ALGANT MASTER THESIS

Bridgeland Stability Conditions For Curves and Surfaces.

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Introduction.

Our aim is to give the notion of a stability condition on a triangulated category and discuss its main properties. The special case we are interested in is the bounded derived category of coherent sheaves over a smooth projective scheme X. Our main focus of study is when X is a curve and a surface. The most interesting feature is that the set of stability conditions $\mathrm{Stab}(\mathcal{D})$ on a fixed triangulated category $\mathcal{D}$ comes with a natural topology. After setting up the necessary definitions, we prove that the space $\mathrm{Stab}(\mathcal{D})$ with its natural topology is a finite dimensional complex manifold.

The motivation for the definition of stability condition came from the work of M.R Douglas on Π-stability of Dirichlet branes. Bridgeland gets the motivation from there and he gave a rigorous mathematical treatment in his paper *Stability condition on triangulated categories* in the year 2007. He also presents a paper which gives full description of a connected component of $\mathrm{Stab}(\mathcal{D})$ where $\mathcal{D}$ is the bounded derived category of coherent sheaves on a K3 surface. A bit after, Arcara, Bertram, Toda and many other mathematicians describe the moduli spaces of Bridgeland stable objects over a K3 surface. In 2008, Kontsevich and Soibelman introduced the concept of support property of a stability condition in the paper *Stabilty structures, motivic Donaldson-Thomas invariants and cluster transformations.* Since then, stability conditions on triangulated categories have been highly influential, due to their connections to physics, mirror symmetry, representation theory, and due to their applications in algebraic geometry, for example to the birational geometry of moduli spaces. The main theorem we are gonna prove is stated below.

Theorem 0.1. *The space of stability conditions* $\mathrm{Stab}(\mathcal{D})$ *on a fixed triangulated category* $\mathcal{D}$ *with the support property is a finite dimensional complex manifold.*

The main reference of this theorem is Bayer's paper titled *A short proof of the deformation property of Bridgeland stabillity conditions.* Such a result has very nice applications. As said before, we look into the special case when $\mathcal{D}$ is the bounded derived category of coherent sheaves over a smooth projective scheme X.

Let us look at the case when X is a curve. Recall that any coherent sheaf E has a unique Harder-Narasimhan filtration with the descending value of slope stability. Let $K(\mathcal{D}) \cong K(\mathrm{Coh}(X))$ be the Grothendieck Group of $\mathcal{D}$. We define a group homomorphism $Z : K(\mathrm{Coh}(X)) \to \mathbb{C}$ by the formula :

$$Z(E) = -\deg(E) + i\,\mathrm{rank}(E)$$

for any non-zero sheaf E. By defining the phase of E as $\phi(E) = 1/\pi \arg(Z(E))$ and extending this defintion to any object in $\mathcal{D}(X)$, we get a generalized Harder-Narasimhan filtration of any object in the bounded dervied category. A stability condition has this similar notion of generalized Harder-Narasimhan Filtration. The concept of t-structures and heart of a t-structure are needed to define the stability conditions and are use as a replacement of the category of coherent sheaves in the example above.

The striking remark is that under the action of a group, this is the only stability condition possible. Thus the stability manifold $\mathrm{Stab}(\mathcal{D})$ in the case of curve is completely understood. When we move to surfaces or higher dimension, this example does not work. The precise result we are going to prove in the thesis is the following:

Proposition 0.1. *Let Y be a smooth projective variety over $\mathbb{C}$ of dimension ≥ 2. There is no numerical stability condition $\sigma \in \mathrm{Stab}(Y)$ with heart $\mathrm{Coh}(Y)$.*

The theorem infers the fact that it is a difficult task to construct the stability condition on the bounded derived categories of smooth projective schemes. The case of dimension 2 is already interesting and highly non-trivial. The third main result in this thesis will be the following:

Theorem 0.2. *Let X be a smooth projective surface over $\mathbb{C}$. Then $\mathrm{Stab}(\mathcal{D}^b(X)) \neq \phi$.*

Now we briefly state the contents of each chapter of the thesis.

The first chapter is a recall of basic definitions of the Chow ring, intersection theory of varieties and Chern classes. We end this chapter by computing Chern classes of some specific sheaves which shall be needed in the later part of the thesis.

The second chapter starts with a brief recall of the definition of the two types of stabilities of a coherent sheaf: Geiseker and μ-stability of sheaves. We also state the two types of filtrations in the category of coherent sheaves:

Harder-Narasimhan and Jordan-Holder filtrations.

The third chapter deals with the concept of derived and triangulated categories. We list all the important properties of derived and triangulated categories. Also we give a brief summary on the construction of the derived category by defining the homotopy category of complexes and the cone of a moprhism of complexes. We then look into the special case of the derived category of coherent sheaves and prove some important lemmas which shall be needed later. The chapter ends with the properties of derived functors on coherent sheaves.

The fourth chapter is about t-structures: a way to construct abelian subcategories of a triangulated category. By proving some basic facts about t-structures, we show that the heart of the t-structure is an abelian subcategory. The chapter ends with an important fact which states the precise condition when an abelian subcategory of a triangulated category can be the heart of a t-structure.

The fifth chapter is the beginning of the main part of the thesis. We introduce the definition of Bridgeland stability function (also called a central charge) on a triangulated category and the definition of Harder-Narasimhan property of such a function. We then define the slicing of a triangulated category which leads to the definition of Bridgeland stability condition. At last, we prove an important theorem which says that having a stability condition is equivalent of having a stability function on the heart of a bounded t-structure. This theorem turns out to be an important tool in constructing stability conditions on curves and surfaces.

The sixth chapter deals with the examples of stability conditions. At first, we deal with the example of stability conditions on a curve (the one mentioned in the beginning). We introduce the action of two groups on the set of stability conditions $\mathrm{Stab}(\mathcal{D})$, namely the group of automorphisms of the triangulated category $\mathcal{D}$ (denoted by $\mathrm{Aut}(\mathcal{D})$) and the universal covering space of $\mathrm{Gl}_2^+(\mathbb{R})$ (denoted by $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$). We prove that in case of curves, the action of $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$ on $\mathrm{Stab}(\mathcal{D})$ is free and transitive. The chapter ends with the proof of Proposition 0.1.

The seventh chapter is about proving Therorem 0.1. At first, we define the support property of a stability condition and state Theorem 0.1 more technically. Before proving the theorem, we define the topology on the space $\mathrm{Stab}(\mathcal{D})$ and introduce the concept of Harder-Narasimhan polygons which will be an important tool needed for the proof. Assuming some facts, we first prove the theorem. The proof ends by showing that we can reduce the theorem under those assumptions.

The eighth chapter deals with problem of constructing the stability conditions on surfaces. We introduce the concept of tilting of abelian categories which helps us to construct a new t-structure from a known t-structure. We use this concept to construct new t- structures from the standard t-structure on $\mathcal{D}(X)$. Finally, we construct a stability condition on any surface thus proving Theorem 0.2 .

Contents

Chapter 1

Chern Classes of Coherent Sheaves.

Chern classes of coherent sheaves are important invariants of a coherent sheaf over a smooth non-singular projective scheme. At first, we recall intersection theory and properties of Chow ring which shall lead to definition of Chern Classes.

1.1 Intersection Theory.

Definition 1.1. Let X be a variety over an algebraically closed field k. A *cycle of codimension* r on X is an element of the free abelian group generated by closed irreducible subvarieties of X of codimension r. We write a cycle as

$$Y = \sum n_i Y_i$$

where Y_i are subvarieties and $n_i \in \mathbb{Z}$.

For defining the Chow ring, we recall the definition of rational equivalence.

Definition 1.2. Given V a subvariety of X, let $f : \tilde{V} \to V$ be the normalization of V. Two cycles W_1, W_2 on X are said to be *rational equivalent* if $W_1 = f_* V_1$ and $W_2 = f_* V_2$, where V_1, V_2 are linearly equivalent Weil Divisors on $\tilde{V}$.

Definition 1.3. Denote $A^r(X)$ be the group of cycles of codimension r on X modulo the rational equivalence. We denote $A(X) = \oplus_{r=0}^{n} A^r(X)$ be the graded group.

Remark.

1. $A^0(X) = \mathbb{Z}$.
2. If X is complete, we have the deg function : $\deg : A^n(X) \to \mathbb{Z}$ given by $\sum n_i p_i \to \sum n_i$ to be a group homomorphism.

3. If $f : X \to X'$ be a morphism of varieties, then we have a map
$$f_* : A(X) \to A(X')$$
as follows. Let Y be a subvariety of X.
   (a) If $\dim f(Y) < \dim Y$, then $f_*(Y) = 0$.
   (b) If $\dim f(Y) = \dim Y$, then $K(Y)$ is a finite extension of $K(f(Y))$ and we define :
$$f_*(Y) = [K(Y) : K(f(Y))].\overline{f(Y)}.$$

Now we want this group $A(X)$ to be a commutative associative graded ring with identity. This is done by the intersection theory.

Definition 1.4. An intersection theory on a given class of varieties $\mathscr{B}$ consists of a pairing $A^r(X) \times A^s(X) \to A^{r+s}(X)$ for each r, s and for each $X \in \mathscr{B}$, satisfiying the axioms below. If $Y \in A^r(X), Z \in A^s(X)$, we denote the intersection cycle as $Y.Z$.

In the axioms mentioned below, we use the following :
If $f : X' \to X$ is a morphism of varieties in $\mathscr{B}$. Assume $X \times X' \in \mathscr{B}$. We define $f^* : A(X') \to A(X)$ as for a subvariety $y' \in X'$, we define :
$$f^*(y') = p_{1*}(\Gamma_f . p_2^{-1}(y'))$$
where Γ_f is the graph of f as a cycle in $X \times X'$ and the other symbols are defined in the diagram below :
$$\begin{tikzcd} & X \times X' \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[rr, "f"] & & X' \end{tikzcd}$$

The axioms are as follows:

1. The intersection pairing makes $A(X)$ to be commutative associative graded ring with identity for every $X \in \mathscr{B}$. It is called the *Chow ring of* X.

2. If $f : X \to X'$ a proper morphism of varieties, then f^*, f_* are group homomorphisms. Also, we have:
$$(f^* \circ g^*) = (g \circ f)^* \;\; ; \;\; g_* \circ f_* = (g \circ f)_*$$
where $g : X' \to X''$ is another morphism.

3. If $f : X \to X'$ is a proper morphism and if $x \in A(X), y \in A(X')$, then
$$f_*(x.f^*y) = f_*(x).y$$

4. (Reduction to diagonal) If y, z are cycles on X, then if $\Delta : X \to X \times X$ is the diagonal morphism, then :
$$y.z = \Delta^*(y \times z)$$

5. If y and z are subvarieties of X which *intersect properly* [that means the codimension of every irreducible component of $y \cap z$ is same as $\mathrm{codim}(y) + \mathrm{codim}(z)$], then we have
$$y.z = \sum i(y.z; w_j)w_j$$
where the sum runs over the irreducible components w_j of $y \cap z$. Also the integer $i(y, z; w)$ depends only on a neighbourhood of a generic point of w_j on X. $i(y, z; w)$ is called the *local intersection multiplicity* of y and z along w_j.

6. (Normalization) If y is a subvariety of X, and z is an effective Cartier divisor meeting y properly, then $y.z$ is just the cycle associated to the Cartier divisor $y \cap z$ in y.

The next theorem states that we can have intersection theory in the specific set of varieties we are interested.

Theorem 1.1. *Let $\mathscr{B}$ be the class of non-singular varieties over a fixed algebraically closed field k. Then there is a unique intersection theory for cycles modulo the rational equivalence of varieties $X \in \mathscr{B}$ which sarisfies the above axioms of Defintion 1.4.*

Remark. In the above theorem, if y and z intersect properly, and if w is an irreducible component of $y \cap z$, the intersection multiplicity defined (due to Serre) as :
$$i(y, z : w) = \sum (-1)^i \,\mathrm{length}\, \mathrm{Tor}_i^A(A/\mathfrak{a}, A/\mathfrak{b}).$$
where A is the local ring $\mathcal{O}_{v,X}$ at the generic point of v of X and $\mathfrak{a}, \mathfrak{b}$ correspond to ideals of y and z in A.

1.2 Properties of the Chow ring.

For any non-singular projective variety X, we now consider the Chow Ring $A(X)$ and list some of its properties:

1. The cycles in codimension 1 are just Weil-Divisors and rational equivalence is same as the linear equivalence of divisors. Also X is non-singular which leads us to have $A^1(X) \cong \mathrm{Pic}(X)$.

2. $A(\mathbb{P}^n) \cong \frac{\mathbb{Z}[h]}{h^{n+1}}$, where h is in degree 1 and is the class of the hyper-plane. It follows from the fact that any subvariety of degree d in $\mathbb{P}^n$ is rationally equivalent to d times a linear space of same dimension.

3. Let $\mathscr{E}$ be a locally free sheaf of rank r on X, let $\mathbb{P}(\mathscr{E})$ be the associated projective space bundle .Let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the projection map. Let $\zeta \in A^1(\mathbb{P}(\mathscr{E}))$ be the divisor corresponding to the line bundle $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$. Then $\pi^* : A(X) \to A(\mathbb{P}(\mathscr{E}))$ makes $A(\mathbb{P}(\mathscr{E}))$ a free $A(X)$ module generated by $1, \zeta, \zeta^2, \cdots, \zeta^{n-1}$.

1.3 Chern classes.

Definition 1.5. Let $\mathscr{E}$ be a locally free sheaf of rank r on a non-singular quasi-projective variety X. For each $i = 0, 1, 2, \cdots, r$,we define the i th *Chern class* $c_i(\mathscr{E}) \in A^i(X)$ by the requirement $c_0(\mathscr{E}) = 1$ and :

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathscr{E}).\zeta^{r-1} = 0$$

in $A^r(\mathbb{P}(\mathscr{E}))$ using then notation of Property 3.

Remark. The above expression makes sense as from property 3, we get that ζ^r can be written as linear combination of $1, \zeta, \zeta^2, ...\zeta^{r-1}$ over $\pi^*(A(X))$ and thus the above expression in the definition makes sense.

Definition 1.6. The Chern polynomial is defined as

$$c_t(\mathscr{E}) = c_0(E) + c_1(\mathscr{E})t + \cdots + c_r(\mathscr{E})t^r$$

using the notation of Definition 1.5.

Definition 1.7. Write

$$c_t(\mathscr{E}) = \prod_{i=1}^{r} (1 + a_i t)$$

where a_i are formal symbols,
Then we define the *Chern character* of $\mathscr{E}$ as :

$$\mathrm{ch}(\mathscr{E}) = \sum_{i=1}^{r} e^{a_i}$$

where $e^x = 1 + x + x^2/2! + \cdots$.

Now we list some properties of Chern class and Chern polynomials:

1. If $\mathscr{E}$ is a line bundle, then $\mathscr{E} \cong Ł(D)$ for a divisor D. Then we have that $\mathbb{P}(\mathscr{E}) \cong X$ and $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(1) \cong Ł(D)$. Thus we have then $\zeta = D$. Hence by definition, we have
$$c_1(E).1 - 1.D = 0.$$
which implies $c_1(E) = D$. So we have
$$c_t(Ł(\mathcal{D})) = 1 + Dt.$$

2. If $f : X' \to X$ is a morphism and $\mathscr{E}$ is a locally free sheaf of X, then we have
$$c_i(f^*\mathscr{E}) = f^*(c_i(E))$$
where the latter f^* is the map $f^* : A(X) \to A(X')$.

3. If
$$0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$$
is an exact sequence of locally free sheaves, then we have :
$$c_t(\mathscr{E}) = c_t(\mathscr{E}').c_t(\mathscr{E}'')$$

4. (Splitting Principle) Given $\mathscr{E}$ a locally free sheaf of rank r on X, there exists a morphism $f : X' \to X$ such that $f^* : A(X) \to A(X')$ is injective and $\mathscr{E}' = f^*\mathscr{E}$ *splits* i.e $\mathscr{E}'$ has filtration :
$$\mathscr{E} = \mathscr{E}_0 \supset \mathscr{E}_1' \supset E_2' \cdots \supset \mathscr{E}_r' = 0$$
where $\mathscr{E}_i'/\mathscr{E}_{i+1}'$ are invertible sheaves for all $1 \le i \le r-1$.

5. Thus we get that if $\mathscr{E}$ splits and has a filtration $Ł_1, Ł_2, \ldots Ł_r$, as quotients, then we have :
$$c_t(\mathscr{E}) = \prod_{i=1}^{r} c_t(Ł_i)$$

6. Using the splitting principle, we can calculate the chern class of tensor product of sheaves. Let $\mathscr{E}$ and $\mathscr{F}$ are locally free sheaves of rank r and s respectively. Write
$$c_t(\mathscr{E}) = \prod_{i=1}^{r}(1 + a_i t) \ ; \ c_t(\mathscr{F}) = \prod_{j=1}^{s}(1 + b_j t).$$
Then we have
$$c_t(\mathscr{E} \otimes \mathscr{F}) = \prod_{i,j}(1 + (a_i + b_j)t).$$

7. By property 3 of this section, we have then :

$$\mathrm{ch}(\mathscr{E}) = \mathrm{ch}(\mathscr{E}') + \mathrm{ch}(\mathscr{E}'')$$

   where $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is an exact sequence of locally free sheaves.

Now we define the Chern character for a general sheaf.

Definition 1.8. Given any sheaf $\mathscr{E}$ on a non-singular quasi-projective variety X, we define

$$\mathrm{ch}(\mathscr{E}) = \sum_{i=0}^{n} (-1)^i \, \mathrm{ch}(\mathscr{F}_i)$$

where $\mathscr{F}_i$ are defined as elements of the exact sequence of the finite free resolution of $\mathscr{F}$ given as :

$$\mathscr{F}_n \to \mathscr{F}_{n-1} \to \mathscr{F}_{n-2} \cdots \mathscr{F}_1 \to \mathscr{F}_0 \to \mathscr{E} \to 0$$

Remark. The above definition asserts the fact that Chern character is additive on short exact sequences. For a coherent sheaf $\mathscr{E}$ over a non-singular projective variety X of dimension n, we usually write

$$\mathrm{ch}(\mathscr{E}) = (\mathrm{ch}_0(E), \mathrm{ch}_1(E), \mathrm{ch}_2(E), \cdots, \mathrm{ch}_n(E))$$

where $\mathrm{ch}_i(E)$ corresponds to elements in codimension i of the expression $\sum e^{a_i}$.

1.4 Computation on Chern classes.

We compute Chern classes of sheaves in special cases which shall be needed afterwards.

1. (Curves). Let X be a curve. Then we have two Chern characters $c\,\mathrm{ch}(E) = (c_0(E), c_1(E))$. The first one turns out to be rank of the sheaf $\mathscr{E}$ and the second one turns out to be the degree of the sheaf $\mathscr{E}$.

2. (Line Bundles) As computed in 1, we have Chern character of line bundle as $(1, D) = e^D$.

3. (Skyscraper sheaf.) Let $\kappa(p)$ be the skyscaper sheaf of a rational point p in a smooth projective scheme X of dimension n over a field $k = \overline{k}$. We have a finite free resolution of $\kappa(p)$ which is a locally a koszul complex given as

$$0 \to \mathcal{O}(-n) \to \cdots \mathcal{O}(-2)^{\oplus \binom{n}{2}} \to \mathcal{O}(-1)^{\oplus n} \to \mathcal{O} \to \kappa(p) \to 0$$

Thus we have by definition

$$\mathrm{ch}(\mathscr{E}) = \sum_{i=0}^{n} (-1)^i \,\mathrm{ch}(\mathcal{O}(-i)^{\oplus \binom{n}{i}}) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} e^{-i\zeta} = (1 - e^{-\zeta})^n = \zeta^n.$$

where ζ is the class of hyperplane in degree 1 . The last equality is due to the fact $\zeta^{n+1} = 0$. Thus $\mathrm{ch}(\kappa(p)) = (0, 0, \cdots, 1)$.

Chapter 2

Stability and filtrations of sheaves.

In this chapter, we recall the two concept of stabilities of sheaves : Gieseker and μ-stability. With the help of these concepts, we recall the concept of two filtrations in sheaves: Harder-Narasimhan and Jordan-Holder filtrations. This filtrations reduces us to study a sheaf in terms of stable sheaves.

2.1 Geiseker and μ-stability.

Let X be a noetherian scheme. By $\mathrm{Coh}(X)$ we mean the category of coherent sheaves on X. At first we define:

Definition 2.1. The *support* of E is the closed set: $\mathrm{Supp}(E) := \{x \in X | E_x \neq 0\}$. Its dimension is called the dimension of sheaf E (denoted by $\dim(E)$)

Definition 2.2. E is *pure* of dimension d if $\dim(F) = d$ for all non trivial coherent subsheaves $F \subset E$

Definition 2.3. The *torsion filtration* of a coherent sheaf E is the unique filtration :

$$0 \subset T_0(E) \subset \cdots\cdots \subset T_d(E) = E$$

where $d = \dim(E)$ and $T_i(E)$ is the maximal subsheaf of dimension $\leq i$

Definition 2.4. A coherent sheaf E on an integral scheme X is called *torsion-free* if for any $x \in X$ and any $s \in \mathcal{O}_{X,x} - \{0\}$, the multiplication map :

$$.s : E_x \to E_x$$

is injective.

Remark. E is torsion-free if $T_{d-1}(E) = 0$ (d = $\dim(E)$). Pure is a generalization of being torsion free.

Definition 2.5. The *saturation* of a subsheaf $F \subset E$ is the minimal subsheaf F' containing F such that E/F' is pure of dimension $d = \dim(E)$ or zero.

Remark. Clearly the saturation of a sheaf F is the kernel of the map :

$$E \to E/F \to (E/F)/T_{d-1}(E/F)$$

Now we do a brief recall of depth and homological dimension of a module M over a local ring A.

Definition 2.6. A set $\{a_1, a_2, \cdots, a_l\} \in \mathfrak{m}$ is an *M-regular sequence* if a_i is not a zero divisor in $M/(a_1, a_2, ...a_{i-1})M$ for all i.

Definition 2.7. The maximal length of a M-regular sequence is called the *depth* of M Its denoted by $\mathrm{depth}(M)$.

Definition 2.8. The maximal length of the projective resolution of M is called the *homological dimension*, denoted by $\mathrm{dh}(M)$

Theorem 2.1. *(Auslander-Buchsbaum Formula) If A is a regular ring we have :*

$$\mathrm{dh}(M) + \mathrm{depth}(M) = \dim A$$

Now we go back to the coherent sheaf setting and we define:

Definition 2.9. For a coherent sheaf E on a noetheriam scheme X , we define $dh(E) := \max\{\mathrm{dh}(E_x) | x \in X\}$.

Remark. For a general sheaf $\mathrm{dh}(E)$ is infinite. By the Auslander-Buchsbaum Formula, as we are on finitely many affine opens (Noetherian Scheme), we have that if the scheme is regular then $\mathrm{dh}(E) \leq \dim(X)$. If it is torsion free then it is bounded by $\dim(X) - 1$

Next we define the version of regularity in terms of sections of cohomology.

Definition 2.10. Let X be a Noetherian Scheme. Let E be a coherent sheaf on X. Let L be a line bundle on X. A section $s \in H^0(X, L)$ is called *E-regular* iff $E \otimes L^v \xrightarrow{.s} E$ is injective. A sequence $s_1, ..s_l$ is called *E- regular* if s_i is regular in $E/(s_1, s_2, \cdots s_{i-1})(E \otimes L^v) \forall i = 1, \cdots l$

Consider X to be a projective scheme over a field k. We have $\chi(E) = \sum(-1)^i h^i(X, E)$ where $h^i = \dim(H^i)$. We fix an ample line bundle $\mathcal{O}(1)$ on X.
The Hilbert Polynomial $P(E)$ is given by

$$m \to \chi(E \otimes \mathcal{O}(m))$$

Lemma 2.1. *Let E be a coherent sheaf of dimension d and let $H_1, H_2, ..H_d \in |\mathcal{O}(1)|$ be an E- regular sequence, then :*

$$P(E,m) := \chi(E \otimes \mathcal{O}(m)) := \sum_{i=0}^{d} \chi(E|_{\cap_{j\leq i} H_j}) \binom{m+i-1}{i}$$

Remark. The proof is done by induction on the dimension of the scheme.

From here we notice that :

$$P(E,m) := \sum \alpha_i(E) \frac{m^i}{i!}$$

For a sheaf of dimension d , we have $\alpha_d(E)$ is positive and is called the *multiplicity* of E.

Definition 2.11. If E is a coherent sheaf of dimension $d = \dim(X)$, we define :

$$\mathrm{rk}(E) := \frac{\alpha_d(E)}{\alpha_d(\mathcal{O}_X)}$$

is called the rank of E.

Remark. On a integral scheme X of dimension d and for any d dimensional coherent sheaf E, there exists an open subset $\subset X$ such that $E|_U$ is locally free. The rank of E is the rank of the vector bundle $E|_U$.

Definition 2.12. The reduced Hilbert Polynomial of a coherent sheaf E is

$$p(E,m) := \frac{P(E,m)}{(\alpha_d(E))}$$

Note: Now we define how to compare two polynomials. We say $f \leq g$ if $f(m) \leq g(m) \forall m >> 0$. Analogously, we say $f < g$ if $f(m) < g(m) \forall m >> 0$.

Definition 2.13. A coherent sheaf E of dimension d is *semistable* if E is pure and for any proper subsheaf $F \subset E$, we have $p(F) < p(E)$. E is called *stable* if the inequality is strict.

We will use () for writing the semi case optional.

Theorem 2.2. *Let E be a coherent sheaf of dimension d and E is pure. TFAE:*

1. *E is (semi)stable.*
2. *$\forall$ proper saturated sheaves $F \subset E$,we have $p(F)(\leq)p(E)$.*
3. *$\forall$ proper quotient sheaves $E \to G$ with $\alpha_d(G) \geq 0$, one has $p(F)(\leq)p(G)$.*

4. *$\forall$ proper purely d dimensional quotient sheaves $E \to G$ one has $p(F)(\leq)p(G)$.*

Sketch of the proof. $1 \implies 2$ and $3 \implies 4$ are obvious by definition.
Now we consider an exact sequence of sheaves :

$$0 \to F \to E \to G \to 0.$$

$(G \cong E/F)$
The properties needed are as follows:

$$\alpha_d(E) = \alpha_d(F) + \alpha_d(G), P(E) = P(F) + P(G).$$

Thus we have :

$$\begin{aligned} \alpha_d(E)p(E) &= \alpha_d(G)p(G) + \alpha_d(F)p(F) \\ \implies (\alpha_d(F) + \alpha_d(G))p(E) &= \alpha_d(G)p(g) + \alpha_d(F)p(F) \\ \implies \alpha_d(F)(p(F) - p(E)) &= \alpha_d(G)(p(E) - p(G)) \end{aligned}$$

So now if 1 is assumed, then if $\alpha_d(G) \geq 0$ then $\alpha_d(E)(\leq)\alpha_d(G)$. This proves $1 \implies 3$.
Now if F is saturated then G is pure and hence the vice versa too. Thus $2 \Leftrightarrow 4$
Now we prove $2 \implies 1$.
Let $F \subset E$. F' be its saturation of F. We use that $\alpha_d(F) = \alpha_d(F')$ We substitute this in the equation and use $p(F)(\leq)p(F')$. We get that $p(F)(\leq)p(E)$.

□

Theorem 2.3. *Let F and G be semistable pure d dimensional coherent sheaves. Then :*

1. *If $p(F) > p(G)$, then $Hom(F,G) = 0$.*

2. *If $p(F) = p(G)$ and let $f : F \to G$ be a non-trivial morphism , then it is injective if F is stable, it is surjective if G is stable.*

3. *If $p(F) = p(G)$ and $\alpha_d(F) = \alpha_d(G)$, then $f : F \to G$ is an isomorphism provided F or G is stable.*

The main corollary that follows from it is :

Corollary 2.1. *If k is algebraically closed and E is a stable sheaf , then $k \cong End(E)$.*

Now we define degree and slope of a coherent sheaf E.

Definition 2.14. Let E be a coherent sheaf on dimension $d = \dim(X)$. Then $\deg(E)$ is defined by

$$\deg(E) := \alpha_{d-1}(E) - \mathrm{rk}(E)\alpha_{d-1}(\mathcal{O}_X).$$

Remark. It can be shown by *Hirzebruch-Riemann-Roch* formula that $\deg(E) = c_1(E).H^{d-1}$ where H is an ample divisor.

Definition 2.15. A coherent sheaf E of $\dim X$ is *μ-(semi)stable* if $T_{d-2}(E) = T_{d-1}(E)$ and $\mu(F)(\leq)\mu(E)\ \forall F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$.

Remark. The condition of the torsion filtration states that any torsion subsheaf of E has codimension at least 2.

Now we state the main theorem which shall be needed in later part of the thesis.

Theorem 2.4. *If E is a pure coherent sheaf of $\dim X = d$, then :*

$$E \text{ is } \mu-\text{stable} \implies E \text{ is stable} \implies E \text{ is semistable} \implies E \text{ is } \mu-\text{semistable}.$$

2.2 Harder-Narasimhan and Jordan-Holder Filtrations.

This section mainly concerns the idea of studying a pure sheaf as building blocks of semistable sheaves and further to stable sheaves. This idea gets generalized to abelian categories which shall be discussed later.

Definition 2.16. Let E be a non-trivial pure sheaf of dimension d over a projective scheme X with a fixed ample line bundle. A *Harder-Narasimhan filtration* is an increasing filtration :

$$0 = \mathrm{HN}_0(E) \subset \mathrm{HN}_1(E) \subset \mathrm{HN}_2(E) \subset \cdots\cdots \subset \mathrm{HN}_l(E) = E$$

such that $\mathrm{gr}_i^{HN} := HN_i(E)/HN_{i-1}(E)$for all $i = 1, \cdots, l$ are semistable sheaves of dimension d with reduced Hilbert Polynomials p_i such that

$$p_{max}(E) := p_1 > p_2 ... p_l = p_{min}(E)$$

Lemma 2.2. *If F, G are pure sheaves of dimension d with $p_{max}(F) > p_{min}(G)$ then $Hom(F, G) = 0$*

The main thoerem regarding this filtration is the following:

Theorem 2.5. *Every pure sheaf E has a unique Harder-Narasimhan Filtration*

Remark. The proof of this in the general setting of stability functions on an abelian category is proved later.

Definition 2.17. Let E be a semistable sheaf of dimension d. A *Jordan-Holder filtration* of E is a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset E_3 \cdots \subset E_l = E$$

such that $gr_i(E) := E_i/E_{i-1}$ are stable with same Hilbert Polynomial $p(E)$.

Unlike the Harder-Narasimhan Filtration, this is not unique filtration but its is unique in a weaker context.

Theorem 2.6. *The Jordan-Holder Filtration of a semistable sheaf always exists. Its is not unique. But*

$$gr(E) := \bigoplus_i gr_i(E)$$

for a semistable sheaf E is unique upto isomorphism.

We hereby end the discussion by stating the definition of S equivalence.

Definition 2.18. Two semistable sheaves E_1 and E_2 are said to be $S-equivalent$ if $gr(E_1) \cong gr(E_2)$.

Thus these two filtrations reduces us to the concept of studying only stable sheaves. As stated before, we shall generalize this for the context of stability functions.

Chapter 3

Derived and Triangulated Categories.

This chapter is devoted to the notion of two special type of categories: triangulated and derived cateogries. At first we list the properties of triangulated categories. We then introduce the category of complexes and introduce the definition of derived category. We show that the derived category of complexes is actually a triangullated category. The chapter ends dealing with the properties of dervied category over coherent sheaves and derived functors on this category.

3.1 Triangulated categories and exact functors.

Definition 3.1. Let $\mathcal{D}$ be an additive category. The structure of a trinagulated category on $\mathcal{D}$ is given by an additive equivalence :

$$T : \mathcal{D} \to \mathcal{D}$$

the *shift functor*, and a set of distinguished triangles :

$$A \to B \to C \to T(A)$$

subject to the axioms **TR1**-**TR4** below. In the axioms, we use the notation $A[1] = T(A)$ and $A[n] = T^n(A)$ for any object $A \in \mathcal{D}$. The axioms are as follows :

TR1 (a) Any triangle of the form :

$$A \xrightarrow{\text{id}} A \to 0 \to A[1]$$

is distinguished.

(b) Any triangle isomorphic to a distinguished triangle is distinguished.

(c) Any morphism $f : A \to B$ can be completed to a distinguished triangle :

$$A \xrightarrow{f} B \to C \to A[1]$$

TR2 The triangle :

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished triangle iff

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is also a distinguished triangle.

TR3 Suppose there exists a commutative diagram of distinguished triangles with vertical arrows f and g :

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dotted, "h"] & A[1] \arrow[d, "f[1]"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1].
\end{tikzcd}$$

Then the diagram can be completed to a commutative diagram by the existence of a morphism $h : C \to C'$

TR4 If we have three distinguished triangles :

$$X \to Y \to Z' \to X[1]; Y \to Z \to X' \to Y[1]; Z \to X \to Y' \to Z'[1]$$

then there exists a distinguished triangle ;

$$Z' \to Y' \to X' \to Z'[1]$$

This is called the octahedral axiom.

We list properties of distinguished triangles in a triangulated category $\mathcal{D}$.

Proposition 3.1. *Let*

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

be a distinguished triangle. We have the following:

1. *$g \circ f = 0$.*
2. *If $A_0 \in \mathcal{D}$, then the following induced sequences are exact:*
$$\mathrm{Hom}(A_0, A) \to \mathrm{Hom}(A_0, B) \to \mathrm{Hom}(A_0, C)$$
and
$$\mathrm{Hom}(C, A_0) \to \mathrm{Hom}(B, A_0) \to \mathrm{Hom}(A, A_0).$$
3. *$A \to B$ is an isomorphism iff $C \cong 0$.*
4. *If h is the zero map, then $B \cong A \oplus C$.*
5. *If we have a morphism of distinguished triangles,*
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & A[1] \arrow[d, "f[1]"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$
then if two of three morphisms f, g, h are isomorphisms, the third one is also an isomorphism.

We introduce the concept of exact functors.

Definition 3.2. An additive functor :
$$F : \mathcal{D} \to \mathcal{D}'$$
between triangulated cateogries $\mathcal{D}$ and $\mathcal{D}'$ is called exact if the following conditions are satisfied.

1. There is a functor isomorphism :
$$F \circ [1] \xrightarrow{\cong} [1] \circ F$$
2. Any distinguished triangle
$$A \to B \to C \to A[1]$$
in $\mathcal{D}$ is mapped to a distinguished triangle :
$$F(A) \to F(B) \to F(C) \to F(A[1])$$
in $\mathcal{D}'$ where $F(A[1])$ is identified with $F(A)[1]$ via the functor isomorphism in (i).

We state the important proposition regarding exact functors which shall be needed in the section of t-structures.

Proposition 3.2. *Let $F : \mathcal{D} \to \mathcal{D}'$ be an exact functor between triangulated categories. Let $H, G : \mathcal{D}' \to \mathcal{D}$ be two additive functors.*
If $F \dashv H$, then H is exact.
If $G \dashv F$, then G is exact.

The next section is about derived categories which shall lead to the derived category of coherent sheaves. It turns out that the derived category of coherent sheaves is also a triangulated category.

3.2 Derived Categories.

We start this section by definition of category of complexes over an *abelian category $\mathcal{A}$*. Recall that a category is *abelian* if:

- it has a zero object,
- it has all binary biproducts,
- it has all kernels and cokernels,
- v and all monomorphisms and epimorphisms are kernels and cokernels of some morphisms respectively.

Definition 3.3. The category of complexes $\mathrm{Kom}(\mathcal{A})$ of an abelian category $\mathbb{A}$ is the category whose objects are complexes $A^\bullet$ in $\mathcal{A}$ and morphisms are morphisms of complexes.

Remark. For any $A \in \mathcal{A}$, the complex $A^\bullet$ with $A^0 = A$ and $A^i = 0$ for all $i \neq 0$ identifies $\mathcal{A}$ as a full subcategory of $\mathrm{Kom}(\mathcal{A})$.

Regarding $\mathrm{Kom}(\mathcal{A})$, we have following important property.

Proposition 3.3. *The category of complexes* $\mathrm{Kom}(\mathcal{A})$ *of an abelian category $\mathcal{A}$ is abelian.*

We introduce the two important features of $\mathrm{Kom}(\mathcal{A})$: the shift and the cohomology functor.

Definition 3.4. Let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ with differential maps d_A^i . Then $A^\bullet[1]$ is the complex defined by $(A^\bullet[1])^i = A^{i+1}$ and $d^i_{A[1]} = -d^i_A$.
The shift $f[1]$ of a morphism of complexes $f : A \to B$ is the map $f[1] : A[1] \to B[1]$ defined by $f[1]^i = f^i$. where $f^i : A^i \to B^i$.

An important corollary related to the shift functor is the following.

Corollary 3.1. *The shift functor $T : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$ given by $A \to A[1]$ defines an equivalence of abelian categories.*

Recall the cohomology functor $H^i(A^\bullet)$ of a complex $A^\bullet = (A^i, d_A^i)$ is the quotient :

$$H^i(A^\bullet) = \frac{\ker(d_A^i)}{\mathrm{im}(d_A^{i-1})} \in \mathcal{A}$$

Remark. Proposition 3.3 allows us to speak exact sequences in $\mathrm{Kom}(\mathcal{A})$. By cohomology functors, we have if :

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

is an exact sequence in $\mathrm{Kom}(\mathcal{A})$. Then we have the corresponding long exact sequence :

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \to H^{i+1}(A^\bullet) \to \cdots$$

We define the notion of quasi-isomorphism.

Definition 3.5. A morphism of complexes $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism if $\forall i \in \mathbb{Z}$, $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism.

Now we want to define the derived category. The main idea is to have quasi-isomorphism complexes as isomorphic objects in the derived category. We have the following existence theorem:

Theorem 3.1. *Let $\mathcal{A}$ be an abelian category. Let $\mathrm{Kom}(\mathcal{A})$ be the category of complexes. Then there exists a category $\mathcal{D}(\mathcal{A})$, the derived category of $\mathcal{A}$ and a functor $Q : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ such that :*

1. *If $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism , then $Q(f)$ is an isomorphism in $\mathcal{D}(\mathcal{A})$.*

2. *Any funcctor $F : \mathrm{Kom}(\mathcal{A}) \to \mathcal{D}$ satisfying property 1 factorizes uniquely over Q. i.e there exists a unique functor $G : \mathcal{D}(\mathcal{A}) \to \mathcal{D}$ which makes the diagram below commutative.*

$$\begin{tikzcd}
\mathrm{Kom}(\mathcal{A}) \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{D} \\
& \mathcal{D}(\mathcal{A}) \arrow[ur, dotted, "!G"'] &
\end{tikzcd}$$

Remark. For construction of $\mathcal{D}(\mathcal{A})$, we want if $C^\bullet \to A^\bullet$ is a quasi-isomorphism in $\mathrm{Kom}(\mathcal{A})$, then it has to be an isomorphism in $\mathcal{D}(\mathcal{A})$.This means that morphisms $C^\bullet \to B^\bullet$ is same as morphisms $A^\bullet \to B^\bullet$. This is the reason that a morphism between $A^\bullet \to B^\bullet$ in $\mathcal{D}(\mathcal{A})$ is given as diagram of the form :

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

To makes sense of such morphisms, we need to define the homotopy category $K(\mathcal{A})$.

Definition 3.6. Two morphisms of complexes $f, g : A^\bullet \to B^\bullet$ are said to be homotopically equivalent $f \sim g$, if there exists a collection of homomorphisms $h^i : A^i \to B^{i-1} \;\; \forall\, i \in Z$ such that :

$$f^i - g^i = h^{i+1} \circ d_A^i + d_B^{i-1} \circ h^i$$

The homotopy category of complexes $K(\mathcal{A})$ has objects $\mathrm{ob}(K(\mathcal{A})) = \mathrm{ob}(\mathrm{Kom}(\mathcal{A}))$ and morphisms $\mathrm{Hom}_{k(\mathcal{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim$.

Remark. We now can define when two morphisms of $A^\bullet \to B^\bullet$ are equivalent. Suppose there are two morphisms A and B given by the diagrams below:

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & A^\bullet & & B^\bullet \end{tikzcd}$$

Then they are said to be equivalent if there are dominated by a diagram of same form in $K(\mathcal{A})$. i.e there exists a diagram

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "qis"] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

which is commutative in $K(\mathcal{A})$.

That means we have $C^\bullet \to C_1^\bullet \to A$ and $C^\bullet \to C_2^\bullet \to A$ are homotopically equivalent. Now the remaining problem is the composition. Given two morphisms $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ given by the diagrams:

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ B^\bullet & & C^\bullet \end{tikzcd}$$

We want the composition to be a diagram of the form

$$\begin{tikzcd} & & C^\bullet \arrow[dl, "qis"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "qis"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & B^\bullet \end{tikzcd}$$

which is commutative in $K(\mathcal{A})$.

The problem lies in existence of such a object $C^\bullet$ and the commutativity of the diagram in $K(\mathcal{A})$. This is resolved by define the cone of a morphism.

Definition 3.7. Let $f : A^\bullet \to B^\bullet$ be a complex morphism. Its mapping cone is the complex $C(f)$ with :

$$C(f) = A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{C(f)} := \begin{bmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{bmatrix}$$

We list the properties of the cone of the morphism:

Proposition 3.4. 1. *Given $f : A^\bullet \to B^\bullet$ a morphism of complexes. There exists natural maps $\tau : B \to C(f)$ and $\pi : C(f) \to A^\bullet[1]$ which makes the following sequence exact in* Kom($\mathcal{A}$).

$$B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1].$$

2. *The composition $A^\bullet \to B^\bullet \to C(f)$ is homotopically equivalent to the trivial map. Thus we have the sequence*

$$A^\bullet \to B^\bullet \to C(f)$$

exact in $K(\mathcal{A})$.

3. *The commutative diagram given below can be completed as follows*

$$\begin{tikzcd}
A_1^\bullet \arrow[r, "f_1"] \arrow[d] & B_1^\bullet \arrow[r] \arrow[d] & C(f_1) \arrow[r] \arrow[d, dotted] & A_1^\bullet[1] \arrow[d] \\
A_2^\bullet \arrow[r, "f_2"] & B_2^\bullet \arrow[r] & C(f_2) \arrow[r] & A_2^\bullet[1]
\end{tikzcd}$$

4. *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $C(f)$ be its mapping cone which comes with natural morphism $\tau : B^\bullet \to C(f), \pi : C(f) \to A^\bullet[1]$. Then we have a complex morphism $g : A^\bullet[1] \to C(\tau)$ which is an isomorphism in $K(\mathcal{A})$. This also makes the following diagram commutative in $K(\mathcal{A})$:*

$$\begin{tikzcd}
B^\bullet \arrow[r, "\tau"] \arrow[d, "="] & C(f) \arrow[r, "\pi"] \arrow[d, "="] & A^\bullet[1] \arrow[r, "-f"] \arrow[d, dotted, "g"] & B^\bullet[1] \arrow[d] \\
B^\bullet \arrow[r, "\tau"] & C(f) \arrow[r, "\tau_\tau"] & C(\tau) \arrow[r, "\pi_\tau"] & B^\bullet[1].
\end{tikzcd}$$

Remark. The part of the proposition above helps us to prove the well-definedness of composition of morphisms in $K(\mathcal{A})$. Thus we have constructed the Derived Category of $\mathbb{A}$.

Proposition 3.5. *$\mathcal{D}(\mathcal{A})$ is an additive category.*

$\mathcal{D}(\mathcal{A})$ is not essentially an abelian category unlike $\text{Kom}(\mathcal{A})$. Notice that parts $1, 2, 3$ of the proposition 3.4 reminds of the axioms of the triangulated category. In fact, we have the following proposition.

Proposition 3.6. *Distinguished triangles given as in definition 3.1 and the natural shift functor of complexes $A^\bullet \to A^\bullet[1]$ make the homotopy category of complexes $K(\mathcal{A})$ and the derived category $\mathcal{D}(\mathcal{A})$ of an abelian category into a triangulated category.*
Morever the natrual functor $Q_\mathcal{A} : K(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is an exact functor of triangulated categories.

By definition, complexes in categories$K(\mathcal{A}), \mathcal{D}(\mathcal{A})$ are unbounded, but often it is more convenient to work with bounded ones.

Definition 3.8. Let $\text{Kom}^*(\mathcal{A})$ with $* =, +, -, b$, be the category of complexes $A^\bullet$ with $A^i = 0$ for $i << 0, i >> 0$, repsectively $|i| >> 0$.

By dividing by homotopy equivalence and quasi isomorphism, one obtains the categories $K^*(\mathcal{A})$ and $\mathcal{D}^*(\mathcal{A})$ with $* = +, -, b$. Consider the natural functors $\mathcal{D}^*(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ by just forgetting the boundedness condition.

Proposition 3.7. *The natural functors $\mathcal{D}^*(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$, where $* = +, -$ or b, define equivalence of $\mathcal{D}^*(\mathcal{A})$ with full triangulated subcategories of all complexes $A^* \in \mathcal{D}(\mathcal{A})$ with $H^i(A^\bullet) = 0$ for $i << 0, i >> 0$, respectively $|i| >> 0$.*

We noew focus on the main example that shall be dealt for the main part of the thesis.

3.3 Derived Category of Coherent Sheaves.

Let X be a scheme. The category of coherent sheaves $\mathrm{Coh}(X)$ is an abelian category. By Proposition 3.7, we get that $\mathcal{D}^b(\mathrm{Coh}(X))$, the bounded derived category of coherent sheaves on X is a triangulated category. We shall use the notation :

$$\mathcal{D}^b(X) := \mathcal{D}^b(\mathrm{Coh}(X)).$$

The main functor we needed in this section is the cohomology functor. For this we consider the category of quasi-coherent sheaves on a noetherian scheme X over a field k, denoted by $\mathrm{Qcoh}(X)$. The category $\mathrm{Qcoh}(X)$ has enough injectives. We considet the right derived functor of the global section functor Γ(which is a left exact functor from $\mathrm{Qcoh}(X)$ to $\mathrm{Vec}(k)$). It is the map $R\Gamma : \mathcal{D}^+(\mathrm{Qcoh}(X)) \to \mathcal{D}^+(\mathrm{Vec}(k))$.
We define $H^i(X, \mathscr{F}^\bullet) = R^i(\Gamma(\mathscr{F}^\bullet))$. It can be seen that we actually get a functor from $\mathcal{D}^b(\mathrm{QCoh}(X)) \to \mathcal{D}^b(\mathrm{Vec}(k))$.
For defining the cohomology of complex of coherent sheaves, we consider the composition $\mathcal{D}^b(X) \to \mathcal{D}^b(\mathrm{QCoh}(X)) \to \mathcal{D}^b(\mathrm{Vec}(k))$.
We start with this important proposition regarding the computation of Ext of two objects in an abelian category having enough injectives.

Proposition 3.8. *Let $\mathcal{A}$ be an abelian category having enough injectives. Let $\mathcal{D} := \mathcal{D}^b(\mathcal{A})$ be the bounded dervied category. Let $A, B \in \mathcal{A}$ considered as complexes in degree 0. Then :*

$$\mathrm{Ext}^i(A, B) \cong \mathrm{Hom}(A, B[i])$$

Let us consider the special case when X is a smooth projective curve C. It turns out to be an interesting property that any object of $\mathcal{D}^b(\mathbb{C})$ can be studied in terms of its cohomologies.

Proposition 3.9. *Let C be a smooth projective curve. Then any object in $\mathcal{D}^b(C)$ is isomorphic to a direct sum $\oplus \mathscr{E}_i[i]$ where $\mathscr{E}_i$ are coherent sheaves on C.*

Sketch of the proof. The proof is done by induction over the length of the complex. Let $\mathscr{E}^\bullet$ be a complex of length k whose $H^i(\mathscr{E}^\bullet) = 0 \ \ \forall i < i_0$. We can find a distinguished triangle of the form :

$$H^{i_0}(\mathscr{E}^\bullet)[-i_0] \to \mathscr{E}^\bullet \to \mathscr{E}_1^\bullet \to H^{i_0}(\mathscr{E}^\bullet)[1 - i_0]$$

with $\mathscr{E}_1^\bullet$ a complex of length $k-1$ with $H^i(\mathscr{E}^\bullet) = 0 \forall i \leq i_0$. If this distinguished triangle splits, then we have $\mathscr{E}^\bullet = \mathscr{E}_1^\bullet \oplus H^{i_0}(\mathscr{E}^\bullet)[-i_0]$. So we are done. Now by property 4 of propostion 3.1, we just need to show that $\mathrm{Hom}(\mathscr{E}_1^\bullet, H^{i_0}(\mathscr{E}^\bullet)[1-i_0]) = 0$. By induction, we have

$$\mathscr{E}_1^\bullet \cong \oplus - i > i_0 H^i(\mathscr{E}_1^\bullet)[-i]$$

. Thus we have :

$$\mathrm{Hom}(\mathscr{E}_1^\bullet, H^{i_0}(\mathscr{E}^\bullet)[1-i_0]) = \oplus \mathrm{Ext}^{1+i-i_0}(H^i(\mathscr{E}_1^\bullet), H^{i_0}(\mathscr{E}^\bullet)) = 0$$

The last line uses that fact over a curve, the homological dimension is one and also Proposition 3.8. So it is proved. □

We end the section by proving a main property of exact sequence in $\mathcal{D}^b(C)$.

Lemma 3.1. *Given a coherent sheaf E on C and a distingushed triangle $A \to E \to B$ where $A, B \in D^b(C)$ .If $\mathrm{Ext}^{\leq 0}(A,B) = 0 \implies A, B \in \mathrm{Coh}(C)$.*

Proof. The main important fact that is that we are working over a curve. If C is a curve , then we have :

$$A = \oplus A_i[-i]; B := \oplus B_i[-i]$$

where $A_i, B_i \in \mathrm{Coh}(C)$. This is by Proposition 3.9. Now taking cohomology of the distinguished triangle we have the exact sequence :

$$0 \to B_{-1} \to A_0 \to E \to B_0 \to A_1 \to 0$$

and $A_i \cong B_{i-1}$(for $i \neq 0, 1$)
Thus by the condition of Hom, if $A_i \neq 0$ we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}(A_i[-i], B_{i-1}[-i]) \\ &\cong \mathrm{Ext}^{-1}(A_i[-i], B_{i-1}[-i+1]) (\text{by Lemma } 3.8) \end{aligned}$$

which is a contradiction.
Thus we have $A := A_0 \oplus A_1[-1], B := B_{-1}[1] \oplus B_0$. Note that uptil this C being of positive genus is not used.
Now we need to prove that $A_1 \cong B_{-1} \cong 0$
To show $A_1 \cong 0$ we show that the map $B_0 \to A_1$ is zero. As we are on positive genus case, the canoncial bundle $\omega_C := \omega_C^0$ which is the dualizing sheaf has non trivial sections. Let $f : B_0 \to A_1$ be the non-zero map. Then

we have a non-zero map $B_0 \to A_1 \otimes \omega_C$. Thus it follows that:

$$\begin{aligned} 0 \neq \mathrm{Hom}(B_0, A_1 \otimes \omega_C^0) \\ \cong \mathrm{Ext}^1(A_1, B_0) \\ \subset \mathrm{Ext}^1(A[1], B) \\ \cong \mathrm{Ext}^1(A, B[-1]) \\ \cong \mathrm{Ext}^0(A, B).(\text{by Lemma } 3.8) \end{aligned}$$

This it is a contradiction. Similarly, it is shown for B_{-1}. So we have $A, B \in \mathrm{Coh}(X)$. □

3.4 Derived Functors in Algebraic Geometry.

This section lists all the important properties of derived functors needed for the technical aspects in the thesis. We have already dealt with the cohomology functors. Let us look at other functors.

Direct Image: Let $f : X \to Y$ be a morphism of noetherian schemes. The direct image functor : $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ is a left exact functor. Thus we have the corresponding right derived functor

$$\mathrm{Rf}_* : \mathcal{D}^+(\mathrm{QCoh}(X)) \to \mathcal{D}^+(\mathrm{QCoh}(Y)).$$

For a complex $\mathscr{F}^\bullet \in \mathcal{D}^+(\mathrm{QCoh}(X))$, we define the higher direct images $\mathrm{R^if}_*(\mathscr{F}^\bullet) = H^i(\mathrm{Rf}_*(\mathscr{F}^\bullet))$.
Our main concern is to consider the functors on $\mathcal{D}^b(X)$. We need more condition on the morphism f. If $f : X \to Y$ is a projective(proper) map of noetherian schemes, then the higher direct images are coherent. Thus we get the induced functor

$$Rf_* : \mathcal{D}^b(X) \to \mathcal{D}^b(Y).$$

Local Hom: Let $\mathscr{F}$ be a quasi-coherent sheaf on a noetherian scheme X. The usual Hom functor $\mathrm{Hom}(\mathscr{F}, \ ) : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X)$ is a left exact functor. We thus have the right derived functor $R\mathcal{H}om : \mathcal{D}^+(\mathrm{QCoh}(X)) \to \mathcal{D}^+(\mathrm{QCoh}(X))$. We define $\mathscr{E}xt^i(\mathscr{F}, \mathscr{E}) = R^i\mathscr{H}om(\mathscr{F}, \mathscr{E})$.
For coherent sheaves, we need X to be regular. If we are on a regular scheme X, then we have the functor

$$R\mathcal{H}om : \mathcal{D}^b(X) \to \mathcal{D}^b(X).$$

Tensor Product: Let $\mathscr{F}$ be a coherent sheaf on a smooth projective scheme X. The functor $\mathscr{F} \otimes (\ ) : \mathrm{Coh}(X) \to \mathrm{Coh}(X)$ is a right exact functor. As X is projective and smooth, the class of locally free sheaves is adapted to the right exactness and hence we get a left functor $\mathscr{F} \otimes^L (\ ) : \mathcal{D}^-(X) \to \mathcal{D}^-(X)$. We define $\mathrm{Tor}_i(\mathscr{F}, \mathscr{E}) = H^{-i}(\mathscr{F} \otimes^L \mathscr{E})$.
Also we get that any coherent sheaf $\mathscr{E}$ has a finite length resolution of locally free sheaves of length n. So $\mathrm{Tor}_i(\mathscr{F}, \mathscr{E}) = 0\ \forall\ i > n$. Thus we get the functor :

$$\mathscr{F} \otimes^L (\ ) : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$$

Also it can be seen that for any complex $\mathscr{F}^\bullet \in \mathcal{D}^b(X)$, then we have the derived functor

$$(\ ) \otimes^L (\ ) : \mathcal{D}^b(X) \to \mathcal{D}^b(X)$$

which is induced from the functor

$$\mathscr{F}^\bullet \times (\ ) : K^-(\mathrm{Coh}(X)) \to K^-(\mathrm{Coh}(X))$$

sending any complex $\mathscr{E}^\bullet$ to the total complex $\mathscr{F}^\bullet \otimes \mathscr{E}^\bullet$. We also have the generalized Tor as

$$\mathrm{Tor}_i(\mathscr{F}^\bullet, \mathscr{E}^\bullet) := H^{-i}(\mathscr{F}^\bullet \otimes^L E^\bullet).$$

Inverse Image: Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then we have the functor $f^* : \mathrm{Sh}_{\mathcal{O}_Y}(Y) \to \mathrm{Sh}_{\mathcal{O}_X}(X)$ which is right exact. We get the left derived functor

$$Lf^* : \mathcal{D}^{-1}(Y) \to \mathcal{D}^{-1}(X)$$

If f is flat, we have Lf^* to be an exact functor.

Compatibilities. We list all the important properties which indicates the compatibility of these functors wtih each other.

1. Let $f : X \to Y$ be a proper morphism of projective scheme over a field k. Let $\mathscr{F}^\bullet \in \mathcal{D}^b(X), \mathscr{E}^\bullet \in \mathcal{D}^b(Y)$. Then there exists a natural isomorphism (*projection formula*):

$$Rf_*(\mathscr{F}^\bullet) \otimes^L \mathscr{E}^\bullet \cong Rf_*(\mathscr{F}^\bullet \otimes^L Lf_*(\mathscr{E}^\bullet)).$$

2. Let $f : X \to Y$ be a morphism of projective schemes and let $\mathscr{F}^\bullet, \mathscr{E}^\bullet \in \mathcal{D}^b(Y)$. Then there exists a natural isomorphism :

$$Lf^*(\mathscr{F}^\bullet) \otimes^L Lf^*(\mathscr{E}^\bullet) \cong Lf^*(\mathscr{F}^\bullet \otimes^L \mathscr{E}^\bullet).$$

3. Let $f : X \to Y$ be a projective morphism. Then Lf^* and Rf_* are adjoint to each other. In other words, it means we have the functorial isomorphism

$$\mathrm{Hom}(Lf^*\mathscr{F}^\bullet, \mathscr{E}^\bullet) \cong \mathrm{Hom}(\mathscr{F}^\bullet, Rf_*\mathscr{E}^\bullet).$$

4. Lrt X be a smooth projective scheme over a field k. Let $\mathscr{F}^\bullet, \mathscr{E}^\bullet, \mathscr{G}^\bullet \in \mathcal{D}^b(X)$, then we have the following isomorphisms

   $R\mathcal{H}om(\mathscr{F}^\bullet, \mathscr{E}^\bullet \otimes^L \mathscr{G}^\bullet) \cong R\mathcal{H}om(\mathscr{F}^\bullet, \mathscr{E}^\bullet \otimes^L \mathscr{G}^\bullet)$.
   $R\mathcal{H}om(\mathscr{F}^\bullet, R\mathcal{H}om(\mathscr{E}^\bullet, \mathscr{G}^\bullet)) \cong R\mathcal{H}om(\mathscr{F}^\bullet \otimes^L \mathscr{E}^\bullet, \mathscr{G}^\bullet)$.
   $R\mathcal{H}om(\mathscr{F}^\bullet, \mathscr{E}^\bullet \otimes^L \mathscr{G}^\bullet) \cong R\mathcal{H}om(R\mathcal{H}om(\mathscr{E}^\bullet, \mathscr{F}^\bullet), \mathscr{G}^\bullet)$.

5. Let X be a smooth projective scheme over a field k.Let $\mathscr{F}^\bullet \in \mathcal{D}^-(X), \mathscr{E}^\bullet \in \mathcal{D}^b(X)$. Then we have

$$H^p(X, \mathscr{E}xt(\mathscr{F}^\bullet, \mathscr{E}^\bullet)) \cong \mathrm{Ext}^{p+q}(\mathscr{F}^\bullet, \mathscr{E}^\bullet).$$

6. Let $f : X \to Y$ be a morphism of projective schemes. Let $\mathscr{F}^\bullet \in \mathcal{D}^-(Y), \mathscr{E}^\bullet \in \mathcal{D}^b(Y)$. Then there exists a natural isomorphism

$$Lf^*R\mathcal{H}om_Y(\mathscr{F}^\bullet, \mathscr{E}^\bullet) \cong R\mathcal{H}om_X(Lf^*\mathscr{F}^\bullet, Lf^*\mathscr{E}^\bullet).$$

7. Consider a fiber product diagram given

$$\begin{tikzcd} X \times_Z Y \arrow[r, "v"] \arrow[d, "g"] & Y \arrow[d, "f"] \\ X \arrow[r, "u"] & Z \end{tikzcd}$$

   with $u : X \to Z$ and $f : Y \to Z$ proper. Then we have the functorial isomorphism :

$$u^*Rf_*\mathscr{F}^\bullet \cong Rg_*v^*\mathscr{F}^\bullet$$

   for any $\mathscr{F} \in \mathcal{D}^b(\mathrm{QCoh}(Y))$.

Chapter 4

t-structures.

As we have seen in the previous chapter, for any projective scheme X, $\mathrm{Coh}(X)$ is an abelian subcategory of the bounded derived category. Now the question arises, is it possible to get other abelian subcategories of $\mathcal{D}^b(X)$? This question is answered in this section with the concept of t-strucutres.

Definition 4.1. Let $\mathcal{D}$ be a triangulated category. Two full subcategories $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ are called a *t- structure* on $\mathcal{D}$ if the following conditions are satisfied.

1. $\mathcal{D}^{\le -1} \subset \mathcal{D}^{\le 0}$ and $\mathcal{D}^{\ge 1} \subset \mathcal{D}^{\ge 0}$.
2. $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $X \in \mathcal{D}^{\le 0}, Y \in \mathcal{D}^{\ge 1}$.
3. For any object X in $\mathcal{D}$, there exists a distinguished triangle
$$X_0 \to X \to X_1 \to X_0[1]$$
where $X_0 \in \mathcal{D}^{\le 0}, X_1 \in \mathcal{D}^{\ge 1}$.

Remark. We will use the notation $\mathcal{D}^{\le n} := \mathcal{D}^{\le 0}[-n]$ and $\mathcal{D}^{\ge n} := \mathcal{D}^{\ge 0}[-n]$.

Definition 4.2. The *heart* of a t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ is the full subcategory $\mathcal{A} := \mathcal{D}^{\ge 0} \cap \mathcal{D}^{\le 0}$.

Example 4.1. The most common example is the standard t-structure on the derived category of an abelian category $\mathcal{A}$. On $\mathcal{D} = D(\mathcal{A})$ we define $\mathcal{D}^{\le 0} := \{E \in \mathcal{D} | H^i(E) = 0, \forall i > 0\}$, $D^{\ge 0} := \{E \in \mathcal{D} | H^i(E) = 0, \forall i < 0\}$. It can be checked that this is a t structure.

The following proposition helps us to define functors similar to the cohomological functor on $\mathcal{D}^b(X)$.

Proposition 4.1. *1. The inclusion $\mathcal{D}^{\leq n} \to \mathcal{D}$ and $\mathcal{D}^{\geq n} \to \mathcal{D}$ admit right adjoint functors $\tau^{\leq n} : \mathcal{D} \to \mathcal{D}^{\leq n}$ and left adjoint functors $\tau^{\geq n} : \mathcal{D}^{\geq n} \to \mathcal{D}$ respectively .*

2. There exists a unique morphism $d : \tau^{\geq n+1}(X) \to \tau^{\leq n}(X)[1]$ and a distinguished triangle :

$$\tau^{\leq n}(X) \to X \to \tau^{\geq n+1}(X) \to \tau^{\leq n}(X)[1]$$

Morever d defines a natural transformation.

Proof. We do the case for $n = 0$. For any n, we define :

$$\tau^{\leq n}(X) := \tau^{\leq 0}(X)[-n], \tau^{\geq n}(X) := \tau^{1}(X)[-n+1].$$

Now given any object $X \in \mathcal{D}$, by the definition of t structure we have

$$X_0 \to X \to X_1 \to X_0[1],$$

a distinguished triangle such that $X_0 \in \mathcal{D}^{\leq 0}, X_1 \in \mathcal{D}^{\geq 1}$. Define :

$$\tau^{\leq 0}(X) := X_0 \text{ and } \tau^{\geq 1}(X) := X_1$$

Given $f : X \to Y$ a morphism in $\mathcal{D}$, we need to define $f_0 : X_0 \to Y_0$ where $Y_0 \to Y \to Y_1 \to Y_0[1]$ is the distinguished triangle for Y.
Apply $\mathrm{Hom}(X_0, -)$ on the triangle above we have the exact sequence

$$\mathrm{Hom}(X_0, Y_0) \to \mathrm{Hom}(X_0, Y) \to \mathrm{Hom}(X_0, Y_1).$$

Now $X_0 \in \mathcal{D}^{\leq 0}, Y_1 \in \mathcal{D}^{\geq 1} \implies \mathrm{Hom}(X_0, Y_1) = 0$. Thus we have $\mathrm{Hom}(X_0, Y_0)$ $\cong \mathrm{Hom}(X_0, Y)$. So we have a map $f_0 : X_0 \to Y_0$ corresponding to the map $X_0 \to X \to Y$. Thus we get the map f_0 and it is evident that $\tau^{\leq 0}$ is a functor. Similarly we get that $\tau^{\geq 1}$ is also a functor (by applying the $\mathrm{Hom}(-, Y_1)$). Also this conclusion shows that these functors are left and right adjoint functors. This proves 1.
For 2, we know d exists. It is unique because of the following lemma:

Lemma 4.1. *Let $\mathcal{D}$ be a triangulated category and assume we are given two distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h_i} X[1]$ for $i = 1, 2$. If $\mathrm{Hom}_{\mathcal{D}}(X[1], Z) = 0$, then $h_1 = h_2$.*

Applying this lemma to the two morphisms $d_1, d_2 : \tau^{>1}(X) \to \tau^{\leq 0}(X)[1]$ and note that there can't exist morphisms on the reverse as there are no morphisms from $\mathcal{D}^{\leq 0} \to \mathcal{D}^{\geq 1}$. So it is unique. The fact that it is a natural transformation follow from the properties of triangulated categories.

□

Corollary 4.1. $X \in \mathcal{D}^{\leq n} \Leftrightarrow \tau^{\geq n+1}(X) = 0$ *and* $X \in \mathcal{D}^{\geq n} \Leftrightarrow \tau^{\leq n-1}(X) = 0$.

Proof. Follows from the exact sequence given above in the proposition. □

Proposition 4.2. *Let* $X' \to X \to X'' \to X'[1]$ *be a distinguished triangle in* $\mathcal{D}$. *If* X' *and* X'' *are in* $\mathcal{D}^{\leq 0}$*(resp* $\mathcal{D}^{\geq 0}$*), then so is* X.

Proof. Suppose $X', X'' \in \mathcal{D}^{\leq 0}$. Apply $\mathrm{Hom}(\tau^{\leq -1}(X), -)$ to the triangle. Note that $\mathrm{Hom}(\tau^{\leq -1}(X), X') = \mathrm{Hom}(\tau^{\leq -1}(X), \tau^{\leq -1}(X')) = 0$ as $\tau^{\leq -1}(X') = 0$. Similarly, we have $\mathrm{Hom}(\tau^{\leq -1}(X), X'') = 0$. Thus by applying Hom on the triangle, we have that $\mathrm{Hom}(\tau^{\leq -1}(X), X) = 0 \implies \mathrm{Hom}(\tau^{\leq -1}(X), \tau^{\leq -1}(X))$ $= 0 \implies \tau^{\leq -1}(X) = 0 \implies X \in \mathcal{D}^{\leq 0}$. Similarly, we prove it for $\mathcal{D}^{\geq 0}$. □

Proposition 4.3. *The heart of the t structure* $\mathcal{A}$ *is an abelian category.*

Proof. (**Sketch of the proof**)
At first, by the previous proposition we see that if $0 \to X' \to X \to X'' \to 0$ is a distinguished triangle, then if $X', X'' \in \mathcal{A} \implies X \in \mathcal{A}$.
Direct products of elements of $\mathcal{A}$ exists in $\mathcal{A}$. Let $X, Y \in \mathcal{A}$. Consider the triangles $X \to X \to 0 \to 0$ and $0 \to Y \to Y \to 0$. We know the direct product of these triangles exists in $\mathcal{D}$ and as $X, Y \in \mathcal{A} \implies X \oplus Y \in \mathcal{A}$ by the previous remark.
Now we need to show every morphism in $\mathcal{A}$ has a kernel and cokernel in $\mathcal{A}$. Let $f : X \to Y$ be a morphism in $\mathcal{A}$. Complete the morphism to a triangle

$$X \xrightarrow{f} Y \to Z \to X[1].$$

First, we see that $Z \in \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq -1}$. This is seen by using the previous proposition on the two rotated triangles:

$$Y \to Z \to X[1] \to Y[1]$$

and

$$Y[-1] \to Z[-1] \to X \to Y$$

And noticing that $Y, X[1] \in \mathcal{D}^{\leq 0}; Y[-1], X \in \mathcal{D}^{\geq 0}$.
Then we claim that

$$\tau^{\geq 0}(Z) = \operatorname{coker} f; \tau^{\leq 0}(Z[-1]) = \ker f$$

The proof of it being kernel and cokernel are done by exact triangles and by applying Hom functor on these triangles showing these satisfy the universal property of cokernel and kernel. Note that $\tau^{\geq 0}(Z) \in \mathcal{A}$. This is because by definition it is in $\mathcal{D}^{\geq 0}$. Now we consider the rotated triangle :

$$Z \to \tau^{\geq 0}(Z) \to \tau^{\leq -1}(Z)[1]$$

and notice that $\tau^{\leq -1}(Z)[1] \in \mathcal{D}^{\leq 0}$. Thus it is in $\mathcal{A}$. Similarly for the other one.
The last thing that remains to be proved is $\operatorname{coim} f \cong \operatorname{im} f$. This is done by chasing triangles and the octahedral axiom.

□

Lemma 4.2. *For any object $X \in \mathcal{D}$. We have : $\tau^{\geq 0}\tau^{\leq 0}(X) \in \mathcal{A}$*

Proof. Already by definition it is in $\mathcal{D}^{\geq 0}$. Consider the distinguished triangle

$$\tau^{\leq -1}(\tau^{\leq 0}(X)) \to \tau^{\leq 0}(X) \to \tau^{\geq 0}(\tau^{\leq 0}X) \to \tau^{\leq -1}(\tau^{\leq 0}(X))[1].$$

Now we see that $\tau^{\leq -1}(\tau^{\leq 0}X)[1] \in \mathcal{D}^{\leq -1}[1]$. This means $\mathcal{D}^{\leq -2} \subset \mathcal{D}^{\leq 0}$ and $\tau^{\leq 0}(X) \in \mathcal{D}^{\leq 0}$ which means $\tau^{\geq 0}(\tau^{\leq 0}X) \in \mathcal{D}^{\leq 0}$. So it is in $\mathcal{A}$. □

Remark. More generally we have for any m, n : $\tau^{\geq m}\tau^{\leq n} \cong \tau^{\leq n}\tau^{\geq m} \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$.

Definition 4.3. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ of $\mathcal{D}$ is *bounded* if there exists $n \in \mathbb{N}$ such that $E \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq -n}$ for all $E \in \mathcal{D}$.

Definition 4.4. Given any t structure, we define the cohomology functor :

$$H^0 : \mathcal{D} \to \mathcal{A}$$

by :

$$E \to \tau^{\geq 0}\tau^{\leq 0}(E)$$

Also we define $H^n(X) := H^0(X[n])$. (Note $H^k := (\tau^{\geq k}\tau^{\leq k})[k])$).

Remark. In a bounded t structure by the lemmas and by the definition, one notes that only finitely many cohomology functors are non-zero especially $H^k(E) = 0 \; \forall -n \leq k \leq n$.This is because as $E \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq -n} \implies \tau^{\geq n+1}(E) = 0$ and $\tau^{\leq -n-1}(E) = 0$ by the corollary 7.1.

Lemma 4.3. *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded t-structure iff*

1. *$\forall k_1 > k_2, \operatorname{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) = 0$*
2. *For every non-zero object $E \in \mathcal{D}, \exists k_1 > k_2 ... > k_n$ integers and a collection of triangles:*

$$\begin{tikzcd}
0 = E_1 \arrow[rr] & & E_2 \arrow[dl] & \cdots & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul] & & \cdots & & A_n \arrow[ul] &
\end{tikzcd}$$

where $A_i \in \mathcal{A}[k_i]$ for all $1 \leq i \leq n$.

Proof. $\Longrightarrow$ **part**
Given the t structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ with its heart $\mathcal{A}$.
Then :
$\mathrm{Hom}_{\mathcal{D}}(A[k_1], B[k_2]) := \mathrm{Hom}_{\mathcal{D}}(A, B[k_2 - k_1])$. Now $A \in \mathcal{A} \subset \mathcal{D}^{\leq 0}, B[k_2 - k_1] \in \mathcal{D}^{\geq 0}[k_2 - k_1]\mathcal{D}^{\geq k_1 - k_2} \subseteq \mathcal{D}^{\geq 1}$. Thus by the property of t structure,this is 0.
The main task is to construct the sequence of objects for 2. Let $E \in \mathcal{D}$.As it is a bounded t structure we have only finitely many cohomology functors to be non zero. We list $k_1 < k_2 < ..k_n$ from minimum to maximum such that $\tau^{\geq k_i}\tau^{\leq k_i}(E) \neq 0 \forall i = 1, ..n$. For each i we have the distingushied triangle:

$$\tau^{\leq k_{i-1}}(E) \to \tau^{\leq k_i}(E) \to \tau^{\geq k_i}\tau^{\leq k_i}(E) \to \tau^{\leq k_{i-1}}(E)[1]$$

Note that $\tau^{\geq k_i}\tau \leq k_i(E) := H^{k_i}(E)[-k_i] \in \mathcal{A}[-k_i]$. This continues uptil the highest one k_n. Notice that as k_n is the highest we shall have $\tau^{\leq k_n}(E) = E$ (this is by the definition of the sequence of k_i's). Thus we have the diagram where $E_i := \tau^{\leq k_i}(E), A_i = \tau^{\geq k_i}\tau^{\leq k_i}(E) \in \mathcal{A}[-k_i]$. So we also have $-k_1 > -k_2 > \cdots - k_n$ as required.
$\Longleftarrow$ **part**
Given such properties of the additive category $\mathcal{A}$, we define the t structure as follows:

$$\mathcal{D}^{\leq 0}(E) := \{E | A_i = 0 \forall k_i < 0\}; \mathcal{D}^{\geq 0}(E) := \{E | A_i = 0 \forall k_i > 0\}$$

Now clearly $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ as $k_i = 0$ is only possible and that means $E \in \mathcal{A}$. We need to verify the properties of the t structure. The first property of Hom being zero follows from contradiction . Indeed,let $E \in \mathcal{D}^{\leq 0}, F \in \mathcal{D}^{\geq 1}$. Let $E \to F$ be a morphism. Now by the decomposition of object of E and F , we shall get a map from $A \to B$ where $A \in \mathcal{A}[k_1]$ and $B \in \mathcal{A}[k_2]$ where $k_1 \geq 0$ and $k_2 < 0$.By the first property of $\mathcal{A}$ this is zero. $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$.
The last thing to prove is that given any object E in $\mathcal{D}$ there exists a distinguished triangle :

$$E_0 \to E \to E_1 \to E_0[1]$$

such that $E_0 \in \mathcal{D}^{\leq 0}, E_1 \in \mathcal{D}^{\geq 1}$.
We use the second condition now. We have a filtration of E. Define $E_0 = E_i$ where k_i is the least positive in the set$\{k_1, k_2, \cdots, k_n\}$. We have non-zero morphism $E_0 = E_i \to E$. By property of triangulated category, we get a distinguished triangle :

$$E_0 \to E \to E_i \in E_0[1].$$

Then by using the decomposition of E_1, we notice that $E_1 \in \mathcal{D}^{\geq 1}$. Thus we get the last property of it being a t sructure.

□

Chapter 5

Bridgeland Stability Conditions.

In this chapter, we introduce the notion of a stability condition on a triangulated category. At first, we define stability functions and Harder-Narasimhan property of a stability function. Then we proceed with the definition of slicing of a triangulated category which leads to defintiion of a stability condition. The chapter ends with stating a connection between stability condition and t-structures.

5.1 Bridgeland Stability Functions and Harder-Narasimhan filtrations.

Let $\mathcal{A}$ be an abelian category. $K(\mathcal{A})$ be its Grothendieck Group.

Definition 5.1. A *stability function* on an abelian category $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that forall $0 \neq E \in A$, $Z(E)$ lies in

$$\{re^{i\pi\phi}; r > 0, 0 < \phi \leq 1\} \subset \mathbb{C}.$$

Definition 5.2. Given a stability function $Z : K(\mathcal{A}) \to \mathbb{C}$. The *phase* of an object $E \in \mathcal{A}$ is defined by $\phi(E) := \frac{1}{\pi} \arg(Z(E))$.

Definition 5.3. Let $Z : K(\mathcal{A}) \to \mathbb{C}$ be a stability function on an abelian category. An object $0 \neq E \in \mathcal{A}$ is said to be *semistable* if $\forall A \subset E$ subobjects, we have $\phi(A) \leq \phi(E)$.

Lemma 5.1. *If A, B are semistable objects and $\phi(A) > \phi(B)$, then* $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$.

Proof. Let $f : A \to B$, then $A \to A/\ker f \cong \mathrm{im}\, f \subset B$ which implies

$$\phi(A) \leq \phi(A/\ker f) = \phi(\mathrm{im}\, f) \leq \phi(B).$$

Thus the if $\phi(A) > \phi(B)$, then $\mathrm{Hom}_{\mathcal{A}}(A, B) = 0$. □

Definition 5.4. Let $Z : K(\mathcal{A}) \to \mathbb{C}$ be a stability function on an $\mathcal{A}$. A *Harder-Narasimhan filtration* of an object E is a finite chain of subobjects.

$$0 = E_0 \subset E_1 \subset E_2 \cdots\cdots \subset E_n = E.$$

such that $F_j := \frac{E_j}{E_{j-1}}$ are semistable objects with :

$$\phi(F_1) > \phi(F_2) > \cdots\cdots > \phi(F_n)$$

Z is said to have a *Harder-Narasimhan property* if every object of $\mathcal{A}$ has it.

Proposition 5.1. *Suppose $\mathcal{A}$ is an abelian category and $Z : K(\mathcal{A}) \to \mathbb{C}$ is a stability function satisfying the chain conditions :*

1. *There doesn't exist an infinite sequence of subobjects in $\mathcal{A}$ of an object :*

$$\cdots\cdots \subset E_j \subset \cdots \subset E_2 \subset E_1 \subset E$$

such that $\phi(E_{j+1}) < \phi(E_j)$ for all j.

2. *There doesn't exist infinite sequence of quotients in $\mathcal{A}$:*

$$E = E_1 \to E_2 \to E_3 \cdots\cdots$$

such that $\phi(E_j) > \phi(E_{j+1})$ for all j.

Then Z has Harder-Narasimhan property.

Proof. First note that if $E \in \mathcal{A}$, then either E is semistable or there is a subobject of $E_1 \subset E$ such that $\phi(E_1) > \phi(E)$. Continuing in this manner we get a chain of subobjects satisfying the condition 1. Thus it should terminate. So any object E has a semistable subobject A whose $\phi(A) > \phi(E)$. Similar argument works for the quotients.

We define the *maximal destabilizing quotient* (mdq) of an object E is a non-zero quotient $E \to B$ such that $E \to B'$ is another quotient, then $\phi(B') \geq \phi(B)$. The equality holds if

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & B' \\ & B \arrow[ur] & \end{tikzcd}$$

commutes.

Lemma 5.2. *If $E \to B$ is an mdq for E, then B is semistable and $\phi(E) \geq \phi(B)$.*

Proof. If B is not semistable, then it has a quotient B' with $\phi(B') > \phi(B)$. Thus we have a quotient map $E \to B \to B'$. By mdq condition, we have $\phi(B') \geq \phi(B)$ which is a contradiction. Hence it is semistable.
The inequality is from the mdq condition applied on the identity morphism $E \to E$. □

Lemma 5.3. *The maximal destabilizing quotient of an object always exists.*

Proof. Take any object E . If E is semistable, then $E \to E$ is an mdq. If not, then there exist a subobject A such that $\phi(A) > \phi(E)$. Thus we have an exact sequence

$$0 \to A \to E \to E' \to 0.$$

At first, we show

Lemma 5.4. *If $E' \to B$ is an mdq for E', then $E \to B$ the composition map is also a mdq.*

Proof. Suppose $E \to B'$ is a quotient map. Let $\phi(B') < \phi(B)$. Thus by mdq property, we have :

$$\phi(B') < \phi(B) \leq \phi(E') < \phi(E) < \phi(A)$$

So there doesn't exist morphism between $A \to B'$. But the diagram below shows there is a non-zero morphism $f_2 \circ f$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & E \arrow[r, "g"] \arrow[d, "f_2"] & E' \arrow[r, "h"] \arrow[d, "f_1"] & 0 \\ & & B' & B & \end{tikzcd}$$

Hence we have a contradiction. Therefore $\phi(B') \geq \phi(B)$.

Suppose $\phi(B) = \phi(B')$. Then this morphism composition $f_2 \circ f$ is zero. Thus this implies $\ker f_2 \supset \operatorname{im} f = \ker g$. Hence there exists a map $f_3 : E' \to B'$ which is a quotient such that $f_2 = f_3 \circ g$. As $E' \to B$ is a mdq, we have $f_3 = g_1 \circ f_1$, where $g_1 : B \to B'$. Thus $f_2 = f_3 \circ g = g_1 \circ f_1 \circ g$. It factors throught B and hence it is a mdq.

□

We replace E by E' another quotient with $\phi(E) > \phi(E')$. Repeating this, we have a chain of quotients and thus by 2 condition , we need this to terminate to a semistable object and thus we get by composition the existence of a mdq of E. □

Back to the main proof: Let E be an object. If it is semistable, $0 \subset E$ is the filtration. Otherwise there is an exact sequence :

$$0 \to A \to E \to E' \to 0$$

with $E \to E'$ mdq. Let $A \to B$ be a mdq. Thus we have the exact sequence

$$0 \to K \to A \to B \to 0.$$

We also get a map from $K \to E$ (from the map $K \to A \to E$) and thus we have an exact sequence of

$$0 \to K \to E \to Q \to 0.$$

We have the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & E \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
 & & E' \arrow[r, "\cong"] \arrow[d] & E' \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0. &
\end{tikzcd}$$

Thus by definition we have $\phi(Q) > \phi(E')$ as $E \to E'$ is a mdq.So we have $\phi(B) > \phi(E')$. Thus we get the same conditions in the exact sequence

$$0 \to K \to A \to B \to 0$$

where $\phi(B) > \phi(E')$ and B is semistable. Repeating this process we get a sequence of subobjects $E_j \subset E_{j-1} .. \cdots E_1 = E$ with $\phi(E_i/E_{i-1})$ being semistable objects and on decreasing order of phase value. This sequence shall terminate by condition 1.So we have the filtration. $\square$

For defining the stability condition, we need to define slicing which is discussed in the next section.

5.2 Slicing of a triangulated category.

We move to the definition of a new object called slicing which has similar resemblance of conditions given in Lemma 4.3.

Definition 5.5. A *slicing* of a triangulated category $\mathcal{D}$ consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$ satisfying :

1. $\mathcal{P}(\phi+1) := \mathcal{P}(\phi)[1]$.
2. If $\phi_1 > \phi_2$ and $A_j \in \mathcal{P}(\phi_j) \implies \mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$.
3. For each non-zero object $E \in \mathcal{D}$,$\exists$ a finite sequence of real nummbers :
$$\phi_1 > \phi_2 > \cdots\cdots > \phi_n$$
and a collection of triangles
$$\begin{tikzcd} 0 = E_1 \arrow[rr] & & E_2 \arrow[dl] & \cdots & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\ & A_1 \arrow[ul] & & \cdots & & A_n \arrow[ul] & \end{tikzcd}$$
such that $A_j \in \mathcal{P}(\phi_j)$ forall j.

Remark. By condition 3, we can write any object E as a finite sequence of extensions by A_j's which is unique upto isomorphism. Thus we define for any object $E \in \mathcal{D}$, $\phi^+_{\mathcal{P}}(E) := \phi_1$ and $\phi^-_{\mathcal{P}}(E) := \phi_n$. So we have $\phi^+_{\mathcal{P}}(E) \geq \phi^-_{\mathcal{P}}(E)$ and the equality holds if $E \in \mathcal{P}(\phi_j)$ for some $\phi_j \in \mathbb{R}$.

Definition 5.6. Forall $I \subset \mathbb{R}$ an interval, we have $\mathcal{P}(I)$ to be the extension closed subcategory of $\mathcal{D}$ generated by $\mathcal{P}(\phi)$ for $\phi \in I$.
Thus $\mathcal{P}((a,b))$ is the category consisting of zero objects of $\mathcal{D}$ together with $E \in \mathcal{D}$ satisfying $a < \phi^-_{\mathcal{P}}(E) \leq \phi^+_{\mathcal{P}}(E) < b$.

Lemma 5.5. *Let $\mathcal{P}$ be a slicing of a triangulated category. Let $I \subset \mathbb{R}$ be an interval of length atmost one. Suppose*
$$\begin{tikzcd} A \arrow[rr] & & E \arrow[dl] \\ & B \arrow[ul] & \end{tikzcd}$$
is a triangle in $\mathcal{D}$ and $A, E, B \in \mathcal{P}(I)$. Then $\phi^+(A) \leq \phi^+(E)$ and $\phi^-(E) \leq \phi^-(B)$.

Proof. We assume $I = (t, t+1)$. If $\phi = \phi^+(A)$. Then there exists an object $A^+ \in \mathcal{P}(\phi)$ such that $f : A^+ \to A$ exists. If $\phi > \phi^+(E)$. then $A^+ \to E$ can't have any morphism.
For this, we consider the diagram of E in the defintion of slicing(condition 3). Let $f : A^+ \to E$ be a non-zero map. Then we have a map $A^+ \to E \to A_n$. If this map is non -zero, we are done as $\phi^+ > \phi_1 > \phi_n$. Assume the map is zero, then this map factors through E_{n-1}. Repeating the same process, we have a

non-zero map $A^+ \to E_{n-1} \to A_{n-1}$. Again we have two cases. Continuing this process, eventually we have a map from $A^+ \to A_1 = E_1$ which has to be non-zero. This can't occur as we have $\phi^+ > \phi_1 = \phi^+(E)$. Thus the morphism doesn't exist.
But in the diagram we have one. Thus the composition of them has to be zero. By the property of distinguished triangles, the map $f : A^+ \to A$ shall factor through $B[-1]$. Thus we have map $A^+ \to B[-1]$. So $\phi^+(B[-1]) \leq \phi^+(A^+) = \phi$. But as $B \in \mathcal{P}((t, t+1)) \implies B[-1] \in \mathcal{P}(\leq t)$. Thus $\phi \leq t$. But $\phi > t$ as $A \in \mathcal{P}((t, t+1))$. We arrive at a contradiction. □

Remark. It is important to note that for any $\phi \in \mathbb{R}$, $\mathcal{P}(> \phi)$ and $\mathcal{P}(\geq \phi)$ are t-structures on $\mathcal{D}$.Their corresponding hearts are $\mathcal{P}((\phi, \phi+1])$ and $\mathcal{P}([\phi, \phi+1))$.

5.3 Quasi-abelian Categories

The categories $\mathcal{P}(I)$ desribed above not be abelian categories but slightly less than that.
Recall that a morphism $f : A \to B$ in a category is called "strict" if $\operatorname{coim} f \cong \operatorname{im} f$

Definition 5.7. A *quasi-abelian category* is an additive category $\mathcal{A}$ with kernels and cokernels such that the pullback of a strict epimorphism is a strict epimorphism and the pushout of a strict monomorphism is a strict monomorphism.

Definition 5.8. A *strict short exact sequence* in a quasi-abelian category is a diagram:

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

such that i is $\ker j$ and j is $\operatorname{coker} i$.

Remark. All other properties of "strict" monomorphisms and "strict" epimorphisms make this quasi-abelian category along with the family of "short" exact sequences an " exact" category.
Also we define the Grothendieck Group on it to be the abelian group on the objects with the relation: $[B] = [A] + [C]$ if there exists a strict short exact sequence $0 \to A \to B \to C \to 0$

Lemma 5.6. *An additive category is quasi-abelian iff there exists abelian categories $\mathcal{A}^\#$ and $\mathcal{A}^b$ and fully faithful embeddings $\mathcal{A} \subset \mathcal{A}^\#, \mathcal{A} \subset \mathcal{A}^b$ such that if:*

1. *If $A \to E$ is a monomorphism in $\mathcal{A}^\#$ with $E \in \mathcal{A}$, then $A \in \mathcal{A}$.*

2. If $E \to B$ is a epimorphism in $\mathcal{A}^b$ with $E \in \mathcal{A}$, then $B \in \mathcal{A}$.

Example 5.1. The most common example of quasi-abelian category is the category of torsion free sheaves over a projective variety.

Lemma 5.7. *Let $\mathcal{P}$ be a slicing of the triangulated category $\mathcal{D}$. For any interval I of $\mathbb{R}$ of length atmost 1, the full subcategory $\mathcal{P}(I)$ is quasi-abelian. Every short exact sequence in it corresponds to the tirangle whose vertice are in $\mathcal{P}(I)$.*

Proof. It follows from the fact $\mathcal{P}((a,b)) \subset \mathcal{P}((a,a+1]), \mathcal{P}((a,b)) \subset \mathcal{P}([b-1,b))$. □

We move to the next section of stability conditions.

5.4 Stability Conditions.

Definition 5.9. A *stability condition* $\sigma = (Z, \mathcal{P})$ on a triangulated category $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ and a slicing $\mathcal{P}$ of the category such that for all $0 \neq E \in \mathbb{P}(\phi)$ we have $Z(E) := m(E)e^{i\pi\phi}$.
Z is called the *central charge.*

Lemma 5.8. *If $\sigma = (Z, \mathcal{P})$ is a stability condition on a triangulated category $\mathcal{D}$, then $\mathcal{P}(\phi) \subset \mathcal{D}$ is abelian.*

Proof. $\mathcal{P}(\phi)$ is a full additive subcategory. To show it is abelian, we need to show a morphism in $\mathcal{P}(\phi)$ has the kernel and cokernel in $\mathcal{P}(\phi)$. Given a morphism $f : E \to F$ we have the kernel in an exact sequence in $\mathcal{D}$. Using lemma 5.5 and comparing ϕ^+ , we get that it is in $\mathcal{P}(\phi)$. Similarly for the cokernel. □

Definition 5.10. Given a stability condition σ, we have a decomposition A_j's of E .They are called the semistable factors of E, We call $m(E) := \sum |Z(A_i)|$ the *mass* of E.

Remark. For any object E in $K(\mathcal{D})$ where $\mathcal{D}$ is a triangulated category with slicing, the argument of E is between $\phi^-(E)$ and $\phi^+(E)$.

We now prove the main theorem which the connects stability conditions and t-structures. It is an important tool in constructing the stability conditions.

Theorem 5.1. *To give a stability condition on a triangulated category is equivalent to give a bounded t-structure and a stability function on its heart with the Harder-Narasimhan property.*

Proof. $\underline{\implies}$
Let $\sigma := (Z, \mathcal{P})$ be the stability condition. Let $\mathcal{D}^{\leq 0} := \mathcal{P}(> 0)$ and $\mathcal{D}^{\geq 1} := \mathcal{P}(\leq 0)$. Thus $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} := \mathcal{P}((0, 1])$ which is the heart. These two define t structures. The inclusions $\mathcal{D}^{\leq -1} \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$ are obvious.Also the condition of Hom of two objects being zero is just from the fact of decomposition of slicing of the two given objects is equivalent to the condition of $\mathrm{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) = 0 \forall \phi_1 > \phi_2$. Now for the existence of the distinguished triangle is the same argument as in the proposition of proving the condition of heart.

Also if $\mathcal{A} = \mathcal{P}((0, 1])$, then $K(\mathcal{A}) \cong K(\mathcal{D})$ (this is a because $\mathcal{A}$ is the heart of a bounded t-structure). So the central charge Z is the stability function on its heart $\mathcal{A}$.

Now we need to define the semistable objects in $\mathcal{A}$. The semistable objects in $\mathcal{A}$ are $\mathcal{P}(\phi), \forall 0 < \phi \leq 1$. We need to see that they are semistable objects. This is because of the Lemma 5.5 and the fact that the exact sequences in $\mathcal{A}$ correspond to exact triangles in $\mathcal{D}$ whose vertices are in $\mathcal{A}$. Now the third condition of slicing shows that for any object $E \in \mathcal{A}$, we have a sequence of subobjects E_i and their quotients A_i which are semistable and also the decreasing order of their phases. Thus this satisfies the Harder-Narasimhan property.

$\underline{\Longleftarrow}$

We have a t structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and the stability function on the heart $\mathcal{A}$ with HN property. As $K(\mathcal{A}) \cong K(\mathcal{D})$, we have a group homomorphism from $K(\mathcal{D}) \to \mathbb{C}$. The main aim is to construct the slicing $\mathcal{P}$. Let $\mathcal{P}(\phi)$ be the full additive subcategory of the semistable objects in phase ϕ. By the first property of slicing, $\mathcal{P}(\phi)$ is defined for all $\phi \in \mathbb{R}$ .The second condition of slicing comes from the property of morphism between semistable objects. The third condition of slicing follows from the HN property of the stability function and the property that exact sequences in $\mathcal{A}$ correspond to exact triangles. Hence we get the pair $\sigma := (Z, \mathcal{P})$ which is a stability condition.

□

Chapter 6

Elementary examples and properties of stability conditions.

In the previous chapter, we saw what a stability condition is. This chapter deals with the examples of stability conditions. At first, we deal the case of a curve and show a nice property about the space of stability conditions $\mathrm{Stab}(\mathcal{D}(X))$ over a curve X. The chapter ends with the case of construction of stability conditions over projective schemes of dimension more than 1 with the standard heart $\mathrm{Coh}(X)$.

6.1 Stability Conditions on curves.

Let X be a non-singular projective curve. Let $\mathcal{A}$ be the category of coherent sheaves on X. Let $\mathcal{D}(\mathcal{A})$ be the bounded derived category of coherent sheaves on X. We need to define a stability condition on $\mathcal{D}(\mathcal{A})$. By Theorem 5.1, we define the stability condition on its heart i.e $\mathcal{A}$ w.r.t the standard t-structure which has the Harder-Narasimhan property.
The stability function Z on $\mathcal{A}$ is defined as :

$$Z(E) := -\deg(E) + i\,\mathrm{rk}(E)$$

Note that the degree of a sheaf E is defined by the formula:

$$\deg(E) := \alpha_0(X) - rk(E)\alpha_0(\mathcal{O}_X)$$

where α_i are coefficients of the Hilbert polynomial of the projective curve. Note that this is a stability function because both rk and deg can't be zero. If $rk = 0$,then by the above formula ,it will be $\alpha_0(X)$ which is always positive. Also by definition $rk \geq 0$. So $Z(E \neq 0)$ lies in the corresponding region for it to be a stability function.

Now we need to see the semistable objects of this function. Those are such sheaves $E \in \mathcal{A}$ such that $F \subset E$ a subsheaf , then $\phi(F) \leq \phi(E)$.

$$\phi(F) \leq \phi(E) \Leftrightarrow \tan^{-1}\left(\frac{1}{-\mu(F)}\right) \leq \tan^{-1}\left(\frac{1}{-\mu(E)}\right)$$

which is same as .

$$\frac{1}{-\mu(F)} \leq \frac{1}{-\mu(E)} \Leftrightarrow -\mu(E) \leq -\mu(F) \Leftrightarrow \mu(F) \leq \mu(E)$$

Thus this is same as μ-semistability condition of E.

Also notice that if a sheaf E is torsion, then $\dim E = 0 \implies \mathrm{Supp}(E)$ has only finitely many points. Thus any subsheaf of E will also have dimension zero. So E is semistable.

Now we need to have the Harder-Narasimhan Property on this ϕ function.

This is true by the theory of semistable sheaves. Every sheaf on X (as it is of dimension one it is pure) has the Harder Narasimhan filtration. The semistable condition is the general semistable condition by the reduced Hilbert Polynomial $p(E)$. Applying Theorem 2.4, we get the Harder Narasimhan Property of ϕ function.

Hence by Theorem 5.1, we get a stability condition on $\mathcal{D}(\mathcal{A})$.

6.2 Action of groups on $\mathrm{Stab}(\mathcal{D})$.

$\mathrm{Stab}(D)$ is defined as the space of stability conditions. Later we shall define this explicitly with the support property. There are two important actions on $\mathrm{Stab}(\mathcal{D})$, namely $\mathrm{Aut}(\mathcal{D})$, the group of automorphisms of $\mathcal{D}$ and $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$, the universal covering space of $\mathrm{Gl}_2^+(\mathbb{R})$.

6.2.1 Action of $\mathrm{Aut}(\mathcal{D})$

Let $\phi \in \mathrm{Aut}(\mathcal{D})$. Let $\sigma := (Z, \mathcal{P})$ be a stability condition. Then the action is defined as $\phi.(Z, \mathcal{P}) := (Z', \mathcal{P}')$ where $Z' := Z \circ \phi^{-1}$ and $\mathcal{P}'(t) := \phi(\mathcal{P}(t))$.

6.2.2 Action of $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$

$\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ is the universal covering space of $\mathrm{GL}_2^+(\mathbb{R})$.

It is defined as a pairs (T, f) where $f : \mathbb{R} \to \mathbb{R}$ an increasing function with the property $f(\phi + 1) = f(\phi) + 1$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an orientation preserving linear isomorphism with the property that when restricted to $S^1 =$

$\mathbb{R} \cup \infty \cong \mathbb{R}/\mathbb{Z}$ is the same as f.

Now the action is defined as : $(T, f) \circ (Z, \mathcal{P}) = (Z', \mathcal{P}')$ where $Z' := T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$

Remark. The above action gains its importance when $\mathcal{D}$ is the bounded derived category of coherent sheaves over a projective scheme, especially when it is over a smooth projective curve of genus ≥ 1. For the proofs below, we assume the assumpotions on Z given in Section 7.1. Notice that on the case of a curve, the numerical Grothendieck group $\mathcal{N}(C) \cong \mathbb{Z} \oplus \mathbb{Z}$ where the map is $[E] \to (\mathrm{rk}(E), \deg(E))$ is an isomorphism.

Let C be a smooth projective curve over $\mathbb{C}$ of genus ≥ 1.

Lemma 6.1. *Given any stability condition on $D^b(C)$, the skyscraper sheaves $\mathcal{O}_c$($c \in C$) and the line bundles $L \in \mathrm{Pic}(C)$ are stable .*

Proof. Let us prove at first that the line bundles and skyscraper sheaves are semistable.

Now let E be the skyscraper sheaf or the line bundle. Consider the first semistable factor in its Harder Narasimhan Filtration of E. Let it be A. We get the complete distinguished triangle :

$$A \to E \to B$$

Now we have $\phi(A) > \phi(B)$.So we have $\mathrm{Hom}(A, B[i]) = 0$ for all $i \leq 0$. So that means $\mathrm{Ext}^i(A, B) = 0$ for all $i \leq 0$ (By Lemma 3.8) . Thus by the previous lemma we have $A, B \in \mathrm{Coh}(C)$.
Thus we have :

$$0 \to A \to E \to B \to 0$$

In the case of a skyscraper sheaf, as it is over a smooth curve we don't have proper subobjects . Thus $A \cong E$ and $B = 0$. So $\mathcal{O}_c$ is semistable.
In the case of line bundle, as A is subobject, it has to be a line bundle .So B is either zero or a torsion sheaf.If it is a torsion sheaf, then it is supported in finitely many points and thus we have $\mathrm{Hom}(A, B) \neq 0$ which is not true. So $B = 0$ and thus $A \cong E$ is a line bundle.
So we have the skyscraper sheaf and the line bundle are semistable. Now we prove that they are stable.
Consider the category $\mathcal{P}(\phi(E))$ the category of objects having the phase $\phi(E)$. Assume E is not stable. Then there exist a stable subobject $S \in \mathcal{P}(\phi(E))$ of E such that $\mathrm{Hom}(S, E) \neq 0$.
Now consider the classes of subobjects $X \subset E$ whose Jordan filtration has all

stable factors isomorphic to S. Let A be the maximal of the collection . Then we have the exact sequence :

$$0 \to A \to E \to B \to 0$$

By construction of A, $\phi(A) = \phi(E)$. Thus $\phi(B) = \phi(A)$ (as $Z(B) = Z(E) - Z(A)$). So then $\phi(A) > \phi(B) - i$ for $i < 0$. So it implies that $\mathrm{Hom}(A, B[i]) = 0, i < 0$ which means $\mathrm{Ext}^{<}(A, B) = 0$.
Also A is maximal and thus B has no proper subobject isomorphic to S. As we know $\mathcal{P}(\phi(E))$ is abelian. So we have $\mathrm{Hom}(S, B) = 0$. Now as A is just made by extension of S so $\mathrm{Hom}(A, B) = 0$. Combining with the conclusion in the previous paragraph, we have $\mathrm{Ext}^{\leq}(A, B) = 0$.Thus by Lemma 3.8, we have $A, B \in \mathrm{Coh}(X)$. By the same method of semistability proof in each case, we have $A \cong E$.
Let A is made by n copies of S. In $K(\mathcal{D})$, then we have $[E] = n[S]$.Now if E is skyscraper sheaf, we have $\deg(E) = 1$. So this implies $n = 1$. In the case of line bundle we have $rk(E) = 1$ which again implies $n = 1$.
So both the skyscraper sheaves and the line bundle are stable in all the stability conditions. □

Now we prove the main theorem:

Theorem 6.1. *The action of* $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ *on* $\mathrm{Stab}(\mathcal{D})$ *is free and transitive. This means :*

$$\mathrm{Stab}(\mathcal{D}) \cong \widetilde{\mathrm{GL}_2^+(\mathbb{R})}$$

Proof. By the previous lemma, we showed that given any stability condition $\sigma := (Z, \mathcal{P})$ the skyscraper sheaves $\mathcal{O}_c$ and the line bundles L are stable.
Now we have the obvious map $L \to \mathcal{O}_c$. So we have $\mathrm{Hom}(L, \mathcal{O}_c) \neq 0$. This means $\phi(L) < \phi(\mathcal{O}_c)$.
Now also we have the following:

$$\begin{aligned} &\mathrm{Hom}(\mathcal{O}_c, L[1]) \\ &\cong \mathrm{Ext}^1(\mathcal{O}_c, L) \\ &\cong \mathrm{Ext}^0(L, \mathcal{O}_c \otimes \omega_C)(\text{Serre Duality}) \\ &\cong \mathrm{Hom}(L, \mathcal{O}_c) \neq 0 \end{aligned}$$

So we have $\phi(\mathcal{O}_c) < \phi(L) + 1$
Thus we have $\phi(\mathcal{O}_c) - 1 < \phi(L) < \phi(\mathcal{O}_c)$
Now we know that the numerical Grothendieck group is generated by the skyscraper and the line bundles(as any coherent sheaf is) .Thus the generators are $(0, 1)$ and $(1, \deg(L))$. Tensoring with $\mathbb{R}$ we have $\mathcal{N}(C) \otimes \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{C}$.So Z is an isomorphism.(as we have assumed Z factors through $\mathcal{N}(C)$).
Now the points $(0, 1)$ and $(1.d)(d = \deg L)$ by Z is mapped to $Z(\mathcal{O}_c)$ and

$Z(L)$ with the arguments $\pi\phi(\mathcal{O}_c)$ and $\pi\phi(L)$. Now the configuration of the diagram shows that the determinant of the map Z is positive and thus it is orientation preserving.
The action is free is because if an element (T, f) fixes a stability condition then f has to be identity and thus by the defintion of T it also has to be identity. So it has to be free.
Now it remains to show that it is transitive.
The way to show this is that upto action of $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ we can change the stability condition σ to the standard stability condition in terms of degree and rank.So for this there are two things to be checked. First of all Z can be written as of the standard form upto action of a matrix $M \in \mathrm{GL}_2^+(\mathbb{R})$ .Second of all, the corresponding heart of the corresponding t structure is $\mathrm{Coh}(C)$

The first part is just done as Z is orientation preserving thus by a matrix $M \in \mathrm{GL}_2^+(\mathbb{R})$ we can assume that $Z(E^\bullet) := -\deg(E^\bullet) + i\,\mathrm{rk}(E^\bullet)$ for $E^\bullet \in D^b(\mathrm{Coh}(C))$

For the second part let $B = \mathcal{P}(0, 1]$ be the heart of t structure associated to σ.At first we claim that all the skyscraper sheaves have same phase.
Let $\mathcal{O}_{c_1}, \mathcal{O}_{c_2}$ be two skyscraper sheaves.As we know Z maps both of them to same value as they are same on $\mathcal{N}(C)$.Thus their arguments differ by an even multiple of 2π. This means that $\phi(\mathcal{O}_{c_1}) - \phi(\mathcal{O}_{c_2}) \in 2\mathbb{Z}$. But we know that :

$$\phi(\mathcal{O}_{c_i}) - 1 < \phi(L) < \phi(\mathcal{O}_{c_i}), i = 1, 2$$

which implies that $\phi(\mathcal{O}_{c_1}) = \phi(\mathcal{O}_{c_2})$.
Now upto composition by a suitable element of $\widetilde{\mathrm{GL}_2^+(\mathbb{R})}$ we can assume that $\phi(\mathcal{O}_c) = 1$. (Notice this composition is done after assuming the stability is in the standrad degree and rank formula). Thus by the inequality, we have $\phi(L) \in (0, 1) \forall L \in \mathrm{Pic}(C)$.

As we are on a curve, every coherent sheaf is generated by line bundles and skyscraper sheaves(torsion and torsion free part). Thus by the property of phase function we have $\phi(E) \in (0, 1) \forall E \in \mathrm{Coh}(C)$ So we have $\mathrm{Coh}(C) \subset B$. Now we have two hearts of corresponding bounded t structures and one is contained in the other.We need to show thet are the same.

Lets prove this. Let $A \subset B$ be the two hearts. Let $Z \in B$,then Z has a filtration whose quotients are in shifts of A . But we know that these quotients will lie in shifts of B.But as $Z \in B$, the filtration will have only one object namely Z and by the filtration done with respect to A , this means $Z \in A$.Thus $A = B$.So we have proved that the action is transitive which completes the proof. $\square$

6.3 Stability conditions over $\dim > 1$ with heart $\mathrm{Coh}(Y)$.

The case of curves is clear. The next theorem is about what happens if we try to construct a stability function on Coh(Y).

Theorem 6.2. *Let Y be a smooth projective variety over $\mathbb{C}$ of dimension ≥ 2. There is no numerical stability condition $\sigma \in \mathrm{Stab}(Y)$ with heart $\mathrm{Coh}(Y)$.*

Proof. Assume that $\sigma := (Z, \mathcal{P})$ a stability condition whose corresponding heart is $\mathrm{Coh}(Y)$.
By definItion of Z, it implies that is an element of dual of $\mathcal{N}(Y) \otimes \mathbb{C}$(here we assume that Z factors through $\mathcal{N}(Y)$ which is the lattice Λ defined in the next chapter). By construction of numerical Grothendieck Group, we have for any sheaf $E \in D^b\,\mathrm{Coh}(Y)$:

$$Z(E) := \sum_{i=0}^{d} (u_i + i v_i).ch_i(E)$$

where $ch_i(E)$ is the ith chern character of E and $u_i, v_i \in H^{2d-2i}(Y, \mathbb{R})$.
Now as it is of dimension $d \geq 2$, there is a surface $S \to Y$ embedded in Y. Thus we have the induced stability function on $K(S) \to K(Y) \to \mathbb{C}$.
So we have reduced this to case of $d = 2$. Thus Y is a surface now. Let $C \subset Y$ be a curve. Let D be a divisor on C. Consider the sheaf $\mathcal{O}_C(D)$. It has rank 0 on Y. So the zeroth chern character is 0. Let us consider the imaginary part of it. We have :

$$\mathscr{I}(Z(\mathcal{O}_C(D))) = v_2(\deg D + ch_2(\mathcal{O}_C)) + v_1.[C] > 0$$

Now as the degree is arbitrary of any divisor, for it to be positive we should have $v_2 = 0$.
Then consider the sheaf $\mathcal{O}_Y(mC), m \in \mathbb{Z}$.For this also then we have

$$\mathscr{I}(\mathcal{O}_Y(mC)) = m v_1.[C] + v_0.1 > 0.$$

As m also can be made sufficiently small, we have then $v_1.[C] = 0$.
Thus we have $\mathscr{I}(Z(\mathcal{O}_C(D))) = 0$.
Thus for it to be a stability condition we need to have

$$\mathscr{R}(Z(\mathcal{O}_C(D))) := u_2(\deg D + ch_2(\mathcal{O}_C)) + u_1.[C] \leq 0.$$

By the same argument, we have $u_2 = 0$.
We know that for any skyscraper sheaf $\mathcal{O}_x, x \in Y$, $\mathrm{ch}(\mathcal{O}_x) = (0, 0, 1)$ (point 3 of Chapter 1, Section 4.) Thus, we have

$$Z(\mathcal{O}_x) := u_2 + i v_2 = 0$$

But $\mathcal{O}_x$ is semistable with respect to the t structure (seen on a curve), thus $Z(\mathcal{O}_x) \neq 0$. So it is a contradiction.

$\square$

Chapter 7

Deformation Property of Stability Conditions.

This chapter entirely focuses on proving Theorem 0.1. At first we setup all the necessary definitions and tools required for proving the theorem. This includes defining the topology of $\mathrm{Stab}_{\Lambda}(\mathcal{D})$ and the concept of Harder-Narasimhan polygons. We prove the theorem under some assumptions. The proof ends by showing that those assumptions can be made.

7.1 Important assumptions

Note: From now on, $\mathcal{D}$ will be a triangulated category with a surjective group homomorphism

$$v : K(\mathcal{D}) \to \Lambda$$

where $\Lambda \cong \mathbb{Z}^m$ for some $m \in \mathbb{N}$.
Also we have assumed that the stability condition $Z : K(\mathcal{D}) \to \mathbb{C}$ factors through Λ via v.

7.2 Support Property

Definition 7.1. Let $Q : \Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R} \to \mathbb{R}$ be a quadratic form. We say that a stability condition $(Z, \mathcal{P})$ satisfies the *support property* with respect to Q if:

1. $\ker Z \subset \Lambda_R$ is negative definite with respect to Q.

2. For all semistable objects E i.e $E \in \mathcal{P}(\phi)$, we have $Q(v(E)) \geq 0$.

7.3 The Statement of Deformation Property

The theorem assumes the construction of topological space $\mathrm{Stab}_\Lambda(\mathcal{D})$ with a metric which is explained in the next section. Also we have the map $\mathscr{Z}$: $\mathrm{Stab}_\Lambda(\mathcal{D}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ given by $(Z, \mathcal{P}) \to Z$.

Theorem 7.1. *Let Q be a quadratic form on $\Lambda \otimes \mathbb{R}$ and assume that the stability condition $\sigma := (Z, \mathcal{P})$ satisfies the support property with respect to Q. Then*

1. *There is an open neighbourhood $\sigma \in U_\sigma \subset \mathrm{Stab}_\Lambda(\mathcal{D})$ such that the map $\mathscr{Z} : U_\sigma \to \mathrm{Hom}(\Lambda, \mathbb{C})$ is a local homeomorphism and Q is negative definite on $\ker Z'$ for all $(Z', \mathcal{P}') \in U_\sigma$.*

2. *All $\sigma \in U_\sigma$ satisfies the support property with respect to Q.*

Remark. At first, notice that $\mathrm{Hom}(\Lambda, \mathbb{C}) \cong \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}) \cong \oplus_{i=1}^m \mathrm{Hom}(\mathbb{Z}, \mathbb{C}) \cong \mathbb{C}^m$. So the theorem states that locally we have that $\mathrm{Stab}_\Lambda(\mathcal{D})$ is homemorphic to $\mathbb{C}^m$ and thus it has a manifold structure. Also it states that any path in $\mathrm{Hom}(\Lambda, \mathbb{C})$ can be lifted to a continuous path of stability conditions in $\mathrm{Stab}_\Lambda(\mathcal{D})$. We shall see that by Lemma 7.5, we get that the collection of all $Z \in \mathrm{Hom}(\Lambda, \mathbb{C})$ such that Q is negative definite on $\ker Z$ is an open subset of $\mathbb{C}^m$.

7.4 Topology on $\mathrm{Stab}_\Lambda(\mathcal{D})$

Notice that $\mathrm{Stab}_\Lambda(\mathcal{D}) \subset \mathrm{Slice}(\mathcal{D}) \times \mathrm{Hom}(\Lambda, \mathbb{C})$ where $\mathrm{Slice}(\mathcal{D})$ is the set of slicings. Thus, at first, we define the topology on $\mathrm{Slice}(\mathcal{D})$ and $\mathrm{Hom}(\Lambda, \mathbb{C})$.

Topology on $\mathrm{Slice}(\mathcal{D})$

Given two slicings $\mathcal{P}, \mathcal{Q}$ we define a distance function. We set :

$$d(\mathcal{P}, \mathcal{Q}) := \sup\{|\phi^\pm_{\mathcal{P}}(E) - \phi^\pm_{\mathcal{Q}}(E)|, 0 \neq E \in \mathcal{D}\}$$

This is a distance function. The following lemma is about that the calculation can be done just considering the semistable objects of $\mathcal{P}$.

Lemma 7.1. *Define:*

$$d'(\mathcal{P}, \mathcal{Q}) := \sup\{\psi^+(E) - \phi, \phi - \psi^-(E), 0 \neq E \in \mathcal{P}(\phi)\}$$

Then we have $d(\mathcal{P}, \mathcal{Q}) = d'(\mathcal{P}, \mathcal{Q})$. (Here $\psi^\pm$ are of $\mathcal{Q}$.)

Proof. Obviously $d'(\mathcal{P},\mathcal{Q}) \leq d(\mathcal{P},\mathcal{Q})$ by definition. We need to show the converse.
Let $E \in \mathcal{D}$. Let A_i be one of the HN factors of E wih respect to $\mathcal{P}$. We have

$$\psi^+(A_i) \leq \phi(A_i) + d'(\mathcal{P},\mathcal{Q}) \leq \phi^+(E) + d'(\mathcal{P},\mathcal{Q}).$$

Thus every A_i satisfies this. Hence, we have $\psi^+(E) \leq \phi^+(E) + d'(\mathcal{P},\mathcal{Q})$. Similarly we can get the inequality for ψ^-, ϕ^- too. Hence $d(\mathcal{P},\mathcal{Q}) \leq d'(\mathcal{P},\mathcal{Q})$ which completes the proof. □

Remark. This metric d is a generalized metric with the additional poprerty that the distance function attains the value infinity. If $d(\mathcal{P},\mathcal{Q}) = \infty$, then we have $\mathcal{P}$ and $\mathcal{Q}$ are on different connected components of $\mathrm{Slice}(\mathcal{D})$.

Topology on $\mathrm{Hom}(\Lambda,\mathbb{C})$

Let $\sigma = (Z,\mathcal{P})$ be a stability condition.
Let $U \in \mathrm{Hom}(\Lambda,\mathbb{C})$. Define

$$||U||_\sigma := \sup\left\{\frac{U(E)}{Z(E)}, \text{E is} \quad \sigma-\text{semistable}\right\}$$

Let us consider V_σ which is defined a:

$$V_\sigma := \{U \in \mathrm{Hom}(\Lambda,\mathbb{C})|, ||U||_\sigma < \infty\}.$$

Now it is easy to see that $||\ ||_\sigma$ defines a finite norm on V_σ. Morever the following lemma tells us that we need not to worry about V_σ when we have the support property.

Lemma 7.2. *If σ satisfies the support property with respect to Q a quadratic form on $\Lambda \otimes \mathbb{R}$, then we have $V_\sigma := \mathrm{Hom}(\Lambda,\mathbb{C})$.*

Proof. By assumption in 7.6, we see that $E \in K^\perp$ where $K = \ker Z$. So $Q(E) = |Z(E)|^2$. Now as $\Lambda_\mathbb{R}$ is of finite dimensional, the collection $\{v(E); E \quad \text{semistable}\ \}$ is finite dimensional. Let $\alpha_1, \alpha_2 \cdots ..\alpha_k$ span the space as an orthogonal basis with respect to Q. Let $[E] = \sum_{i=1}^k a_i\alpha_i$. Thus $Q(E) = \sum_{i=1}^k a_i^2$. Let $||.||$ be a norm on $\Lambda_\mathbb{R}$. Then

$$||E|| = ||\sum_{i=1}^k a_i\alpha_i|| \leq \sqrt{\sum_{i=1}^k a_i^2}.||\sum_{i=1}^k \alpha_i|| \leq \sqrt{Q(E)}.||\sum_{i=1}^k \alpha_i||$$

Thus we get a constant C such that

$$||[E]|| \leq C|Z(E)|.$$

Let $U \in \text{Hom}(\Lambda, \mathbb{C})$. Then we have a constant D such that $|U(E)| \leq D.|[E]|$ .Combining this with above inequality, we have $|U(E)| < C.D|Z(E)|$. Thus we have $||U||_\sigma < \infty$.

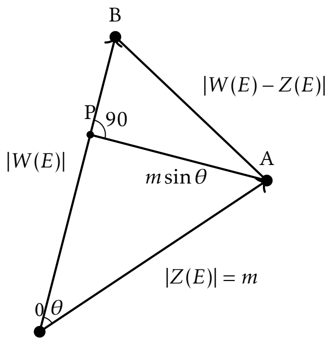
□

We arrive at the definition of the topology on $\text{Stab}_\Lambda(\mathcal{D})$.
Let $\sigma = (Z, \mathcal{P})$ be a stability condition, we define the balls :

$$B_\epsilon(\sigma) = \{\tau = (W, \mathcal{Q}) \mid ||W - Z||_\sigma < \sin(\pi\epsilon),\ d(\mathcal{P}, \mathcal{Q}) < \epsilon\}.$$

We have the following remarks.

Remark. 1. The conditions $||W-Z||_\sigma < \sin(\pi\epsilon)$ and $d(\mathcal{P}, \mathcal{Q}) < \epsilon$ are compatible in the sense that $||W - Z||_\sigma < \sin(\pi\epsilon) \implies d(\mathcal{P}, \mathcal{Q}) < \epsilon$. This is evident from the diagram below.



2. These form the basis of the topology on $\text{Stab}_\Lambda(\mathcal{D})$. The main idea behind the proof of this statement is if $\tau \in B_\epsilon(\sigma)$, then $||.||_\sigma \sim ||.||_\tau$. This implies that in a connected component of $\text{Stab}_\Lambda(\mathcal{D})$, the topology is same as the subspace topology of the product topology $\text{Slice}(\mathcal{D}) \times \text{Hom}(\Lambda, \mathbb{C})$.

We have an important lemma which proves that the map $\mathscr{Z} : \text{Stab}_\Lambda(\mathcal{D}) \to \text{Hom}(\Lambda, \mathbb{C})$ given by $(Z, \mathcal{P}) \to Z$ is locally injective.

Lemma 7.3. *If* $\sigma := (Z, \mathcal{P}), \tau := (Z, \mathcal{Q})$ *are two stability conditions with* $d(\mathcal{P}, \mathcal{Q}) < 1$. *Then* $\sigma = \tau$.

This gives us the immediate corollary.

Corollary 7.1. *The map* $\mathscr{Z} : \text{Stab}_\Lambda(\mathcal{D}) \to \text{Hom}(\Lambda, \mathbb{C})$ *is locally injective.*

7.5 Harder-Narasimhan polygons.

Let $\mathcal{A}$ be an abelian category and Z be a stability function on $\mathcal{A}$.

Definition 7.2. The *Harder-Narasimhan polygon* $HN^Z(E)$ of an object $E \in \mathcal{A}$ is the convex hull of the central charges $Z(A)$ of all the subobjects $A \subset E$ of E.

Definition 7.3. We say that the Harder-Narasimhan polygon $HN^Z(E)$ of an object $E \in \mathcal{A}$ is *polyhedral on the left* if the set has finitely many extremal points $0 = z_0, z_1..z_m = Z(E)$ such that $HN^Z(E)$ lies right to the path $z_0, z_1, \cdots, z_m$. See the figure below.

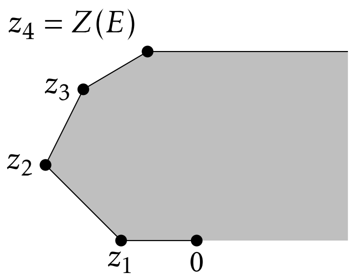


Figure 7.1: HN polygon polyhedral to the left

The following theorem relates the connections between HN polygons and HN filtrations.

Theorem 7.2. *An object E has HN filtration w.r.t Z iff the HN polygon of $Z(E)$ is polyhedral to the left.*

This leads to the following important corollary which is important for the main proof.

Corollary 7.2. *Given $E \in \mathcal{A}$, assume there are only finitely many classes $v(A)$ of subobjects $A \subset E$ with $\Re Z(A) < \max\{0, \Re Z(E)\}$. Then E admits a HN filtration.*

7.6 Main Proof

Important Assumption:
Assume that Q has signature $(2, \operatorname{rk}\Lambda - 2)$. It will be proved later that this assumption can be made.

The proof at first includes two lemmas to reduce the Theorem 7.1 to a special case.

Lemma 7.4. *Upto the action of* $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$ *on* $\mathrm{Stab}_\Lambda(\mathcal{D})$*, we may assume that we are in the following situation:*
There is a norm $\|.\|$ *on* $\ker Z$ *such that if* $p : \Lambda_{\mathbb{R}} \to \ker Z$ *denoted the orthogonal projection with respect to* Q *, then :*

$$Q(v) := |Z(v)|^2 - \|p(v)\|^2.$$

Proof. Let $K^\perp$ denoted the orthogonal complement of $K = \ker Z$ w.r.t Q. We know that $\ker Z$ is negative definite w.r.t Q.
Considering the map $Z : \Lambda_R \to \mathbb{C}$ as a map of real vector space, we get that $\dim K \geq rk\Lambda - 2$. So $\dim K^\perp \leq 2$. Now we use the assumption. The assumption says that $\dim K^\perp \geq 2$. So $\dim K^\perp = 2$.
As $Z|_{K^\perp} : K^\perp \to \mathbb{C}$ is injective, by dimension argument this is an isomrphism of real vector spaces. By action of $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$ we can assume that this map is an isometry.
Let $\|.\|^2 = -Q$ on K
So now we write Q in terms of signature:

$$Q(v) = Q \circ (1-p)v + Q \circ p(v) \implies |Z(v)|^2 - \|p(v)\|^2$$

□

Consider the subset of central charges in $\mathrm{Hom}(\Lambda, \mathbb{C})$ whose kernel is negative definite with respect to Q. Denote $\mathcal{P}_Z(Q)$ be its connected component starting from Z.

Lemma 7.5. *Assuming that we are in the situation of the previous lemma. Upto the action of* $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$ *,we can assume that* $Z' \in \mathcal{P}_Z(Q)$ *is of the form*

$$Z' := Z + u \circ p$$

where $u : \ker Z \to \mathbb{C}$ *with operator norm* $\|u\| < 1$

Proof. We have the isomorphism $Z'|_{K^\perp} = Z|_{K^\perp}$ where $K = \ker Z$ by the previous lemma.Thus we have $g \in \mathrm{Gl}_2(\mathbb{R})$ such that $Z'|_{K^\perp} = gZ|_{K^\perp}$. Now $\mathcal{P}_Z(Q)$ is connected, so we have $g \in \mathrm{Gl}_2^+(\mathbb{R})$. Now we define $u : \ker Z \to \mathbb{C}$ the restriction of Z' to $\ker Z$. Now we see that the equation $Z' = g \circ Z + u \circ p$ is valid in K and $K^\perp$. Thus it is valid in $\Lambda_{\mathbb{R}}$. So up to the action of $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$,we have :

$$Z' = Z + u \circ p$$

□

The reduction is now as follows:

Lemma 7.6. *In order to prove Theorem 7.1, it is enough to show the following: Given any stability condition $\sigma_0 = (Z_0, \mathcal{P}_0)$ and a path of central charges $t \to Z_t = Z_0 + t.u \circ p$ for $t \in [0,1]$ where $u : \ker Z \to \mathbb{R}$ a linear map to the real numbers with $\|u\| < 1$, there exists a continuous life $t \to \sigma_t$ to the space of stability conditions. Morever all σ_t satisfy the support property with respect to the same quadratic form Q.*

Proof. We know that before the map is locally injective by corollary 7.1. So it is enough to prove the existence of lift of any given path. Also we can replace the path in $\mathcal{P}_Z(Q)$ by a homotopic one.
Suppose that this holds. By $\widetilde{\mathrm{Gl}_2^+(\mathbb{R})}$ action, this shall hold for u to be purely imaginary. Now for our case write $u = \mathscr{R}(u) + i\mathscr{I}(u)$, as $\|u\| < 1 \implies \|\mathscr{R}u\|, \|\mathscr{I}(u)\| < 1$. So by our assumption at first we can construct the path from Z to $Z + \mathscr{R}u \circ p = Z'$. Then from Z' we can lift it to $Z_1 = Z' + i\mathscr{I}u \circ p = Z' + u \circ p$. Composing these two paths, we get the required path.

□

Let Z_0 be the stability condition initially given with the heart $\mathcal{A} = \mathcal{P}(0,1]$. We need to show that Z_t is also a stability condition with the same heart $\mathcal{A}$. It is enough to show that $Z_1 = Z + u \circ p$ is a stability condition as for other ones, it is just the same.
The first part is regarding the proof of it being the stability function.

Lemma 7.7. *Let Z, u as before, then $Z_1 = Z + u \circ p$ is a stability function on $\mathcal{A}$.*

Proof. . Let $E \in \mathcal{A}$. If $\mathscr{I}(Z(E)) = \mathscr{I}(Z_1(E)) > 0$, then it is fine. Otherwise we have then $Z(E) \in \mathbb{R} < 0$. We can see that E has to be semistable. We know that $Q(v(E)) \geq 0$. This implies :

$$|Z(E)|^2 - \|p(E)\|^2 \geq 0 \implies (Z(E) - \|p(E)\|)(Z(E) + \|p(E)\|) \geq 0.$$

Now the first term is already negative. This implies that $\|p(E)\| \leq -Z(E)$. Thus we have

$$Z_1(E) \leq Z(E) + \|u\|.\|p(E)\| < Z(E) - Z(E) = 0.$$

Hence it is a stability function. □

Now we need to show that it satisfies the Harder-Narasimhan Property. At first we define mass of an object E.

Definition 7.4. The mass $m^Z(E)$ of an object E with respect to Z is the length of the boundary $HN^Z(E)$ from 0 to $Z(E)$.

The Harder-Narsimhan property shall now be proved by using several lemmas and the main corollary at the end of HN polygon section.

Lemma 7.8. *For all $E \in \mathcal{A}$ we have $\|p(E)\| \leq m^Z(E)$.*

Proof. If E is semistable, then $|Z(E)| = m^Z(E)$. Thus we have $(m^Z(E))^2 - \|p(E)\|^2 = Q(v(E)) \geq 0$ which proves the inequality.
Now if E is not semistable, write E as semistable factors E_i with respect to Z. So we have :

$$\|p(E)\| \leq \sum \|p(E_i/E_{i-1})\| \leq \sum |Z(E_i/E_{i-1})| \leq |Z(E_i) - Z(E_{i-1})| = Z(E).$$

.The last equality follows from the plotting of the Harder-Narasimhan polygon.

□

Lemma 7.9. *If $A \subset E$, then $HN^Z(A) \subset HN^Z(E)$.*

Proof. The proof is obvious as subobjects of A are subobjects of E and thus convex hull of A will be inside of convex hull of E. □

Lemma 7.10. *Given a subobject $A \subset E$, we have*

$$m^Z(A) - \mathscr{R}(Z(A)) \leq m^Z(E) - \mathscr{R}(Z(E)).$$

Proof. We see the picture below. We choose $x > \mathscr{R}Z(A), \mathscr{R}Z(E)$. Let $a = x + i\mathscr{I}Z(A)$ and $e = x + i\mathscr{Z}(E)$. The paths γ_A, γ_E are defined in the figure. Now we see that :

$$|\gamma_A| = m^Z(A) + x - \mathscr{R}Z(A) \qquad |\gamma_E| = m^Z(E) + x - \mathscr{R}Z(E)$$

It follows from the picture that ;

$$|\gamma_A| \leq |\gamma_I| \leq |\gamma_E|$$

and hence the result follows. □

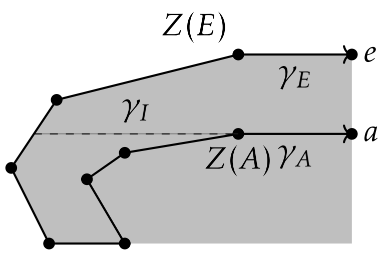


Figure 7.2: Proof of Lemma 7.10

Lemma 7.11. *Given $C \in \mathbb{R}$, there are only finitely many subobjects $A \subset E$ such that $\mathscr{R}(Z + u \circ p)(A) < C$.*

Proof. We have

$$C > \mathscr{R}(Z + u \circ p)(A) \geq \mathscr{R}(Z(A)) - \|u\|.\|p(A)\| \geq \mathscr{R}(Z(A)) - \|u\| m^Z(A).$$

We rewrite the last expression as

$$(1 - \|u\|)\mathscr{R}(Z(A)) - \|u\|(m^Z(A) - \mathscr{R}^Z(A)).$$

Using Lemma 7.10, we have it is greater than equal to

$$(1 - \|u\|)\mathscr{R}(Z(A)) - \|u\|(m^Z(E) - \mathscr{R}(Z(E))).$$

As $\|u\| < 1$, we have the bound of $\mathscr{R}(Z(A))$. Now if the real part of $Z(A)$ is bounded and as $Z(A) \in HN^Z(E)$, then we have that $Z(A)$ lies in the compact region of $\mathbb{C}$. So $|Z(A)|$ is bounded .
By Lemma 7.10, we have $m^Z(A)$ is bounded. By Lemma 7.8, we have that $\|p(A)\|$ is bounded.
Thus then $Q(v(A))$ is bounded by $|Z(A)|^2$ and $|Z(A)|^2 - \|p(A)\|^2$. So in the topology on $\Lambda_{\mathbb{R}}$ which is $\mathbb{R}^m$, the norm is defined by $|v| := \sqrt{|Q(v)|}$. According to this topology and by the previous arguments, the class $v(A)$ of all such A satisfying the required conditions is closed and bounded in $\mathbb{R}^m$. This is in $\Lambda \subset \mathbb{R}^m$ which is discrete topology. Hence we have a compact subset in Λ and thus it is finite.

□

So by Corollary 7.2, we have that $(Z_1, \mathcal{A})$ satisfies the Harder-Narasimhan property.

We need to show that the lift of any path is a continuous map in the space of stability conditions. This shall show that the inverse map is still continuous.

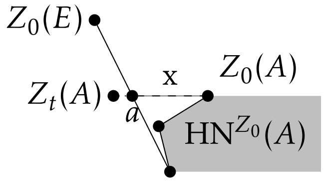
Lemma 7.12. *The map $t \to \sigma_t = (Z_t, \mathcal{A})$ is a continuous path in the space of stability conditions.*

Proof. By definition of the topology on $\mathrm{Stab}_\Lambda(\mathcal{D})$, If t is small, we shall show that $d(\mathcal{P}_0, \mathcal{P}_t)$ is small where $\mathcal{P}_t$ is the assciated slicing of Z_t.
Let E be a semistable object of $\mathcal{A}$. We need to show that

$$\psi^+(E) - \phi(E)$$

is sufficiently small.(where ψ, ϕ are the phases of Z_t, Z_0).The same can be shown for ψ^-
To compute $\psi^+(E)$, let A be the leading semistable factor of E with respect to Z_t. We have $\psi^+(E) = \psi(A)$.

Write $Z_0(A) = a + x$. where $a \in \mathbb{C}$ has same phase as $Z_0(E)$ and $x \geq 0$.
Now $m^{Z_0}(A) \leq |a| + x$ (Figure below).



Therefore we have :
$\mathscr{R}(Z_t(A)) \geq \mathscr{R}(Z_0(A)) - t\|u\|.\|p(A)\| \geq \mathscr{R}(Z_0(A)) - tm_0^Z(A) \geq \mathscr{R}(Z_0(A)) - tx - t|a| \geq \mathscr{R}(a) - t|a|$.
Note that :

$$\pi.(\psi^+(E) - \phi(E)) = \arg \frac{Z_t(A)}{a}.$$

Now let us prove a small lemma at first :

Lemma 7.13. *If* $z_1 = a + ib, z_2 = c + id$, *then*

$$\arg\left(\frac{z_1}{z_2}\right) \leq \sin^{-1} \frac{|z_1 - z_2|}{|z_2|}.$$

Proof. By elementary calculations, we have :

$$\arg\left(\frac{z_1}{z_2}\right) = \sin^{-1} \frac{bc - ad}{|z_1||z_2|}$$

So we need to show that

$$bc - ad \leq |z_1||z_1 - z_2| = \sqrt{a^2 + b^2}\sqrt{(b-d)^2 + (c-a)^2}.$$

This is true by the Cauchy-Schwarz inequality.

□

Now we apply this to $z_1 = Z_t(A), z_2 = a$. We get

$$\pi.(\psi^+(E) - \phi(E)) \leq \sin^{-1} \frac{|Z_t(A) - a|}{|a|}$$

Now note $\mathscr{I}(Z_t(A)) = \mathscr{I}(Z_0(A)) = \mathscr{I}(a)$. Thus $|Z_t(A) - a| = \mathscr{R}a - \mathscr{R}(Z_t(A)) \leq t|a|$.
Hence the argument is less than $\sin^{-1}(t)$.
Similarly we do this for ψ^-.

Thus we then have $d(\mathcal{P}_0, \mathcal{P}_t) \leq \frac{1}{\pi}t$.

□

So we have a continuous map in $\mathrm{Stab}_\Lambda(\mathcal{D})$.

7.6.1 Preservation of Quadratic Reciprocity

By the previous subsection, we have the local homeomorphim by lemma 7.12. Its remains to show that all the stability conditions in the specific above path satisfy the support property.
That is we need to show that :

$$\forall t \in [0,1] \quad Q(v(E)) \geq 0 \text{ where E is semistable object of } Z_t$$

We show that the stability condition Z_1 satisfies this. For all other t, the proof remains the same. The main idea of the proof is to proceed by contradiction. If E doesn't satisfy the required condition, then it has to be Z_0 unstble. The main idea is to get a t such that E is strictly Z_t semistable.

Assumption

We are assuming the notion of Jordan-Holder Filtration of a semistable object with respect to a stability condition. It is similar as of the filtration in case of sheaves.

Main proof

Before the main proof we have three lemmas and definition about truncated HN polygons.

Lemma 7.14. *Let $\sigma = (Z, \mathcal{P})$ be a stability condition. Assume that Q is non-degenerate quadratic form on $\Lambda_{\mathbb{R}}$ of signature $(2, \operatorname{rk}\Lambda - 2)$ such that Q is negative definite on* $\ker Z$*. If E is strictly $\sigma-$ semistable and admits a Jordan Holder Filtration with factors $E_1, E_2, ...E_m$ and if $Q(v(E_i)) \geq 0$ for $i = 1, 2 \cdots m$ then $Q(v(E)) \geq 0$.*

Proof. We apply $Q(v) = |Z(v)|^2 - \|p(v)\|^2$ from lemma 7.4. So $Q(v) \geq 0 \implies Z(v) \geq \|p(v)\|$. We obtain :

$$|Z(E)| = \sum |Z(E_i)| \geq \sum \|p(v(E_i))\| \sum \|p(v(E_i))\| = \|p(v(E)\|).$$

Thus we have $Q(v(E)) \geq 0$. □

Lemma 7.15. *Given two objects $A, E \in \mathcal{A}$ denote their phases with respect to Z_t by $\phi_t(A), \phi_t(E)$. If the set of $t \in [0,1]$ with $\phi_t(A) \geq \phi_t(E)$ is non-empty. Then it is a closed subinterval of $[0,1]$ containing one of its endpoints.*

Proof.

$$
\begin{aligned}
& \phi_t(A) \geq \phi_t(E) \\
\implies & \frac{\mathscr{I}(Z_t(A))}{\mathscr{R}(Z(_tA))} \geq \frac{\mathscr{I}(Z_t(E))}{\mathscr{R}(Z_t(E))} \\
\implies \frac{\mathscr{I}(Z(A))}{\mathscr{R}(Z(A)) + tu \circ p(A)} \geq & \frac{\mathscr{I}(Z(E))}{\mathscr{R}(Z(E)) + tu \circ p(E)}
\end{aligned}
$$

which is a linear inequality in t and thus it shall have one of the endpoints. □

Remark. This lemma is called the wall crossing. We get a wall such that the condition holds.

We define the truncated HN polygons.

Definition 7.5. Let Z_0 be a stability condition and $E \in \mathcal{A}$. The polygon formed by the external points of $HN^{Z_0}(E)$ on the left is called the *truncated HN polygon.*(7.3)

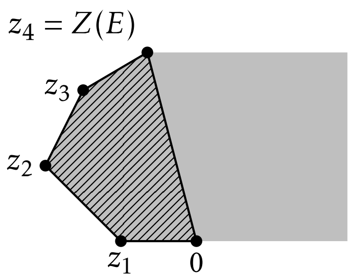


Figure 7.3: A truncated HN polygon

Remark. 1. If $A \subset E$ with $\phi_0(A) \geq \phi_0(E)$ iff $Z_0(A)$ in the truncated HN polygon of E. This is evident from the diagram. See 7.4.

2. Now if $Z_0(A)$ is in the truncated HN polygon of E, then we know that $\mathscr{R}(Z_0(A))$ and $Z(A)$ are bounded. Thus by the Lemma 7.10, we have

$$m^{Z_0}(A) \leq \mathscr{R}(Z_0(A)) + m^{Z_0}(E) - \mathscr{R}(Z_0(E))$$

Also then $\|p(v(A))\| \leq m^Z(A)$ (Lemma 7.8)is also bounded. Thus in the topology of $\Lambda_{\mathbb{R}}$ the classes $v(A)$ form a compact set in $\mathbb{Z}$. Thus there are finitely many such classes of objects.

Now we prove the main result.

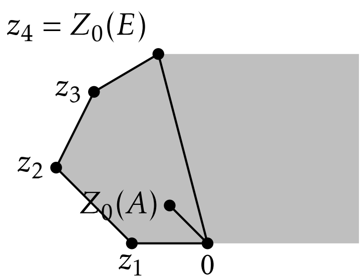


Figure 7.4: Remark 1

Lemma 7.16. *Every Z_1 semistable object $E \in \mathcal{A}$ satisfies $Q(v(E)) \geq 0$.*

Proof. Suppose it is not, then by assumption of the support property of Z_0 we have E is Z_0 unstable. By the remark 2, we get that for any $t \in [0,1]$ there are only finitely many subobjects $v(A)$ which destabilise E with respect to Z_t.
Let $A_1, ..A_m$ are destabilizing objects of Z_0. For each those of A_i's, by applying Lemma 7.15, we get $p_i \in [0,1]$ such in $[0, p_i]$ the destabilizing inequality is satisfied with respect to Z_{p_i}'s. Take $1 > t_1 > \max\{p_1, p_2, \cdots p_m\}$.
The claim is E is stricitly semistable with respect to Z_{t_1}. Suppose it is not. Then we have a subobject $A \subset E$ such that $\phi_{t_1}(A) > \phi_{t_1}(E)$. By Lemma 7.15 we get a range where this occurs. Note that the endpoint considered in this inequality can't be 1 as E is semistable with respect to Z_1. So it implies by the inequality that E has to be Z_0 unstable. Thus $A = A_i$ for some i and thats a contradiction by the construction of t_1.

So E is strictly semistable w.r.t Z_{t_1} and thus admits a Jordan-Holder filtration. By Lemma 7.14, we have then as $Q(v(E)) < 0 \implies Q(v(F_1/G_1)) < 0$ where $G_1 \to F_1 \to E$ are subobjects E of the same phase such that F_1/G_1 is stable.

Applying the same logic above to F_1/G_1 instead of E in the interval $[0, t_1]$, we get $t_2 \in (0, t_1)$ such that F_1/G_1 is stricitly semistable w.r.t Z_{t_2} and corespondingly subobjects $G_1 \subset G_2 \subset F_2 \subset F_1$ and $Q(v(F_2/G_2)) < 0$ and$G_2/G_1, F_2/G_1, F_1/G_1$ are of the same phase. Continuing like this, we get an infinite chain of subobjects of E :

$$G_1 \subset G_2 \subset G_3 \cdots \cdots F_3 \subset F_2 \subset F_1 \subset E$$

and a sequence $t_1 > t_2 > t_3 \cdots \cdots$
Now we know $\phi^{t_1}(F_1) = \phi^{t_1}(E) \implies$ the set of t such that $\phi^t(F_1) \geq \phi^t(E)$ is non empty and by Lemma 7.15 it is a closed subinterval of the form$[0, x]$ (if it contains 1 as endpoint, then it shall contradict the fact that E

is semitable w.r.t Z_1). Thus $\phi^{t_2}(F_1) \geq \phi^{t_2}(E)$ and similarly $\phi^{t_2}(G_1) \geq \phi^{t_2}(E)$. Now we know that :

$$\phi^{t_2}(F_1/G_1) = \phi^{t_2}(F_2/G_1)$$

We perform the following manipulation:

$$\begin{aligned}
\phi^{t_2}(F_1/G_1) &= \phi^{t_2}(F_2/G_1) \\
\implies \frac{Z_{t_2}(F_1/G_1)}{\mathcal{I}(Z_{t_2}(F_1/G_1))} &= \frac{Z_{t_2}(F_2/G_1)}{\mathcal{I}(Z_{t_2}(F_2/G_1))} \\
\implies \frac{Z_{t_2}(F_1) - Z_{t_2}(G_1)}{\mathcal{I}(Z_{t_2}(F_1/G_1))} &= \frac{Z_{t_2}(F_2) - Z_{t_2}(G_1)}{\mathcal{I}(Z_{t_2}(F_2/G_1))} \\
\implies \frac{\mathcal{I}(Z_{t_2}(F_2/G_1))Z_{t_2}(F_1) - \mathcal{I}(Z_{t_2}(F_2/G_1))Z_{t_2}(G_1)}{\mathcal{I}(Z_{t_2}(F_1/G_1))} + Z_{t_2}(G_1) &= Z_{t_2}(F_2) \\
\implies Z_{t_2}(F_2) = \frac{\mathcal{I}(Z_{t_2}(F_2) - Z_{t_2}(G_1))Z_{t_2}(F_1) + \mathcal{I}(Z_{t_2}(F_1) - Z_{t_2}(F_2))Z_{t_2}(G_1)}{\mathcal{I}(Z_{t_2}(F_1) - Z_{t_2}(G_1))}
\end{aligned}$$

This expression shows that $Z(F_2)$ is in the line segment joining $Z(F_1)$ and $Z(G_1)$ and thus similarly for G_2. Thus we get that $\phi^{t_2}(F_2) \geq \phi^{t_2}(E)$ and $\forall t \in [0, t_2]$, we have $\phi^t(F_2) \geq \phi^t(E)$ and similarly for G_2. Continuing with t_3 and so on with the same argument we get

$$\phi^0(F_i) \geq \phi^0(E); \phi^0(G_i) \geq \phi^0(E)$$

for all i.
Remark 1 of Truncated HN polygon says that $Z_0(F_i), Z_0(G_i)$ are in truncated HN polygon of $Z_0(E)$. But there are finitely many classes of such objects and thus the process should terminate.(Remark 2 of truncated HN polygon). □

Thus this ends our whole proof under the Assumption 7.6 of the quadratic form. The next section will be on proving that the assumption can be made.

7.7 Reduction to the Assumption

This proof is done into two parts. First we show that we can assume that the quadratic form to be non-degenerate. Secondly, we show that the signature can be assumed to be $(2, \operatorname{rk}\Lambda - 2)$.

Lemma 7.17. *Assume that the quadratic form Q on $\Lambda_{\mathbb{R}}$ is degnerate. Then there exists an injection $\Lambda_{\mathbb{R}} \to \overline{\Lambda}$ of real vector spaces and a non-degenerate quadratic form $\overline{Q}$ on $\overline{\Lambda}$ extending $\mathbb{Q}$ such that the central charge $Z : \Lambda_{\mathbb{R}} \to \mathbb{C}$ whose kernel is negative definite with respect to Q extends to $\overline{Z} : \overline{\Lambda} \to \mathbb{C}$ whose kernel is negative definite with respect to $\overline{Q}$.*

Proof. Let $N \hookrightarrow \Lambda_{\mathbb{R}}$ be the null space of Q. By iterating the process, we assume that $\dim_{\mathbb{R}} N = 1$.
We write :$\Lambda_{\mathbb{R}} = N \oplus C$. Now we define $\overline{\Lambda} = N^v \oplus N \oplus C$. Let B be the corresponding billinear form of Q. We define the quadratic form $\overline{Q}$ on $\overline{Z}$ as $\overline{Q} = q \oplus Q|_C$ where q is the quadratic form on $N \oplus N^\vee$ as $q(x+y) = 2xy$ where $x \in N, y \in N^\vee$ It is easily seen that this is a quadratic form. The corresponding billinear form on $N \oplus N^\vee$ is $b(x_1 + x_2, y_1 + y_2) = x_1y_2 + x_2y_1$ where$x_1, y_1 \in N, x_2, y_2 \in N^\vee$.
Now we show that the $\overline{Q}$ is non-degenerate. Let $\overline{B}$ be the corresponding billinear form .Let $v = x_1 + x_2 + x_3; x_1 \in N, x_2 \in N^\vee, x_3 \in C$ be the degeneracy vector of $\overline{B}$.
We have $\overline{B}(x_1+x_2+x_3, x_1) = b(x_1+x_2, x_1)+B(x_3, x_1) = x_1x_2$ So either $x_1 = 0$ or $x_2 = 0$. If $x_1 = 0$. Let $w \in C$, we have $\overline{B}(x_2+x_3, w) = \overline{B}(x_2, 0)+\overline{B}(x_3, w) = \overline{B}(x_3, w)$. Thus $x_3 = 0$ as it is true for any $w \in \mathbb{C}$.
If $x_2 = 0$,then $w' \in \Lambda_{\mathbb{R}}$. Then $\overline{B}(x_1 + x_3, w') = \overline{B}(x_1, w') + \overline{B}(x_3, w') = \overline{B}(x_3, w')$. Thus this implies $x_3 = 0$.
Now in either case we reduce to $v = x_1$ or $v = x_2$. Then by definition of b we can have $x \in N \oplus N^v$ such that $b(x_1, x) \neq 0$ or $b(x_2, x) \neq 0$. So $v = 0$. Thus $\overline{B}$ is non-degenerate.
Now we define $\overline{Z}$.
As $\ker Z$ is negative definite with respect to Q, this means $Z|_N$ is injective. So we can assume Z maps N to the real line.(as the image is of dimension is 1). Let $n \in N$ such that $Z(n) = 1$. Let $n^\vee$ be the dual vector of n in $N^\vee$. We define $\overline{Z}(n^\vee) = \alpha$ and $\alpha >> 0$ such that it satisfies the desired property.
Let $K = \ker Z$. The kernel of $\overline{Z}$ is contained in $N \oplus N^\vee \oplus K$ (Notice that $K \subset C$). It is given by vectors of the form $a.n - \frac{a}{\alpha}n^\vee + k$ where $n \in N, k \in K$. Now the main condition is that these vectors are negative definite with respect to $\overline{Q}$.
Now

$$\begin{aligned}\overline{Q}\left(a.n - \frac{a}{\alpha}n^\vee + k\right) &= \overline{B}\left(a.n - \frac{a}{\alpha}n^\vee + k, a.n - \frac{a}{\alpha}n^\vee + k\right)\\ &= b\left(a.n - \frac{a}{\alpha}n^\vee, a.n - \frac{a}{\alpha}n^\vee\right) + B(k,k) + 2B\left(a.n - \frac{a}{\alpha}n^\vee, k\right)\\ &= -\frac{2a^2}{\alpha} - \frac{2a(n^\vee, k)}{\alpha} + Q(k).\end{aligned}$$

This is quadratic function in a with a negative constant term. Thus for the function to be negative, we need the discriminant to be negative, i.e

$$\frac{4(n^\vee,k)^2}{\alpha^2}+\frac{2Q(k)}{\alpha}<0 \implies \alpha > \frac{(n^\vee,k)^2}{-Q(k)}$$

for all $k \in K$. Now we need to choose the α the maximum of all such quantities. We need to show that it is possible. Let $k_1, ..k_m$ be an orthogonal basis of K with respect to B.
Let $k=\sum_{i=1}^m a_i k_i$. Now we see that :

$$\begin{aligned}
&\frac{(n^\vee,k)^2}{-Q(k)}\\
&=\frac{(n^\vee,\sum_{i=1}^m a_ik_i)^2}{-Q(\sum_{i=1}^m a_ik_i)}\\
&=\frac{\left(\sum_{i=1}^m a_i\sqrt{-Q(k_i)}\frac{(n^\vee,k_i)}{-\sqrt{-Q(k_i)}}\right)^2}{\sum_{i=1}^m -a_i^2Q(k_i)}\\
&\le\frac{(\sum_{i=1}^m -a_i^2Q(k_i))\left(\sum_{i=1}^m \frac{(n^\vee,k_i)^2}{-Q(k_i)}\right)}{\sum_{i=1}^m -a_i^2Q(k_i)}\\
&=\sum_{i=1}^m\frac{(n^\vee,k_i)^2}{-Q(k_i)}.
\end{aligned}$$

where the inequality used is Cauchy-Schwarz. We choose $\alpha > \sum_{i=1}^m \frac{(n^\vee,k_i)^2}{-Q(k_i)}$ and it shall work.
Thus we define α such that it satisfies the condition of negative definite property.

□

Now thus we can assume the quadratic form to be non-degenerate. Now we need to prove the assumption of signature to be $(2, \operatorname{rk}\Lambda-2)$.
Now as $\ker Z$ is negative definite, we have the signature of Q to be $(p, \operatorname{rk}\Lambda - p)$ where $p\in\{0,1,2\}$. We have the following lemma.

Lemma 7.18. *Assume that Q is non-degenerate and of signature $(p, \operatorname{rk}\Lambda-p)$ for $p\in\{0,1\}$. Let $\Lambda=\Lambda_{\mathbb{R}}\oplus\mathbb{R}$. We define $\overline{Q}(v,\alpha)=Q(v)+\alpha^2$ for $v\in\Lambda_{\mathbb{R}}, \alpha\in\mathbb{R}$. Then any central charge on Z whose kernel is negative definite with respect to Q extends to $\overline{Z}$ on$\Lambda_{\mathbb{R}}$ whose kernel is negative definite with respect to $\overline{Q}$.*

Proof. Let $K\subset\Lambda_{\mathbb{R}}$ be the kernel of Z. Let $K^\perp$ be the orthogonal complement with respect to Q. We know $Z|_{K^\perp}$ is injective. Thus from signature of Q, rank of $K^\perp$ is of one or of signature $(1,-1)$. So we have a one dimensional basis of $K^{\perp}$ where Q is negative definite. This means we have $v' \in K^{\perp}$ such that $Q(v') < -1$. Let $Z(v') = z$.Now notice that if $v \in \Lambda_{\mathbb{R}}$ exists such that $Z(v) = z$. This implies $v = w + v'$ where $w \in \ker Z$. Thus $Q(v) < Q(v') < -1$. We define $\overline{Z}(v, \alpha) = Z(v) + \alpha z$ .If $(v, \alpha) \in \ker \overline{Z}$, then $Z(v) = -\alpha z$.By the statement regarding Z, we get that $Q\left(\frac{v}{-\alpha}\right) < -1 \implies Q(v) < -\alpha^2$. Thus $\overline{Q}(v, \alpha) < 0$. This completes our proof. □

Now with Lemma 7.18, notice that the signature of $\overline{Q}$ on $\overline{\Lambda}$ changes to $(p + 1, \operatorname{rk} \Lambda - p - 1)$. Thus we reach to the stage of $p = 2$ which is our assumption.
This ends the whole proof of Theorem 7.1 along with the proof of Assumption 7.6. Hence we get the deformation property of stability conditions.

Chapter 8

Stability condition on surfaces.

For defining stability conditions on surfaces as we have seen before, the standard t structure doesn't work. So we need to define another heart from the previous heart. The method that we do is called tilting. The chapter starts with the definition of tilting which helps us to construct stability function on a surface. At last, we prove the Harder-Narasimhan property of the stability function, thus proving Theorem 0.2.

8.1 Tilting of Abelian Categories.

At first we define torsion pairs. Let $\mathcal{A}$ be an abelian category.

Definition 8.1. Let $(\mathcal{T}, \mathcal{F})$ be a pair of full subcategories in $\mathcal{A}$. This is said to be a **torsion pair** in $\mathcal{A}$ if the following conditions are satisfied.

1. $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ for all $T \in \mathcal{T}$ and $F \in \mathcal{F}$.
2. For all $X \in \mathcal{A}$, there exists a short exact sequence

$$0 \to t(X) \to X \to X/t(X) \to 0$$

   where $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.

Example 8.1. Let $\mathcal{A} = \mathrm{Coh}(X)$ where X is a smooth projective scheme. Then if $\mathcal{T}$ is the category of torsion sheaves and $\mathcal{F}$ is the category of torsion-free sheaves, we have $(\mathcal{T}, \mathcal{F})$ to be a torsion pair.

The following proposition relates the concept of t-structures and torsion pairs.

Proposition 8.1. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair in an abelian category $\mathcal{A}$.*
Let $\mathcal{D}^{\leq 0} = \{X^\bullet | H^i(X^\bullet) = 0; i > 0, H^0(X^\bullet) \in \mathcal{T}\}$
and $\mathcal{D}^{\geq 0} = \{X^\bullet | H^i(X^\bullet) = 0; i < -1, H^{-1}(X^\bullet) \in \mathcal{F}\}$.
Then $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t structure on $D^b(\mathcal{A})$.

Sketch of the proof. We verify the conditions of the t structure one by one (Definition 4.1).

Condition 1: $\mathrm{Hom}(X^\bullet, Y^\bullet) = 0$ for all $X^\bullet \in \mathcal{D}^{\leq 0}, Y^\bullet \in \mathcal{D}^{\geq 1}$.

Let $f \in \mathrm{Hom}(X, Y) \neq 0$. Then in $\mathcal{D}^b(\mathcal{A})$, we have $f = (f^\bullet, s^\bullet)$ and we have the diagram

$$\begin{tikzcd} & Z^\bullet & \\ X^\bullet \arrow[ur, "f^\bullet"] & & Y^\bullet \arrow[ul, "s^\bullet"'] \end{tikzcd}$$

where $s^\bullet$ is a quasi isomorphism.

Thus $Z \in \mathcal{D}^{\geq 1}$. Then we use the functors $\tau_{\leq 0}, \tau_{\geq 1}$ on X and Z (by Proposition 4.1) and we get the following diagram.

$$\begin{tikzcd} \tau_{\leq 0}(X^\bullet) \arrow[r] \arrow[d, "\tau_{\leq 0} f^\bullet"] & X^\bullet \arrow[r] \arrow[d, "f^\bullet"] & \tau_{\geq 1} X^\bullet \arrow[r] \arrow[d, "h^\bullet"] & \tau_{\leq 0} X^\bullet[1] \arrow[d] \\ \tau_{\leq 0}(Z^\bullet) \arrow[r] & Z^\bullet \arrow[r] & \tau_{\geq 1} Z^\bullet \arrow[r] & \tau_{\leq 0} Z^\bullet[1] \end{tikzcd}$$

With the help of this diagram, we get that $\tau_{\leq 0} f^\bullet \neq 0$. But by further diagram chasing, we also get that $\tau_{\leq 0} f^\bullet = 0$ and arrive at a contradiction. Thus $f = 0$.

Condition 2: $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$.

This is evident from the definition of $*(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$.

Condition 3 : For $X^\bullet \in \mathcal{D}^b(\mathcal{A})$ we have a distinguished triangle :

$$X_0^\bullet \to X^\bullet \to X_1^\bullet \to X_0^\bullet[1]$$

where $X_0^\bullet \in \mathcal{D}^{\leq 0}, X_1^\bullet \in \mathcal{D}^{\geq 1}$.

For this, we start from the object $H^0(X^\bullet)$ and use the definition of the torsion pair and get an exact sequence

$$0 \to t(H^0(X^\bullet)) \xrightarrow{\mu} H^0(X^\bullet) \to \frac{H^0(X^\bullet)}{t(H^0(X^\bullet)} \to 0.$$

Let $X^\bullet = (X^i, d^i)$ Now we consider the diagram :

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \operatorname{Im} d^{-1} \arrow[r, "\rho"] \arrow[d, equal] & E \arrow[r, "\mu'"] \arrow[d] & t(H^0(X^\bullet)) \arrow[r] \arrow[d, "\mu"] & 0 \\
0 \arrow[r] & \operatorname{Im} d^{-1} \arrow[r] & \ker d^0 \arrow[r] \arrow[d] & H^0(X^\bullet) \arrow[r] \arrow[d] & 0 \\
 & & H^0(X^\bullet)/t(H^0(X^\bullet)) \arrow[r, equal] \arrow[d] & H^0(X^\bullet)/t(H^0(X^\bullet)) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

With the help of this diagram, we define the complex $X_0^\bullet$ of X as $X_0^i = 0$ for all $i > 0, X_0^0 = E$ and $X_0^i = X^i$ for all $i \leq -1$. We define $d'^i_{X_0^\bullet} = d^i$ for all $i < -1$ and $d'^{-1}_{X_0^\bullet} = \rho \circ d^{-1}$ and $d'^i_{X_0^\bullet} = 0$ for all $i \geq 0$. It is seen easily that that $X_0^\bullet \in \mathcal{D}^{\leq 0}$. Then we define $X_1^\bullet = X^\bullet / X_0^\bullet$ to be the quotient complex. Finally, we show that $X_1^\bullet \in \mathcal{D}^{\geq 1}$. ☐

Now we get the following important corollary as follows.

Corollary 8.1. *Let $\mathcal{A}$ be an abelian category and $(\mathcal{T}, \mathcal{F})$ be a torsion pair in $\mathcal{A}$. Then the following hold:*

1. *$\mathcal{B} = \{X^\bullet \in \mathcal{D}^b(\mathcal{A}), H^i(X^\bullet) = 0 \ \forall i \neq 0, -1$ and $H^0(X^\bullet) \in \mathcal{T}, H^{-1}(X^\bullet) \in \mathcal{F}\}$ is an abelian category.*

2. *The pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories of $\mathcal{X} = \mathcal{F}[1]$ and $\mathcal{Y} = \mathcal{T}$ is a torsion pair in $\mathcal{B}$.*

Sketch of the proof. :

1. This part is evident from Proposition 8.1 and from Proposition 4.3.

2. We verify the two conditions of the torsion pair. Firstly, we see that $\operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(\mathcal{F}[1], \mathcal{T}) = \operatorname{Ext}^{-1}_{\mathcal{A}}(F, \mathcal{T}) = 0$. Thus we got the first part.
   Now we come to the second part. At first, we show that $Z^\bullet \in \mathcal{B}$ is isomorphic to $Z'^\bullet \in \mathcal{D}^b(\mathcal{A})$ where $Z'^i = 0$ for all $i \neq 0, -1$.
   Then if we can write Z like that by the commutative diagram below, we get our condition:

$$\begin{tikzcd}
\cdots 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H^{-1}(Z^\bullet) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \cdots \\
\cdots 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Z^{-1} \arrow[r, "d^{-1}"] \arrow[d] & Z^0 \arrow[r] \arrow[d] & 0 \cdots \\
\cdots 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & H^0(Z^\bullet) \arrow[r] & 0 \cdots
\end{tikzcd}$$

where the first row is in $\mathcal{F}[1]$ and the third row is in $\mathcal{T}$.

□

This corollary shall help us construct the stability function for surface.

8.2 Construction of stability function on a surface.

Let D, F be $\mathbb{R}$ divisors on a smooth projective surface S with F ample. We know F- slope of a torsion free coherent sheaf E is defined by

$$\mu_F(E) := \frac{c_1(E).F}{\operatorname{rk} E}.$$

Here we use . means the intersection cycle of subvarieties.
We also known that such coherent sheaves have a Harder Narasimhan filtration (refer to Chapter 2, Section 2)

$$E_0 \subset E_1 \subset ...E_n = E$$

where we define $\mu_i = \mu_F(E_i/E_{i-1})$. Also, we have

$$\mu_{F-max}(E) = \mu_1 > \mu_2 > \cdots > \mu_n(E) = \mu_{F-min}(E).$$

Remember that each semistable sheaf has a filtration of stable sheaves of the same slope. We want to apply Corllary 8.1, so at first we need to define the torsion pairs on the standard heart $\operatorname{Coh}(S)$.

Definition 8.2. Let $\mathcal{A} = \operatorname{Coh}(S)$. We define :

$$\mathcal{T} = \{\text{Torsion Sheaves}\} \cup \left\{ E| \ \ \mu_{F-min}(E) > D.F \right\}$$

and

$$\mathcal{F} = \left\{ E| \ \ \mu_{F-max}(E) \leq D.F \right\}$$

Remark. The pair $(\mathcal{T}, \mathcal{F})$ is a torsion pair as this is beacuse it is just the truncation of the HN filtration of E in two parts. For example, the first condition of torsion pair of the Hom group to be zero is because of the fact that if A, B are $F-$ semistable, then $\mu_F(A) > \mu_F(B) \implies \operatorname{Hom}(A, B) = 0$.

Now by 8.1, we get that

$$\mathcal{A}^{\#}_{(D,F)} = \{E^\bullet \in D^b(S) | H^i(E^\bullet) = 0, \forall i \neq 0,1, H^{-1}(E^\bullet) \in \mathcal{F}, H^0(E^\bullet) \in \mathcal{T}\}$$

is an abelian category and thus the heart of a bounded t structure(by **??**). Now we need to define $Z_{(D,F)}$, the central charge on $\mathcal{A}^{\#}_{(D,F)}$.

Definition 8.3. For a sheaf $E \in \mathrm{Coh}(S)$, we define:

$$Z_{(D,F)}(E) = -(e^{-(D+iF)}.ch(E))$$

Now for extending it to $\mathcal{A}^{\#}_{(D,F)}$, we define :

$$Z_{(D,F)}(E^\bullet) = Z_{(D,F)}(H^0(E^\bullet)) - Z_{(D,F)}(H^{-1}(E^\bullet))$$

Explicitly, the definition of $Z_{(D,F)}(E)$ where $E \in \mathrm{Coh}(S)$ is :

$$\begin{aligned} Z_{(D,F)}(E) &= -(e^{-D-iF}.ch(E)) \\ &= (-1, D+iF, -(D^2/2 - F^2/2 + iD.F)).(\mathrm{rk}\, E, c_1(E), \mathrm{ch}_2(E)) \\ &= -\mathrm{ch}_2(E) - \mathrm{rk}(E).(D^2/2 - F^2/2 + iD.F) + (D+iF).c_1(E) \\ &= -\mathrm{ch}_2(E) - \mathrm{rk}(E)(D^2/2 - F^2/2) + D.c_1(E) + iF.(c_1(E) - \mathrm{rk}(E).D). \end{aligned}$$

Our main aim is to show that this is a stability function.
Before the start of the proof, let us recall the statements of *Hodge-Index Theorem* and *Bogomolov-Gieseker Inequality.*

Theorem 8.1. *(Hodge-Index Theorem) If D is an $\mathbb{R}$ divisor on S and F is an ample $\mathbb{R}$ divisor, then :*

$$D.F = 0 \implies D^2 \leq 0.$$

Theorem 8.2. *(Bogomolov-Gieseker Inequality) If E is a F-stable torsion-free sheaf, then we have :*

$$\mathrm{ch}_2(E) \leq \frac{c_1^2(E)}{2.\mathrm{rk}(E)}.$$

Now we prove the main aim.

Corollary 8.2. *$Z_{(D,F)}$ is a Bridgeland stability function on $\mathcal{A}^{\#}_{(D,F)}$.*

Proof. We need to prove for all $E \in \mathcal{A}^{\#}_{(D,F)}$, we need to show that $Z(E) \in \mathscr{H}$ where $\mathscr{H}$ is the strict upper half plane plus the negative semiline.
Now notice that by 2, we have the exact sequence :

$$0 \to H^{-1}(E)[1] \to E \to H^0(E) \to 0$$

Thus if we prove the condition satisfies for $H^{-1}(E)[1] \in \mathcal{F}[1]$ and $H^0(E) \in \mathcal{T}$, it shall satisfy for E as Z is additive on exact sequences (the map is from $K(\mathcal{A}^{\#}_{(D,F)})$). Thus it boils down to the following choice of E:

1. $E = T$, where T is torsion sheaf on S.
2. E where E is F stable torsion-free sheaf with $\mu_F(E) > D.F$.
3. $E[1]$ where E is F stable torsion-free sheaf with $\mu_F(E) \leq D.F$.

Note 2 and 3 follow from the fact of filtration of sheaf in semistable and then into stable factors.

We deal with each case separately. We use the explicit formula of $Z_{(D,F)}$ in the proof.

1. We have T a torsion sheaf supported on a curve, so $\mathrm{rk}(E) = 0$. Thus $\mathscr{I}(Z_{(D,F)}(T)) = D.c_1(T) > 0$ as $c_1(T)$ is an effective divisor. Now if this is zero, then we have $c_1(T) = 0 \implies T$ is supported in dimension 0. Thus in that case

$$\mathscr{R}(Z_{(D,F)}(T) = -\mathrm{ch}_2(T) < 0$$

as $\mathrm{ch}_2(T)$ just counts number of points in the support of $\mathcal{T}$. Thus $Z_{(D,F)}(T) \in \mathscr{H}$.

2. Now in this case we rewrite the imaginary part of $Z_{(D,F)}(E)$. We write as :

$$\mathscr{I}(Z_{(D,F)}(E)) = (c_1(E).F - \mathrm{rk}(E)D.F) = \mathrm{rk}(E)(\mu_F(E) - D.F) > 0$$

by definition of E in this case. Thus $Z_{(D,F)}(E) \in \mathscr{H}$.

3. Using the same formula above, we have if $\mu_F(E) < D.F$, then we have $\mathscr{I}(Z_{(D,F)}(E)) < 0$. Now ,by definition of $Z_{(D,F)}$, we have $Z_{(D,F)}(E[1]) = -Z_{(D,F)}(E)$. Thus $\mathscr{I}(Z(E[1])) > 0$.
Now it is just left to deal when $\mu_F(E) = D.F$. In that case, the imaginary part is zero. We use the Bogomolov Inequality in the real part of $Z_{(D,F)}$:

$$\begin{aligned}
\mathscr{R}(Z_{(D,F)}(E)) &= -\mathrm{ch}_2(E) - \mathrm{rk}(E)(D^2/2 - F^2/2) + D.c_1(E) \\
&\geq \frac{-c_1^2(E)}{2\,\mathrm{rk}(E)} - \mathrm{rk}(E)(D^2/2 - F^2/2) + D.c_1(E) \\
&= -\frac{c_1^2(E) - 2\,\mathrm{rk}(E)D.c_1(E) + (\mathrm{rk}(E)D)^2}{2.\,\mathrm{rk}(E)} + \mathrm{rk}(E)F^2/2 \\
&= -\frac{(c_1(E) - \mathrm{rk}(E)D)^2}{2\,\mathrm{rk}(E)} + \mathrm{rk}(E)F^2/2.
\end{aligned}$$

Thus now $\mu_F(E) = D.F$ gives us from the imaginary part of $Z_{(D,F)}(E)$ that $F.(c_1(E) - \mathrm{rk}(E)D) = 0$. Apply Hodge-Index Thoerem to this condition. We get that $(c_1(E) - \mathrm{rk}(E)D)^2 \leq 0$. Thus we have

$$\mathscr{R}(Z_{(D,F)}(E)) > 0 \implies \mathscr{R}(Z_{(D,F)}(E[1])) < 0$$

So in this case also, we have $Z_{(D,F)}(E) \in \mathscr{H}$.

□

This completes the whole proof of the function being a stability function.

8.2.1 Harder Narsimhan Property of $Z_{(D,F)}$

In this part, we prove the HN property of the stability function. We give a sketch of the proof.

Theorem 8.3. *If $D, F \in NS(S) \otimes \mathbb{Q}$ and F ample, then $Z_{(D,F)}$ satisfies the HN property.*

Sketch of the proof. As D, F are $\mathbb{Q}$ divisors, the image of $Z = Z_{(D,F)}$ turns to be a discrete subgroup of $\mathbb{C}$. The idea of this proof is to prove the conditions of Proposition 5.1.
At first we let we have an infinite chain of subobjects of an object $E \in \mathcal{A}^{\#}_{(D,F)}$:

$$\cdots E_{i+1} \subset E_i \subset \cdots E_i \subset E_0 = E.$$

with $\phi(E_{i+1}) > \phi(E_i)$.
Thus we have the exact sequences

$$0 \to E_{i+1} \to E_i \to F_i \to 0.$$

So we have $\mathscr{I}(Z(E_{i+1})) \leq \mathscr{I}(Z(E_i))$ as we are in $\mathcal{A}^{\#}_{(D,F)}$.
As the image of Z is discrete, we get that $\mathscr{I}(Z(E_i))$ is constant after some stage. Thus in that stage we get $\mathrm{Im}(Z(F_i)) = 0 \implies \mathscr{R}(Z(F_i)) < 0 \implies \mathscr{R}(Z(E_i)) < \mathscr{R}(Z(E_{i+1}))$. But for such i, as the imaginary parts are same, we have $\phi(E_{i+1}) < \phi(E_i)$ which contradicts the assumption.
Thus the first part is proved.
Now we prove the second condition of Proposition 5.1, interestingly, if we follow the same argument as before, we don't get a contradiction. For this part, we need to do a lot more work.
Let

$$E = E_0 \twoheadrightarrow E_i \twoheadrightarrow E_2 \cdots E_i \twoheadrightarrow E_{i+1} \cdots.$$

be a sequence of quotients of E where $\phi(E_{i+1}) < \phi(E_i)$.
We consider the sequence in cohomology :

$$H^0(E) \to H^0(E_1) \to \cdots ..H^i(E) \cdots ..$$

We know that the category of sheaves is noetherian. Thus this sequence terminates. After eliminating finitely many terms, we assume : $H^0(E) \cong H^0(E_i)$ for all i .
We consider the exact sequences

$$0 \to L_i \to E_0 \to E_i \to 0$$

where $\mathscr{I}(Z(L_i)) = 0$(This is because we are using the previous argument of the first part of the proof).
We get a sequence of L_i's and prove that that $L_i = L_{i+1}$.
One has the following exact sequences :

$$0 \to L_{i-1} \to L_i \to B_i \to 0 \cdots (1).$$

We then have the sequence of objects :

$$\cdots H^{-1}(L_{i-1}) \subset H^{-1}(L_i) \subset \cdots H^{-1}(E).$$

Now this chain terminates, so we can assume(after eliminating finitely many terms) that $H^{-1}(L_{i-1}) \cong H^{-1}(L_i)$ for all i.
Now every L_i fits into an exact sequence :

$$0 \to H^{-1}(L_i)[1] \to L_i \to H^0(L_i) \to 0.$$

As $\mathscr{I}(Z(L_i)) = 0$ which implies $\mathscr{I}(H^0(Z(L_i))) = 0$. By the definition of Z, we have that $H^0(L_i)$ is torsion and supported in dimension 0.
Taking cohomology of the sequence (1), we get the sequence

$$0 \to H^{-1}(B_i) \to H^0(L_{i-1}) \to H^0(L_i) \to H^0(B_i) \to 0.$$

Now by definition of Z, we have that $H^{-1}(A)$ is torsion-free for any object A in the abelian category. Thus we have a non-zero map between a torsion-free object $H^{-1}(B_i)$ and a torsion object $H^0(L_{i-1})$ which is impossible. So we have $H^{-1}(B_i) = 0$.
Now if we show that $H^0(B_i) = 0$, then we shall have $B_i = 0 \implies L_{i-1} = L_i$.
Now showing $H^0(B_i) = 0$ is equivalent to showing that $H^0(L_{i-1}) = H^0(L_i)$.
In other words, we should give a bound to the finite length sheaves $H^0(L_i)$(Note that these sheaves are torsion supported on dimension 0).
Let us go back to the original exact sequence :

$$0 \to L_i \to E \to E_i \to 0.$$

Remember that we have $H^0(E) \cong H^0(E_i)$ for all i. Thus taking cohomology, we have

$$0 \to H^{-1}(L_i) \xrightarrow{g} H^{-1}(E) \xrightarrow{f} H^{-1}(E_i) \to H^0(L_i) \to 0.$$

Let Q be the image of the map f. At first, notice that Q is independent of i as $Q = \operatorname{im} f = \ker g$ where $g : H^{-1}(L_i) \cong H^{-1}(L_{i+1}) \to H^{-1}(E)$. Then we have the exact sequence :

$$0 \to Q \to H^{-1}(E) \to H^0(L_i) \to 0.$$

We know that $H^{-1}(E)$ is torsion-free, thus we have that Q is torsion-free. By diagram chasing, we have that $H^0(L_i)$ is a subsheaf of the finite length sheaf Q^{**}/Q. Thus the length of $H^0(L_i)$ is bounded. So we have $H^0(B_i) = 0$ and thus we have $L_{i-1} = L_i$. Thus all the conditions of Proposition 5.1 are satisfied.
Hence, we have the HN property of $Z_{(D,F)}$.

□

Thus overall we get $(\mathcal{A}^{\#}_{(D,F)}, Z_{(D,F)})$ to be the stability condition on a smooth projective surface S which proves the following theorem.

Theorem 8.4. *If S is a smooth projective surface and $\mathcal{D} = \mathcal{D}^b(S)$, then the space of stability conditions* $\mathrm{Stab}(\mathcal{D})$ *is non-empty.*

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