ALGANT Master Thesis in Mathematics

Classification of Complex Algebraic Surfaces

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Introduction

The classification of varieties with respect to various criteria is a classical problem of mathematics. The one we will cover in this thesis is the birational classification of complex algebraic surfaces. Its analoguous in dimension 1, the birational classification of curves, is a relatively easy problem as birational morphisms of curves are isomorphisms.
The classification of complex algebraic surfaces up to birational equivalence, which is the subject of this thesis, is not a recent result. In fact, the birational classification of complex algebraic surfaces was already completed early in the 20th century with results by Castelnuovo, Enriques and M. Noether among other people. Since then, however, the notions of cohomology, sheaves and schemes have been introduced, and with these tools we will be able to have a different and more modern approach.
As a historical note, the classification of varieties of dimension 3 was given by S. Mori with tools more refined than the ones we are going to use and earned him a Fields medal in 1990. The classification of varieties of dimension 4 and all higher dimensions is an open problem which is being actively worked upon.
The main characteristic we will use is the Kodaira dimension of a surface which, as we will show, is a birational invariant, and work with it on a specific class of surfaces, called minimal surfaces, but to do all of this we will first need to show some results.
In the first part, we will discuss divisors on a surface. In particular, we will give the definition of Weil divisor and Cartier divisor, and show that on the surfaces we are considering they coincide. We are also going to show that invertible sheaves on a surface correspond to Cartier divisors, and we will use this fact in the next section to define the intersection number of two divisors by extending the usual intersection number of two distinct curves. We will conclude the second section by introducing some tools to work with, including Riemann-Roch's theorem for surfaces.
In the next section, we will give the definition of birational maps and morphisms, which we will follow up the definition of a particular birational morphism, the blow-up, and with some results about their structure, and in particular we will see how we can always lift a birational map to a birational morphism. At the end of this section we will also state and give a proof for Castelnuovo's contractibility criterion.
After this last result, we will be able to define what a minimal surface is, and we will define some characteristics associated to a surface that are invariants under birational morphisms and we will get into the core of the classification by observing how specific values for these invariants determine specific min-

imal surfaces. We will then close this section by giving the definition of the Kodaira dimension of a variety.
After giving some more results about birational invariants and how they determine specific minimal surfaces, we will finish by giving the classifcation of complex algebraic surfaces up to birational equivalence with respect to their Kodaira dimension.

1 Divisors

1.1 Weil and Cartier divisors

Let X be a noetherian integral separated scheme such that every local ring $\mathcal{O}_{X,x}$ of dimension 1 is regular. We define a prime divisor on X to be a closed subscheme of X of codimension 1.
We may define the free abelian group $\operatorname{Div} X$ whose generators are the prime divisors and whose elements $D \in \operatorname{Div} X$ are called Weil divisors. Weil divisors can be expressed as a formal sum $\sum n_i D_i$, where the sum is taken over all prime divisors D_i and the n_i are integer coefficients, only finitely many of which are non zero. We will say a divisor is effective if all the n_i are non negative.
Furthermore, let f be a non-zero rational function on X. Let us define the divisor associated to f to be $(f) = \sum v_{D_i}(f) \cdot D_i$, where the sum is over all the prime divisors D_i on X. We have that $v_{D_i} \geq 0$ for all D_i, but only a finite amount of them are not zero. We will say that any divisor equal to a divisor associated to a function is principal.
Consider two divisors D and D' over X. We will say that D and D' are linearly equivalent if $D - D'$ is a principal divisor, and we will denote this by $D \sim D'$. This is an equivalence relation on $\operatorname{Div} X$, and we denote the group of classes of divisors under this equivalence as $\operatorname{Cl} X$.
Since this definition of divisors only works for schemes with specific traits, we will now give a different definition of divisor which works for any arbitrary scheme. Let X be a scheme and let us denote by $\mathcal{O}_X$ the sheaf of rings of the scheme X. For all affine open U, let $S(U)$ be the set of elements in $\Gamma(U, \mathcal{O}_X)$ that are not divisors of zero in $\mathcal{O}_{X,x}$ for all points $x \in U$. This way, we obtsin a presheaf of rings, which is defined on the open U by $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$, and we may consider the associated sheaf, which we will denote $\mathcal{K}_X$.
A Cartier divisor on X is defined as a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$. Let $(U_i)_{i\in I}$ be an open cover of X and D a Cartier divisor. We may give a local description of D by giving for every U_i some function $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$, such that the quotient functions f_i/f_j are in $\Gamma(U_i \cap U_j, \mathcal{O}_X^*)$. A Cartier divisor is said to be principal if it is in the image of the natural projection map $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. Also, two Cartier divisors are said to be linearly equivalent if their difference, which is obtained by taking the quotient of the respective functions on every open U_i, is a principal divisor.
An invertible sheaf $\mathcal{L}$ on X is defined as a locally free $\mathcal{O}_X$-module of rank 1, and recall that for every such $\mathcal{L}$ there exists an invertible sheaf $\mathcal{L}^{-1}$ such that $\mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{O}_X$. Isomorphisms of invertible sheaves are an equivalence relation, so we can consider the set of classes of invertible sheaves under this

equivalence relation, which is a group under the $\otimes$ operation. We call this group the Picard group of X, and we denote it by $\operatorname{Pic} X$.

If $D = (U_i, f_i)_{i \in I}$ be a Cartier divisor on the scheme X, we may define the sheaf associated to D, denoted $\mathcal{O}_X(D)$, to be the $\mathcal{O}_X$-module defined locally by $f_i^{-1}\mathcal{O}_{U_i}$. Notice that since f_i/f_j is an invertible function on $U_i \cap U_j$, then f_i and f_j define the same $\mathcal{O}_X(U_i \cap U_j)$-module. Also $\mathcal{O}_X(D)$ is actually an invertible sheaf since every f_i is in $\Gamma(U_i, \mathcal{K}_x^*)$ and so the map $\mathcal{O}_{U_i} \to \mathcal{O}_X(D)_{|U_i}$ defined by $1 \mapsto f_i^{-1}$ is actually an isomorphism. These invertible sheaves associated to divisors also have the following properties:

Proposition 1.1.1. *Let D, D' be Cartier divisors, then:*

1. *The map $D \mapsto \mathcal{O}_X(D)$ is a 1-1 correspondance between Cartier divisors and invertible subsheaves of $\mathcal{K}_X$*
2. $\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \otimes \mathcal{O}_x(D')^{-1}$
3. $D \sim D' \iff \mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ *as invertible sheaves.*

Proof.

1. Let $\mathcal{L}$ be an invertible sheaf on X. We can obtain a divisor from $\mathcal{L}$ by taking, for every open U_i, a local generator f_i of $\mathcal{L}$. Then, the family $(U_i, f_i)_{i \in I}$ describes a Cartier divisor D, and since they coincide on every open U_i we have $\mathcal{L} = \mathcal{O}_X(D)$, and so we get the 1-1 correspondance

2. Let D, D' be two divisors, locally given by f_i and by g_i respectively. Then $D - D'$ is locally given by $f_i g_i^{-1}$, and in turn $\mathcal{O}_X(D - D')$ is locally generated by $f_i^{-1} g_i$. It follows $\mathcal{O}_X(D - D') = \mathcal{O}_X(D) \cdot \mathcal{O}_X(D')^{-1}$, which is canonically isomorphic to $\mathcal{O}_X(D) \otimes \mathcal{O}_X(D')^{-1}$

3. By the previous point, $\mathcal{O}_X(D) = \mathcal{O}_X(D') \otimes \mathcal{O}_X(D - D')$, so it is enough to prove that $D - D'$ is principal if and only if $\mathcal{O}_X(D - D') \cong \mathcal{O}_X$. First, let us assume $D - D'$ is principal. Then by definition there exists some $f \in \Gamma(X, \mathcal{K}_X^*)$ such that D is the image of f through the canonical projection $\Gamma(X, \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$.
   Using the above construction and the fact that f is globally defined, we can define a map $\mathcal{O}_X \to \mathcal{O}_X(D - D')$ by $1 \mapsto f^{-1}$ and it is an isomorphism. Conversely, if $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$, the image of 1 through the isomorphism gives some $f^{-1} \in \Gamma(X, \mathcal{K}_X^*)$, whose inverse defines a principal Cartier divisor that corresponds to $D - D'$.

□

At this point we have a correspondance between invertible sheaves and Cartier divisors, but we still have given no link between Weil divisors and Cartier divisors. What we will now see is that under some conditions Weil divisors and Cartier divisors are equivalent definitions.

Theorem 1.1.2. *Let X be a noetherian integral separated scheme such that every local ring is an UFD. Then* $\operatorname{Div} X \cong \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$*, that is the group of Weil divisors is isomorphic to the group of Cartier divisors. Furthermore, principal Weil divisors correspond to principal Cartier divisors.*

Proof. First, we want to check we can actually define Weil divisors on X. We already have that the scheme is noetherian, integral and separated, and we just need for every local ring of dimension 1 to be regular. As every local ring on our scheme is an UFD, then any of them which is of dimension 1 is also a PID, and in particolar it is regular, so $\operatorname{Div} X$ is defined on X.
As the scheme X is integral, the sheaf $\mathcal{K}_X$ is a constant sheaf and it corresponds to the function field K of X. Let $D = (U_i, f_I)_{i\in I}$ be a Cartier divisor, where $(U_i)_{i\in I}$ is an open of X. For each prime divisor Y on X, we may consider for every $i \in I$ such that $U_i \cap Y \neq \varnothing$, the valuation $v_Y(f_i)$.
If there exist two indexes i, j such that $Y \cap U_i$ and $Y \cap U_j$ are both non-empty, then the quotient function f_i/f_j is invertible on the intersection $U_i \cap U_j$ by the definition of Cartier divisor, and so it is obviously also invertible on $U_i \cap U_j \cap Y$. Then, $v_Y(f_i/f_j) = 0$, which in turn implies $v_Y(f_i) = v_Y(f_j) = v_Y(D)$, and since X is noetherian, only a finite amount of v_Y will be non-zero.
We may now define a Weil divisor $D = \sum_Y v_Y Y$, which from the previous discussion is well-defined and is induced canonically from the Cartier divisor we considered.
Conversely, suppose we have a Weil divisor D and we want to obtain a Cartier divisor. Let $p \in X$ be a point, and consider the induced Weil divisor D_p, which is on the local scheme $\operatorname{Spec} \mathcal{O}_{X,p}$. Since $\mathcal{O}_{X,p}$ is an UFD, it follows that D_p is principal and in particular for some function f_p we get $D_p = (f_p)$. The principal divisor defined by (f_p) on X has the same restriction to $\operatorname{Spec} \mathcal{O}_{X,p}$ as D, and so they may only differ by prime divisors which do not pass through p. Then we may take an open neighborhood U_p of p where D and (f_p) have the same restriction.
Giving an open cover $(U_i)_{i\in I}$ of X, where the open subset are obtained like U_p, naturally results in giving a Cartier divisor (U_i, f_i) which is well-defined as if two funtions f and f' define the same Weil divisor in an open U, we would have that $f/f' \in \Gamma(U, \mathcal{O}_X^*)$, that is they give the same Cartier divisor, and so this map of divisors is well defined.
We now have a construction of a Cartier divisor starting from a Weil divisor and one of a Weil divisor starting from a Cartier divisor, and we want to

show one is the inverse of the other. If we take a Cartier divisor $(U_i, f_i)_{i\in I}$, we can consider the Weil divisor D it defines. As before, around every point of U_i, the divisor D_p defined D on $\operatorname{Spec} \mathcal{O}_{X,p}$ is a principal divisor (f_i'). As the map is well-defined and the valuation of f_i' has to coincide with the valuation of f_i for every prime divisor by construction of D, we may actually take $f_i' = f_i$ and so we get that the Cartier divisor we construct starting from D is $(U_i, f_i)_{i\in I}$, and so the first map is the inverse of the second. □

1.2 Divisors on a surface

Recall that a variety is an integral separated scheme of finite type over an algebraically closed field k, and that varieties of dimension 1 are called curves and varieties of dimension 2 are called surfaces.
Our intent is to work with complex algebraic varieties, and so whenever we consider curve, surface or variety, if not specified we will always assume they are over $\mathbb{C}$ and projective, unless specified. Furthermore, we will similarly always assume a surface is smooth if not specified.
In a smooth variety every local ring is regular, and every regular local ring is an UFD and so whenever we will work on smooth surfaces we will be in a situation where Weil divisors correspond to Cartier divisors. In this case, we will use the term divisor to indicate either structure.
In particular, closed subschemes of codimension 1 on a surface are actually varieties of dimension 1, that is curves, so we have that the prime divisors on a surface are irreducible curves and in general we may describe any divisor on a surface as a formal sum of curves.
We may also state and prove a small result which we will use repeatedly later on:

Lemma 1.2.1. *Let X be a surface and C a smooth irreducible curve on X. Let $i : C \to X$ be the associated closed immersion. The sequence of sheaves*

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \stackrel{i^\#}{\to} i_*\mathcal{O}_C \to 0$$

is exact.

Proof. Let (U_i, f_i) be a description of the curve C as cartier divisor. The sheaf $\mathcal{O}_X(-C)$ is defined on every U_i to be the ideal $f_i\mathcal{O}_{U_i}$, and so we naturally have a morphism of sheaves $\mathcal{O}_X(-C) \to \mathcal{O}_X$. Also, as a sequence of sheaves is exact if and only if it is exact at level of stalks, then we may check exactness locally.
Let p be a point in X. If $p \notin C$ then $(i_*\mathcal{O}_C)_p = 0$, and for any open U_i containing p the function f_i is locally invertible in U_i, so $f_i\mathcal{O}_{X,p} = \mathcal{O}_{X,p}$ and

we have that the sequence is exact at $p \notin C$.
Let now $p \in C$ instead, and let f_i be an equation defining C locally around p. In this case, $f_i \mathcal{O}_{X,p}$ is a proper ideal of $\mathcal{O}_{X,p}$. Since f_i defines C locally, then the value of f_i in all the points of C around p is 0, and so the image of the ideal $f_i \mathcal{O}_{X,p}$ in $(i_* \mathcal{O}_C)_p$ is 0.
It remains to show that $\mathcal{O}_{X,p} \to (i_* \mathcal{O}_C)_p$ is surjective, which is true since i is a closed immersion. The given sequence is then exact. □

For simplicity we will write $\mathcal{O}_C$ instead of $i_* \mathcal{O}_C$ when using this sequence. Furthermore, it is useful to notice that, as invertible sheaves are locally free, tensoring this sequence by an invertible sheaf $\mathcal{L}$ on X will still yield an exact sequence.
It may also be sometimes useful to consider linear systems of divisors. Let D be a divisor on a surface X. We denote by $|D|$ the set of all effective divisors linearly equivalent to D, and we will say it is the complete linear system of D. Any linear subspace of $|D|$ will be called a linear system on X. Let P be such a linear subspace, the dimension of P is its dimension as a projective space.
We may also see that every non-zero global section of the sheaf $\mathcal{O}_X(D)$ defines an effective divisor linearly equivalent to D, that is an element of $|D|$. Viceversa, we can associate a non-zero sglobal section of $\mathcal{O}_X(D)$ to every element of the linear system $|D|$. This gives us an identification betweeen the linear system $|D|$ and the vector space $H^0(X, \mathcal{O}_X(D))$. Furthermore, a generic linear system P on X is associated to a vector subspace of $H^0(X, \mathcal{O}_X(D))$.
Let P be a linear system on the surface X. We will say that a curve, which is a prime divisor on the surface, is a fixed component of the linear system if for every divisor $D \in P$ the index of C in D is not zero. The formal sum of every fixed component defines the fixed part of a linear system, which really is some divisor F. We may also define the complete linear system $|D - F|$ which from the previous discussion can see has no fixed part.
Similarly, a point $p \in X$ is said to be fixed in the linear system P if for every $D \in P$ we have $p \in D$. In this moment, we still can not give any information on the number of fixed points of a linear system on a surface, but we are going to discuss this later on.
We close this section by recalling that an invertible sheaf $\mathcal{L}$ on a projective scheme X is said to be very ample if there exists an immersion $i : X \to \mathbb{P}^n$ such that, if we denote by $\mathcal{O}(1)$ the bundle on $\mathbb{P}^n$ whose sections are homogeneous polynomials of degree 1, then $\mathcal{O}_X(1) = i^* \mathcal{O}(1) \cong \mathcal{L}$. We call $\mathcal{O}_X(1)$ the tautological bundle on X. We will also say that a divisor H such that $\mathcal{O}_X(H) \cong \mathcal{L}$ is a very ample divisor, and if it effective we will say it is an hyperplane section.

2 Riemann-Roch's theorem for surfaces

2.1 Intersections on a surface

Let X be a surface, and let C, C' be two distinct irreducible curves on X that intersect, and let $p \in C \cap C'$ be a point in the intersection. If we let f and g be two local equations for C and C' respectively around p, we define their intersection multiplicity at p to be $m_p(C \cap C') = \dim_{\mathbb{C}} \mathcal{O}_{X,p}/(f,g)$.
We may also define the intersection number of C and C' as

$$(C.C') = \sum_{p \in C \cap C'} m_p(C \cap C').$$

This definition only works for two distinct curves, but we want to extend this definition so that it works for any two divisors, not necessarily distinct.
To do so, we are going to define a symmetric bilinear form $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$ such that $(\mathcal{O}_X(C).\mathcal{O}_X(C')) = (C.C')$ for any two distinct curves C and C'.
Let us recall the Euler-Poincaré characteristic of a sheaf $\mathcal{L}$:

$$\chi(\mathcal{L}) = \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{L}) \tag{2.1.1}$$

where we use the notation $h^i(X, \mathcal{L}) = \dim_{\mathbb{C}}(H^i(X, \mathcal{L}))$.
For C and C' two distinct irreducible curves we also define the skyscraper sheaf $\mathcal{O}_{C \cap C'} = \mathcal{O}_X/(\mathcal{O}_X(-C) + \mathcal{O}_X(-C'))$, which is a sheaf on X whose stalks are $\mathcal{O}_{C \cap C', p} = \mathcal{O}_{X,p}/(f,g)$. We may see that

$$(C.C') = \dim H^0(X, \mathcal{O}_{C \cap C'}).$$

Also, since a skyscraper sheaf $\mathcal{F}$ has $H^i(X, \mathcal{F}) = 0$ for $i \geq 1$, then we have $\chi(\mathcal{F}) = h^0(X, \mathcal{F})$, which combined with what we just said implies that $(C.C') = \chi(\mathcal{O}_{C \cap C'})$.
Now, we will give a definition of the extension of the intersection number which works for every $D, D' \in \operatorname{Div} X$, by giving it as a bilinear form on the Picard group.

Theorem 2.1.2. *For $\mathcal{L}, \mathcal{L}' \in \operatorname{Pic} X$, define*

$$(\mathcal{L}.\mathcal{L}') = \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{L}'^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{L}'^{-1}) \tag{2.1.3}$$

This is a symmetric bilinear form $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$ that for $C \neq C'$ satisfies $(C.C') = (\mathcal{O}_X(C).\mathcal{O}_X(C'))$.

Proof. it is immediate to show that the form is symmetric, as the tensor product is commutative.
First we are going to show that if C and C' are two distinct irreducible curves on X, then $(C.C') = (\mathcal{O}_X(C).\mathcal{O}_X(C'))$, and so our form is a consistent definition of an extensions of the intersection number of two curves.
We claim that the sequence

$$0 \to \mathcal{O}_X(-C-C') \to \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-C') \to \mathcal{O}_X \to \mathcal{O}_{(C\cap C')} \to 0,$$

is exact.
Since a sequence of sheaves is exact if and only if it exact at level of stalks, we can check everything locally around a point $p \in S$. So, let f and g be two functions in $\mathcal{O}_{X,p}$ define C and C' locally around p respectively.
We then get the sequence

$$0 \to fg\mathcal{O}_{X,p} \longrightarrow f\mathcal{O}_{X,p} \oplus g\mathcal{O}_{X,p} \longrightarrow \mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/(f,g) \to 0$$

where the map $f\mathcal{O}_{X,p} \oplus g\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}$ is given by mapping the pair (s,s') to $s-s'$, and the other two maps are the standard inclusion and the standard quotient.
Injectivity of the first map is trivial, and the image of this map is also obviously contained in the kernel of the second one. We may say that it actually is the kernel as the curves are taken to be irreducible and distinct so are the functions f and g defining it locally. We may also observe that the image of $f\mathcal{O}_{X,p} \oplus g\mathcal{O}_{X,p}$ is the ideal (f,g) of $\mathcal{O}_{X,p}$ and $\mathcal{O}_{X,p} \to \mathcal{O}_{X,p}/(f,g)$ is trivially surjective, so the sequence to be exact.
If we apply the Euler-Poincaré characteristic and use the fact that it is additive with respect to direct sum of sheaves, we get

$$\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) - \chi(\mathcal{O}_X(-C')) + \chi(\mathcal{O}_X(-C-C')) = \chi(\mathcal{O}_{C\cap C'})$$

which by using the properties in 1.1.1 and that the only non-trivial cohomology group is the one indexed 0 as stated before, this gives us

$$(\mathcal{O}_X(C).\mathcal{O}_X(C')) = \chi(\mathcal{O}_{C\cap C'}) = (C.C')$$

which means the form satisfies the condition on the intersection of curves we asked for.
We claim that if we take an irreducible curve C, for every invertible sheaf $\mathcal{L}$ we have that

$$(\mathcal{O}_X(C).\mathcal{L}) = \deg(\mathcal{L}_{|C})$$

where $\mathcal{L}_{|C} = \mathcal{L} \otimes \mathcal{O}_C$.
To prove this, we apply the Euler characteristic to the exact sequence of

lemma 1.2.1 and the same sequence tensored by $\mathcal{L}^{-1}$. Explicitly we have that the short exact sequences

$$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and

$$0 \to \mathcal{L}^{-1} \otimes \mathcal{O}_X(-C) \to \mathcal{L}^{-1} \to \mathcal{L}^{-1} \otimes \mathcal{O}_C \to 0$$

give respectively

$$\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_C)$$

and

$$\chi(\mathcal{L}^{-1}) - \chi(\mathcal{L}^{-1} \otimes \mathcal{O}_X(-C)) = \chi(\mathcal{L}^{-1}_{|C})$$

which substituted in the definition of the intersection form give in turn

$$(\mathcal{O}_X(C).\mathcal{L}) = \chi(\mathcal{O}_C) - \chi(\mathcal{L}^{-1}_{|C}) = -\deg(\mathcal{L}^{-1}_{|C}) = \deg(\mathcal{L}_{|C}),$$

where the second equality is due from Riemann-Roch's theorem on curves. In particular, as $\deg((\mathcal{L} \otimes \mathcal{L}')_{|C}) = \deg(\mathcal{L}_{|C}) + \deg(\mathcal{L}'_{|C})$, then

$$(\mathcal{O}_X(C).\mathcal{L} \otimes \mathcal{L}') = (\mathcal{O}_X(C).\mathcal{L}) + (\mathcal{O}_X(C).\mathcal{L}'^{-1}).$$

Furthermore, if $\mathcal{L} \cong \mathcal{L}'$ then we also have $\deg(\mathcal{L}_{|C}) = \deg(\mathcal{L}'_{|C})$ which in turn gives $(\mathcal{O}_X(C).\mathcal{L}) = (\mathcal{O}_X(C).\mathcal{L}')$. In particular if D and D' are two linearly equivalent divisors we may define two isomorphic invertible sheaves $\mathcal{O}_X(D)$ and $\mathcal{O}_X(D')$ to which we can apply this result. Similarly, if C and C' are two smooth curves, and $C \sim C'$, then for any invertible sheaf $\mathcal{L}$, $\deg(\mathcal{L}_{|C}) = \deg(\mathcal{L}_{|C'})$.
Recall that by a theorem of Serre ([6]) for every divisor D on X and for an hyperplane section H, there exists a large enough integer $n \geq 0$ such that there exist two smooth curves A and B which satisfy $A \sim D+nH$ and $B \sim nH$. In particular we have $D \sim A - B$.
So, let $\mathcal{L}, \mathcal{L}'$ be two invertible sheaves. We may write $\mathcal{L} = \mathcal{O}_X(D)$ for some divisor D, and we may take A, B two smooth curves like above so that $D \sim A - B$.
We want to show that $(\mathcal{L}.\mathcal{L}') = (\mathcal{O}_X(A-B).\mathcal{L}') = (\mathcal{O}_X(A).\mathcal{L}')+(\mathcal{O}_X(-B).\mathcal{L}')$. From the previous discussion we have

$$(\mathcal{O}_X(-B).\mathcal{L}') = (\mathcal{O}_X(-B).(\mathcal{L}' \otimes \mathcal{O}_X(A))) - (\mathcal{O}_X(-B).\mathcal{O}_X(A))$$

and so if we denote $\mathcal{L}' \otimes \mathcal{O}_X(A)$ by $(\mathcal{L}'(A))$ we have to prove

$$(\mathcal{L}.\mathcal{L}') = (\mathcal{O}_X(A).\mathcal{L}') + (\mathcal{O}_X(-B).\mathcal{L}'(A)) - (\mathcal{O}_X(-B).\mathcal{O}_X(A)).$$

By the fact that $\mathcal{O}_X(D)^{-1} = \mathcal{O}_X(-D)$ for any divisor D we have $\mathcal{L}^{-1} = \mathcal{O}_X(-A+B)$ too, by applying the intersection form we get

$$(\mathcal{O}_X(A).\mathcal{L}') = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-A)) - \chi(\mathcal{L}'^{-1}) + \chi(\mathcal{L}'(A)^{-1})$$

to which we will have to add

$$(\mathcal{O}_X(-B).\mathcal{L}'(A)) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(B)) - \chi(\mathcal{L}'(A)^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{L}'^{-1})$$

and then subtract

$$(\mathcal{O}_X(A).\mathcal{O}_X(-B)) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-A)) - \chi(\mathcal{O}_X(B)) + \chi(\mathcal{L}^{-1}).$$

Then, we compute and obtain

$$(\mathcal{O}_X(A).\mathcal{L}') + (\mathcal{O}_X(-B).(\mathcal{L}' \otimes \mathcal{O}_X(A))) - (\mathcal{O}_X(-B).\mathcal{O}_X(A)) =$$

$$= \chi(\mathcal{O}_X) - \chi(\mathcal{L}^{-1}) - \chi(\mathcal{L}'^{-1}) + \chi(\mathcal{L}^{-1} \otimes \mathcal{L}'^{-1}),$$

which is the definition of $\mathcal{L}.\mathcal{L}'$ so we have proven the statement. □

Observe also that since for any divisor D we have $\mathcal{O}_X(D) \otimes \mathcal{O}_X(-D) = \mathcal{O}_X$ and it follows from the definition that $\mathcal{O}_X.\mathcal{L} = 0$, then $\mathcal{O}_X(D).\mathcal{L} + \mathcal{O}_X(-D).\mathcal{L} = \mathcal{O}_X.\mathcal{L} = 0$ for any invertible sheaf $\mathcal{L}$, that is $\mathcal{O}_X(D).\mathcal{L} = -(\mathcal{O}_X(-D).\mathcal{L})$.
To go back to why we wanted to extend the intersection form, if we let D and D' be two divisors, we denote by $D.D'$ the intersection number $(\mathcal{O}_X(D).\mathcal{O}_X(D'))$, and since D and D' may be identical, we will denote the intersection number $D.D$ by D^2.

Lemma 2.1.4. *Let C be an irreducible curve on X such that $C^2 \geq 0$, and let D be an effective divisor. Then $D.C \geq 0$.*

Proof. Let $n \geq 0$ be the coefficient of C in D, and write $D = D' + nC$. The divisor D' is also effective, and does not contain C as a component. Then, this gives $D'.C \geq 0$, and then $D.C = (D' + nC).C = D'.C + nC^2 \geq 0$. □

We are interested in seeing if there is any way in which intersection theory is preserved in morphisms between surfaces. In a more general setting, let X and Y be two smooth varieties, and let $f : X \to Y$ be a morphism of varieties. From this morphism, we may induce an homomorphism $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$, and if f is surjective we may use this homomorphism to define the inverse image of a divisor.
Let D be a divisor, we take the inverse image of D through f, denoted f^*D, to be the divisor associated to the invertible sheaf on X given by $f^*\mathcal{O}_Y(D)$, that is $\mathcal{O}_X(f^*D) = f^*\mathcal{O}_Y(D)$. Since we have an identification

between classes of linearly equivalent divisors on a scheme and the Picard group of the scheme, then may notice that if D and D' are two divisors and $D \sim D'$, then $f^*D \sim f^*D'$.
As we are interested in the particular case of morphisms of surfaces, let $f : X \to Y$ be a morphism of surfaces which is generally finite of degree d. Let C be an irreducible curve on X, then we can define the direct image of C, denoted $f_*(C)$ to be 0 in the case $f(C)$ is a point, or $r\Gamma$ if $f(C)$ is a curve Γ on Y and the induced surjective morphism of curves $f_{|C} : C \to \Gamma$ is of degree r.
Furthermore, for every divisor $D = \sum n_i C_i$ on X we may define by linearity $f_*D = \sum n_i f_* C_i$.
To give an application of this, let us consider a smooth curve C, and a surjective morphism $\pi : X \to C$. Let F be a fibre of the morphism, that is $F = \pi^{-1}(p)$ for some $p \in C$. We have that $F^2 = F.F = 0$. We may observe there exists on C a divisor A, which is linearly equivalent to the divisor $1 \cdot p$ on C, such that p has multiplicity 0 in A. In particular, this means none of the fibers which compose $\pi^*(A)$ intersect F, and since the fibers are curves and so $\pi^*(A)$ is also a sum of curves, we may compute $(F.\pi^*(A)) = 0$ directly, which gives $F^2 = 0$.
We may now also show that linear systems without fixed part have a finite number of fixed points. Let P be a linear system without fixed part on a surface X. Then, we may put a bound on the number of fixed points in P. If we take two effective divisors $D, D' \in P$ then D^2 is equal to $D.D'$ as they define the same element in $\operatorname{Pic} X$. Since these are both effective divisors, and we may assume they have no common component as P has no fixed part, we may easily compute their intersection number $D.D'$ by linearity of the intersection number.
Specifically, D and D' can both be given as a finite formal sum of curves, and we may assume that all curves appearing in D do not appear in D'. Then, by taking the intersection number of every curve in D with every curve in D', and eventually considering multiplicites, we get $D.D'$ is a finite sum of integers, so finite. Every fixed point of P has to be contained in $D.D'$ trivially, so we get that D^2 is a bound for the number of fixed points in P.
We also have the following result regarding morphisms of surfaces.

Proposition 2.1.5. *Let $g : X \to Y$ be a morphism of surfaces of degree d, and let D, D' be two divisors on Y. Then $(g^*(D).g^*(D')) = d(D.D')$*

Proof. As in the proof of 2.1.2, we may write the divisor D as difference of two smooth curves on Y by taking an hyperplane section H and setting for some n positive big enough $A \sim D + nH$ and $B \sim nH$, we have $D \sim A - B$ and let us do the same for D' with some other hyperplane section H', and write

$D' = A' - B'$. So, if we show the statement for D and D' hyperplane sections, we may extend it by linearity, and so it will follow for the general case.
Since there exists an open subset U of Y over which the morphism g is étale, we may take H and B' so that the divisors linearly equivalent to D and D' they define are such that their intersection is normal and lies entirely in U. We then have $g^*(D) \cap g^*(D') = g^{-1}(D \cap D')$ and, since g is étale over U, this last term is composed of d copies of $D \cap D'$, so $(g^*(D).g^*(D')) = d(D.D')$. □

2.2 Riemann-Roch's theorem and related results

Let us recall that Serre duality tells us that if ω_X is the sheaf of differentials 2-forms on X we have $\chi(\mathcal{L}) = \chi(\mathcal{L}^{-1} \otimes \omega_X)$ for any invertible sheaf $\mathcal{L}$ on X. We now state and prove Riemann-Roch's theorem for surfaces:

Theorem 2.2.1 (Riemann-Roch for Surfaces)**.** *Let X be a surface and let $\mathcal{L} \in \operatorname{Pic} X$. Then*

$$\chi(\mathcal{L}) = \chi(\mathcal{O}_X) + \frac{1}{2}(\mathcal{L}^2 - \mathcal{L}.\omega_X) \tag{2.2.2}$$

Proof. Let us consider the intersection number of $\mathcal{L}^{-1}$ and $\omega_X^{-1} \otimes \mathcal{L}$, and so we have

$$(\mathcal{L}^{-1}.\omega_X^{-1} \otimes \mathcal{L}) = \chi(\mathcal{O}_X) - \chi(\mathcal{L}) - \chi(\omega_X \otimes \mathcal{L}^{-1}) + \chi(\omega_X)$$

and by Serre duality we have that $\chi(\mathcal{O}_X) = \chi(\omega_X)$, so substituting this in the right hand side gives

$$(\mathcal{L}.\omega_X) = 2(\chi(\mathcal{O}_X) - \chi(\mathcal{L}))$$

which by linearity of the intersection and rearranging terms gives (2.2.2), □

Since we have a strong correspondance between invertible sheaves and divisors, we may give a different formulation of this using divisors. Let D be a divisor associated to $\mathcal{L}$, and K be a divisor associated to the canonical sheaf ω_X. We will call any such K the canonical divisor of X, which we will sometimes denote by K_X if we are going to use many surfaces. For simplicity of notation, by $h^i(D)$ we will indicate the number $\dim H^i(X, \mathcal{O}_X(D))$ as in (2.1.1).
The surface X is a variety of dimension 2, so any cohomology group $H^i(X, \mathcal{L})$ will be 0 if $i \geq 3$ for any invertible sheaf $\mathcal{L}$.
We may now rewtite Serre duality in terms of divisors, and so we get that $h^i(D) = h^{2-i}(K - D)$ for $i = 0, 1, 2$. By definition we also have that $\mathcal{L}^2 = D^2$ and $(\mathcal{L}.\omega_X) = (D.K)$.

Then, we have $\chi(\mathcal{L}) = h^0(D)+h^1(D)+h^2(D)$ and by the previous observations we may write Riemann-Roch's theorem for surfaces as

$$h^0(D) + h^0(K - D) - h^1(D) = \chi(\mathcal{O}_X) + \frac{1}{2}(D^2 - D.K) \tag{2.2.3}$$

and since sometimes we may not know much about $h^1(D)$ except that it is trivially positive, we will write this as an inequality

$$h^0(D) + h^0(K - D) \geq \chi(\mathcal{O}_X) + \frac{1}{2}(D^2 - D.K). \tag{2.2.4}$$

Now, let us state a corollary of Riemann-Roch's theorem on surfaces.

Corollary 2.2.5. *Let C be a curve of arithmetic genus $g(C)$ on the surface X. Recall that $g(C) = h^1(C, \mathcal{O}_C)$. Then,*

$$g(C) = 1 + \frac{1}{2}(C^2 + C.K) \tag{2.2.6}$$

Proof. The sequence $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ is exact, and using the Euler characteristic on it gives $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C))$ like in the proof of 2.1.3.
By Riemann-Roch $\chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_X) + \frac{1}{2}(\mathcal{O}_X(-C)^2 - \mathcal{O}_X(-C).\omega_X)$ that corresponds, in terms of divisors, to $\chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_X) + \frac{1}{2}(C^2 - C.K)$.
We may also notice $H^i(C, \mathcal{O}_C) = 0$ for $i \geq 2$, as a curve has dimension 1. Also, $h^1(C, \mathcal{O}_C)$ is $g(C)$ by definition, and the global sections on a curve are the constant functions, so $h^0(C, \mathcal{O}_C) = 1$.
Then, we have

$$\chi(\mathcal{O}_C) = 1 - g(C) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-C)) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X) - \frac{1}{2}(C^2 + C.K),$$

from which we immediately get the desired formula. □

The following result, which is in nature analytical, will be also useful. Let ${}^h\mathcal{O}_X$ be the sheaf of holomorphic functions on the surface X considered as an analytical manifold, and let $e : {}^h\mathcal{O}_X \to {}^h\mathcal{O}_X^*$ be the exponential map, which is locally surjective. The kernel of this map is trivially the sheaf of locally constant functions that take values in $2\pi i\mathbb{Z}$. Then, we get an exact sequence of sheaves

$$0 \to \mathbb{Z} \to {}^h\mathcal{O}_X \xrightarrow{e} {}^h\mathcal{O}_X^* \to 0.$$

As X is a compact surface, $H^0(X, {}^h\mathcal{O}_X) = \mathbb{C}$ and $H^0(X, {}^h\mathcal{O}_X^*) = \mathbb{C}^*$. So, if we consider the long cohomology sequence associated to the exact sequence above, we may start from $H^1(X, \mathbb{Z})$:

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, {}^h\mathcal{O}_X) \to H^1(X, {}^h\mathcal{O}_X^*) \to H^2(X, \mathbb{Z}) \to ... \tag{2.2.7}$$

An important fact is that the analytic Picard group is canonically identified with $H^1(X, {}^h\mathcal{O}_X^*)$, but from Serre's GAGA theorem ([7]), this is the same as the Pic we are considering here, $H^1(X, \mathcal{O}_X^*)$.
Notice the above sequence also means we have a map $c : \operatorname{Pic} X \to H^2(X, \mathbb{Z})$. We want to induce the intersection form we defined before on $H^2(X, \mathbb{Z})$. Observe that by Poincaré duality we already have a pairing $H^2(X, \mathbb{Z}) \times H^2(X, \mathbb{Z}) \to \mathbb{Z}$. If we consider an irreducible curve C on X, we may consider the restriction $H^2(X, \mathbb{Z}) \to H^2(C, \mathbb{Z}) \cong \mathbb{Z}$, which gives a linear form on $H^2(X, \mathbb{Z})$ which by Poincaré duality means that it actually defines an element $c(C) \in H^2(X, \mathbb{Z})$. We may now define $c(D)$ for any divisor D by linearity, and $c(D).C(D') = D.D'$ as we wanted.
Before moving to the next part, we will state the following formula, valid for any variety X, due to M. Noether, without giving proof:

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K_X^2 + \sum(-1)^i b_i) \tag{2.2.8}$$

where $b_i = \dim_{\mathbb{R}} H^i(X, \mathbb{R})$. Notice that $\sum(-1)^i b_i$ is the topological Euler-Poincaré characteristc, denoted $\chi_{top}(\mathcal{O}_X)$.
When we are working on a surface, using Poincaré duality we have $b_0 = b_4 = 1$ and $b_3 = b_1$, so we get the following formula for the topological Euler-Poincaré characteristic:

$$\chi_{top}(\mathcal{O}_X) = 2 - 2b_1(X) + b_2(X).$$

3 Birational maps

3.1 Rational maps and the blow-up morphism

Let X and Y be two varieties, and let X be irreducible. A rational map ϕ between X and Y, denoted $\phi : X \dashrightarrow Y$, is defined as an equivalence class of morphisms from an open subset U of X to Y, where the equivalence relation is given by having two morphisms be equivalent if they are equal on the intersection of the open subsets on which they are respectively defined.
If we glue all the morphisms in a class together, we may see a rational map as a morphism ϕ from an open subset U of X such that for any open subset $V \subseteq X$ containing U, ϕ can not be extended to V. If p is a point in U, we will say ϕ is defined at p.
The subset of X over which ϕ is not defined is trivially a closed subset, as U is an open. The set of points over which a rational map is not defined in a variety is a closed subset of codimension ≥ 2, so if X is a smooth surface we have that the set of points over which ϕ is not defined $X - U$ is a finite subset F.
Let C be an irreducible curve on X. We have that ϕ is defined over the set $C - F$, and we may obviously consider its image $\phi(C - F)$. We may define the image of C through ϕ, denoted with a slight abuse of notation $\phi(C)$, to be the closure of $\phi(C - F)$, and analogously, we define the image of X through ϕ to be $\phi(X) = \overline{\phi(X - F)}$.
We will now state and prove a result in a more general setting, which we will then apply to surfaces.

Proposition 3.1.1. *Let X be a noetherian integral separated scheme, let Z be a proper closed subset of X and let $U = X - Z$. Then, there exists a surjective morphism $\operatorname{Cl} X \to \operatorname{Cl} U$, which is an isomorphism if Z has codimension in X greater or equal than 2.*

Proof. Consider the restriction map from X to U, and let Y be a prime divisor on X. Then, $Y \cap U$ is either empty or a prime divisor on U. Let f be a rational function on X, and $(f) = \sum n_i Y_i$ the divisor associated to it.
If we consider the restriction $f_{|U}$ as a rational function on U, we obrain a divisor $(f_{|U}) = \sum n_i (Y_I \cap U)$. Then we have an homomorphism $\operatorname{Cl} X \to \operatorname{Cl} U$ defined by $\sum n_i Y_i \mapsto \sum n_i (Y_I \cap U)$, which is surjective because every prime divisor on U is the restriction of some prime divisor on X. We may also notice that if Z is of codimension ≥ 2 then since divisors only depend on codimension 1 the homomorphism is actually an isomorphism. □

While we will not use this immediately, we may also observe that if Z is of codimension 1 in X, there exists an exact sequence

$$0 \to \mathbb{Z} \to \operatorname{Cl} X \to \operatorname{Cl} U \to 0. \tag{3.1.2}$$

This is because, due to how we defined the morphism $\operatorname{Cl} X \to \operatorname{Cl} U$ before, the kernel is composed of all divisors in X contained in Z, but Z is a prime divisor as it is of codimension 1 and so the only divisors on X satisfying this property are the multiples of Z. In particular, the map $\mathbb{Z} \to \operatorname{Cl} X$ in the sequence is the one defined by $n \mapsto n \cdot Z$.
Returing to surfaces, since the set F is of codimension 2, we have an isomorphism $\operatorname{Cl}(X) \to \operatorname{Cl}(X - F)$. In turn, this gives us another isomorphism $\operatorname{Pic} X \to \operatorname{Pic}(X - F)$, so we may define the inverse image under the rational map ϕ of divisors and invertible sheaves.
If we have two rational maps $\phi : X \dashrightarrow X'$ and $\psi : X' \dashrightarrow X$, such that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity on the respective sets of definition, we will say that X and X' are birationally equivalent, and that the map ϕ is birational (and so is ψ). If ϕ is a morphism, we will say that it is a birational morphism. Observe that if we have two birational maps their composition is again a birational map, so two surfaces being birationally equivalent is really an equivalence relation.
Now, we will construct a particular morphism of surfaces that will be useful to lift birational maps to birational morphisms. Let X be a surface, and $p \in X$ a point. Take an open neighborhood U of p, in which we can take some local coordinates x and y centered at p such that the curves $y = 0$ and $x = 0$ on U only intersect in the point p. Consider a variety $U \times \mathbb{P}^1$, take the subvariety defined by $xA - yB = 0$ in it, where x, y are the coordinates on U and A, B the homogeneous coordinates on $\mathbb{P}^1$. Denote this subvariety $\widehat{U}$.
Naturally, we have a projection morphism $\widehat{U} \to U$, which induces an isomorphism between $U - p$ and its inverse image. The inverse image of p is instead the curve $(0, 0) \times \mathbb{P}^1$. We may say there exists a surface $\widehat{X}$ and a morphism $\epsilon : \widehat{X} \to X$ which is an extension of the morphism $\widehat{U} \to U$ described, such that this morphism induces an isomorphism from $\widehat{X} - \epsilon^{-1}(p)$ to $X - p$. This morphism is birational, and it is called the blow-up morphism of X at p, while $E = \epsilon^{-1}(p)$ is called the exceptional curve or the exceptional divisor of the blow-up, and we observe that $E \cong \mathbb{P}^1$.
We have that ϵ also induces a morphism $\epsilon^* : \operatorname{Pic} X \to \operatorname{Pic} \widehat{X}$, which in turn induces a morphism on divisors. We will now study the inverse image of a curve through the blow-up of a surface. Trivially, if the curve does not pass through p, then its inverse image is isomorphic to itself.
If instead, C passes through p with multiplicity m, then $\epsilon^{-1}(C)$ contains E, but $\epsilon^{-1}(C - p)$ is a curve minus a point and we may take its closure, which

is then a curve, and we will call this curve the strict transform of the curve C, denoted $\widehat{C}$.

Proposition 3.1.3. *Using the above notation,* $\epsilon^* C = \widehat{C} + mE$

Proof. First, we may notice that $\epsilon_*(\widehat{C}) = C$, and that $\epsilon_*(E) = 0$. This gives us that, since away from p the blow-up is an isomorphism and so it is of degree 1 generally, and we will necessarily have $\epsilon^* C = \widehat{C} + kE$ for some integer k that we have to determine.
Let f be a function defining the divisor C in an open neighborhood U of p. Let x, y be the coordinates on U centered in p and without loss of generality assume that p is the only solution for $x = 0$ in $C \cap U$, and that $x = 0$ is not a tangent of C at p.
Then, since p is of multiplicity m in C, we can write the function f locally as $f(x, y) = f_m(x, y) + g(x, y)$, where f_m is a homogeneous polynomial of degree m, and g is the sum of homogeneous polynomials of degree $> m$.
As in the construction of the blow-up morphism, consider the open subset $\widehat{U} \subseteq U \times \mathbb{P}^1$, which is the subset given by the equation $xY = yX$, and let us restrict to the affine subset of $\mathbb{P}^1$ defined by $X \neq 0$. In this setting, $y = Yx$, and so we can write

$$\epsilon^* f = f_m(x, Yx) + g(x, Yx) = x^m (f_m(1, Y) + x g'(x, Y)).$$

Now, $x = 0$ is a local equation for E, and $f_m(1, Y)$ only has a finite amount of zeroes in E. Then, E occurs with multiplicity m in the equation defining $\epsilon^* C$ locally, which means $\epsilon^* C = \widehat{C} + mE$. □

We will now give some properties on how the blow-up morphism determines some results of the intersection theory on $\widehat{X}$ by giving rise to the intersection theory on X

Proposition 3.1.4. *Let X be a surface, $p \in X$ a point, $\epsilon : \widehat{X} \to X$ the blow-up of X at p and $E = \epsilon^{-1}(p)$*
Then, $\operatorname{Pic} \widehat{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$, *and this isomorphism is given, with divisors, by mapping* $(D, n) \to \epsilon^*(D) + nE$ *where* $D \in \operatorname{Div} X$.
Furthermore, for D, D' divisors on X, the intersection number form defined on $\widehat{X}$ is such that $(\epsilon^*(D).\epsilon^*(D')) = (D.D')$ and for D divisor on X and D' divisor on X' we have $(\epsilon^* D.D') = (D.\epsilon_* D')$, which in particular gives $(\epsilon^*(D).E) = 0$. Also, $E^2 = -1$.

Proof. From proposition 3.1.1, we have that $\operatorname{Div}(X - p) \cong \operatorname{Div} X$. Also, from the construction of the blow-up morphism we can say that $X - p$ is isomorphic to $\widehat{X} - E$, and in particular $\operatorname{Cl}(X - p) \cong \operatorname{Cl}(\widehat{X} - E)$. Then, since E is a curve on $\widehat{X}$ and so it is a closed subset of codimension 1, from (3.1.2) we get an exact sequence

$$0 \to \mathbb{Z} \to \mathrm{Cl}\,\widehat{X} \to \mathrm{Cl}\,X \to 0,$$

and so $\mathrm{Cl}\,\widehat{X} \cong \mathrm{Cl}\,X \oplus \mathbb{Z}$, where the isomorphism is the one in the statement of the proposition.

Now we want to show $(\epsilon^*(D).\epsilon^*(D')) = (D.D')$. We may see immediately that if D and D' do not contain p, then obviously the formula holds as the blow-up is an isomorphism around the divisors. If either of them passes through p instead, by the theorem of Serre used in the proof of 2.1.2 we may take linearly equivalent divisors obtained as difference of curves which do not pass through p.

To prove $(\epsilon^*D.D') = (D.\epsilon_*D')$ we may use the same method, and so assume that D does not pass through p, which means all the intersections between ϵ^*D and D' lie in $\widehat{X}-E$, and so the equation is quickly proven by isomorphism. Note that for the particular case where D' the exceptional curve E, the direct image of E is 0, as the image of E through ϵ is a point, and so $(\epsilon^*(D).E) = 0$.

Let C be a curve passing through p with multiplicity m. Then, by the previous result $\epsilon^*C.E = 0$, and we have that $\epsilon^*C = \widehat{C} + mE$ from 3.1.3. We may observe that the equation of the curve $\widehat{C}$, which using the same notation as in 3.1.3 we may write as $(f_m(1,Y) + xg'(x,Y))$, has m solutions for $x = 0$, which is an equation for the curve E, and so $\widehat{C}.E = m$, then by linearity we get $E^2 = -1$. □

The Neron-Severi group of the surface X, denoted $NS(X)$, is defined as the quotient of classes of linearly equivalent divisors by the set of divisors of degree zero.

Since $\mathrm{Cl}(\widehat{X}) \cong \mathrm{Cl}(X) \oplus \mathbb{Z}E$, it follows that $NS(\widehat{X}) \cong NS(X) \oplus \mathbb{Z}E$. In particular, a blow-up morphism decreases the rank of the Neron-Severi group, which is finite as a result of Severi's theorem of the base.

Furthermore, as the canonical sheaves on $X-p$ and $\widehat{X}-E$ are the same, then we obviously have that $K_{\widehat{X}} = \epsilon^*K_X + kE$ for some $k \in \mathbb{Z}$. Furthermore, we may use the genus formula, and the fact that $g(E) = 0$ to get $-2 = E.(E + K_{\widehat{X}})$, and so it follows from the results of the previous proposition that $k = 1$.

Before stating the next result, we give a bijection between the following two sets:

- The set of rational maps $\phi : X \dashrightarrow \mathbb{P}^m$ such that $\phi(X)$ is not contained in an hyperplane.
- Linear systems on X without fixed part of dimension m.

To be specific, let ϕ be a rational map $X \dashrightarrow \mathbb{P}^m$, and let $|H|$ be the complete linear system of hyperplanes in $\mathbb{P}^m$. Then, for every divisor $D \in |H|$, we may consider the inverse image $\phi^* D$, and so we define $\phi^*|H|$ to be the linear system on X given by all the inverse images $\phi^* D$. Observe that since $\phi(X)$ is contained in no hyperplane, then $|H|$ has no fixed part, and so does $\phi^*|H|$. Viceversa, let P be a linear system on X with no fixed part, and let P^* be the projective space dual to P, which has a natural structure of projective variety. We can then define a rational map $X \dashrightarrow P^*$ that is not defined at the fixed points of P, and that elsewhere is defined by sending a point $x \in X$ to the point in P^* corresponding to the hyperplane of divisors in P passing through x.

3.2 Factorization of birational maps

Now, we are going to show how the blow-up morphism can be used to lift rational maps from a surface to rational morphisms.

Theorem 3.2.1. *Let $\phi : X \dashrightarrow Y$ be a rational map from a surface to a projective variety. Then, there exists a surface X' and a birational morphism of surfaces $\eta : X' \to X$ and a morphism $f : X' \to Y$ such that the diagram*

$$\begin{array}{ccc} & X' & \\ {\scriptstyle\eta}\swarrow & & \searrow{\scriptstyle f} \\ X & \overset{\phi}{\dashrightarrow} & Y \end{array}$$

is commutative. Also, η is up to an isomorphism the composition of a finite number of blow-ups.

Proof. Since Y is a projective variety, in particular it lies in some projective space $\mathbb{P}^m$, we may just assume $Y = \mathbb{P}^m$, and we may further assume that $\phi(X)$ is not contained in any hyperplane of $\mathbb{P}^m$.
Then, ϕ corresponds to some linear system P of dimension m on X, and P has no fixed components. Then, we know that the linear system may only have a finite amount of fixed points, and if this number is 0, then we may just take $X' = X$ as the linear system already defines a morphism $X \to Y$. We may then assume that P has at least one base point p.
We may consider the blow-up of X at p, which gives a surface $\widehat{X}$, then consider the induced linear system $\epsilon^* P$. As p is a fixed point for P, then $\epsilon^* P$ has a fixed component, which is the exceptional curve E of the blow-up with some multiplicity k. As the linear system remains unchanged away from E, there are no other fixed components and so the fixed part of $\epsilon^* P$ is kE.

Then we may construct the linear system whose elements are by $D - kE$, for all divisors $D \in \epsilon^* P$, which we will denote $\epsilon^* P - kE$, and we may observe it has no fixed part. Then, it corresponds to a rational map $\widehat{X} \dashrightarrow \mathbb{P}^n$. This rational map coincides with $\phi \circ \epsilon$. Recall that thThee amount of fixed points in P is finite and bounded by D^2, where D is a divisor in P, and observe that $(\epsilon^* D - kE) - (\epsilon^* D - kE) = D^2 - k^2$ otherwise their image through ϵ would also have to be a base point in P this map is a morphism.
Then, we may construct inductively a chain of blow-ups using the above procedure, and at each step the amount of base points in the linear system gets smaller. We will eventually find a base point free linear system, and so a rational morphism of surfaces $X' \to Y$ which makes the diagram commute. □

Another result is that every birational morphism may be expressed as a composition of blow-ups, but before showing this result we have to give a couple of intermediate results.

Lemma 3.2.2. *Let X be a possibly singular irreducible surface, X' a smooth surface and let $f : X \to X'$ a birational morphism, where the inverse rational map $f' : X' \dashrightarrow X$ is not defined at some point $p \in X'$. Then $f^{-1}(p)$ is a curve on X.*

Proof. This question is local on X, and so we may suppose X to be affine. So, there exists an embedding $i : X \hookrightarrow \mathbb{A}^n$. With these assumptions, we may consider the rational map $i \circ f' : X' \dashrightarrow \mathbb{A}^n$, which is defined by rational functions $g_1, ..., g_n$ in each coordinate, and at least one of which is not defined at p, without loss of generality let one be g_1. Then, we may write $g_1 = u/v$, for some $u, v \in \mathcal{O}_{X',p}$ coprime functions such that $v(p) = 0$.
Consider the curve D on X defined by $f^* v = 0$. We have that $f^* u = x_1 f^* v$, where x_1 is the function associated to the first coordinate on $\mathbb{A}^n$. It follows $f^* u = f^* v = 0$ on D. and so D is the inverse image of the subset of X' defined by $u = v = 0$. Up to shrinking, since u and v are coprime, we way assume that this set contains only one point, that is p and so $f^{-1}(p)$ is the curve D. □

Lemma 3.2.3. *Let $\phi : X \dashrightarrow X'$ be a rational map between surfaces, such that ϕ^{-1} is not defined at some point $p \in X'$. Then, there exists some curve C on X such that $\phi(C) = p$.*

Proof. Let U be the open on which the rational map is defined, and consider the subset of $X \times X'$ defined as the closure of the set of points $(x, \phi(x))$ $\forall x \in U$, and let us denote this set $\widetilde{X}$. The set $\widetilde{X}$ can also be seen as an irreducible surface, and we can consider the natural projections from $\widetilde{X}$ to X and to X', denoted π_1 and π_2 respectively.
We get the following commutative diagram:

$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ X & \overset{\phi}{\dashrightarrow} & X' \end{array}$$

As the diagram is commutative, and ϕ^{-1} is not defined at p, then π_2^{-1} is also not defined at p. By the previous lemma, there exists a curve $\widetilde{C}$ on $\widetilde{X}$ such that $\pi_2(\widetilde{C}) = p$. This also implies that the projection of $\widetilde{C}$ on the other coordinate then has to be a curve, $\pi_1(\widetilde{C}) = C$. By commutativity, we conclude that $\phi(C) = p$. □

Now, using these two lemmas, we can prove the following result

Proposition 3.2.4. *Let $f : X \to Y$ be a birational morphism of surfaces, and suppose the rational map f^{-1} is not defined at some point $p \in Y$. Then, if we take $\epsilon : \widehat{Y} \to Y$ the blow-up of Y at p, we can factorize f as $\epsilon \circ g$ where g is some birational morphism $X \to \widehat{Y}$.*

Proof. We may consider the birational map $g : X \dashrightarrow \widehat{Y}$ defined $g = \epsilon^{-1} \circ f$, and assume that there exists a point $q \in X$ at which it is not defined. Then, by lemma 3.2.3, there exists a curve C on Y such that $g^{-1}(C) = q$, and applying f to both sides of this last equation we get $\epsilon(C) = f(q)$, and $f(q)$ can only be a point. Then, C is a curve contracted by the blowup, and so it is the exceptional divisor E. This also means that $f(q) = p$.
Let $\mathfrak{m}_q$ be the maximal ideal of the local ring $\mathcal{O}_{X,q}$, and let (x, y) be some local coordinate system around p on Y. Assume $g^*y \notin \mathfrak{m}_q^2$, which means that g^*y defines a local equation for $g^{-1}(p) = C$, as there it vanishes with multiplicity 1.
We may say that, for some $u \in \mathcal{O}_{X,q}$, we have $g^*x = ug^*y$. We may now choose a local coordinate t by setting $t = x - u(q)y$, and this is such that $g^*t = (u - u(q))g^*t$, which is in $\mathfrak{m}_q^2$.
Let $p' \in E$ be a point at which the map g^{-1} is defined. We have that $(g^{-1})^* g^* t = \epsilon^* t \in \mathfrak{m}_{p'}^2$. This holds for all points in E, except the finite set on which the rational function is not defined. But, by construction of the blow-up morphism, $\epsilon^* t$ is a local coordinate for all points in E, except one, which gives a contradiction. □

We may now use this proposition to prove that, up to an isomorphism, every birational morphism is given by composition of blow-ups.

Proposition 3.2.5. *Let $f : X \to Y$ be a birational morphism. Then, $f = \epsilon_1 \circ \ldots \circ \epsilon_n \circ u$, where u is an isomorphism, and the ϵ_i are blow-up morphisms.*

Proof. Assume there exists a point $p \in Y$ in which f^{-1} is not defined, otherwise f would already be an isomorphism, and so this case is trivially solved. By the previous proposition, f factors through the blow-up of Y at p, which we will denote ϵ_1. as $f = \epsilon_1 \circ f_1$.
Again, f_1 is a birational morphism, and so once more we either have an isomorphism, or apply the previous proposition inductively. If this procedure stops, then the statement is proven, and so we will show that we can not have an infinite number of blowups.
Let n be the number of curves on X contracted by f, which is finite because there are only finitely many points at which f^{-1} is not defined, and if we let q be one of them, the inverse image $f^{-1}(q)$ is a closed subset in X with a finite amount of irreducible components, that is the curves contracted to that specific point.
Obviously, any curve contracted by f_1 is also contracted by f, and the exceptional divisor of the blow-up is not contracted by f_1 but it is contracted by f, so if we denote n_1 the number of curves contracted by f_1, we can say $n_1 < n$. Then, we have that after a finite number of blow-ups, we will have that the morphism f_m will contract a negative number of curves, which is absurd. Then, we have that the preocedure stops, and so a finite amount of blow-ups. □

Another result which we will now prove is that if on a surface there is a divisor with the same properties as the exceptional divisor of a blow-up, then it really is the exceptional divisor of a blow-up.

Theorem 3.2.6 (Castelnuovo). *Let X be a surface, E a curve on X such that $E \cong \mathbb{P}^1$ and $E^2 = -1$. Then, there exists a morphism, called contraction morphism, $\mu : X \to Y$, where Y is a non-singular projective surface, such that $\mu(E)$ is a point $p \in Y$ and $\mu(X - E) = Y - p$.*
This morphism is identical to the blow-up of Y at p.

Proof. Let H be a very ample divisor on X such that $H^1(X, \mathcal{O}_X(H)) = 0$, which we may obtain by taking a high enough multiple of an ample section on X, and let $k = H.E$ the intersection number. We may consider the divisor $H' = H + kE$, observe $H'.E = 0$, and denotethe invertible sheaf $\mathcal{O}_X(H')$ by $\mathcal{L}$.
Let i be an integer such that $0 \le i \le k$. First thing we will prove is that $H^1(X, \mathcal{O}_X(H + iE)) = 0$. This is true by hypotesis for $i = 0$, and we may construct the exact sequence

$$0 \to \mathcal{O}_X(H + (i-1)E) \to \mathcal{O}_X(H + iE) \to \mathcal{O}_E \otimes \mathcal{O}_X(H + iE) \to 0$$

We may compute the intersection number $(H+iE).E = k-i$, and so we get $\mathcal{O}_E \otimes \mathcal{O}_X(H+iE) \cong \mathcal{O}_{\mathbb{P}^1}(k-i)$. If we take the associated long cohomology sequence and using the fact that $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = 0$ for all $m \geq 0$, we get

$$0 \to H^0(X, \mathcal{O}_X(H+(i-1)E)) \to H^0(X, \mathcal{O}_X(H+iE)) \to H^0(E, \mathcal{O}_E(k-i)) \to$$
$$\to H^1(X, \mathcal{O}_X(H+(i-1)E)) \to H^1(X, \mathcal{O}_X(H+iE)) \to 0,$$

and taking $i=1$ we get $H^1(X, \mathcal{O}_X(H+(k-1)E)) = 0$, which in turn gives us $H^1(X, \mathcal{O}_X(H+kE)) = 0$, and by induction it gives the desired result for all $i \geq 0$.

Since H is very ample the linear system $|H+kE|$, which corresponds to the invertible sheaf $\mathcal{L}$, has no base points away from E, and so there it is generated by global sections. The previous discussion about cohomology also tells us that the map $H^0(X, \mathcal{O}_X(H+KE)) \to H^0(E, \mathcal{O}_X(H+kE) \otimes \mathcal{O}_E)$ is surjective. We can check this by taking $i=k$, so we have the exact sequence

$$0 \to H^0(X, H+(i-1)E) \to H^0(X, H+iE) \to H^0(E, \mathcal{O}_E) \to 0$$

and that $(H+kE).E = 0$.

It follows that $\mathcal{L} \otimes \mathcal{O}_E = \mathcal{O}_X(H+kE) \otimes \mathcal{O}_E \cong \mathcal{O}_{\mathbb{P}^1}$, which is generated by the global section 1. Then, if we take a lift of the section 1 to $H^0(X, \mathcal{L})$ then, since Nakayama's lemma says that the lift of the generators of the quotient will generate the module, $\mathcal{L}$ is also generated by global section on every point of E, too. Then at every point in X, $\mathcal{L}$ is generated by global sections.

The invertible sheaf $\mathcal{L}$ is associated to a linear system of divisors, which determines a morphism $f : X \to \mathbb{P}^N$. Let X' be the image of X through this morphism and we have that $f^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathcal{L}$. Since the degree of $\mathcal{L} \otimes \mathcal{O}_E$ is 0, then $f(E)$ has to be a point p, and since the linear system $|H-kE|$ separates points on E from points not on E, the map is an isomorphism between $X-E$ and $X'-p$.

Now, we have to show that the surface X' is non-singular at p. We will now show that $f_*\mathcal{O}_X = \mathcal{O}_{X'}$. Since the question is local on X', we may as well assume it is affine $X' = \operatorname{Spec} A$, and since $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$-algebras, $\Gamma(Y, f_*\mathcal{O}_X)$ is a finitely generated A-module. Now, A and $\Gamma(Y, f_*\mathcal{O}_X)$ are two integral domains with identical quotient field, and A is integrally closed, therefore $A = \Gamma(Y, f_*\mathcal{O}_X)$ and $f_*\mathcal{O}_X = \mathcal{O}_{X'}$.

Let $\mathfrak{m}_p$ be the maximal ideal at p, and let X_n be the closed subscheme of X defined by $\mathfrak{m}_p^n\mathcal{O}_X$. By the theorem on formal functions, ([3], III.11.1) we have $\widehat{\mathcal{O}}_{X',p} \cong \varprojlim H^0(X_n, \mathcal{O}_{X_n})$, where $\widehat{\mathcal{O}}_{X',p}$ is the completion of $\mathcal{O}_{X',p}$. Note that $X_1 = E$, and $H^0(E, \mathcal{O}_E) = \mathbb{C}$. For every $n \geq 1$ we also have an exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to \mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n} \to 0.$$

Since $E \cong \mathbb{P}^1$ and $E^2 = -1$, then $\mathcal{I}^n/\mathcal{I}^{n+1} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, and for every $n \geq 2$, we have $\mathcal{I}^n/\mathcal{I}^{n+1} \cong \mathcal{O}_{\mathbb{P}^1}(n)$. This gives a cohomology sequence

$$0 \to H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \to H^0(X_{n+1}, \mathcal{O}_{X_{n+1}}) \to H^0(X_n, \mathcal{O}_{X_n}) \to 0$$

Let x, y be a basis of the vector space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. Then, $H^0(X_2, \mathcal{O}_{X_2})$ is isomorphic to the ring $\mathbb{C}[[x, y]]/(x, y)^2$.
We can also see that, lifting x and y, we can write a basis for $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n))$ with the monomials of degree n in the variables x, y, and now we can say that if $H^0(X_{n-1}, \mathcal{O}_{X_{n-1}}) \cong \mathbb{C}[[x, y]]/(x, y)^{n-1}$, then by induction we also have $H^0(X_n.\mathcal{O}_{X_n}) \cong \mathbb{C}[[x, y]]/(x, y)^n$.
Now, we have that $\widehat{\mathcal{O}_p} \cong \varprojlim H^0(X_n, \mathcal{O}_{X_n}) \cong \mathbb{C}[[x, y]]$ which is a regular local ring, and so $\mathcal{O}_{X',p}$ is regular, which means p is not singular.
It remains only to see that this morphism really is the blow-up of X' at p. We have a rational morphism $X \to X'$ such that the inverse is not defined at p, and so we can factor this morphism through the blow-up by p. But, seeing as no other curves are contracted by construction, then the resulting birational morphism $X \to \widehat{X'}$ has to be an isomorphism. □

4 Minimal surfaces

4.1 Minimal surfaces and birational invariants

Let X, X' be two surfaces. We say that X dominates X' if there exists a birational morphism $X \to X'$. A surface X is said to be minimal if every birational morphism $X \to X'$, where X' is any other surface, is actually an isomorphism. This is equivalent to saying that X only dominates surfaces isomorphic to itself.

Theorem 4.1.1. *Let X be a surface. There exists a minimal surface X' such that X dominates X'.*

Proof. If X is already minimal, there is nothing to prove in this case. So assume that there exists a surface Y such that there exists a birational morphism $\phi : X \to Y$. By 3.2.5, this is the composition of an isomorphism and some blow-ups. Again, if Y is minimal, the statement is proven, so let us restrict to the case where Y is not minimal. We have to show that by iterating this process a finite number of times, we will necessarily get a minimal surface.
As up to isomorphism the rational map $X \to X'$ is the composition of a finite number of blow-ups, and we have seen before that blow-ups decrease the rank of the Neron-Severi group by 1, due to rank of the Neron-Severi group $NS(X)$ being finite, we can say that iterating we will eventually find a minimal surface □

So, for any surface X we consider, we may consider a minimal surface Y birationally equivalent to it and we may also say that X is obtained by blowing up Y some number of times. An important consequence, is that birational invariants associated to a surface have to take the same value for any associate minimal surface, and we may use this to later give a classification of surfaces.
We will principally consider the following values associated to the surface X: the irregularity $q(X)$, defined by $q(X) = h^1(X, \mathcal{O}_X)$, for all $n \geq 1$ the plurigenuse $P_n(X) = h^0(X, \mathcal{O}_X(nK))$ and the geometric genus which is defined as $p_g(X) = h^2(X, \mathcal{O}_X)$.
By Serre duality we have $p_g = h^0(\mathcal{O}_X(K)) = h^0(X, \Omega^2_X) = P_1$, and by Hodge theory we have that $q(X) = h^0(X, \Omega^1_X) = \frac{1}{2}b_1(X)$, where $b_1 = \dim_R H^0(X, \mathbb{R})$.

Theorem 4.1.2. *The values q, p_g and P_n for all $n \geq 0$ are birational invariants.*

Proof. Let $\phi : X' \dashrightarrow X$ be a birational map, let F be the set of points of X' on which ϕ is not defined, and let $f : X' - F \to X$ be the underlying morphism,

and recall F is a finite set. Let $\omega \in H^0(X, \Omega^1_X)$ be a 1-form on X. Then, $f^*\omega$ may be extended to a rational 1-form on X' which may only have poles in the finite set F.
The poles of a differential form are the support of a divisor, which on a surface may not be a finite set, we may say that the extension of $f^*\omega$ has no poles on X' and so is holomorphic. Then, we may define an injective map $H^0(X, \Omega^1_X) \to H^0(X', \Omega^1_{X'})$.
Since ϕ is birational, then there exists some rational map ϕ^{-1}, and so we may also construct an inverse to this injective map with an analogous procedure, which gives us $H^9(X, \Omega^1_X) \cong H^0(X', \Omega^1_{X'})$, and so $q(X) = q(X')$.
We show that the plurigenuses are birational invariants using the same argument, and in particular so is the geometric genus. ☐

If we consider $\chi(\mathcal{O}_X)$ and substitute the previous invariants in (2.1.1) we may notice that for a surface X we have $\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$.
If we consider a blow-up morphism $\epsilon : \widehat{X} \to X$, where $K_{\widehat{X}} = \epsilon^* K_X + E$ as we have shown before. From this formula it follows immediately that $K^2_{\widehat{X}} = K^2_X - 1$, which means K^2 is not a birational invariant. Also, as b_1 and χ are both birational invariants, it follows immediately from (2.2.8) that b_2 is also not a birational invariant.

4.2 Minimal ruled surfaces

Let C be a smooth curve. We say that a surface X is geometrically ruled over C if there exists a smooth morphism $\pi : X \to C$ such that each fibre of the morphism is isomorphic to $\mathbb{P}^1$.

Theorem 4.2.1. *Let X be a surface, C a smooth curve, $\pi : X \to C$ a morphism. Suppose there exists a point $p \in C$ such that π is smooth over p and we have $F = \pi^{-1}(p) \cong \mathbb{P}^1$.*
Then, there exists an open $U \subseteq C$ of the Zariski topology such that $p \in U$ and an isomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{P}^1$ such that, if we consider the projection on the first coordinate $\pi_1 : U \times \mathbb{P}^1 \to U$, we obtain the following commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times \mathbb{P}^1 \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_1} & \\ & U & \end{array}$$

Proof. First, as F is a fibre, we have that $F^2 = 0$, and by (2.2.6) we also get $F.K = -2$. Asumme that there exists an effective divisor D in $|K|$. Then, we

would have $D.F = -2$ by linear equivalence, and $D.F \geq 0$ by 2.1.4 ando so we have a contradiction. Therefore, by Serre duality we can conclude that $H^2(X, \mathcal{O}_X) = 0$.
From this we get that the map $\operatorname{Pic} X \to H^2(X, \mathbb{Z})$ defined as in (2.2.7) is surjective. Let f be the class of F in $H^2(X, \mathbb{Z})$. We know that the set $\{(a.f) | a \in H^2(X, \mathbb{Z})\}$ is an ideal of $\mathbb{Z}$, and so for some integer $d \geq 0$ it is $d\mathbb{Z}$. We may then define a linear form on $H^2(X, \mathbb{Z})$ as $a \mapsto \frac{1}{d}(a.f)$. By Poincaré duality, we have that the product $H^2(X, \mathbb{Z}) \otimes H^2(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ is a duality, and so the associated map $H^2(X, \mathbb{Z}) \to \operatorname{Hom}(H^2(X, \mathbb{Z}), \mathbb{Z})$ is surjective. In particular, there exists an element $f' \in H^2(X; \mathbb{Z})$ such that the linear form on $H^2(X, \mathbb{Z})$ defined by $(a.f')$ is equal to $\frac{1}{d}(a.f)$. As $f^2 = 0$ it follows from this formula that $f'^2 = 0$.
We may observe that if we let k be the class of K and let c be the class of any curve, then $c^2 + c.k$ is an even integer, and since this formula is additive modulo 2, we have that it is even for the class of any divisor. Finally we get $f'^2 + f'.k = f'.k = (f.k)/d = -2/d$, and so since it has to be an even integer we have $d = 1$.
Then, there exists a divisor H on X such that if h is its class, then $h.f = 1$, and so $H.F = 1$. By tensoring (1.2.1) with $\mathcal{O}_X(H)$ and $\mathcal{O}_X(F)$, and using the fact that $\mathcal{O}_F \otimes \mathcal{O}_X(F) \cong \mathbb{C}$ and that $H.F = 1$ by the previous point, we obtain the exact sequence

$$0 \to \mathcal{O}_X(H + (r-1)F) \to \mathcal{O}_X(H + rF) \to \mathcal{O}_F(1) \to 0$$

for any $r \in \mathbb{Z}$, which induces a long exact cohomology sequence, where we use the fact that $F \cong \mathbb{P}^1$ and so $H^1(F, \mathcal{O}_F(m)) = 0$ for $m \geq 0$,

$$\begin{aligned} \ldots \to H^0(X, \mathcal{O}_X(H + rF) \xrightarrow{a_r} H^0(F, \mathcal{O}_F(1)) \to H^1(H + (r-1)F) \to \\ \xrightarrow{b_r} H^1(X, \mathcal{O}_X(H + rF)) \to 0. \end{aligned}$$

We may observe that since b_r is surjective, $h^1(H + (r-1)F) \geq h^1(H + rF)$ naturally. But, since $h^1(D)$ is a finite dimensional vector space for all divisors D, we will eventually get that $h^1(H + (r-1)F) \geq h^1(H + rF)$ for all r large enough, and so b_r is bijective for r large enough, and in particular injective, so this implies a_r is surjective.
Then, we may take a vector subspace V of $H^0(X, \mathcal{O}_X(H + rF))$ of dimension 2 such that $a_r(V) = H^0(F, \mathcal{O}_F(1))$, and let P be the linear system of dimension 1 on X associated to V.
All the fixed components of P must be contained in fibres $F_{x_1}, \ldots, F_{x_q}$ over some points $x_1, \ldots, x_q$, and as P has no base points in F then all these fibres are distinct from F.

Similarly, any base point of P not in the fixed part is contained in fibres $F_{x_{q+1}}, ..., F_{x_l}$ distinct from F. Also, denote by $F_{x_{l+1}}, ..., F_{x_m}$ all the reducible fibres.of π.
Set $U = C - \{x_1, ..., x_m\}$. The restriction of P to $\pi^{-1}(U)$, which we will denote P', is then base point free. Let t parametrize P', and let C_t be a curve in the linear system, which contains a section of π and some fibres possibly. We may also take another curve $C_{t'}$ with $t \neq t'$and check that if $C : t$ contained any fibre, then C'_t would intersect C_t and since the linear system has dimension 1 it would have base points.
Also, since P' has no base points it defines a morphism $g : \pi^{-1}(U) \to \mathbb{P}^1$, whose fibres are $g^{-1}(t) = C_t$. Then, we may also construct a morphism $h : \pi^1(U) \to U \times \mathbb{P}^1$ by mapping $x \mapsto (\pi(x), g(x))$.
We now conclude observing that $h^{-1}((y,t)) = F_y \cap C_t$ is a point, and so h is an isomorphism. □

Let C be a smooth curve, and let X be a surface birationally equivalent to $C \times \mathbb{P}^1$. Then, we say that X is ruled over C. In particulat, we have that a surface X satisfying the conditions of the previous theorem is ruled.
Recall that a variety Y of dimension n is said to be rational if there exists a birational map $\mathbb{P}^m \dashrightarrow Y$. Then, we may observe that if the surface X is ruled over $\mathbb{P}^1$, then it is rational as the map $U \subseteq \mathbb{P}^2 \to V \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ defined on the homogeneous coordinates as $[s,t,1] \mapsto [s,1] \times [t,1]$ induces a birational map between $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$.

Lemma 4.2.2. *Let π be a surjective morphism from a surface X to a smooth curve C, and let π have connected fibres. Let $F = \sum n_i C_i$ be a reducible fibre of π, where the sum is indexed over $i \in I$. Then, $C_i < 0$ for all $i \in I$.*

Proof. Fix $i \in I$. We have $C_i.n_iC_i = C_i.(F - \sum n_j C_j)$, where this last sum is taken over all $j \in I$ such that $j \neq i$. As we may substitute F with another lineally equivalent fibre not including C_i, and $C_i.C_j \geq 0$ for $i \neq j$ we get $nC_i^2 \leq 0$. We conclude by observing that, since F is connected, there exists C_j such that $C_i \cap C_j \neq \varnothing$, and so $n_iC_i^2 < 0$. □

Lemma 4.2.3. *Let X be a minimal surface and C be a smooth curve, and let $\pi : X \to C$ be a morphism with generic fibre isomorphic to $\mathbb{P}^1$, then X is a geometrically ruled surface over C where the structure is given by the morphism π.*

Proof. Let F be a fibre of π. As in the proof of 4.2.1 we have that $F^2 = 0$ and $F.K = -2$, a.
Assume F is reducible, and so $F = \sum n_i C_i$. By the previous lemma $C_i^2 < 0$, and the genus formula tells us that we have that $K.C_i = -2 + 2g(C_i) - C_i^2$,

and in particular we have $K.C_i \geq -1$. In the previous inequality, equality is satisfied if and only if $g(C) = 0$ and so $C \cong \mathbb{P}^1$. This also implies $C_i^2 = -1$, and so C_i is an exceptional curve on X, which contradicts the fact that X is minimal, and so we may now restrict to $K.C_i \geq 0$ for all C_i. In this case we wpuld have $K.F \geq 0$, which is another contradiction, and so F may not be reducible.
Also, by recycling the argument used in the proof of 4.2.1, we may say that F may not be the multiple of another fibre. We may then compute the genus of F using (2.2.6), and we get $g(F) = 0$, and so $F \cong \mathbb{P}^1$ and p is smooth over F so by 4.2.1 we conclude that X is ruled over C. □

Theorem 4.2.4. *Let X be a geometrically ruled surface over an irrational curve C. Then X is a minimal surface.*
In particular, any minimal model of the surface $\mathbb{P}^1 \times C$ is a geometrically ruled surface over C.

Proof. Let $\pi : X \to C$ be the morphism giving the ruling of the surface. To show X is minimal, we have to show there is no exceptional curve E in it.
First, we may exclude that any fibre F of π is an exceptional curve as $F^2 = 0$, and we require $E^2 = -1$.
Then, as E is not a fibre we must have that $\pi(E) = C$. In particular, as a morphism of curves decreases the genus by Riemann-Hurwitz and $g(E) = 0$, then we also have $g(C) = 0$, and so $C \cong E \cong \mathbb{P}^1$. Then C is rational, which contradicts the statement, and so there is no exceptional divisor on X, which is minimal.
To show the second part of the theorem, assume Y is a minimal surface associated to $C \times \mathbb{P}^1$. Then, there exists a rational map $\phi : Y \dashrightarrow C \times \mathbb{P}^1$. Let $\pi_1 : C \times \mathbb{P}^1 \to C$ be the standard projection onto C. Then, we have a rational map of varieties $\pi_1 \circ \phi : Y \dashrightarrow C$, and by theorem 3.2.1 we obtain a morphism of surfaces $f : Y' \to C \times \mathbb{P}^1$ and a commutative diagram

$$\begin{array}{ccc} & Y' & \\ {}^{\epsilon}\swarrow & & \searrow^{\pi \circ f} \\ Y & \overset{\pi_1 \circ \phi}{\dashrightarrow} & C \end{array}$$

where ϵ is the composition of a finite amount of blow-ups, say n, which we can assume to be minimal.
Assume $n > 0$, and let E be the exceptional curve on Y' contracted by the first blow-up. Using the same argument as before, we have that have $f(E) \neq C$ otherwise C would be rational. Then, there exists a point p in C such that $f(E) = p$. Then, by 3.2.4, the morphism f would factor through the first

blow-up, and so n would not be minimal.
We may now conclude that $n = 0$, and so $f = \pi_1 \circ \phi$, and the generic fibre is isomorphic to $\mathbb{P}^1$. The previous lemma then lets us say that $f : Y \to C$ gives Y the structure of a geometrically ruled surface over C. □

Proposition 4.2.5. *Let C be a curve and X be a surface ruled over C. Then $q(X) = g(C)$ and $P_n(X) = 0$ for all $n \geq 1$*

Proof. Recall that by Hodge theory, $q(X) = h^0(X, \Omega^1_X)$. Since X is ruled over C, we may assume $X = C \times \mathbb{P}^1$. Let $\pi_1 : X \to C$ and $\pi_2 : X \to \mathbb{P}^1$ be the canonical projections.
If we denote by ω_C and $\omega_{\mathbb{P}^1}$ the sheaves of differentials for the curves, we have an isomorphism from $\pi_{1*}(\pi_1^*\omega_C \otimes \pi_2^*\omega_{\mathbb{P}^1})$ to $\omega_C \otimes \pi_{1*}\pi_2^*\omega_{\mathbb{P}^1}$ by the projection formula, which also gives that $\pi_{1*}\pi_2^*\omega_{\mathbb{P}^1} = \mathcal{O}_C \otimes_{\mathbb{C}} H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1})$, so the canonical morphism $H^9(C, \omega_C) \otimes H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}) \to H^0(C \times \mathbb{P}^1, \pi_1^*\omega_C \otimes \pi_2^*\omega_{\mathbb{P}^1})$ is actually an isomorphism.
Then, let x be a local coordinate for C and y a local coordinate for $\mathbb{P}^1$. We get a local system of coordinates $x.y$ is a local system of coordinates for X, and so we may say $\Omega^1_X \cong \pi_1^*\omega_C \otimes \pi_2^*\omega_{\mathbb{P}^1}$.
We get $q(X) = h^0(X, \Omega^1_X) = h^0(C, \omega_C) + h^0(\mathbb{P}^1, \omega_{\mathbb{P}^1}) = g(C) + g(\mathbb{P}^1) = g(C)$.
Also, since $P_n(X) = h^0(X, \mathcal{O}_X(nK))$ and $\mathcal{O}_X(K) \cong \omega_X$, we may reformulate this as $P_n = h^0(X, \omega_X^{\otimes n}) = \dim_{\mathbb{C}}(H^0(\mathbb{C}, \omega_C^{\otimes n}) \otimes H^0(\mathbb{P}^1, \omega_{\mathbb{P}^1})^{\otimes n}) = 0$. □

Combining Noether's formula (2.2.8) and that $\chi(\mathcal{O}_X) = 1 - q + p_g$, we get that if X is ruled over C, we have that $1 - g(C) = \frac{1}{12}(K^2 + 2 - 2b_1 + b_2)$.
As 4.2.4 gives us no result about surfaces ruled over a rational curve C, that is rational surfaces, we will now study them separately. First, we need to give another characterization of geomtrically ruled surfaces.

4.3 Ruled surfaces as vector bundles

Let C be a curve, and let $\mathcal{E}$ be a locally free invertible sheaf of rank 2 over C. Then, we may define the projective space bundle $\mathbb{P}(\mathcal{E})$ as a surface fibred over C such that for each point $p \in C$ the fibre over p is the projective space associated to $\mathcal{E}_p$. Since $\mathcal{E}$ is locally free, this fibre is isomorphic to $\mathbb{P}^1$ and so $\mathbb{P}(\mathcal{E})$ is locally isomorphic to $C \times \mathbb{P}^1$. Then, $\mathbb{P}(\mathcal{E})$ has the structure of a surface ruled over C.
We now prove that we can characterize every geometrically ruled surface over a curve C as $\mathbb{P}(\mathcal{E})$ for some locally free sheaf $\mathcal{E}$ of rank 2 over C..

Proposition 4.3.1. *Let X be a geometrically ruled surface over a curve C and $\pi : X \to C$ the structural morphism. There exists a locally free sheaf $\mathcal{E}$*

of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E})$ over C.
Furthermore, two such sheaves $\mathcal{E}$ and $\mathcal{E}'$ define two isomorphic geometrially ruled surfaces over C if and only if there exists $\mathcal{L} \in \operatorname{Pic} C$ such that $\mathcal{E} = \mathcal{E}' \otimes \mathcal{L}$.

Proof. By 4.2.1, the fibration π is locally trivial and so there exists an open cover $(U_i)_{i\in I}$ of C such that $t_i : \pi^{-1}(U_i) \tilde{\to} U_i \times \mathbb{P}^1$.
We may define a sheaf of groups $\mathcal{G}$ on C. Set $\mathcal{G}(U)$ to be the group of automorphisms of $U \times \mathbb{P}^1$ that keep U fixed, that is $\operatorname{Aut}_U(U \times \mathbb{P}^1)$. This is compatible with glueing and uniqueness, and so it really defines a sheaf $\mathcal{G}$, and we may also identify every $\mathcal{G}(U)$ with the morphisms of U into the projective linear group $\mathrm{PGL}_2(\mathbb{C})$.
We may also identify the set of isomorphism classes of locally free sheaves of rank 2 on C with the cohomology set $H^i(C, \mathcal{G})$ as composing a trivialization t_i and the inverse of a trivialization T_j^{-1} on $\pi^{-1}(U_i \cap U_j)$ gives us a Cech cocycle $(i,j) \mapsto g_{ij} \in \mathcal{G}(U_i \cap U_j)$, and so a class in $H^1(C, \mathcal{G})$.
Recall that if we consider the linear group $\mathrm{GL}_2(\mathbb{C})$, by definition we have $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^* = \mathrm{PGL}_2(\mathbb{C})$, and so we have an exact sequence

$$1 \to \mathbb{C}^* \to \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C}) \to 1.$$

If we denote by $\mathrm{PGL}_2(\mathcal{O}_C)$ the sheaf $\mathcal{G}$, we then get an exact sequence of sheaves

$$1 \to \mathcal{O}_C^* \to \mathrm{GL}_2(\mathcal{O}_C) \to \mathrm{PGL}_2(\mathcal{O}_C) \to 1$$

which gives a long cohomology sequence

$$... \to H^1(C, \mathcal{O}_C^*) \to H^1(C, \mathrm{GL}_2(\mathcal{O}_C)) \to H^1(C, \mathrm{PGL}_2(\mathcal{O}_C)) \to H^2(C, \mathcal{O}_C^*) \to ...$$

and we know $H^1(C, \mathcal{O}_C^*) \cong \operatorname{Pic} C$. Furthermore, $H^1(C, \mathrm{GL}_2(\mathcal{O}_C))$ is the set of isomorphism classes of rank 2 locally free sheaves on C, and $\operatorname{Pic} C$ acts by tensoring on this set.
Also, as we are working over $\mathbb{C}$ which is algebraically closed and $\dim C = 1$, for all sheaves $\mathcal{F}$ on C we have $H^2(C, \mathcal{F}) = 0$ by Tsen's theorem ([2], 6.2.8), so the map $H^1(C, \mathrm{GL}_2(\mathcal{O}_C)) \to H^1(C, \mathrm{PGL}_2(\mathcal{O}_C))$ is surjective, which means that the isomorphism classes of locally free sheves of rank 2 on C map to all of the isomorphism classes of geometrically ruled surfaces, and in particular for any fibration $\pi : X \to C$ there exists a locally free sheaf $\mathcal{E}$ of rank 2 such that $X \cong \mathbb{P}_C(\mathcal{E})$- □

If C is a smooth curve, we may identify a locally free sheaf of rank 2 with a vector bundle of rank 2, and we will use both definitions equivalently.
Let C be a curve and let $\mathcal{E}$ be a rank 2 locally free sheaf on C. We will denote $\deg(\mathcal{E}) = \deg(\wedge^2 \mathcal{E})$ and by $h^i(\mathcal{E})$ we will mean $\dim_{\mathbb{C}} H^i(C, \mathcal{E})$.

Observe that if we consider the sheaf $\mathcal{E} \otimes \mathcal{L}$ for some $\mathcal{L} \in \operatorname{Pic} C$, we have that $\deg(\mathcal{E} \otimes \mathcal{L}) = \deg(\mathcal{E}) + 2\deg(\mathcal{L})$, and so the parity of $\deg(\mathcal{E})$ is invariant up to tensoring by an invertible sheaf.

Proposition 4.3.2. *Let C be a curve and let $\mathcal{E}$ be a rank 2 locally free sheaf on C. Then, there exists $\mathcal{L}, \mathcal{M} \in \operatorname{Pic} C$ such that there is an exact sequence*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$$

Proof. Tne sheaf $\mathcal{E}$ is a coherent $\mathcal{O}_C$-module, and so up to replacing $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{O}_C(n)$ for n large enough, by a theorem of Serre we may assume there exists a non-zero section s of $\mathcal{E}$. From s we get a non-zero morphism from $\mathcal{E}^*$, the dual of $\mathcal{E}$, to $\mathcal{O}_C$, whose image is an ideal of $\mathcal{O}_C$, and in particular for some effective divisor D on C it is $\mathcal{O}_C(-D)$. Then, the kernel of the surjective morphism $\mathcal{E}^* \to \mathcal{O}_C(-D)$ is then an invertible sheaf, and so by duality we conclude. □

Recall $\chi(\mathcal{E}) = \chi(\mathcal{L}) + \chi(\mathcal{M})$. Applying Riemann-Roch's theorem for curves, which says that $\chi(\mathcal{L}) = 1 - g(C) + \deg(\mathcal{L})$, we get the equation $\chi(\mathcal{E}) = \deg(\mathcal{L}) + \deg(\mathcal{M}) + 2(1 - g(C))$ and by additivty of degree we obtain Riemann-Roch's theorem for rank 2 vector bundles on a curve

$$\chi(\mathcal{E}) = \deg(\mathcal{E}) + 2 - 2g(C). \tag{4.3.3}$$

Moreover, whenever we have such an exact sequence, we will say that $\mathcal{E}$ is an extension of $\mathcal{M}$ by $\mathcal{L}$. Two such extensions are said to be isomorphic if there exists an isomorphism of the relative exact sequences inducing the identity on $\mathcal{L}$ and $\mathcal{M}$.
Recall that the isomorphisms classes of extensions of $\mathcal{M}$ by $\mathcal{L}$ are identified with the elements of the group $\operatorname{Ext}^1(\mathcal{M}, \mathcal{L})$, where $\operatorname{Ext}(\mathcal{M}, \cdot)$ is the right derived functor of $\operatorname{Hom}(\mathcal{M}, \cdot)$.
Observe that the exactness of the sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$ does not imply that $\mathcal{E} \cong \mathcal{L} \oplus \mathcal{M}$. When this will be the case, we will say the extension of $\mathcal{M}$ by $\mathcal{L}$ is trivial. Notice that by the splitting lemma we may have $\mathcal{E} = \mathcal{L} \oplus \mathcal{M}$ if and only if the sequence splits.
If we tensor the sequence by $\mathcal{M}^{-1}$, we obtain another exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{M}^{-1} \to \mathcal{E} \otimes \mathcal{M}^{-1} \to \mathcal{O}_C \to 0$$

that splits if and only if $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$ splits, and we get a long exact cohomology sequence

$$\ldots \to H^0(C, \mathcal{E} \otimes \mathcal{M}^{-1}) \to H^0(C, \mathcal{O}_C) \xrightarrow{\partial} H^1(C, \mathcal{L} \otimes \mathcal{M}^{-1}) \to \ldots$$

Recall there is an isomorphism $H^1(X, \mathcal{L} \otimes \mathcal{M}^{-1}) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{M}^{-1} \otimes \mathcal{L})$, which as $\mathcal{M}$ is locally free is in turn isomorphic to $\mathrm{Ext}^1(\mathcal{M}, \mathcal{L})$, which is identified with the group of isomorphism classes of extensions of $\mathcal{M}$ by $\mathcal{L}$.
Observe that the sequence splits if and only if there exists a global section of $\mathcal{E} \otimes \mathcal{M}^{-1}$, that is an element of $H^0(C, \mathcal{E} \otimes \mathcal{M}^{-1})$, such that its image in $H^0(C, \mathcal{O}_C)$ is the global section 1_C. By the exactness of the sequence. we necessarily have $\partial(1_C) = 0$. Viceversa, $\partial(1_C) = 0$ implies there is a global section that is mapped to 1_C and so the extension is trivial.
In general, $\partial(1)$ is called the class of the extension $\mathcal{L} \to \mathcal{E} \to M$, and if two extensions have the same class up to a non-zero multiplicative constant, then they are isomorphic.

Proposition 4.3.4. *Let $\mathcal{E}$ be rank 2 locally free sheaf on $\mathbb{P}^1$. then, $\mathcal{E}$ is isomorphic to the direct sum of two invertible sheaves.*

Proof. Up to tensoring the sequence by some $\mathcal{E}$ by $\mathcal{E} \otimes \mathcal{L}$ for some $\mathcal{L} \in \mathrm{Pic}\, \mathbb{P}^1$, we may assume $\deg(\mathcal{E}) = d$ is either 0 or -1. Then, by Riemann-Roch we have $h^0(\mathcal{E}) \geq deg(\mathcal{E}) + 2 \geq 1$, and so there exists a section in $H^0(C, \mathcal{E})$.
Then, there exists an exact sequence

$$0 \to O_{\mathbb{P}^1}(k) \to \mathcal{E} \to \mathcal{O}_{\mathbb{P}^1}(d-k) \to 0$$

where $k \geq 0$. The class of this extension is in $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2k-d))$, which is always zero, hence the extension is trivial. □

A consequence of this is that for some $n \geq 0$ we obtain that tensoring the sequence by $\mathcal{O}_{\mathbb{P}^1}(d-k)^{-1} = \mathcal{O}_{\mathbb{P}^1}(k-d)$, we obtain $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ where we set $n = 2d - k$, and we know n is non-negative. We will denote the surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$ by $\mathbb{F}_n$.
If we let X be a geometrically ruled surface, $\pi : X \to C$ the fibration,we may assume without loss of generality that for some vector bundle $\mathcal{E}$ of rank 2 over C we have $\mathbb{P}_C(\mathcal{E}) = X$. We may canonically define a line bundle $\mathcal{M}$ as a sub-bundle of $\pi^*(\mathcal{E})$. Let $x \in X$ be a point, and let $D \subset E_{\pi(x)}$ be the line corresponding to x, we then define $\mathcal{M}$ by setting $\mathcal{M}_x = D$.
Let $\mathcal{O}_X(1)$ be the tautological bundle on X, that is the dual of the Serre twisting sheaf. By the definition of $\mathcal{M}$, we have an exact sequence

$$0 \to \mathcal{M} \to \pi^* E \xrightarrow{u} \mathcal{O}_X(1) \to 0.$$

If we let $\sigma : C \to X$ be a section of π, we may obtain a line bundle on C by $\mathcal{L} = \sigma^* \mathcal{O}_X(1)$. Then, we also have a surjective morphism $\sigma^* u = \mathcal{E} \to \mathcal{L}$.
Viceversa, let $v : \mathcal{E} \to \mathcal{L}$ be aurjective morphism of sheaves. Then, we may define a morphism $\sigma : C \to X$ by associating to a point $p \in C$ the line

$\ker(\mathcal{E}_p \to \mathcal{L}_p)$. This construction gives a section, and we may also observe that it is the inverse of the previous one, so we have that giving a section of the structure morphism of a geometrically ruled surface $\pi : \mathbb{P}_C(\mathcal{E}) \to C$ is equivalent to giving a quotient line bundle for the associate vector bundle $\mathcal{E}$. We may also give the following results about the Picard group of a geometrically ruled surface.

Proposition 4.3.5. *Let X be a geometrically ruled surface over a curve C and let $\pi : X \to C$ be the structure morphism. Denote by h the class of the sheaf $\mathcal{O}_X(1)$ in $\operatorname{Pic} X$. Then we have $\operatorname{Pic} X \cong \pi^* \operatorname{Pic} C \oplus \mathbb{Z}h$.*
Furthermore, $H^2(X, \mathbb{Z}) \cong \mathbb{Z}h \oplus \mathbb{Z}f$, and if we let $\mathcal{E}$ be a vector bundle generating X, we have that $h^2 = deg(\mathcal{E})$ and the class of the canonical divisor K is $[K] = -2h + (deg(\mathcal{E}) + 2g(C) - 2)f$.

Proof. Let F be a fibre of π, and let H be a divisor associated to h, which has to be a section of π. Then, $F.H = 1$ as every section of π has to intersect each fibre exactly once. This means that for every divisor in $D' \in \operatorname{Div} X$ we have $D'.F = m$ for some m, and so $D' \sim D + mH$ for some divisor $D \in \operatorname{Div} X$ which satisfies $D.F = 0$. To complete the first part, it is now enough to show that any such D is the pullback of a divisor on C and so a fibre.
Set $D_n = D + nF$. It follows immediately that $D_n^2 = D^2$ and $D_n.F = 0$, and since $F.K = -2$ we have $D_n.K = D.K - 2n$. Thus, for n large enough we have $h^0(K - D_n) = 0$.
If we apply Riemann-Roch's theorem to D_n we then obtain that $h^0(D_n) \geq n+c$ for some constant c. Then, for n large enough the linear system $|D_n|$ is non-empty, and we may take an effective divisor E in it. Since $E.F = D_n.F = 0$, then E is obtained a sum of fibres of π, and so E is the inverse image by π of a divisor on C.
$H^2(X, \mathbb{Z})$ is a quotient of $\operatorname{Pic} X$, and two points of C have the same cohomology class in $H^2(C, \mathbb{Z}) \cong \mathbb{Z}$. So $H^2(X, \mathbb{Z})$ is generated by f and h, which are linearly independent as $f^2 = 0$, $f.h = 1$.
Let $\mathcal{E}$ be a bundle generating X as a geometrically ruled surface over C. We know there exists an exact sequence $0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{M} \to 0$ of sheaves on C by 4.3.2, and we may consider the pull-back of this sequence through π, and compute the intersection number $(\pi^*\mathcal{L}.\pi^*\mathcal{M})$, which by the definition of the intersection form in $\operatorname{Pic} X$ and the fact that $(\mathcal{L}.\mathcal{M}) = (\mathcal{L}^{-1}.\mathcal{M}^{-1})$ is $\chi(\mathcal{O}_X) - \chi(\mathcal{L}) - \chi(\mathcal{M}) + \chi(\mathcal{L} \otimes \mathcal{M})$. We also have $\chi(\mathcal{L}) + \chi(\mathcal{M}) = \chi(\mathcal{E})$ and $\chi(\mathcal{L} \otimes \mathcal{M}) = \chi(\wedge^2\mathcal{E})$, and so $(\pi^*\mathcal{L}.\pi^*\mathcal{M}) = \chi(\mathcal{O}_X) - \chi(\mathcal{E}) + \chi(\wedge^2\mathcal{E})$.
This means that the number $\pi^*L.\pi^*\mathcal{M}$ does not depend on the choice of $\mathcal{L}$ and $\mathcal{M}$ and so is a characteristic of $\pi^*\mathcal{E}$.
The exactness of the sequence $0 \to \mathcal{N} \to \pi^*\mathcal{E} \to \mathcal{O}_X(1) \to 0$ gives $\mathcal{O}_X(1).\mathcal{N} = 0$. We get an isomorphism $\mathcal{N} \otimes \mathcal{O}_X(1) \cong \pi^* \wedge^2 \mathcal{E}$. If we denote the class of $\wedge^2\mathcal{E}$

in $\operatorname{Pic} C$ by e, then this last formula gives that the class of $\mathcal{N}$ in $\operatorname{Pic} C$ is $[\mathcal{N}] - h + \pi^* e$.
Then $\mathcal{N}.\mathcal{O}_X(1) = -h^2 + h.\pi^* e = 0$, and so $h^2 = \deg(\mathcal{E})$.
Now, we know that there exist some a,b integers such that $[K] = ah + bf$ in $H^2(X, \mathbb{Z})$. As we have $[K].f = -2$ we get $a = -2$. We may take a section of π, say $s : C \to X$, such that $[s(C)] = h + rf$ in $H^2(X, \mathbb{Z})$ for some integer r. The genus formula for $s(C)$ may be written

$$2g(C) - 2 = (h + rf)^2 + (h + rf).(-2h + bf)$$

which we can rewrite $\deg(\mathcal{E}) + b$, and so by rearranging we get the equation in the statement. □

4.4 Minimal rational surfaces

Proposition 4.4.1. *The surfaees $\mathbb{F}_n$ are minimal if $n \neq 1$, and $\mathbb{F}_n$ is not isomorphic to $\mathbb{F}_m$ unless $n = m$.*

Proof. From 4.3.5, we have that if f and h are the classes in $\operatorname{Pic} \mathbb{F}_n$ of a fibre and of the tautological bundle respectively, then $\operatorname{Pic} \mathbb{F}_n \cong \mathbb{Z}f \oplus \mathbb{Z}h$, $f.h = 1$ and $h^2 = n$.
Consider the section s of the projection $\pi : \mathbb{F}_n \to \mathbb{P}^1$ which corresponds to $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n) \to \mathcal{O}_{\mathbb{P}^1}$, and denote by B the curve $s(C)$, and by b the class of B in $\operatorname{Pic} \mathbb{F}_n$, which we may express as $h + rf$ for some $r \in \mathbb{Z}$.
As we have that $s^* \mathcal{O}_{\mathbb{F}_n}(1) = \mathcal{O}_{\mathbb{P}^1}$, then we also have $h.b = 0$ which in turn implies $r = -n$. In particular, we get $b^2 = -n$.
Let C be another curve on $\mathbb{F}_n$ such that $C^2 < 0$, and set $c = \alpha h + \beta f$ to be its class in $\operatorname{Pic} \mathbb{F}_n$. We have $c.f \geq 0$, which means we must have $\alpha \geq 0$, and since $c.b \geq 0$ as it is the intersection of two curves, and $h.b = 1$ from before, we have $\beta \geq 0$ too. Thus, $c^2 = \alpha^2 n + 2\alpha\beta \geq 0$, which means that B is the unique curve with negative self-intersection on $\mathbb{F}_n$.
Then we have found that if $n \neq 1$ there is no curve on $\mathbb{F}_n$ that may be contracted, and so $\mathbb{F}_n$ is minimal. Also, if we have any rational surface, we may determine uniquely n by searching for the unique curve with negative self-intersection.
To show that $\mathbb{F}_1$ is not minimal, consider the surface S obtained by blowing up $\mathbb{P}^2$ at a point p.
We may construct a map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ by taking for any point $q \neq p \in \mathbb{P}^2$ the line passing through q and p. As the set of lines through a point in $\mathbb{P}^2$ can be identified with $\mathbb{P}^1$ this is a good definition. This map is not defined at p, obiously, but then by 3.2.5 we may extend it to a morphism $S \to \mathbb{P}^1$, which then gives S the structure of a surface geometrically ruled over

$\mathbb{P}^1$. We may conclude by saying that every surface geometrically ruled over $\mathbb{P}^1$ is isomorphic to one of the $\mathbb{F}_n$ by 4.3.4 and we saw before that n is determined uniquely on such a surface, we then have $S \cong \mathbb{F}_1$ which is then not minimal. □

Lemma 4.4.2. *Let X be a minimal surface such that $K^2 < 0$. Then, for all $n > 0$ there exists an effective divisor $D \in \operatorname{Div} X$ such that $K.D \leq -n$ and $|K + D| = \varnothing$.*

Proof. If there exists an effective divisor D that satisfies $K.D < 0$, then one of its components C is such that $K.C < 0$ necessarily.
By 2.2.5 we have, $C^2 \geq -1$, and notice that $C^2 = -1$ may only happen if the curve C is actually an exceptional divisor, which is not possible as X is minimal, so $C^2 \geq 0$.
For m large enough, we have that $(nC + mK).C < 0$, then it follows there is no effective divisor linearly equivalent to $(nC + mK)$, that is $|nc + mK| = \varnothing$, by lemma 2.1.4.
Let m be the largest integer such that the linear system $|nC + mK|$ is not empty, and let $D' \in |nC + mK|$.
We may compute that $K.D' = K.(nC + mK) \leq -n$ and by linear equivalence we also obtain $|K + D'| = |nC + (m+1)K| = \varnothing$. It is now sufficient to find that there exists an effective divisor D such that $K.D < 0$.
Let H be an hyperplane section of X. Obviously, if $K.H < 0$, we may take $D = H$, and if $K.H = 0$ for some integer m large enough the linear system $|K + mH|$ will be non-empty, and so we may take D as an element of this linear system.
Assume now $K.H > 0$. Then, the rational number $\frac{K.H}{-(K^2)} = q$ is positive. Then, $(H + qK).K = 0$, and so it follows

$$(H + qK)^2 = H^2 + q^2K^2 + 2qK.H = H^2 + qK.H > 0.$$

Then, we may find a rational number $r = a/b > q$ such that $(H + rK).K < 0$, $(H + rK)^2 > 0$. Then $D' = b(H + rK)$ is a divisor with integer coefficients, such that $D'^2 > 0$ and $D'.K < 0$. We may now observe that if we apply (2.2.4) to mD' for $m \in \mathbb{Z}$, we have that for $m \to +\infty$, $h^0(mD') + h^0(H - mD') \to +\infty$. Now, for m large enough we have that $(K - mD').H < 0$ and so $|mD'|$ is non-empty, and we may just take in it the divisor D we are looking for. □

Proposition 4.4.3. *Let X be a minimal surface such that $q(X) = 0$ and $P_2(X) = 0$. Then there exists on X a smooth rational curve C such that $C^2 \geq 0$.*

Proof. It is sufficient to show that there exists an effective divisor D on X such that $K.D < 0$ and $|K + D| = \varnothing$, as then for a component C of D we will necessarily have $K.C < 0$ and $|K+C| = \varnothing$. We then have $h^0(X, \mathcal{O}_X(K+C)) = 0$ and from applying (2.2.4) to $|K + C|$ we get

$$0 \geq 1 + \frac{1}{2}(C^2 - C.K) = g(C),$$

so C has genus 0 and is therefore a smooth rational curve. It follows by the genus formula $C^2 \geq -1$. We may als notice that if $C^2 = -1$, then by 3.2.6 C is an exceptional curve on X which contradicts minimality of X, and so $C^2 \geq 0$. Now, we are to find that such an effective divisor D exists. We divide the problem in three cases, corresponding to the value of K^2 with respect to 0. If $K^2 < 0$, the previous lemma tells us that the effective divisor we are looking for exists.
Assume now $K^2 = 0$. As $P_2 = 0$, applying (2.2.4) to $-K$ we get

$$h^0(-K) + h^0(2K) = h^0(-K) \geq 1 + K^2,$$

and so there exists an effective divisor $D \in |-K|$.
Let H be an hyperplane section of X, and let n be the maximum integer such that $|H + nK| \neq \varnothing$. If we take a divisor D in this linear system, then $|K + D| = \varnothing$ necessarily, and $D.K = (H + nK).K = H.K < 0$ since $H \in |-K|$.
Then, only the case $K^2 > 0$ remains. Using the same formula as before, we get $h^0(-K) \geq 2$. So, let us take a divisor $D \in |-K|$ and assume it is reducible as $D = A + B$. As $D \sim -K$, we get $D.K < 0$, which implies that $A.K < 0$ or $B.K < 0$. We may now without loss of generality let $A.K < 0$. It follows $|K + A| = |-B| = \varnothing$. We msy then assume that there we only take irreducible divisors in $|-K|$, and let $D \in |-K|$.
So let H be an effective divisor, and let n be the largest integer such that the linear system $|H + nK|$ is not empty, and we may take $D' \in |H + nK|$ such that $D' = \sum n_i C_i \not\sim 0$.
Since $D \in |-K|$, we have that $K.D' = -D.D' \leq 0$, as by 2.1.4 we may say $D.D' \geq 0$ as D is irreducible. It now follows that for some component C of D', we have $K.C \leq 0$.
From the first part of the proof we get $|K + C| = \varnothing$ and that C is a smooth rational curve, that is $g(C) = 0$.
We also get $C^2 = -2 - K.C$, and so if $K.C \leq -2$ we have $C^2 \geq 0$, which proves the statement. Also, if $K.C = -1$ then $C^2 = -1$ and so it is the exceptional divisor of a blow-up, which is an absurd as X is a minimal surface.
The only case remaining is $K.C = 0$, where we have $C^2 = -2$. As we have that $h^0(K + C) = 0$ then $h^0(2K + C) \leq 0$, and so by Riemann-Roch and the

intersection numbers we found before we get

$$h^0(-K-C) \geq 1 + \frac{1}{2}((K+C)^2 + K.(K+C)) = 1 + \frac{1}{2}(C^2 + 3K.C + 2K^2) = K^2.$$

In particular, $h^0(-K-C) \geq 1$. As $C^2 = -2$, then $C \notin |-K|$- Then there exists some non-zero effective divisor $A \in |-K-C|$, and then $A+C \in |-K|$. We now found a reducible divisor in $|-K|$, which contradicts our previosu hypotesis. Now, consider the case where $|H+nK| = 0$. Then, every effective divisor is linearly equivlent to a multiple of K, and so $\mathrm{Cl}\,X \cong \mathbb{Z}$. If we consider the exact sequence in (2.2.7), we may observe that $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$, and so $b_2 = 1$. By Poincaré duality, we get $K^2 = 1$, and so by Noether's formula (2.2.8)

$$\chi(\mathcal{O}_X) = \frac{1}{12}(K^2 + 2 - 2b_1 + b_2),$$

which gives us $b_1 = -4$, which is impossible. Then, this last case is an absurd as by definition we have $q = \frac{1}{2}b_i = 0$. □

As a corollary of this last statement, we get Castelnuovo's rationality criterion.

Theorem 4.4.4. *Let X be a surface with $q(X) = 0$ and $P_2 = 0$. Then X is a rational surface.*

Proof. Assume X is minimal. By the previous proposition, we have a smooth rational curve C on X such that $C^2 \geq 0$. Then, we may consider the following exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_C(C) \to 0$$

and the associated long cohomology sequence. As $H^1(X,\mathcal{O}_X) = q = 0$, we get that

$$0 \to H^0(X,\mathcal{O}_X) \to H^0(X,\mathcal{O}_X(C)) \to H^0(C,\mathcal{O}_C(C)) \to 0$$

is an exact sequence, and so $h^0(X,\mathcal{O}_X(C)) = h^0(X,\mathcal{O}_X) + h^0(C,\mathcal{O}_C(C))$. Observe that $h^0(X,\mathcal{O}_X) = 1$ and that by Riemann-Roch's theorem on curves

$$h^0(C,\mathcal{O}_C) = 1 - g(C) + deg(\mathcal{O}_C(C)) + h^1(C,\mathcal{O}_C(C)) = 1 + C^2$$

and $h^1(C,\mathcal{O}_C(C)) = h^0(\mathcal{O}_C(K) \otimes \mathcal{O}_C(-C)) = 0$ as $C \cong \mathbb{P}^1$ and the sheaf $\mathcal{O}_C(K) \otimes \mathcal{O}_C(-C)$ has negative degree on C.
Then, $h^0(X,\mathcal{O}_X(C)) = 2 + C^2$, which is ≥ 2 as $C^2 \geq 0$ and we may take a divisor $D \in |C|$ such that $D \neq mC$ for any $m \geq 0$. Then, the linear system P generated by C and D has no fixed component, and determines a rational map $X \dashrightarrow \mathbb{P}^1$, which we msy extend to a rational morphism $X' \to \mathbb{P}^1$, where

X' is obtained by blowing up X at all the base points of P. One of the fibers of the morphism is then isomorphic to C, and so by 4.2.1 the surface X' is rational, and so is X.
If we remove the assumption of minimality, by 4.1.1 there is a birational morphism from X to a minimal surface Y, and the birational invariants are preserved. So we may find a birational map $\mathbb{P}^m \dashrightarrow X$ by composing $\mathbb{P}^m \dashrightarrow Y$ and $Y \dashrightarrow X$. □

Proposition 4.4.5. *Let X be a minimal rational surface. Then, we either have $X \cong \mathbb{P}^2$ or $X \cong \mathbb{F}_n$, where $n \neq 1$*

Proof. Let H be an hyperplane section of X, and let A be the set of smooth rational curves C on X such that $C^2 \geq 0$. By 4.4.3, there exists at least one such curve C.
Let us consider all the curves $C \in A$ such that $C^2 = m$ is minimal, and fix a curve C among these such that $C.H$ is minimal. Let $D = \sum n_i C_i \in |C|$ be a generic effective divisor in the linear system.
Observe that then $C.C_i \geq 0$ for all i, as if $C_i = C$ we have fixed $C^2 \geq 0$, and if C_i is distinct from C this is the intersection of two distinct curves. From 2.1.4 we have $|K+D| = \varnothing$, and so $h^0(X, \mathcal{O}_X(K+D)) = h^0(X, \mathcal{O}_X(K+C)) = 0$, and then for every C_i we get $h^0(K + C_i) = 0$, which means all the curves C_i are rational, as in the proof of 4.4.3.
As we have $K.D = K.C < 0$, we can find a C_i such that $K.C_i < 0$, which we can assume to be C_0 without loss of generality. Then, since X is assumed to be minimal, we necessarily have $C_i^2 \geq 0$.
Observe we may write $D = n_0 C_0 + D'$ where D' is the remainder of the sum, which is still effective, and $C_0.D' \geq 0$. Also, $C.D' \geq 0$ follows from $C.C_i \geq 0$ for all i.
Since $C \sim D = n_0 C_0 + D'$, we have $C^2 = D^2$, and we may rewrite D^2 as $(n_0 C_0 + D').D$, and so we get $n_0 C_0.(n_0 C_0 + D') + D.D' = C^2$. Then, using the fact $D.D' = C.D' \geq 0$ as they are linearly equivalent, this equation gives us an inequality $n_0^2 C_0^2 \leq (n_0 C_0 + D')^2 = C^2 = m$.
We chose C so that m was minimal thus as to not have a contradiction we have that $n_0 = 1$ and $C_0^2 = m$.
Furthermore, $H.C = H.C_0 + H.D'$ and since $H.C$ was supposed to also be minimal among all curves with $C^2 = m$ and $H.D' \geq 0$ as it is intersection of very ample divisor and an effective divisor, we get $H.D' = 0$ and so $D' = 0$, that is $C_0 = C$ and so we conclude all the divisors in $|C|$ are smooth rational curves.
Let $p \in X$ be a point, and let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. We observe that since $\dim(O_{X,p}/\mathfrak{m}_p^2) \geq 3$ then the linear system of curves passing through p with multiplicity at least 2 has codimension ≤ 3 in $|C|$. Since by what we

said before all the curves in $|C|$ are rational, then this has to be empty, and so $\dim |C| \leq 2$.
So let $C_0 \in |C|$ be a curve distinct from C, which is smooth and rational and has $C_0.C = m$. We obtain from the usual exact sequence in 1.2.1 an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C) \to \mathcal{O}_{C_0}(m) \to 0$$

and since we know $H^1(X, \mathcal{O}_X) = 0$, then from the cohomology we obtain $h^0(C) = m + 2$, and $|C|$ has no base points on C_0. It follows that $|C|$ has no fixed point and so it defines a morphism. Recall that $|C| \leq 3$, and $m \geq 0$, so we only have two cases ro check, $m = 0$ and $m = 1$.
If $m = 0$, the morphism $X \to \mathbb{P}^1$ defined by $|C|$ has fibres which are smooth rational curves, and so X is ruled over $\mathbb{P}^1$, and so is one of the $\mathbb{F}_n$, where $n \neq 1$ as $\mathbb{F}_1$ is not minimal.
If instead $m = 1$, then we have a morphism $X \to \mathbb{P}^2$, and for every point in $p \in \mathbb{P}^2$ the fibre over p is the intersection over two rational curves in $|C|$, and so it is a point. Then, this map is actually an isomorphism. □

4.5 The Albanese map

Recall that if V is a complex vector space, V^* its dual, Γ is a lattice in V and $T = V/\Gamma$ is a torus, there is an isomorphism $\delta : V^* \to H^0(T, \Omega^1_T)$ since for $x^* \in V^*$ we have $x^*(v + \gamma) = x^*(v) + c$ for all $v \in V, \gamma \in \Gamma$ and where c is a constant, then the differential dx^* induces a form δx^* on the torus.
We also have an identification $\Gamma = H_1(T, \mathbb{Z})$ as $V \to T$ is an universal cover pf T, and we can an isomorphism $h : \Gamma \to H^1(T, \mathbb{Z})$ by $\gamma \mapsto c_\gamma$, where c_γ is the path $t \mapsto t\gamma$.
Observe that we have $\int_{h_\gamma} \delta x^* = \int_0^1 d\langle x^*, t\gamma\rangle = \langle x^*\gamma\rangle$ using the same notation as before, and recall any morphism of complex tori $u : T_1 = V_1/\Gamma_1 \to T_2 = V_2/\Gamma_2$ is the composition of a translation and a group morphism $a : T_1 \to T_2$. There also exists a linear map $\widetilde{a} : V_1 \to V_2$ that induces a, and such that $\widetilde{a}(\Gamma_1) \subseteq \Gamma_2$. With the identification δ given above, $V_! \cong H^0(T_1, \Omega^1_{T_1})$ and $V_2 \cong H^0(T_2, \Omega^1_{T_2})$ so we may identify the form a^* with the transpose of $\widetilde{a}$. In particular we may determine a by $u^* : H^0(T_2, \Omega^1_{T_2}) \to H^0(T_1, \Omega^1_{T_1})$

Proposition 4.5.1. *Let X be a smooth projective variety. Then there exists an Abelian variety A and a morphism $\alpha : X \to A$ with the universal property that for any complex torus T and any morphism $f : X \to T$, there exists an unique morphism $\widehat{f} : A \to T$ such that $\widehat{f} \circ \alpha = F$.*
Furthermore, α induces an isomorphism $\alpha^* : H^0(A, \Omega^1_A) \to H^0(X, \Omega^1_X)$.

Proof. Let $i : H^1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^*$ be the map defined by $\langle i(\gamma), \omega\rangle = \int_\gamma \omega$.

By Hodge theory, the image of i in $H^0(X,\Omega^1_X)$ is a lattice, and the quotient is an Abelian variety. Let us denote this lattice by H, and let us set $A = H^0(X,\Omega^1_X)^*/H$, with the canonical projection $\pi : H^0(X,\Omega^1_X)^* \to A$.
Fix now a point $p \in X$, and let c_x be a path connecting p with another point $x \in X$. We may define a linear form $a(c_x)$ on $H^0(X,\Omega^1_X)$ by $\omega \mapsto \int_{c_x} \omega$. If we take another path from p to x, denoted c'_x, we may observe $\int_{c_x} \omega - \int_{c'_x} \omega \in H$. Then, by continuity $a(c_x)$ and $a(c'_x)$ only differ by an element of H and therefore have the same class in A, which we denote by $\alpha(x)$.
We want to show α is analytic in a neighborhood of a point $q \in X$. We may choose a path c from p to q and a neighborhood U of q such that U is isomorphic to a ball in $\mathbb{C}^n$.
We now define for all points $x \in U$ a path c_x by composing c and the segment from q to u, and define a by setting $a(x) = a(c_x)$. Now, $\alpha_{|U} = \pi \circ a$, and $\alpha(p) = 0$, and if we change p then α is altered by translation in A.
We may define an identification δ as above from $H^0(X,\Omega^1_X)$ to $H^0(A,\Omega^1_A)$. and take $\omega \in H^0(X,\Omega^1_X)$. Let $\omega \in H^0(X,\Omega^1_X)$, and as locally we can write $\alpha = \pi \circ a$ we obtain $\alpha^*(\delta\omega) = \alpha^*\pi^*(\delta\omega) = a^* d(\langle \omega, \cdot \rangle)$.
If we take a point $x \in X$, we have $d(\langle \omega, a(x)\rangle) = d(\int_p^x \omega) = \omega(x)$, and so we have $a^*(\delta\omega) = \omega$. Then since δ is an isomorphism we have that α^* is then an isomorphism $H^0(A,\Omega^1_A) \to H^0(X,\Omega^1_X)$ as required.
Let V be a complex vector space, Γ a lattice in V and let $T = V/\Gamma$ be a complex torus. Let $f : X \to T$ be a morphism, and construct the commutative diagram

$$\begin{array}{ccc} H^0(T,\Omega^1_T) & \xrightarrow{\widetilde{f}^*} & H^0(A,\Omega^1_A) \\ & \searrow^{f^*} & \downarrow{\scriptstyle \alpha^*} \\ & & H^0(X,\Omega^1_X) \end{array}$$

where $\widetilde{f}^*$ is determined by the isomorphism α^*. Then, $\widetilde{f}$ is determined up to translation, and as we require $\widetilde{f}(0) = f(p)$, then such an f is unique.
Consider the composition homomorphism $u = f^* \circ \delta$ satisfies ${}^t u(H) \subset T$ and let $\gamma \in H_1(X,\mathbb{Z})$ and $v \in V^*$. Then

$$\langle {}^t u(i(\gamma)), v^* \rangle = \langle i(\gamma), u(v^*) \rangle = \int_\gamma f^*(\delta v^*) = \int_{f_*\gamma} \delta v^*$$

and $\int_{f_*\gamma} \delta v^* = \langle h^{-1}(f_*\gamma), v^* \rangle$. So ${}^t(u(i(\gamma))) = h^{-1}(f_*\gamma) \in H$, from which follows ${}^t u(H) \subset T$ and so the proposition is proven. $\square$

We have that $\dim \mathrm{Alb}(X) = h^0(X, \Omega^1_X) = q(X)$, and so if $q = 0$ then every morphism from X to a complex torus is trivial. Furthermore, the Albanese map is functorial n nature. Let X, Y be two varieties, and $f : X \to Y$ a morphism between them. Then, by universal property there exists an unique morphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$ making the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathrm{Alb}(X) & \xrightarrow{F} & \mathrm{Alb}(Y) \end{array}$$

commutative.
The following results is a corollary of Zariski's main theorem ([3], III.11.4)

Theorem 4.5.2 (Stein factorization)**.** *Let $f : X \to Y$ be a proper morphism of projective schemes. Then f factorizes as $f : X \overset{p}{\to} \widetilde{Y} \overset{g}{\to} Y$ where g is a finite morphism and p is a surjective morphism with connected fibres.*

We will use this result to prove the next one.

Lemma 4.5.3. *Let X, Y be smooth projective varieties and $\alpha : X \to \mathrm{Alb}(X)$ the Albanese map of X. Suppose there exists two morphisms $f : X \to Y$ and $g : Y \to \mathrm{Alb}(X)$ such that $g \circ f = \alpha$. Then, the map $\widetilde{g} : \mathrm{Alb}(Y) \to \mathrm{Alb}(X)$ is an isomorphism.*

Proof. The Albanese map is functorial, and so it provides a morphism $F : \mathrm{Alb}(X) \to \mathrm{Alb}(Y)$, and we have a diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & & \\ \downarrow{\scriptstyle \alpha_S} & & \downarrow{\scriptstyle \alpha_T} & \overset{g}{\searrow} & \\ \mathrm{Alb}(X) & \xrightarrow{F} & \mathrm{Alb}(Y) & \xrightarrow{\widetilde{g}} & \mathrm{Alb}(X) \end{array}$$

From this diagram, we can get that $\widetilde{g} \circ F$ is the identity by universal property, and since F is surjective both F and $\widetilde{g}$ are isomorphisms. □

Proposition 4.5.4. *Let X be a surface, and $\alpha : X \to \mathrm{Alb}(X)$ the Albanese map. Suppose $\alpha(X)$ is a curve C. Then C is smooth, $g(C) = q(X)$ and the fibres of α are connected.*

Proof. Let N be the normalization of the curve C. As X is normal, there exists morphism $f : X \to N$ and $g : N \to \mathrm{Alb}(X)$ such that $\alpha = g \circ f$, and by the previous lemma we have that $\widetilde{g} : \mathrm{Alb}(N) \to \mathrm{Alb}(X)$ is an isomorphism.

The map $N \to \mathrm{Alb}(N)$ is an embedding, and then so is g. Then, $N = C$, that is C is already normal and so it is smooth, and has arithmetic genus $g(C) = q(X)$.
It remains to prove the fibres are connected. By Stein factorization, which we stated before, a proper morphism of schemes $f : X \to Y$ factorizes as a surjective morphism with connected fibres followed by a finite morphism, and so we may take $\widetilde{C}$ such that we have a finite morphism $\phi : \widetilde{C} \to C$ and a surjective morphism $p : X \to \widetilde{C}$ with connected fibres such that $\phi \circ p = \alpha$
We may now assume without loss of generality $\widetilde{C}$ is smooth as we can replace it with its normalization.
Then, ϕ induces an isomorphism $\eta : \mathrm{Alb}(\widetilde{C}) \to \mathrm{Alb}(C)$ such that the diagram

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{\phi} & C \\ \downarrow{\scriptstyle \alpha_{\widetilde{C}}} & & \downarrow{\scriptstyle \alpha_C} \\ \mathrm{Alb}(\widetilde{C}) & \xrightarrow{\eta} & \mathrm{Alb}(C) \end{array}$$

is commutative, so ϕ is an isomorphism and from this we get the fibres of α are connected. □

Lemma 4.5.5. *Let X be a surface with $p_g = 0$ and $q \geq 1$, and consider the Albanese map $\alpha : X \to \mathrm{Alb}(X)$. Then $\alpha(X)$ is a curve.*

Proof. Let uis denote $\mathrm{Alb}(X)$ by A. If $\alpha(X)$ is a surface, then we have that the induced morphism $\alpha : X \to \alpha(X)$ is generally finite, and so we may take an open $U \subseteq \alpha(X)$ over which α is étale.
Take a point $p \in U$, and let $x_1, ..., x_n$ be local coordinates for A centered at p, and choose them so that $\alpha(X)$ is defined by $x_i = 0$ for $3 \leq i \leq n$.
As A is an abelian variety,its tangent bundle is trivial, and so there exists a 2-form $\omega \in H^0(A, \Omega^2_A)$ such that ω and $dx_1 \wedge dx_2$ have the same value at p. Then, $\alpha^*\omega$ is a global 2-form on X and it is non-zero above p. This contradicts that $p_g = h^0(X, \Omega^2_X) = 0$, and so □

Notice that if we combine this result with the previous one, we obtain that the curve $\alpha(X)$ has genus ≥ 1.

4.6 Minimal non-ruled surfaces

Theorem 4.6.1. *Let $\phi : X \dashrightarrow X'$ be a birational map between two minimal non-ruled surfaces. Then, ϕ is an isomorphism*

Proof. By 3.2.1, there exists a surface Y such that we get a commutative diagram

$$\begin{array}{ccc} & Y & \\ {\scriptstyle \epsilon}\swarrow & & \searrow{\scriptstyle f} \\ X' & \overset{\phi}{\dashrightarrow} & X \end{array}$$

where f is a birational morphism and ϵ is the composition of a finite amount of blow-ups, say $\epsilon = \epsilon_1 \circ ... \circ \epsilon_n$. If $n = 0$, then we have $f = \phi$ and so the case is trivially solved. Assume then that $n \geq 1$.
Let E be the curve on Y which is the exceptional divisor of the blow-up ϵ_n. As n is minimal by construction, we have that $f(E)$ has to be a curve C on X, otherwise by 3.2.4 we could factorize f through ϵ_n, so n would not be minimal and we would get a contradiction.
Also, 3.2.5 tell us that the morphism f is the composition of a finite number of blow-ups too. Let $\epsilon : V \to X$ be the blow-up at a point p, and E' the exceptional divisor of this blow-up, and let m be the multiplicity of the curve C in p. As $\epsilon^* K_X = K_V + E$, we have that

$$K_V.E = (\epsilon^* K.E).(\epsilon^* C + mE) = K_X.C - m.$$

In particular, if $m = 0$, and so C does not pass through the blow-up point we have $K_V.E = K_X.C$.
Inductively, we get that $K_Y.E \geq K_X.C$, and as $K_Y.E = -1$ we get an upper bound for $K_X.C$. Observe that the relation is satisfied with equality if and only if C does not pass through any blown up point, which means the restriction of f to E is an isomorphism. But, C would now be a rational curve such that $K_X.C = -1$, and so it would be contractible through a rational morphism, which would contradict that X is a minimal surface.
We now necessarily have $K_X.C \leq -2$, and so $C^2 \geq 0$ by (2.2.6). We may also notice that $|nK_X| = \varnothing$ for all $n \geq 1$, as otherwise by lemma 2.1.4 we would have that $K_X.C = D.C/n \geq 0$, where $D \in |nK_X|$.
If $q(X) = 0$, the surface X would then be rational, and so we may assume $q \geq 1$. In this case, the Albanese map gives a surjective morphism $\phi : X \to B$ which has connected fibres, where B is a smooth curve of genus q. Since C is rational, then it is contained in a fibre F of ϕ.
As we already have $C^2 \geq 0$ from before, 4.2.2 gives us F can not be a reducible fibre, and so has to be an integer multiple of C, say rC. So, $0 = F^2 = r^2 C^2$, which implies $C^2 = 0$, hence $C.K = -2$. We may observe that by the genus formula we get $r = 1$ and $g(F) = 0$, and so by 4.2.1 we have X is ruled, which gives a contradiction. □

From this last result, we obtain that, up to isomorphism, the minimal surface associated to any non-ruled surface is unique up to isomorphism. We have now described the minimal surfaces associated to ruled and rational surfaces, and observed that there is an unique one in the remaining cases, and we may begin the process of classifying minimal surfaces.

Definition 4.6.2. Let X be any smooth projective variety over a field k and K the canonical divisor of X. The Kodaira dimension of the surface X, denoted $\kappa(X)$, is defined as the maximum of the dimension of the image of X through the morphism associated to the linear system $|nK|$, which we will denote as $\phi_{|nK|}$, for $n \geq 1$.

We can easily see that the Kodaira dimension of a variety is bounded by the dimension of the variety. In particular, for a surface X we may only have $\kappa(X) = -\infty, 0, 1, 2$, where by $\kappa(X) = -\infty$ we denote the case $\phi_{|nK|} = \varnothing$ $\forall n \geq 1$. Recall that there is an identification between $H^0(X, \mathcal{O}_X(D))$ and the linear system $|D|$, so we the plurigenus P_n can be seen as the dimension of the linear system $|nK|$.
Observe also that the morphism associated to a linear system is well defined, so $\phi_{|nK|}(X) = \varnothing$ if and only if the linear system $|nK!$ is composed only of fixed points, that is $\dim |nK| = P_n = 0$, and so $\kappa(X) = -\infty$ implies $P_n(X) = 0$ for all $n \geq 0$.
By this reasoning, we can also see that if $\kappa(X) \geq 0$, then there must exist an n such that $P_n(X) \geq 1$. In particular, if $\kappa(X) = 0$ this means that for a given n such that $\mathbb{P}_n \geq 1$ that there exists only 1 divisor passing through a non-fixed point of the linear system, and so $|nK| = 1$, and so $P_n = 1$.
It also follows that if $P_n \leq 1$ for all n then $\kappa(x) \leq 0$, and so $\kappa(X) = 1$ requires that for some n we have $P_n \geq 2$ but $\phi_{|nK|}(X)$ is at most a curve for all n. Finally, we are not able to say much about $\kappa(X) = 2$, but we have that for some n we have $\phi_{|nK|}(X)$ is a surface.
Let us consider a blow-up morphism $\widehat{X} \to X$. Notice that by 3.1.4 we have $K_{\widehat{X}} = \epsilon^* K_X + E$ and that E is not linearly equivalent to any other curve. Thus, the linear system $|nK_{\widehat{X}}|$ has a fixed component nE. If we consider the associated morphism, we may then remove the fixed part and obtain that it is the morphism defined by the linear system $|\epsilon^*(nK_X)|$, which by linearity is $\epsilon^*|nK_X|$. Thus, we obtain that the image of the linear maps that define the Kodaira dimension are invariant for blow-up morphisms and so is the Kodaira dimension, which is then a birational invariant.

5 Surfaces with $\kappa = -\infty$

5.1 Minimal surfaces with $p_g = 0$ and $q \geq 1$

We have seen that the Kodaira dimension of a variety is a birational invariant, and so we want to classify the surfaces over $\mathbb{C}$ by the Kodaira dimension of the minimal surface (or surfaces) they are birationally equivalent to. We will now work towards giving a complete classification of the first case, $\kappa(X) = -\infty$.
So, let X be a surface with $\kappa(X) = -\infty$. We know $P_n(X) = 0$ for all $n \geq 1$, and so $P_2 = 0$ in particular. Then, if the surface has $q = 0$ we may just apply 4.4.4 and say the surface X is rational, so we may restrict ourselves to consider surfaces where $q \geq 1$.
In particular, we have the following characterization:

Lemma 5.1.1. *Let X be a minimal surface with $K^2 < 0$. Then $p_g(X) = 0$ and $q(X) \geq 1$.*

Proof. Suppose $p_g \neq 0$. Then, as by Serre duality $p_g = P_1$, we may choose a non-zero $D \in |K|$ and we can write $D = \sum n_i C_i$, where all $n_i > 0$ as D is effective.-
By the statement we have $K.D = K^2 < 0$, and so one of the components C_i satifies $K.C_i < 0$. Since $K.C_i = D.C_i$, and $C_i.C_j > 0$ for any $i \neq j$, as it is the intersection of distinct curves, then by the same reasoning we have that C_i must also satisfy $C_i^2 < 0$. In particular C_i would be an exceptional curve, and so X would not be minimal. It follows that $p_g = 0$.
Repeating the argument for P_n with $n \geq 2$ shows that there is no divisor in any $|nK|$, and so $P_n = 0$ for all n. Then, the Kodaira dimension of X is $\kappa(X) = -\infty$.
We conclude observing that, since $q = 0$ and $P_2 = 0$ mean that X is a rational surface by 4.4.4 and rational surfaces have $q(X) = g(\mathbb{P}^1) = 0$ which implies $K^2 = 8$ or $K^2 = 9$ depending on b_2. Then we are forced to have $q \geq 1$. □

While the converse of this last statement is not true, a surface having $q \geq 1$ and $p_g = 0$ still proves to give a strong condition on the canonical divisor

Proposition 5.1.2. *Let X be a surface with $q \geq 1$ and $p_g = 0$. Then, $K^2 \leq 0$.*

Proof. By (2.2.8) and (2.1.1) we have $12 - 12q = K^2 + 2 - 2b_1 + b_2$, and by Hodge theory $b_1 = 2q$. Then, we obtain the equation $K^2 = 10 - 8q - b_2$.
Since both q and b_2 are non-negative, it is immediate that if $q \geq 2$ then $K^2 < 0$. Let now $q = 1$ and consider the Albanese map $\alpha : X \to \mathrm{Alb}(X)$. Observe that $\alpha(X)$ is an elliptic curve, and let f be the class of a fibre of α

in $H^2(X,\mathbb{Z})$ and let h be the class of an hyperplane section in $H^2(X,\mathbb{Z})$. We have that $f^2 = 0$ and that $f.h > 0$, and so they are linearly independent in $H^2(X,\mathbb{Z})$. Then, $b_2(X) \geq 2$, and from $K^2 = 10 - 8q - b_2$ fwe finally obtain $K^2 \leq 0$.- □

Notice that if we assume $q \geq 1$, then $K^2 = 0$ in the previous proposition may only occur if $q = 1$ and $b_2 = 2$. If we exclude this case, we have that a minimal surface with $p_g = 0$ and $q \geq 1$ has $K^2 < 0$ and viceversa.

Lemma 5.1.3. *Let $\pi : X' \to X$ be an étale map of surfaces of degree n. Then, $\chi_{top}(X') = n\chi_{top}(X)$, $K^2_{X'} = nK^2_X$ and $\chi(\mathcal{O}_{X'}) = n\chi(\mathcal{O}_X)$.*

Proof. The first equation is proved by using a topological argument. Take a triangulation of X', and let $f_i(X)$ be the number of faces of dimension i. Then, $\chi_{top}(X) = \sum(-1)^i f_i(X)$, and since the faces are simply connected their inverse image through π gives a triangulation of X', and since π is étale $f_i(X') = nf_i(X)$.
The second equation is immediate from the fact that $K_{X'} \cong \pi^* K_X$, as then we may apply 2.1.5 to get $K^2_{X'} = nK^2_X$. Finally, $\chi(\mathcal{O}_{X'}) = n\chi(\mathcal{O}_X)$ follows immediately from (2.2.8) and the previous two equations. □

We may use the above lemma to prove the following result, which gives some more information about part of the surfaces with $p_g = 0$ and $q \geq 1$.

Proposition 5.1.4. *Let X be a minimal surface with $K^2 < 0$. Then X is ruled.*

Proof. From 4.5.5 The Albanese map $\alpha : X \to \mathrm{Alb}(X)$ has connected fibres and $\alpha(X) = B$ is a smooth curve.
Assume there exists an irreducible curve C such that $K.C < 0$ and $|K+C| = \varnothing$, so we have $h^0(K+C) = 0$. If we apply Riemann-Roch to $K+C$, we get

$$0 = h^0(K+C) \geq \chi(\mathcal{O}_X) + \frac{1}{2}(C^2 + C.K).$$

As we have $p_g(X) = 0$, we get that this is equal to $1 - q(X) + g(C) - 1$, and so $0 \geq g(C) - q(X)$.
We have X is minimal, so $C^2 \geq 0$ necessarily. If we assume $C^2 = 0$, then C can not be contained in a reducible fibre by lemma 4.2.2, and if C were a fibre, we would have $C^2 = 0$, $C.K = -2$ and $g(C) = 0$. Then, using 4.2.1, we may say that the surface X is ruled, and so $q(X) = 0 = g(C)$.
Now assume $\alpha(C) = \mathrm{Alb}(X)$, and let N be the normalization of C. The morphism α induces a ramified cover of $N \to \mathrm{Alb}(X)$ of a certain degree d and let r be the number of branch points of the cover. By Riemann-Hurwitz's

theorem, we get $g(N) = 1 + d(g(B) - 1) + \frac{r}{2}$.
Normalizing a curve may not increase its genus, so we get the following chain of inequalities

$$q(X) \geq g(C) \geq g(N) \geq 1 + (q(X) - 1)d$$

which can only be satisfied if $q = 1$ and C is already normal, or $d = 1$, and again we have $g(C) = q(X)$.
Assume that X is not ruled. Then, if such a C existed, the restriction of α to C would then be an étale morphism of curves. Also, if $q \geq 2$, then $d = 1$ necessarily and so we get C is a section of α.
By 4.4.2 there is an effective divisor $D = \sum n_i C_i$ such that $|D + K| = \varnothing$ and $D.K < -1$.
Take C_i in the components such that $C_i.K < 0$, and assume its coefficient n_i which is positive as D is effective, is ≥ 2. Then, $|K + 2C_i| = \varnothing$, and so by Riemann-Roch, we get $0 \geq 1 - q + 2C_i^2 + C_i.K$, which we can rewrite as $1 - q + 4g(C) - 4 + C_i.K$, and since $g(C) = q(X)$ we have $0 > 3q - 4$, and so $q < 1$ which is a contradiction.
Now, assume that there are at least two different components in C, and let them be C_1 and C_2. We would then have $|K + C_1 + C_2| = \varnothing$. So, by using once again Riemann-Roch, we get

$$h^0(K + C_1 + C_2) = h^1(K + C_1 + C_2) + 1 - q + \frac{1}{2}((C_1 + C_2)^2 + (C_1 + C_2).K)$$

which we can rewrite as

$$h^1(K + C_1 + C_2) + 1 - q + \frac{1}{2}(C_1^2 + C_1.K) + \frac{1}{2}(C_2^2 + C_2.K) + C_1.C_2$$

and as C_1 and C_2 are irreducible curves, $g(C_1) = g(C_2) = q$, and from this we obtain $(q - 1) + C_1.C_2 + h^1(K + C_1 + C_2) = 0$.
The three terms in this last equation are all non-negative, so they must all be 0a and in particular we must have $C_1 \cap C_2 = \varnothing$, and so we have an exact sequence

$$O \to \mathcal{O}_X(-C_1 - C_2) \to \mathcal{O}_X \to \mathcal{O}_{C_!} \oplus \mathcal{O}_{C_2} \to 0.$$

If we consider the associated long cohomology sequence $H^1(X; \mathcal{O}_X(-C_1-C_2))$ has to be non-trivial, and so $h^1(X, \mathcal{O}_X(K + C_1 + C_2))$ has to be positive too, which gives a contradiction. From this discussion we deduce that D is an irreducible curve on X.
Assume D is a section of α. By Riemann-Roch, we have $h^0(D) \geq 1 - q + \frac{1}{2}(D^2 - D.K) = -K.D \geq 2$, so D moves in its linear equivalence class. If we let F be a generic fibre of α, then $D \cap F$ is a point and it moves linearly on F, and so F is rational, then by 4.2.1 the surface X is ruled. Then, assume

$q = 1$. and so $\alpha_{|D}$ is étale. The inclusion $i : D \hookrightarrow X$ defines a section in the fibred product $e : D \hookrightarrow X \times_B D$.
Then, $e(D) = D'$ is contained in a connected component of the fibred product which we will denote X'. If we consider the projection π of X' onto X, which is étale, we see that $\Omega^1_{X'} \cong \pi^* \Omega^1_X$.and $K_{X'} = \pi^* K_X$.
Now the following chain of equalities holds:

$$K_{X'}.D' = \deg(\epsilon^* K_{X'|D'}) = \deg(i^* K_{|D}) = K.D$$

and $K.D < -1$. We also have the morphism of surfaces π has finite degree, say n. Then, by the previous lemma, we get $\chi(\mathcal{O}_{X'}) = 0$ and so we then have that $h^0(D') \geq 2$ and like before F now has to be rational, which is impossible and so the surface is ruled. □

After this result, only the case where $K^2 = 0$, that is $q = 1$ and $b_2 = 2$, remains. To study this case, we are going to need some more results.

Lemma 5.1.5. *Let X be a surface, B a smooth curve and $\pi : X \to B$ a surjective morphism. Denote by S the subset of B composed of all points where π is not smooth, and let $\eta \in B$ be a point over which π is smooth. Then*

$$\chi_{top}(X) = \chi_{top}(B)\chi_{top}F_\eta + \sum_{x \in S}(\chi_{top}(F_x) - \chi_{top}(F_\eta)),$$

where F_p denotes the fibre above the point p.

Proof. For any topological space Y and $F \subset Y$ closed, there exists a long exact cohomology sequence

$$... \to H^i_c(Y - F, \mathbb{Z}) \to H^i(Y, \mathbb{Z}) \to H^i(F, \mathbb{Z}) \to H^{i+1}_c(Y - F, \mathbb{Z}) \to ...$$

where H^i_c is the cohomology with compact support, from which we obtain $\chi_{top}(Y) = \chi_{top}(F) + \chi_{top,c}(Y - F)$.
Then, we may apply this to the closed subset $Z = \cup_{p \in S} F_p$ of X, and we may observe that $\chi_{top}(Z) = \sum \chi_{top}(F_p)$. Then, if we set $U = X - Z$ we get $\chi_{top}(X) = \sum \chi_{top}(F_s) + \chi_{top,c}(U)$.
Also, S is a closed subset of B, so again we may apply this to B and S, from which we get

$$\chi_{top}(B) = \chi_{top,c}(B - S) + \chi_{top}(S)$$

and since S is finite $\chi_{top}(S) = \#(S)$.
Also, the restriction of π to U gives a fibre bundle $U \to (B - S)$, from which we get $\chi_{top,c}(U) = \chi_{top,c}(B - S)\chi_{top}(F_\eta)$. By combining all these results, and expanding $\chi_{top}(S)\chi_{top}(F_\eta)$ as $\sum_{x \in S} \chi_{top}(F_\eta)$, we obtain the equation in the statement. □

If C is a smooth curve, by Hodge theory we have $H^0(C,\mathbb{C}) \cong H^0(C,\mathcal{O}_C)$, $H^1(C,\mathbb{C}) \cong H^0(C,\mathcal{O}_C) \oplus H^0(C,\omega_C)$ and $H^2(C,\mathbb{C}) \cong H^0(C,\omega_C)$, and by Serre duality we have $H^i(C,\mathcal{O}_C) \cong H^{1-i}(C,\omega_C)$ for $i = 0,1$. Then, for a smooth curve C we obtain that $2\chi(\mathcal{O}_C) = \chi_{top}(C)$.
We can give a weaker version of this statement that is true for any curve.

Lemma 5.1.6. *Let C be a curve, then $\chi_{top}(C) \geq 2\chi(\mathcal{O}_C)$.*

Proof. Consider the normalization N of C and the corresponding morphism $f : N \to C$. We have a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{C}_C & \longrightarrow & f_*\mathbb{C}_N & \longrightarrow & \epsilon & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\phi} & & \\ 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & f_*\mathcal{O}_N & \longrightarrow & \delta & \longrightarrow & 0 \end{array}$$

where $\mathbb{C}_C$ is the constant sheaf with values in $\mathbb{C}$ on C, and ϵ and δ are skyscraper sheaves concentrated at the singular points of C which are defined so to make the respective sequences exact. In particular, if they are 0 then we have $N = C$
We have that any local section of $f_*\mathcal{O}_N$ that sits in both the images of the morphisms $f_*\mathbb{C}_N \to f_*\mathcal{O}_N$ and $\mathcal{O}_C \to f_*\mathcal{O}_N$ is more generally in the image of the composition $\mathbb{C}_C \to f_*\mathcal{O}_N$. By diagram chasing, this is equivalent to saying that the morphism of skyscraper sheaves $\phi : \epsilon \to \delta$ is injective and so we have $h^0(\delta) \geq h^0(\epsilon)$.
Applying χ_{top} and χ in the above diagram we get $\chi_{top}(N) = \chi_{top}(C) + h^0(\epsilon)$ and $\chi(\mathcal{O}_N) = \chi(\mathcal{O}_C) + h^0(\delta)$. We also have $\chi_{top}(\mathcal{O}_N) = 2\chi(\mathcal{O}_N)$, and so by linear combination of the equations we get

$$\chi_{top}(C) = 2\chi(\mathcal{O}_C) + 2h^0(\delta) - h^0(\epsilon) = 2\chi(\mathcal{O}_C) + h^0(\delta) + (h^0(\delta) - h^0(\epsilon))$$

and $h^0(\delta)$ and $(h^0(\delta) - h^0(\epsilon))$ are both positive, so we get $\chi_{top}(C) \geq 2\chi(\mathcal{O}_C)$ as desired. □

We may observe that the above proof gives us $\chi_{top}(C) = 2\chi(\mathcal{O}_C)$ if and only if $h^0(\delta) = h^0(\epsilon) = 0$, that is $C = N$.

Proposition 5.1.7. *Let X be a minimal surface with $p_g = 0$, $K^2 = 0$ and $q = 1$. Let $\alpha : X \to \mathrm{Alb}(X)$ be the Albanese map and let g be the genus of a generic fibre of α. Then, if $g \geq 2$, α is smooth. Instead if $g = 1$, the fibres of α are of the form nC, where C is a smooth elliptic curve.*

Proof. Observe first that by 4.5.5 $\alpha(X)$ is an elliptic curve, which we will denote by B.

Suppose there exists a fibre which is not irreducible, and so assume it is such that it has two different irreducible components F_1 and F_2, and as they are contained in a fibre $F_1.F$ and $F_2.F = 0$ for any fibre F. Let H be a hyperplane section of X. We know that hyperplane sections and fibres of a morphism are linearly independent in $H^2(X,\mathbb{Z})$. Assume H, F_1 and F_2 are linearly dependent, and let h, f_1 and f_2 be their respective classes in $H^2(X,\mathbb{Z})$. Let a, b, c be some coefficients such that $ah + bf_1 + cf_2 = 0$. Let f be the class of a generic fibre. Assume $a \neq 0$, then $0 = (ah + bf_1 + cf_2).f = ah.f$ and so $h.f = 0$, which is a contradiction. Then, $a = 0$, and so for some number r we have $f_1 = rf_2$. It follows $0 < f_1.h = rf_2.h$ and since $f_2.h > 0$ we get $r > 0$. But, by 4.2.2 we have $f_i^2 < 0$, and so $rf_2.f_1 < 0$, but since $f_2.f_1 > 0$ as it is intersection of effective divisors, we have $r < 0$ and so we have now obtained a contradiction.
We msy now say that h, f_1 and f_2 are linearly independent in $H^2(X,\mathbb{Z})$. From 5.1.2 we know that $b_2(X) = 2$, and in the case described above we would have $b_2 \geq 3$, which gives a contradiction. Then, we have that no fibre may be reducible. This however does not exclude that a fibre F is a multiple, say $F = nC$ for some curve C and $n \geq 1$. Also, let F' be a smooth fibre.
In this case, we would have $C^2 = 0$ and $\chi_{top}(F) = \chi_{top}(C)$, but we have $\chi_{top}(C) \geq 2\chi(\mathcal{O}_C)$ by the previous lemma. Furthermore,

$$2\chi(\mathcal{O}_C) = -C^2 - C.K = -\frac{1}{n}F.K = -\frac{1}{n}F'.K = \frac{2}{n}\chi(\mathcal{O}_F),$$

which is $\frac{1}{N}\chi_{top}(F)$ as the fibre is smooth.
Since $g \geq 1$, we have $\chi_{top}(F) \geq \chi_{top}(F')$, where equality holds if and only if $2\chi(\mathcal{O}_C) = \chi_{top}(C)$ and $\frac{1}{n}\chi_{top}(F') = \chi_{top}(F')$, which means either $n = 1$ or $g(C) = 1$.
So $\chi_{top}(F) - \chi_{top}(F') \geq 0$, and in case F is a singular fibre equality golds if and only if C is a smooth elliptic curve, and $g(F') = 1$ too.
Then, applying 5.1.5 we get $\chi_{top}(X) = 2 - 2b_1 + b_2 = 0$, and $\chi_{top}(B) = 0$, and we also get that for every singular fibre F_s we have $\chi_{top}(F_s) - \chi_{top}(F') = 0$. We may now conclude that $F_s = nC_s$ where C_s is an elliptic curve. □

Proposition 5.1.8. *Let $\pi : X \to B$ be a smooth morphism from a surface to a curve, and F a fibre of π- Assume either $g(F) = 1$ or $g(B) = 1$ and $g(F) \geq 1$. Then, there exists an étale cover $B' \to B$ such that the fibration $\pi' : X \times_B B' \to B'$ is trivial.*
Furthermore, we may take $B' \to B$ to be a Galois cover with Galois group G, so that $X \cong (B' \times F)/G$.

The proof for this proposition uses results on moduli spaces which are out of the scope of this work, so we are only going to give an idea of proof.

If we take a smooth morphism from a surface to a curve $X \to B$ with fibres of genus g, then we may find an étale cover B' of B and a surface $X' = X \times_B B'$ such that the fundamental group of B' at the point p acts trivially on $H^1(X'_p, (\mathbb{Z}/n\mathbb{Z})_{X'_p})$, where $n \geq 3$ is fixed. One can show that, for every g and n, there exists a morphism of varieties $U_{g,n} \to T_{g,n}$, where $U_{g,n}$ and $T_{g,n}$ are unique up to isomorphism, that satisfies the previous condition and such that we get a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & U_{g,n} \\ \downarrow & & \downarrow \\ B' & \longrightarrow & T_{g,n} \end{array}$$

which means that every such X' can be expressed as the pull-back of $U_{g,n} \to T_{g,n}$. It can also be shown that if we assume $g = 1$ every analytical morphism $X' \to T_{1,n}$ has to be constant and if $g \geq 2$ then all the analytical morphisms $\mathbb{C} \to T_{g,n}$ are also constant, and $\mathbb{C}$ is an universal cover of B', and so the statement follows from this argument.

Lemma 5.1.9. *Let X be a surface, B a curve and $\alpha : X \to B$ a smooth morphism between them whose fibres are either smooth or multiples of smooth curves. Then, there exists a ramified Galois cover $q : B' \to B$ with Galois group G, a surface X' and a commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{q'} & X \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ B' & \xrightarrow{q} & B \end{array}$$

such that the action of G on B' lifts to X', q' induces an isomorphism $X'/G \tilde{\to} X$ and α' is smooth.

Proof. if α has no multiple fibres, then the problem is trivially solved. Also, if we can eliminate every multiple fibre successive branched covers, the problem is solved. To do so, we may consider a local description of the problem, and use the following argument.
Let $\Delta \subset \mathbb{C}$ be the unit disk, and U a smooth analytic surface, and so not necessarily compact, and let $\phi : U \to \Delta$ be a morphism which is smooth everywhere except over 0, and such that $\phi^* 0 = nC$ for a smooth curve C and a positive integer n. Let $\psi : \Delta \to \Delta$ be the morphism defined by $z \mapsto z^n$, and let $\widehat{U} = U \times_\Delta \Delta$ obtained by the morphisms ϕ and ψ. Let U' be the normalization

of $\widehat{U}$, and π_1 and π_2 the projections of U' on Δ and U respectively.
The group of n-th roots of unity G acts on Δ, and this action induces an action on $\widehat{U}$, which induces an action on U'.
Since we are in a local setting, we may assume there exists coordinates x, y on U such that $\phi(x, y) = x^n$, and C is defined by $x = 0$.
Then, if we let (x, y, z) be coordinates for $\widehat{U}$, $(x, y) \in U$, $z \in \Delta$, we obtain that $\widehat{U} = \{(x, y, z) | x^n = z^n\}$. Then, we may express U as union of $(U_g)_{g \in G}$, where $U_g = \{(x, y, gx\} | (x, y) \in U\}$.
The U_g are isomorphic to U and we can see that $\widehat{U}$ is their glueing along the line C. Then, U' is the disjoint union of the U_g and G acts on U' by switching the U_g around. We have that on each U_g, the restriction of π_1 is defined as $(x, y) \mapsto gx$. Now we have that $\pi_2 : U'/G \to U$ is an isomorphism, and the fibres of $\pi_1 : U'/G \to \Delta$ are smooth and π_2 induces an isomorphism $U'/G \to U$.
We have that the cover $q : B' \to B$ exists and the map $q' : X' \to X$ is étale by the previous result, and this lets us conclude the proof. □

Let X be a non-ruled minimal surface with $p_g = 0$, $q = 1$ and $K^2 = 0$. Then by these last results, there are two curves B and F which have positive genus, and a group G of automorphisms of B that induces an action of $B \times F$ compatible with the action on B, and $X \cong (B \times F)/G$.
Moreover, if $g(F) \geq 2$, then $g(B) = 1$ and so B is elliptic and G is a group of translations of B.
We want to find a characterization for this type of surfaces in terms of its Kodaira dimension.

Lemma 5.1.10. *Let B, F be curves of genus ≥ 1, and G a group of automorphisms of B acting on $B \times F$ compatibly with its action on B. Then:*
- *If $g(F) = 1$ there is an étale cover B' of B a group H acting on B' and F such that $B'/H \cong B/G$ and $(B' \times F)/H \cong (B \times F)/G$.*
- *If $g(F) \geq 2$, then G acts on F and $g(b, f) = (gb, gf)$ for $g \in G$, $b \in B$ and $f \in F$.*

Proof. For $g \in G$ and $b \in B$ we necessarily have that $g(b, f) = (gb, \phi_g(b)f)$ where $\phi_g(b)$ is an automorphism of F depending continuously on B. In the case $g(F) \geq 2$, its group of automorphisms is finite and so $\phi_g(b)$ does not actually depend on b, which is enough to finish the proof of this case.
We may now assume $g(F) = 1$, that is F is elliptic, and fix a point F to be the 0 of the group structure on the points of F. Then, $\phi_g(b) : f \mapsto a_g(b)f + t_g(b)$ where $a_g(b)$ is an automorphism of F that keeps 0 fixed and $t_g(b)$ is a translation. The group of automorphisms of F that keep 0 fixed is finite, so $a_g(b)$ does not depend on B, and for $g, h \in G$ we have $a_g a_h = a_{gh}$.

Then, we may give a group homomorphism $a : G \to Aut_0(F)$ such that $a(g) = a_g$.
Denote by $\mathrm{Div}_0(F)$,the group of divisors of degree 0 on F, and recall that if we consider the group the group structure on the elliptic curve F we have a canonical group isomorphism $F \to \mathrm{Div}_0(F)$ defined by $p \mapsto [p] - [0]$, where $[p]$ is the divisor on F associated to point p.
Then, if we have r points on F, say $p_1, ..., p_r$, we have $\sum [p_i] = (r-1)[0]+[\sum p_i]$, where the sum of points is defined by the group structure on F.
Let $u \in Aut(F)$ and write it as $u(p) = a(p) + t$ for $a \in Aut_0(F)$ and $t \in F$, and let $D = (n-1)[0] + [p]$. Then,

$$u^* D = (n-1)u^*[0] + u^*[p] = (n-1)[a^{-1}(t)] + [a^{-1}p - a^{-1}t]$$

which we may rewrite as $(n-1)[0] + [a^{-1}p - na^{-1}t]$ by the previous formula.
Let H be a hyperplane section of $B \times F$. We may define a line bundle $\mathcal{L} = \mathcal{O}_{B\times F}(\sum_{g\in G} g^* H)$ that is invariant under the action of G. We also denote $\mathcal{L}_b = \mathcal{L} \otimes \mathcal{O}_{\{b\}\times F}$ for all points $b \in B$.and observe it is a line bundle on F of positive degree.
As $\mathcal{L}$ is invariant under the action of G, then we may use the equivalent definition $\mathcal{L}_b = (g^*\mathcal{L}) \otimes \mathcal{O}_{\{b\}\times F} = \phi_g(b)^* \mathcal{L}_{gb}$.
$\mathcal{L}_b$ corresponds to some divisor on F, and so we say $\mathcal{L}_b = \mathcal{O}_F((n-1)[0]+\rho(b))$ for some ρ. This way, we define a morphism $\rho; B \to F$ such that for some integer n we have $\rho(gb) - a_g\rho(b) = nt_g(b)$ for $g \in G$.
We may now define $u \in Aut(B\times F)$ by $u(b, f) = (b, f - \rho(b))$. Then, assuming $n = 1$, we have

$$ugu^{-1}(b, f) = (gb, a_g f)$$

and so we have that u induces an isomorphism $B\times F/G \to (B\times F)/H$, where $H = uGu^{-1}$ acts on $B \times F$ by acting on the two factors separately. Then, if we impose $n = 1$ the remaining case is proven.
In the general case, we have an étale cover $\pi : B' \to B$ induced by the pullback

$$\begin{array}{ccc} B' & \xrightarrow{\rho'} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle n} \\ B & \xrightarrow{\rho} & F \end{array}$$

Notice that B' may be not connected. B' is stable under the action of G, and the group F_n of points of order n on F also acts on B' by $\epsilon(b, f) = (b, f + \epsilon)$, so we may define a subgroup H of $\mathrm{Aut}(B')$ by $H = \langle G, F_n \rangle$, and $g\epsilon g^{-1} = a_g\epsilon$.
Then, there exists a split exact sequence $1 \to F_n \to H \stackrel{v}{\to} G \to 1$.
Then, we may define an action of H on $B' \times F$ by $g(b', f) = (hb', \phi_{vh}(\pi b').f)$.

Since $B'/F_n \cong B$, we have $B'/H \cong B/G$ and $(B' \times F)/H \cong (B \times F)/G$.
We may take $u \in \mathrm{Aut}(B' \times F)$ defined by $u(b', f) = (b', f - \rho'(b'))$. Then,

$$uhu^{-1}(b', f) = (hb', a_{vh} f + \theta_h(b'))$$

where $\theta_h(b') = a_{vh}.\rho'(b') - \rho'(hb') + t_{vh}(\pi b)$.
We now get $n.\theta_h(b') = 0$ and so $\theta_h(b') \in F_n$ and is thus independent of b, and the action of H on $B' \times F$ is of the required form.
If B' is not connected,we may consider a connected component B_0 and the subgroup of H preserving B_0 which we denote H_0. Then, we have a canonical isomorphism $(B' \times F)/H \cong (B_0 \times F)/H_0$. □

Lemma 5.1.11. *Let X be a smooth variety and G a finite subgroup of* $\mathrm{Aut}\, X$*, Let $\pi : X \to X/G$ be the natural projection, and assume $Y = X/G$ is smooth. Then, the G-invariant k-fold p-forms in $H^0(X, (\Omega_X^p)^{\otimes k})$ are the pullback of the k-fold rational p-form on Y such that $\pi^*\omega$ is regular on X.*

Proof. We will prove the lemma for 1-forms, and the general case will follow immediately.
Let $(dy_1, ..., dy_n)$ be a basis for the the space of rational 1-forms $M\Omega_Y^1$ on Y as a $\mathcal{K}_Y$-vector space, where $\mathcal{K}_Y$ the function field of Y. Then, $(\pi^* dy_1, ...\pi^* dy_n)$ is a basis for $M\Omega_X^1$ as a $\mathcal{K}_X$-vector space.
Let α be a rational 1-form on X, and write $\alpha = \sum A_i \pi^* dy_i$ for $A_i \in \mathcal{K}_X$. The form is G-invariant if and only if the A_i are, that is for all i, $A_i = \pi^* B_i$ for some $B_i \in \mathcal{K}_Y$. We then get $\alpha = \pi^*\omega$, where $\omega = \sum B_I dy_i$.
We now have π^* is an isomorphism between $M\Omega_Y^1$ and $(M\Omega_X^1)^G$, which proves the lemma. □

If π is étale, that is G acts freely, this lemma tells us π^* is an isomorphism between $H^0(Y, (\Omega_Y^p)^{\otimes k})$ and $H^0(X, (\Omega_X^p)^{\otimes k})^G$.
In a more general case, let $p \in Y$ be a branch point of π. Then G acts transitively on $\pi^{-1}(p)$, and denote the points in $\pi^{-1}(p)$ by $q_1, ..., q_s$. The order of the stabilizer of the q_i, which are conjugate, is the ramification index e_p, and in particular $s \cdot e_p = \#(G) = \deg \pi$. Then, we may give a local coordinate y on Y and x_i on X around q_i such that $\pi^* y = x_i^{e_p}$ ina neighborhood q_i.
Around p we may then write $\alpha = A y^{-r} (dy)^{\otimes k}$ for some integer r and function A such that $A(p) \neq 0$.
At q_i we then get $\pi^*\alpha = A_i x_i^{-re_p + k(e_p - 1)} (dx_i)^{\otimes k}$ for some A_i that is not 0 at q_i. we then obtain $\pi^*\alpha$ is regular if and only if $-re_p + k(e_p - 1) \geq 0$ and so we get an isomorphism

$$\pi^* : H^0(Y, \omega_Y^{\otimes k}(\sum_{p \in Y} [k(1 - \frac{1}{e_p})])) \to H^0(X, \omega_X^{\otimes k})^G,$$

where $[n]$ is the integral part of n.
Observe that even if we take the previous sum over all points $p \in Y$, the contribution to the sum of all the points where the cover does not ramify is 0.

Proposition 5.1.12. *Let X be a minimal non-ruled surface with $p_g = 0$ and $q \geq 1$. Then, X is isomorphic to $(B \times F)/G$, where B and F are smooth irrational curves, G is a finite group which acts faithfully on both B and F, B/G is elliptic and F/G is rational. Furthermore, either of the following conditions has to hold:*
- *B is elliptic and G is a group of translations of B*
- *F is elliptic and G acts freely on $B \times F$*

Conversely, any such surface $(B \times F)/G$ is minimal, has $p_g = 0$, $q = 1$, $K^2 = 0$ and is non-ruled.

Proof. Let X be a minimal non-ruled surface with $p_g = 0$ and $q \geq 1$. Then, by 5.1.4 we have that $K^2 \geq 0$, and by 5.1.2 we actually get X must have $K^2 = 0$ and $q = 1$-
By the previous lemma we have that $X \cong (F \times B)/G$ where G is a group acting on the curves B and F, and B/G is elliptic. Moreover, either B or F is elliptic, and the projection $\pi : B \times F \to X$ is étale, and this gives us our two cases.
Now we want to prove the converse. First, observe that if there exists a rational curve C on X, then $\pi^{-1}(C) \subset B \times F$ would be a union of rational curves, each of which would map surjectively to either B or F, meaning that either B or F is rational, giving an absurd. Then, X is minimal and non-ruled.
Set $X' = B \times F$, then $H^0(X', \Omega^1_{X'}) \cong H^0(B, \omega_B) \oplus H^0(F, \omega_F)$, which gives $q(X') = g(B) + g(F)$.
Furthermore, we also have $H^0(X', \Omega^2_{X'}) \cong H^0(B, \omega_B) \otimes H^0(F, \omega_F)$, which gives us $p_g(X') = g(B)g(F)$. We then also get $\chi(\mathcal{O}_{X'}) = \chi(\mathcal{O}_B)\chi(\mathcal{O}_F)$, and since either B or F is elliptic $\chi(\mathcal{O}_{X'}) = 0$.
Also, if B is elliptic, then $\Omega^2_{X'} \cong \pi_2^*(\omega_F)$ where $\pi_2 : X' \to F$ is the canonical projection on the second coordinate. Similarly if F is elliptic, we obtain an isomorphism $\Omega^2_{X'} \cong \pi_1^*(\omega_B)$.Then $K^2_{X'} = 0$, and by 5.1.3 we obtain that $K^2_X = 0$ and $\chi(\mathcal{O}_X) = 0$.
Since π is étale, there exists an isomorphism $H^0(X, \Omega^1_X) \to H^0(X', \Omega^1_{X'})^G \cong H^0(B, \omega_B)^G \oplus H^0(F, \omega_F)^G = H^0(B/G, \omega_{B/G}) \oplus H^0(F/G, \omega_{F/G})$-
We know B/G is elliptic and F/G is rational, so $q(X) = 1$. Then, $p_g(X) = 0$ since $\chi(\mathcal{O}_X) = 0$. □

Before stating the next result, let us recall without giving proof the following fact about automorphisms of elliptic curves.

Proposition 5.1.13. *Let C be an elliptic curve on which we fixed a group structure. Every automorphism of C is the composition of a translation amd an automorphism of the group structure. Furthermore, for a general elliptic curve the only non-trivial group automorphisms is the symmetry $x \mapsto -x$. There are more non-trivial automorphisms only in the following cases:*
- *If $C \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$ there are automorphisms $x \mapsto \pm ix$.*
- *Let ρ denote a primitive 3rd root of unity. If $C \cong \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}_\rho)$ there are automorphisms $x \mapsto \pm\rho x$ and $x \mapsto \pm\rho^2 x$.*

Proposition 5.1.14. *Let $X = B \times F/G$ be a surface satisfying the conditions of proposition 5.1.12. Then:*
- *$P_4 \neq 0$ or $P_6 \neq 0$ (which implies $P_{12} \neq 0$).*
- *If B or F is not elliptic, then there exists an increasing sequence of integers n_i such that $P_{n_i} \to +\infty$. for $i \to +\infty$*
- *If B and F are both elliptic, $4K_X \sim 0$ or $6K_X \sim 0$. In particular, $12K \cong 0$.*

Proof. First, we may observe that the third property comes naturally from the first one, as if B and F are both elliptic, $K_{B\times F}$ is trivial, so if $D \in |K_X|$ then $\pi^* D = 0$, and so $D = 0$.
Denote $Y = B \times F$. Let us consider the case where G acts on B by translations first. Then, we have that

$$H^0(X, \Omega_X^2)^{\otimes k} \cong (H^0(Y, \Omega_Y^2)^{\otimes k})^G \cong [H^0(B, \omega_B^{\otimes k}) \otimes H^0(F, \omega_F^{\otimes k})]^G$$

and $H^0(B, \omega_B^{\otimes k})$ is G-invariant as G acts on B by translation and regular 1-forms on B are invariant by translation.
We denote by $\mathcal{L}_k$ the sheaf $\omega_{F/G}^{\otimes k}(\sum_{p \in F/G}[k(1 - \frac{1}{e_p})]p)$, and we have

$$P_k(S) = \dim H^0(F, \omega_f^{\otimes k})^G = \dim H^0(F/G, \mathcal{L}_k).$$

Since $F/G \cong \mathbb{P}^1$, then $\mathcal{L}_k$ is determined by its degree, which is

$$\deg \mathcal{L}_k = -2k + \sum_p [k(1 - \frac{1}{e_p})].$$

From Riemann-Hurwitz's formula, we get

$$2g(F) - 2 = -2n + \sum_p n(1 - \frac{1}{e_p}). \tag{5.1.15}$$

If we denote the number of ramification points by r, we get that

$$\deg \mathcal{L}_k \geq -2k + \sum_p (k(1 - \frac{1}{e_p}) - 1) = k\frac{2g(F) - 2}{n} - r.$$

Then, if we assume that $g(F) \geq 2$ we have $P_k \to \infty$ as k grows, so the second property is proven in this case.
We now want to prove the first one, which as said before implies the first one. Write the ramification indexes e_i in increasing order. By Riemann-Hurwitz's formula we obtain $\sum(1 - \frac{1}{e_i}) \geq 2$. If we can show that $\deg \mathcal{L}_k \geq 0$ for some k dividing 12, the statement is proven.
By (5.1.15) we must have at least 3 ramification points for $2g(f)-2 \geq 0$, so we may assume $r \geq 3$. Also, if we let $r \geq 4$, as we know $e_1 \geq 2$ and so $2(1-\frac{1}{e_i}) \geq 1$ for all i, then $\deg \mathcal{L}_2 \geq 0$. Assume then $r = 3$, and so $1/e_1 + 1/e_2 + 1/e_3 \leq 1$. If $e_1 \geq 3$, then the condition is satisfied as the ramification indexes are taken to be ascending, so $\deg \mathcal{L}_3 \geq 0$. Assume then $e_1 = 2$ and $1/e_2 + 1/e_3 \leq \frac{1}{2}$. Using similar reasoning, we may see that $e_2 \geq 4$ gives $\deg \mathcal{L}_4 \geq 0$, and so assume $e_2 = 3$. We now have the condition is satisfied if and only if $e_3 \geq 6$, and so $\deg \mathcal{L}_6 \geq 0$. This finishes all the possible cases, and proves the first property.
We now assume instead F is an elliptic curve and G acts freely on $B \times F$. We may observe that

$$H^0(X, {\Omega_X^2}^{\otimes k}) \cong [H^0(B, \omega_B^{\otimes k}) \otimes H^0(F, \omega_F^{\otimes k})]^G.$$

If we let ω be a regular 1-form on F, then $\omega^{\otimes m}$, where m is the order of the group of automorphisms which by 5.1.13 is 2, 4 or 6, is invariant under $\operatorname{Aut} F$. Let k be a multiple of m, then we have

$$P_k(X) = \dim H^0(B, \omega_B^{\otimes k})^G = h^0(B/G, \mathcal{O}_{B/G}(\sum_p [k(1 - \frac{1}{e_p})]p))$$

as B/G is an elliptic curve. This expression is non-zero, and so tends to ∞ as $k \to \infty$ if $g(B) \geq 2$, so the properties also hold in this case and the statement of the proposition is proven. □

We will say that a surface $X = (B \times F)/G$ where B, F are both elliptic curves and G is a finite group of translations of B whose action on F is such that $F/G \cong \mathbb{P}^1$ is a bielliptic surface.
The following theorem, by Enriques, completes the classification of surfaces with $\kappa(X) = -\infty$.

5.2 Classification of surfaces with $\kappa = -\infty$

Theorem 5.2.1 (Enriques). *Let X be a surface with $P_4 = 0$ and $P_6 = 0$. Then, X is ruled.*

Proof. If $q = 0$, we may apply 4.4.4 since P_2 is 0. If, instead $q \geq 1$, then assume X is not ruled, and so the surface satisfies the conditions of 5.1.12.

Then we may apply 5.1.14 which would give $P_4 \neq 0$ or $P_6 \neq 0$ if X were to be not ruled, which is a contradiction. □

if $\kappa(X) = -\infty$ as we observed previously $P_n(X) = 0$ for $n \geq 1$, and so we may use this theorem to say that any such surface X has to be ruled.
We may actually give a slightly stronger version of this statement

Proposition 5.2.2. *Let X be a surface. The following are equivalent:*

1. *X is ruled*
2. *There exists a curve C on X that is not exceptional and such that $K.C < 0$.*
3. *For every $D \in \operatorname{Div} X$, $|D + nK| = \varnothing$ for n large enough.*
4. *$P_n = 0$ $\forall n \geq 1$.*
5. *$P_{12} = 0$.*

Proof. $1 \implies 2$: Since X is ruled, there exists a minimal surface Y and a rational morphism $f : X \to Y$. We know minimal models for ruled surfaces are either geometrically ruled surfaces by 4.2.4 or $\mathbb{P}^2$. Then, let F be a fibre of Y is ruled, or any line on Y if $Y = \mathbb{P}^2$, over which f is an isomorphism. Then, intersection theory gives $f^*F.K_X = F.K_M$, which is $= -2$ if Y is geometrically ruled and $= -3$ if Y is $\mathbb{P}^2$.
$2 \implies 3$: C is not exceptional, then $C^2 \geq 0$. As $K.C < 0$, we also may find for any divisor D an integer n large enough that $D + nK.C < 0$, and we may conclude by using Lemma 2.1.4.
$3 \implies 4$: if we can take $D = 0$ and observe that if $|nK| = \varnothing$ for n large enough say $n \geq m$, then $|nK| = \varnothing$ for all positive n, otherwise we could just take $D' \in |lK|$ for some l, and for n large enough we have $ln \geq m$ and $lD' \in |lmK|$ which is a contradiction.
$4 \implies 5$ is trivial, and $5 \implies 1$ also is, as it is equivalent to Enriques's theorem. The proposition is then proven. □

From this result, we actually get that every ruled surface has $\kappa = -\infty$, and observe that this last result also gives that if $\kappa(X) \neq -\infty$, then for every divisor D on X we have $D.K \geq 0$.

6 Surfaces with $\kappa = 0, 1, 2$

6.1 Classification of surfaces with $\kappa = 0$

At this point, giving basic classification results for the remaining cases does not require much work. Observe that all non-ruled surfaecs are equivalent up to isomorphism to an unique minimal surface. We will start with $\kappa = 0$.

Proposition 6.1.1. *Let X be a minimal surface with $\kappa(X) = 0$. Then, $K^2 = 0$, $\chi(\mathcal{O}_X) \geq 0$ and if the plurigenuses P_n and P_m are both 1 for some integers n, m, then $P_{(n,m)} = 1$.*

Proof. We know that for $\kappa(X)$ to not be $-\infty$, at least one of the linear systems $|nK|$ has to be non-empty. Let $D \in |nK|$ be an effective divisor. We also have that $D.K \geq 0$ otherwise by 5.2.2 we would have $\kappa(X) = -\infty$, and it follows $K^2 = 0$. Also, we apply (2.2.4) to the divisor nK, and we obtain

$$h^0(nK) + h^0((1-n)K) \geq \chi(\mathcal{O}_X) + \frac{n(n-1)}{2}K^2.$$

Assume $K^2 > 0$ and observe that for $n \geq 2$ we have $|(1-n)K| = \varnothing$ otherwise there would exist an effective divisor $D' \sim (1-n)K$ such that $D'.K < 0$, which is again a contradiction, so we have $P_n = h^0(nK) \geq \chi(\mathcal{O}_X) + \frac{n(n-1)}{2}K^2$. At this point, we may observe that the right hand side grows to ∞ as $n \to \infty$, so we get another contradiction as $P_n \leq 1$, so $K^2 = 0$.
We may now see that by (2.2.8) we get $12\chi(\mathcal{O}_X) = \chi_{top}(X) = 2 - 4q + b_2$, which we may rewrite as $8\chi(\mathcal{O}_X) = -2 - 4p_g + b_2 \geq -2 - 4p_g \geq -6$, where the last inequality follows from that we necessarily have $p_g \leq 1$. We observe that $\chi(\mathcal{O}_X)$ has to be an integer, so this formula actually gives $\chi(\mathcal{O}_X) \geq 0$, as desired.
To prove the last part of the statement, let $d = (n, m)$, so that $n = dn'$, $m = dm'$ and let $D \in |nK|$ and $D' \in |mK|$.
We notice that $|dn'm'K|$ contains both $m'D$ and $n'E$, and since $P_{dn'm'} \leq 1$ this linear system has dimension 0, so $m'D = n'E$. This means, we can find an effective divisor D'' such that $D = n'D''$ and $D' = m'D''$.
We now get that $n'(D'' - dK) \sim 0$ and $m'(D'' - dk) \sim 0$ and since $(n', m') = 1$, then we have $D'' - dK \sim 0$, so $D'' \in |dK|$, and so $P_d = 1$. □

In the following results, we will encounter divisors with coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$. These divisors may be defined simply by extending the definition of divisor with coefficient in $\mathbb{Z}$.

Proposition 6.1.2. *Let X be a surface, C_i irreducible curves on X and m_i positive integers. Set $F = \sum m_i C_i$ and suppose for each i we have $F.C_i \leq 0$.*

Let also $D = \sum r_i C_i$ with $r_i \in \mathbb{Z}$ be a non-zero divisor. Then $D^2 \leq 0$ and if F is connected and $D^2 = 0$, $D = rF$ for some $r \in \mathbb{Q}$, and $F.C_i = 0$ for all i

Proof. Let $G_i = m_i C_i$ and $s_i = r_i/m_i$, so that we may write $F = \sum G_i$ and $D = \sum s_i G_i$. Observe that $G_i^2 = G_i.(F - \sum_{i \neq j} G_j)$. Then, we have

$$D^2 = \sum_i s_i^2 G_i^2 + 2 \sum_{i<j} s_i s_j G_i G_j = \sum_i s_i^2 G_i.F - \sum_{i<j} (s_i - s_j)^2 G_i.G_j$$

Notice now that all the $(s_i - s_j)^2 G_i.G_j$ are positive and that by definition $F.C_i \leq 0$, so $F.G_i \leq 0$ too and so we have $D^2 \leq 0$.
Assume now $D^2 = 0$, and F connected. If two curves C_i and C_j intersect, we have to have $s_i = s_j$ for the above formula to be 0. As F is connected, we can always connect two curves with a chain of curves, so by transitivity all the s_i are the same. Moreover, we also get $F.C_i = 0$ for all C_i by setting the remaining part of the formula equal to 0. □

The following result follows from the previous one

Proposition 6.1.3. *Let X be a surface and $g : X \to X'$ a surjective morphism to a possibly singular surface X'. Let C_i be irreducible curves of X such that $g(C_i) = p \in X'$. Then, for any $D = \sum r_i C_i$ one has $D^2 < 0$.*

Proof. We may assume that $\cup C_i$ is connected, as we may just apply the proof to every connected component otherwise.
We may also replace g by its Stein factorization, and so assume without loss of generality it has connected fibres. We then have $g^{-1}(p) = \cup C'_i$, where the C'_i include the curves C_i.
Let H be a hyperplane section of X' passing through p. We may write $g^*H = \widetilde{H} + F$, where all the components C'_i are connected in F. Then, $g^*H.C'_i = 0$ for all i, as we can move H away from p by the theorem of Serre. and s $F.C'_i = -\widetilde{H}.C'_i \leq 0$, with at least one i satisfying $F, C'_i < 0$ as H intersects the fibre $g^{-1}(p)$. □

Proposition 6.1.4. *Let X be a minimal surface with $\kappa(X) = 0$.Then X belongs to one of the following cases:*
- $p_g = 0$, $q = 0$. *Then $2K \sim 0$. This type of surface is called Enriques surface*
- $p_g = 0$, $q = 1$. *Then X is a bielliptic surface.*
- $p_g = 1$, $q = 0$. *Then $K \sim 0$. This type of surface is called $K3$ surface*
- $p_g = 1$, $q = 2$. *Then X is an Abelian surface.*

Proof. By the relation between Kodaira dimension and plurigenuses, we necessarily have $p_g = P_1 \leq 1$.
Suppose first $p_g = 0$. Then, if $q = 0$, for X to not be a rational surface by

4.4.4, $P_2 \geq 1$, which really means $P_2 = 1$ as $\kappa(X) = 0$, so $|2K| \neq \varnothing$.
By the previous result, we have $K^2 = 0$, and so applying Riemann-Roch to $3K$ we get $h^0(3K) + h^0(-2K) \geq 1$.
Since $P_1 = p_g = 0$ and $P_2 = 1$, if we assumed $P_3 = 1$ we would get a contradiction in that $P_{(2,3)} = P_1 = 1$, and so $P_3 = h^0(3K) = 0$. Then we have $h^0(-2K) \geq 1$ and as $h^0(2K) \geq 1$ too we have $2K \sim 0$.
If, instead $q \geq 1$, by proposition 5.1.12 it follows that $\kappa(X) = 0$ implies $q = 1$ and the surface X has to be bielliptic.
We may now assume $p_g = 1$. First, we know $\chi(\mathcal{O}_X) \geq 0$, so checking $\chi(\mathcal{O}_X) = 1 - q + p_g$ gets us $q \leq 2$.
Setting $q = 0$ and computing Riemann-Roch for $2K$ gives us $h^0(2K) + h^0(-K) \geq 2$, hence $h^0(-K) \geq 1$, which means $K \sim 0$.
Now suppose $q = 1$. Then, there is a divisor $D \in \operatorname{Div} X$ such that $D \nsim 0$, but $2D \sim 0$. In particular, $D.D' = 0$ for all divisors D' on X, and we have $h^0(D) = h^0(-D) = 0$, so Riemann-Roch gives $h^0(K - D) \geq 1$.
Take two divisors D_1 and D_2 in $|K - D|$ and $|K|$ respectively. Then, $2D_1$ and $2D_2$ both are in $|2K|$, giving $2D_1 \sim 2D_2$ as $P_2 \leq 1$. This really means $D_1 \sim D_2$, which contradicts, $D \nsim 0$, so there is no surface in this case.
What is left now is the case $p_g = 1$, $\kappa = 0$, $q = 2$. Let K be an effective divisor linearly equivalent to the canonical divisor, which exists as $p_g = 1$, If $K \neq 0$, we may write $K = \sum n_i C_i$ for some C_i irreducible curves and n_i positive integers. For every C_i we also have $K.C_i \geq 0$, but since $K^2 = 0$ we get $K.C_i = 0$, Computing $K.C_i$ gives $n_i C_i^2 + \sum_{j \neq i} n_j C_i C_j = 0$, so we have $C_i^2 \leq 0$. If $C_i^2 = 0$ then $C_i.C_j = 0$ for all $j \neq i$, and so C_i is a smooth elliptic curve. Else, if $C_i^2 < 0$, $C_i^2 = -2$ and $C_i \cong \mathbb{P}^1$ is a rational curve with a double point, and is a connected component of $\cup C_i$.
Consider the image of X through the Albanese map $B = \alpha(X)$. We will check different cases based on whether or not $K \sim 0$ and B is a curve.
Assume B is a curve and $K \nsim 0$. From 4.5.4, B is smooth, has genus 2, and $\alpha : X \to B$ has connected fibres. No curve of genus < 2 may map surjectively to B, so every connected component D of K is contained in a fibre F_b, for some $b \in B$, of α and so $D^2 = 0$. It follows from 6.1.2 that $D = rF_b$ for some $r \in \mathbb{Q}$. Then $nD = rnF_b = p^*(rn[b])$ where $rn \in \mathbb{Z}$. Therefore, $h^0(nD) \to \infty$ for $n \to \infty$, and so does $h^0(nK)$, which contradicts $\kappa = 0$, so this case is excluded.
Let now $K \sim 0$, and let B still be a curve. Take an étale cover B' of B of degree ≥ 2, and consider $X' = X \times_B B'$. Since the Albanese map has connected fibres, X' is also connected. Also, 5.1.3 gives $K'_X \equiv \pi^* K_X \equiv 0$ and $\chi(\mathcal{O}'_X) = 0$. Then $q(X') = 2$, but $q(X') \geq g(B') \geq 3$, where the last inequality comes from the HurwiTz formula, and so this case is also impossible.
So, $\alpha(X)$ is a surface, and $\alpha : X \to \operatorname{Alb}(X)$ is surjective. If $K \nsim 0$, take a

connected component D of K, and $D^2 = 0$ as before, then by 6.1.3 D is not contracted to a point by α, and so it is not union of rational curves. Then, from the previous discussion $D = nE$ for some smooth elliptic curve E. Denote $\alpha(E)$ by E', and notice it is a smooth elliptic curve in $\mathrm{Alb}(X)$, and assume without loss of generality that it is an abelian subvariety of $\mathrm{Alb}(X)$. Also, let us denote $\mathrm{Alb}(X)$ by B.
Let F be the quotient curve $\mathrm{Alb}(X)/E'$, and let $f : X \to F$ be the composition of α and the standard projection $\mathrm{Alb}(X) \to F$. Consider the Stein factorization of f, $S \overset{g}{\to} B \to F$, and observe that E is contained in a fibre F_b of g which by 6.1.2 means it is $\frac{1}{m}F_b$ for some integer m. We may observe now that as before, we may observe tat $h^0(nD) \to \infty$ as n does, so we get a contradiction.
We now have to study the case where α is surjective and $K \sim 0$, which is the last one remaining. Let η_1 and η_2 be two forms which give a basis of $H^0(A, \Omega^1_A)$, and set $\omega_1 = \alpha^*\eta_1$ and $\omega_2 = \alpha^*\eta_2$. We have that α is étale at a point $p \in X$ if and only if the form $\omega_1 \wedge \omega_2$ does not vanish at p. Since we are in characteristic zero and α is étale almost everywhere, then we may assume that $\omega_1 \wedge \omega_2$ is not identically 0, and since $K \sim 0$ it actually never takes value 0, and so α is an étale cover of an Abelian variety, which is again an Abelian variety, so X is an Abelian surface. □

6.2 Classification of surfaces with $\kappa = 1$

We now move on to surfaces with $\kappa = 1$.

Lemma 6.2.1. *Let X be a minimal non-ruled surface. If $K^2 > 0$, there exists an integer m such that ϕ_{nK} is a birational map from X to its image for all $n \geq m$.*
If $K^2 = 0$ and $P_r \geq 2$, if we write $rK = F + M$ where F is the fixed part of $|rK|$. Then, $K.F = K.M = F^2 = F.M = M^2 = 0$.

Proof. Let H be a hyperplane section of X. Since $K^2 < 0$, Riemann-Roch gives us $h^0(nK - H) + h^0(H + (1-n)K) \to \infty$ as $n \to \infty$. Since X is non-ruled, we have $H.K > 0$ and so for n large enough $(H + (1-n)K).H < 0$, which means that $h^0(H + (1-n)K) = 0$. Then, we may fix an integer m such that $h^0(nk - H) \geq 1$ for all $n \geq m$. Take an effective divisor $D \in |nK - H|$, and observe that $|nK| = |H + E|$, and that $|H + E|$ separates points and tangents to points of $S - E$, and so the restriction of ϕ_{nK} to $S - E$ is an embedding, and so induces a birational map from X to its image.
We also have $rK^2 = K.F + K.M = 0$, and both $K.F$ and $K.M$ are to be non-negative, and so are forced to be zero. Furthermore, since M is not fixed, both

M^2 and $F.M$ are non negative, and by checking $rK.M = F.M + M^2 = 0$ we get they are also zero. We conclude by observing that $F^2 = (rK.M)^2 = 0$. □

Proposition 6.2.2. *Let X be a minimal surface with $\kappa(X) = 1$. Then, $K^2 = 0$ and there exists a smooth curve B and a surjective morphism $p : X \to B$ whose generic fibre is an elliptic curve.*

Proof. By the previous lemma, we have $K^2 \leq 0$ otherwise we would have $\kappa = 2$, and since $K^2 < 0$ implies X is a ruled surface by 5.1.1 we get $K^2 = 0$. Since $\kappa(X) = 1$, there is an integer m such that $P_m \geq 2$. Let Z be the fixed part of the linear system $|mK|$, so that $rK = Z + M$. By the previous lemma we get $M^2 = M.K = 0$, so $|M|$ has no fixed part trivially and no fixed point either, so it defines a morphism $f : X \to \mathbb{P}^n$ whose image is a curve C. By Stein factorization f factors $f : X \to B \to C$, through a morphism $p : X \to B$, which has connected fibres. Let F be a fibre of p. Since M is a sum of fibres and $K.M = 0$, then $K.F$ is also 0. It follows that $g(F) = 1$ so the smooth fibres of p are elliptic curves. □

A surface X such that there exists a smooth curve B and a surjective morphism $X \to B$ whose fibres are elliptic curves is called an elliptic surface. All surfaces with $\kappa = 1$ are elliptic, but the converse is not true. A trivial example of this is the ruled surface $E \times \mathbb{P}^1$ where E is an elliptic curve.
Despite this, we may actually give a result that separates elliptic surfaces with $\kappa = 1$ and ones with different κ.

Proposition 6.2.3. *Let X be a minimal elliptic surface, $p : X \to B$ the elliptic fibration. For a point $b \in B$ denote $F_b = p^*[b]$. Then, $K_X^2 = 0$.*
X is either ruled over an elliptic curve, has $\kappa = 0$ or has $\kappa = 1$. In particular, if $\kappa = 1$ there exists a positive integer d such that $dK \sim \sum n_i F_{b_i}$ and for a large enough integer r the linear system $|rdK|$ is base points free.
The morphism $X \to \mathbb{P}^N$ defined by $|rdK|$ factorizes through p and an embedding of B in $\mathbb{P}^N$.

Proof. Observe that if X is ruled over a curve C then the fibres must map surjectively onto C, so C is either rational or elliptic, and so $K^2 \geq 0$. In particular, we then have that $K^2 \geq 0$ for all minimal elliptic surfaces.
Suppose there exists an integer n such that $|nK|$ is not empty, and take a divisor D in it. Since for all b we have $K.F_b = D.F_b$ the components of D are contained in the fibres of p, and since $K^2 \geq 0$ we have $D = \sum r_i F_{b_i}$ for some (non-negative) $r_i \in \mathcal{Q}$. thus, $D^2 = 0$ and $K^2 = 0$ too.
Let now $K^2 > 0$, then $|nK| = \varnothing$ for all $n < 0$ and Riemann-Roch gives a contradiction in $h^0(nK) + h^0((1-n)K) \to \infty$ as $n \to \infty$, so $K^2 = 0$.
Also, since $K.F_b = 0$, the images of the maps ϕ_{nK} have dimension ≤ 1 as they

contract the fibres F_b, and so $\kappa = -\infty, 0$ or 1.
Assume X has $\kappa = 1$. Then, we may choose an integer n such that $P_n \geq 1$, and let $D \in |nK|$, which as before is a finite sum or fibres with rational coefficients. Then, we may find a multiple of D where all the coefficients are integers, and so we may write $dK \sim p^*A$ for some effective divisor A on B. Furthermore, for a large enough integer r, the system $|rA|$ has no base points and so defines an embedding of B in $\mathbb{P}^N$, and so the linear system $|rdK| = p^*|rA|$ is also without base points and defines a morphism $X \to P^N$ which factorizes through p. □

6.3 Classification of surfaces with $\kappa = 2$

Finally, we may easily assume that all remaining surfaces have $\kappa = 2$. A surface with $\kappa = 2$ is called a surface of general type, and again as every surface not included in the previous cases is of general type, we will not give a strong characterization. We may, however, give some equivalent conditions.

Proposition 6.3.1. *Let X be a minimal surface. The following are equivalent:*

1. $\kappa(X) = 2$
2. $K_X^2 > 0$ *and X is irrational*
3. *There exists an integer n_0 such that $\phi_{|nK|}$ is a birational map of X to its image for all $n \geq n_0$.*

Proof. 2 $\implies$ 3 is true by 6.2.1 as an irrational surface with $K^2 > 0$ is non-ruled, and 3 $\implies$ 1 by definition of Kodaira dimension.
1 $\implies$ 2: If $K^2 = 0$, by 6.2.1 if we denote the mobile part of $|nK|$ by M, we have $M^2 = 0$ and so $\dim \phi_{nK}(X) \leq 1$. This is true for any n, and so this concludes the proof. □

Surfaces with $\kappa = 2$ are quite diverse, so let us conclude with an example of one.
Let C and D be two curves with genus ≥ 2, and consider the surface $X = C \times D$. Let $\pi_1 : X \to C$ and $\pi_2 : X \to D$ be the canonical projections, then we may express $\Omega_X^2 = \pi_1^*\omega_C \otimes \pi_2^*\omega_D$. Then

$$H^0(X, (\Omega_X^2)^{\otimes n}) = H^0(C, \omega_C^{\otimes n}) \otimes h^0(D, \omega_D^{\otimes n})$$

which we can rewrite using canonical divisors as

$$H^0(X, \mathcal{O}_X(nK_X)) = H^0(C, \mathcal{O}_C(nK_C)) \otimes H^0(D, \mathcal{O}_D(nK_D)),$$

and so we may factor the map ϕ_{nK_X} through

$$X = C \times D \xrightarrow{(\phi_{nK_C}, \phi_{nK_D})} \mathbb{P}^{n_1} \times \mathbb{P}^{n_2}.$$

and the Segre embedding $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \hookrightarrow \mathbb{P}^N$ and that a curve has genus ≥ 2 if and only if the image of its canonical morphism is a curve, and so we have that for all $n \geq 1$ the image of X through ϕ_{nK} is the cartesian product of two curves, that is a surface, and so $\kappa(X) = 2$.

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