^7. We will now prove that each of these bundles is, in fact, homeomorphic to a 7-sphere.

The following is a well-known important result in Morse theory, known as Reeb's theorem

Theorem 6.3. *Let M be a n-dimensional closed, oriented, smooth manifold which admits a smooth function having exactly two critical points, both of which are non-degenerate. Then there exists a homeomorphism from M to S^n. Moreover, this homeomorphism is a diffeomorphism except at (possibly) a single point.*

Proof. **Sketch** Using the implicit function theorem we can construct a closed neighbourhood of both critical points which is homeomorphic to the n dimensional closed disk and these can be extended so both disks cover the manifold and intersect along their boundary. The manifold is then clearly homeomorphic to the n-sphere. Details can be found in any book on Morse theory, for example [11] pg. 25 $\square$

Lemma 6.4. *The sphere bundles B_{hj} constructed earlier are homeomorphic to the 7-sphere whenever $h + j = 1$.*

Proof. Let B_{hj}, $h+j = 1$ be the sphere bundle described about. Let $(u, v), u, v \in \mathbb{H}$ be coordinates of the sphere bundle over a trivialising neighbourhood and define a function

$$f : \mathbb{H} \times \mathbb{H} \to \mathbb{R} \tag{15}$$

$$(u, v) \mapsto \frac{Re(v)}{(1 + \|u\|^2)^{1/2}} \tag{16}$$

This has critical points only at $(0, 1)$ and $(0, -1)$.

In the coordinates of the second chart $\tilde{u} = u^{-1}$, $\tilde{v} = \frac{(u^h v u^{1-h})}{\|u\|}$ such a function is given by

$$f(\tilde{u}, \tilde{v}) \mapsto \frac{Re(\tilde{u}\tilde{v})}{(1 + \|\tilde{u}\tilde{v}\|^2)^{1/2}} \tag{17}$$

This is smooth on the second chart and has no critical points. Therefore by Reeb's theorem the manifold that we have created is homeomorphic to the 7-sphere. $\square$

6.6 Milnor's λ invariant

We have a collection of disk bundles D_{hj} with boundaries B_{hj} homeomorphic to S^7 for $h + j = 1$ (to be proven later). Following the discussion in the introduction, we can attempt to attach an 8-cell to D_{hj} along B_{hj} in such a way as to get a smooth, closed, orientable 8-manifold M_{hj}. This is possible when the boundary is the differentiable 7-sphere but for certain combinations of h, j satisfying $h+j = 1$, despite B_{hj} being homeomorphic to the 7-sphere, constructing such an 8-manifold is impossible. Whence this bundle cannot diffeomorphic to the standard 7-sphere.

To achieve this we relate the characteristic classes and signature of the hypothetical M_{hj} to the characteristic classes and signature of the components of the glued manifold. They will be expressed in terms of an invariant λ of the boundary manfolds B_{hj} although defined in terms of the manifolds with boundary D_{hj}, turns out to depend only on the B_{hj}.

Definition. *Let B be a closed, oriented 7-manifold which is the boundary of a smooth oriented 8-manifold. Fix an orientaion μ of B and let ν be a generating element of $H^n(M,B)$ which induces the orientation μ on B. Define the relative signature of the pair (M,B) as the signature $\sigma(M,B)$ of the bilinear form*

$$\begin{aligned} H^4(M,B;\mathbb{R}) \times H^4(M,B;\mathbb{R}) &\to \mathbb{R} \\ (\alpha,\beta) &\mapsto \langle \nu, \alpha \cup \beta \rangle \end{aligned}$$

where $\alpha \cup \beta$ is the relative cup product.

We assume from now on that B is a closed, oriented 7-manifold which is the boundary of a smooth oriented 8-manifold, denoted M, and that $H^3(B)$ and $H^4(B)$ vanish (as is the case for our topological 7-spheres). By the long exact sequence for relative cohomology, $\imath : H^4(B,M) \tilde{\to} H^4(B)$ is an isomorphism. This allows us to define the following

Definition. *Define the relative Pontrjagin number of the pair (M,B)*

$$q(M,B) = \langle \nu, i^{-1}(p_1(B))^2 \rangle$$

Where i is the isomorphism given in the exact sequence for relative cohomology

Now we assume that we have two oriented smooth manifolds M_1 and M_2 both with boundary B. We form a closed oriented manifold C by gluing along the common boundary. Note that the orientation on C inducs opposite orientations of the pair of 'B's bounding the M_1 and M_2. The signature $\sigma(C)$ and Pontrjagin numbers of the glued manifold can be expressed as a combination of the relative signature and relative Pontrjagin number of the components.

Lemma 6.5. *Let M_1, M_2 be manifolds bounded by B and form a smooth, closed, oriented manifold $C = M_1 \cup M_2$ by gluing along the common boundary. Then we have*

$$\begin{aligned} \sigma(C) &= \sigma(M_1,B) - \sigma(M_2,B) \\ q(C) &= q(M_1,B) - q(M_2,B) \end{aligned}$$

where $\sigma(C)$ is the signature of the glued manifold and $q(C)$ is the Pontrjagin number $q(C) = \langle \nu, p_1(C)^2 \rangle$

Proof. Choose an orientation ν of C which induces the orientation ν_1 on M_1, where ν_i induces the chosen orientation μ on the boundary B. Then ν induces the orientation $-\nu_2$ on M_2.

The Mayer Vietoris sequence for relative cohomology gives an isomorphism $H^n(C,B) \cong H^n(M_1,B) \oplus H^n(M_2,B)$ for all n. As $H^3(B)$ and $H^4(B)$ vanish the long exact sequence of relative cohomology gives an isomorphism $H^4(C,B) \cong H^4(C)$. Therefore a cohomology class in $H^4(C)$ is the image of classes $(\alpha_1, \alpha_2) \in H^n(M_1,B) \oplus H^n(M_2,B)$.

$$\begin{tikzcd}
H^n(C,B) \arrow[r, "h"] \arrow[d, "j"] & H^n(M_1,B) \oplus H^n(M_2,B) \arrow[d, "i_1 \oplus i_2"] \\
H^n(C) \arrow[r, "k"] & H^n(M_1) \oplus H^n(M_2)
\end{tikzcd}$$

Likewise for $n = 4$ we have the dual isomorphisms on homology $H_4(C) \cong H_4(C,B)$ and $H_4(M_1,B) \oplus H_4(M_2,B) \cong H_4(C,B)$ In particular for the signature of C we have

$$\langle \nu, \alpha^2 \rangle = \langle \nu, j(h^{-1}(\alpha_1 \oplus \alpha_2))^2 \rangle = \langle \nu_1 \oplus -\nu_2, \alpha_1^2 \oplus \alpha_2^2 \rangle = \langle \nu_1, \alpha_1^2 \rangle + \langle \nu_1, \alpha_1^2 \rangle$$

and the quadratic form on $H^2(C)$ is the direct sum of the quadratic forms on the components. This implies that the signature of the manifold C is the relative signature of (M_1, B) less the relative signature of (M_2, B).

Similarily we have that the class $p_1(C)$ is the image of classes in $H^n(M_1,B) \oplus H^n(M_2,B)$ under the isomorphisms $j \circ h^{-1}$ and the conclusion for the Pontrjagin number follows in a similar manner to the calculation of the signature. $\square$

We are now ready to define the λ-invariant of B.

Definition.
$$\lambda(B) = 2q(M,B) - \sigma(M,B) \mod 7$$

An easy consequence of the above lemma is that this depends only on the boundary manifold B and so defines an invariant of these manifolds.

Lemma 6.6. *Let M_1, M_2 be manifolds bounded by B, where as before B is a closed, oriented manifold such that $H^3(B)$ and $H^4(B)$ vanish. Then $2q(M_1,B) - \sigma(M_1,B) = 2q(M_2,B) - \sigma(M_2,B) \mod 7$*

Proof. By the Hirzebruch signature theorem

$$\sigma(C) = \langle \nu, \frac{1}{45}(7p_2(C) - p_1(C)^2 \rangle$$

and so

$$45\sigma(C) + \langle \nu, p_1(C)^2 \rangle \equiv 2q(C) - \sigma(C) \mod 7$$

so that $2(q(M_1, B) - q(M_2, B)) - (\sigma(M_1, B) - \sigma(M_2, B)) \equiv 0 \mod 7$.

$\square$

Note that the above proof relies implicitly on the idea that we can glue manifolds with diffeomorphic boundaries to get a smooth, closed manifold. If two manifolds B_1, B_2 have λ invariants which are not inverses $\mod 7$ then the Hirzebruch signature theorem gives a fractional value for $\langle \nu, p_2 \rangle$ upon gluing manifolds bounded by B_1, B_2 and so B_1 cannot be diffeomorphic to B_2 (with the opposite orientation).

Note that we have proved that λ depends on the classes of tangent bundle of B and so it is invariant under diffeomorphisms of B. We now wish to compute this invariant for the sphere bundles $B_{hj}, h+j = 1$ built earlier. These bundles fulfill the conditions for the λ-invariant to be defined and bound the disk-bundles D_{hj}. As the λ invariant is independent of the choice of bounded manifold, we may as well calculate the λ explicitly using the D_{hj}.

Lemma 6.7. *For the B_{hj} given earlier we have*

$$\lambda(B_{hj}) = (h - j)^2 - 1 \mod 7 \tag{18}$$

Proof. The mappings $D_{hj} \hookrightarrow E_{hj} \xrightarrow{\pi} S^4$ are clearly homotopy equivalences and so the pullbacks of cohomology groups under these mappings are isomorphisms. The cohomology group $H^4(D_{hj})$ is then generated by the cohomology class $\alpha = i^*(\pi^* e)$, where as before e is the Euler class of the tautological bundle over $\mathbb{H}P^1$, which is a generator of $H^4(S^4)$. We fix an orientation on D_{hj} and B_{hj} by chosing the generator of $H^8(D_{hj}, B_{hj})$ for which $q(M, B) = \langle \nu, i^{-1}(\alpha)^2 \rangle = 1$. Then the signature of the form $\sigma(B, M)$ for this orientation is 1.

The disk bundle D_{hj} is embedded in the vector bundle E_{hj} and so $TD_{hj} = TE_{hj}|_{D_{hj}}$. Therefore to find the Pontrjagin classes of the bundle TD_{hj} it is sufficient to study the classes of the bundle TE_{hj}. The tangent space of a vector bundle E_{hj} splits as bundles tangent to the fibers and normal to the fibers i.e

$$TE_{hj} = \pi^*(E_{hj}) \oplus \pi^*(TS^4) \tag{19}$$

As $H^*(E_{hj})$ has no torsion we have from the Whitney sum formula $p(\pi^*(E_{hj}) \oplus \pi^*(TS^4)) = p(\pi^*(E_{hj}))p(\pi^*(TS^4))$ and in particular

$$\mathrm{p}_1(TE_{hj}) = \pi^*(p_1(TS^4)) + \pi^*(p_1(E_{hj})) = 0 + \pi^*(2(h-j)e) = 2(h-j)\pi^*e$$

Pull back this class under the inclusion $i : D_{hj} \to E_{hj}$ to get $p_1(D_{hj}) = i^*(2(h-j)\pi^*e) = 2(h-j)\alpha$. Then

$$q(M,B) = \langle \nu, i^{-1}(2(h-j)\alpha)^2 \rangle = 4(h-j)^2 = \langle \nu, i^{-1}(\alpha)^2 \rangle = 4(h-j)^2 \quad (20)$$

So we have

$$\lambda(B_{hj}) = 2q - 1 = 8(h-j)^2 - 1 = (h-j)^2 - 1 \mod 7 \quad (21)$$

□

Whence we have the conclusion

Corollary 6.8. *For* $(h-j)^2 \neq 1(\mod 7)$ *the differentiable stucture on these* B_{hj} *cannot be the same as the canonical one on* S^7*.*

Proof. We have shown that λ is invariant under diffeomorphism. As $\lambda(S^7) \equiv 0 \mod 7$ the result follows. □

Corollary 6.9. *For* $(h-j)^2 \neq 1(\mod 7)$ *the manifold* B_{hj} *admits no orientation reversing diffeomorphism to itself*

Proof. An orientation reversing diffeomorphism on B_{hj} reverses the sign of the Pontrjagin numbers and signature of the manifold and so of the invariant λ. As is invarient under diffeomorphisms, such an orientation reversing diffeomorphism can only exist if $\lambda \equiv 0$ □

There are four possible values of $(h-j)^2 - 1 \mod 7$ and so we have constructed four topological spheres which are not diffeomorphic to the standard sphere. We get three more by taking *connected sums*, formed by removing a disk from each sphere and gluing the resulting manifolds along the common boundary.

The question of how many exotic spheres exist in a given dimension has been addressed by Milnor and Kervaire[8]. Investigation into the geometry of these objects is ongoing, a very recent development being the discovery by Petersen and Wilhelm[15] (after quite a long search) of a metric on an exotic sphere which has everywhere positive sectional curvature. More generally Milnor's discovery stimulated research into manifolds with exotic differentiable structures with exciting results such as the discovery of an infinitude of differentiable structures on $\mathbb{R}^4$ by Donaldson[2] ($\mathbb{R}^n$ has a unique differentible structure for all other values of n), and the first discovery of a manifold which does not admit any smooth structure by Kervaire[7].

7 Appendix

7.1 Orientation of Manifolds

An orientation of a real vector space is an assignment of "positive" or "negative" to each ordering of basis elements of $\mathbb{R}^n$ preserved by rotations and reversed by reflections. To make such a choice we can fix an orientation on a chosen frame and use that all other frames of the vector space are the image of the chose frame under a combination of rotations and relflections. However there is another way to make this choice which more readily extends to the idea of a "local" orientation on a manifold.

Definition. *An orientation a real vector space $\mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is a choice of generator of the infinite cyclic group $H_n(\mathbb{R}^n, \mathbb{R}^n - x)$.*

By looking at the natural isomorphisms $H_n(\mathbb{R}^n, \mathbb{R}^n - x) \cong H_{n-1}(\mathbb{R}^n - x) \cong H_{n-1}(S^{n-1})$ we see that such a choice induces a choice of generator of the top homology group of spheres in $\mathbb{R}^n$ centered at x. Now a rotation of $\mathbb{R}^n$ about x induces a degree 1 mapping of such a sphere to itself and reflection induces a degree -1 mapping of the sphere to itself. Thus such a choice satisfies the conditions for this to be a reasonable definition of "orientation". A similar argument shows that a choice of generator of the relative cohomology $H^n(\mathbb{R}^n, \mathbb{R}^n - x)$ also corresponds to a choice of orientation. A local orientation on the vector space is a choice of orientation at all points in an open ball which is "consistent", in the sense that we require the choice of generator of each point be the image of a chosen generator $\mu \in H^n(\mathbb{R}^n, \mathbb{R}^n - U) \cong \mathbb{Z}$ under the isomorphism $H^n(\mathbb{R}^n, \mathbb{R}^n - U) \xrightarrow{\sim} H^n(\mathbb{R}^n, \mathbb{R}^n - x)$.

To extend this definition to a (topological) connected manifold we note that the local constructions above can be performed equally well in a neighbourhood $x \in M$ homeomorphic to an ball in $\mathbb{R}^n$. We can now define the concept of orientation for an arbitrary manifold.

Definition. *An orientation on a manifold M is a choice of orientation at all points of M such that these generators are the image a choice of generator in $\mu_i \in H^n(M, M - U_i) \cong \mathbb{Z}$ where the U_i are (topological) balls of finite radius in M which cover M. If it is possible to make such a choice the manifold is said to be orientable.*

If the manifold is also compact then we can use the following theorem to define a single orientation class on M which will induce the local orientations at each point.

Theorem 7.1. *Let M be an n-dimensional compact topological manifold and ∂M the boundary of M. If M is connected and orientable then for each x in the interior of M the natural inclusion $H_n(M, \partial M) \to H_n(M, M - x)$, is an isomorphism.*

In particular $H_n(M, \partial M) \cong \mathbb{Z}$ and a choice of generator of this group induces an orientation at all points in the interior of M. If the manifold is closed then the above isomorphism is an isomorphism from absolute cohomology $H_n(M, \partial M) \to H_n(M, M - x)$. We define

Definition. *The fundamental class of M is a choice of generator $H_n(M) \cong \mathbb{Z}$ denoted $[M]$. If M is a manifold with boundary then we define the fundamental class of M to be a choice of genertor $H_n(M, \partial M)$.*

Note that in the second case the image of $[M]$ under the boundary map $\partial : H_n(M, \partial M) \to H_{n-1}(\partial M)$ induces an orientation on the boundary.

Given two orientable manifolds A, B with the same boundary $\partial A = \partial B = M$ we can form a manifold C as the disjoint union of A and B by identifying points along the common boundary. The resulting manifold is orientable as an orientation can be given by "compatible" orientations $\mu_A \in H_n(A, M)$ and $\mu_B \in H_n(B, M)$. "Compatible" is defined as follows: Let x be a point on M and consider the n-dimensional ball about x in C with M intersecting the ball as an $n-1$ ball. An orientation on C induces a local orientation at x and so on the ball about x. It is clear that the two halves of the ball induce opposite orientations on the $n-1$ dimensional intersection with M. So two orienations of A and B are compatible and induce and orientation on the glued manifold if $\partial\mu_A + \partial\mu_B = 0$.

7.2 The Oriented Bordism Ring

Attempting to classify manifolds up to diffeomorphism is a provably hopeless task. We can however attempt to classify manifolds up to a much courser equivalence relation that of *bordism*

Two n-manifolds are said to be cobordant or in the same bordism class if their disjoint union is the boundary of a smooth $(n+1)$-manifold. This can be seen to be an equivalence relation. Two oriented manifolds are said to be in the same oriented bordism class if their disjoint union bounds an oriented $(n+1)$-manifold and the orientation on this manifold induces the chosen orientations on the boundary components. The oriented bordism classes of manifolds form a graded ring (graded with respect to the dimension of the manifolds in the equivalence class) with the following operations

- an addition operation given by the disjoint union of manifolds
- a product operation given by taking the Cartesian product of manifolds

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