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# Arithmetic of special values of triple product $L$-functions


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*"Salvation through Zahlentheorie"*

Roger Godement, *"Notes on Jacquet–Langlands' theory"*, §3.1

*"Apparent rari nantes in gurgite vasto"*

Publio Virgilio Marone, *"Eneide"*, Liber I, 118

# Contents

# Introduction

In this Master Thesis we study two remarkable formulas for the central values of the triple product $L-$function, namely the *Gross–Kudla formula* in section (4) and the *Ichino's formula* in section (6). These two formulas have found fruitful applications and generalizations. Important results have been achieved considering $p-$adic analogous formulas and $L-$functions, for example, in the work of Bertolini and Darmon [**BD07**], which is strictly related to the Birch and Swinnerton-Dyer conjecture, and that of Bertolini, Darmon and Prasanna [**BDP13**].

In section (1) we start reviewing some basic results about quaternion algebras, beginning with the definition of quaternion algebras over local and global fields, studying the notion of orders and then considering an adèlic formulation which will be fundamental in the following sections. We try to give all the discussion a computational flavour as the two formulas are meant for both a theoretic and a computational approach.

The second section (2) is devoted to the study of relations among algebraic groups, quaternion algebras and normalization of measures. As a consequence, we study how the group of invertible elements of a quaternion algebra can be thought as an algebraic group and how it is possible to equip its quotient on the center with a normalized Haar measure, the Tamagawa measure.

Section (3) contains a brief overview of the ideas of Jacquet and Langlands in [**JL70**]. Again we try to focus mostly on cases of our interest, having in mind the *Gross–Kudla* formula and a mildly computational point of view. The first half of the section deals with the representation theory on $GL_2$ while the second one reproduces the analogous theory for quaternion algebras and establishes the correspondence.

The hearth of the thesis is indeed section (4) in which we present the *Gross–Kudla formula.* We define the notion of Brandt matrices and give a brief insight on the work of Gross [**G87**]. In particular, we compute two examples of application of the formula, verifying two values of the tables in the paper by Gross and Kudla [**GK92**]. Notably all the procedures can be automated with the computation of such Brandt matrices, as noticed in Pizer's paper [**P80**].

The next section (5) is divided into two parts, namely the one describing the notion of *triple product $L-$functions* and the one dealing with the *Jacquet conjecture*. The former is a naive introduction to the Langlands dual and the Langlands $L-$functions. This topic has a remarkably deep description which is beyond our purposes. For this reason we just consider examples arising from modular forms, characterized by a more explicit construction, although the functorial and algebraic flavour has been therefore lightened. Mainly it allows us to consider the triple product $L-$function in [**GK92**] in a more general setting. The latter part introduces the so-called *Jacquet conjecture*, which is a well known theorem by Harris and Kudla, [**HK91**] and [**HK04**].

Section (6), the last one, presents the result of Ichino, [**I08**]. He provided a formula relating the trilinear form in the *Jacquet conjecture*, with the central value of the triple product $L-$function. The *Ichino's formula* is certainly more abstract and general than that in section (4). For a better comprehension of it and to show its generality, we describe how to recover the *Gross–Kudla formula* starting with the *Ichino's formula.*

# 1 Quaternion algebras

## 1.1 Basic notions about quaternion algebras

### 1.1.1 Definitions

In this section we recall some basic definitions and properties about quaternion algebras over a field.

**Definition 1.1.1** *Let $K$ be a field. $B$ is a* (division) quaternion algebra *over $K$ if it is a central simple (division) algebra over $K$ of rank 4. Moreover, if $K$ is a field of characteristic different from 2, the above definition is equivalent to the following one. $B$ is a quaternion algebra over $K$ if it is a $K-$algebra with basis $\{1, i, j, ij\}$ for $i, j \in B$ satisfying*

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji$$

*for certain $a, b \in K^\times$. In such case, sometimes we write $\{a, b\}_K$ instead of $B$.*

Let $K$ be a field of characteristic different from 2. Writing each element of $B = \{a, b\}_K$ as $b' = x + yi + zj + tk$, where $k = ij$, we have the $K-$endomorphism of *conjugation*

$$b' \longmapsto \overline{b'} = x - yi - zj - tk$$

satisfying, for $\alpha, \beta \in K$, $b', b'' \in B$,

$$\overline{\alpha b' + \beta b''} = \alpha\overline{b'} + \beta\overline{b''}, \qquad \overline{\overline{b'}} = b', \qquad \overline{(b'b'')} = \overline{b''}\,\overline{b'}$$

We define also the *reduced trace* of $b'$

$$t(b') = b' + \overline{b'} = 2x,$$

the *reduced norm* of $b'$

$$n(b') = b'\overline{b'} = x^2 - ay^2 - bz^2 + abt^2$$

and in the end, the minimal polynomial of $b' \in B \smallsetminus K$ over $K$, as

$$X^2 - t(b')X + n(b').$$

Such maps give a first description of the structure of $B$, in particular it holds the

**Lemma 1.1.2 (Lemme 1.1 [V80])** *$(i)$ The invertible elements of $B$ are exactly the elements of $B$ with non-zero reduced norm.*

*$(ii)$ The reduced norm defines a multiplicative homomorphism from $B^\times$ to $K^\times$.*

*$(iii)$ The reduced trace is $K-$linear and the association $(b, b') \longmapsto t(bb')$ defines a non-degenerative symmetric bilinear form on $B$.*

**Example 1.1.3** The first example of a quaternion division algebra is that of the Hamilton's quaternions, $\mathbb{H}$, i.e. the quaternion algebra over $\mathbb{R}$ with $\{a, b\} = \{-1, -1\}$. It can be represented as

$$\mathbb{H} = \left\{ \begin{pmatrix} z & z' \\ -\overline{z'} & \overline{z} \end{pmatrix}, z, z' \in \mathbb{C} \right\}$$

with usual sum and product of matrices.

The second main example is that of $2 \times 2$ matrices with coefficients in some field $K$, $M_2(K)$. The reduced trace is the usual trace of matrices and the reduced norm is the determinant. We identify $K$ in $M_2(K)$ as the space $K\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Explicitly we have

$$M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \;\Rightarrow\; \overline{M} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right),\;\; t(M) = a + d,\;\; n(M) = ad - bc.$$

**Proposition 1.1.4 ([V80], Ch. 1, §1)** *If the field $K$ is separabily closed the unique quaternion algebra over $K$, up to isomorphism, is the matrix algebra $M_2(K)$.*

It is important noticing that taken $K \hookrightarrow F$ a field extension and $B$ a quaternion algebra over $K$, then the tensor product $B \otimes_K F$ is (still) a quaternion algebra, but with base field $F$.

**Definition 1.1.5** *We say that a quaternion algebra, $B$, over $K$ is split if it is isomorphic to the matrix algebra. Moreover we say that $B$ is split over $F$, for $K \hookrightarrow F$ a field extension, if $B \otimes_K F$ is split.*

By the above proposition we deduce immediately that every quaternion algebra admits a field on which it is split (e.g. the separable closure of the base field).

### 1.1.2 Orders

Fix now the following notation. Let $R$ be a Dedekind ring, $K$ its fraction field and take $B$ a quaternion algebra over $K$. We are interested in the case in which $K$ is either a local or global field and $R$ is its ring of integers. Recall that a $R-$lattice $L$ in a $K-$vector space $V$ (e.g. in $B$), is a finitely generated $R-$module contained in $V$ and that the lattice is complete if moreover $L \otimes_R K \cong V$ (or equivalently $LK = V$).

**Definition 1.1.6** *An* ideal *in $B$ is a complete $R-$lattice and an* order *in $B$ is an ideal which is also a ring. We say that an order is maximal if it is not properly contained in any other order.*

**Remark 1.1.7** Intersection of complete $R$-lattices in a finite-dimensional $K-$vector space is indeed a complete $R-$lattice. First of all, notice that $L$ is a complete $R-$lattice in $V$ if and only if it is a finitely generated $R-$module in $V$ which contains a $K-$basis of $V$. Hence taken $L$ and $M$ complete lattices, we have that $N := L \cap M$ is a finitely generated $R-$module (as $R$ is Noetherian). Moreover, we can find a $K-$basis of $V$ in $N$ and so $N$ contains the $R-$module generated by this basis, which is a complete $R$-lattice in $V$. Since $- \otimes_R K$ is exact (as localization on $R \smallsetminus \{0\}$), thus $N$ is a complete $R-$lattice.

**Example 1.1.8 (of lattices)** If $L$ is a finitely generated free $R-$modules then it is a lattice and if moreover $rk_R(L) = dim_K(V)$, it is a complete lattice. By the Structure Theorem for modules over P.I.D. those are the unique lattices and complete lattices if $R$ is a P.I.D. as the condition of being contained in a vector space means that such modules are torsion free. For example $R = \mathbb{Z}$ and $K = \mathbb{Q}$ or, more generally, $K$ a non-archimedean local field and $R = O_K$ its ring of integers.

Now it makes sense taking intersection of orders since intersection of subrings is a subring and so we can consider the

**Definition 1.1.9** *With the notation as above, an* Eichler order *in $B$ is an order which is intersection of two maximal orders in $B$. In particular, a maximal order is a (maximal) Eichler order.*

We can associate (canonically) two order to each ideal $I \subset B$, i.e.

- Left order: $O_l = O_l(I) := \{b \in B \mid bI \subset I\}$;
- Right order: $O_r = O_r(I) := \{b \in B \mid Ib \subset I\}$.

Note that they have the structure of $R-$module and of ring induced by $I$; the only thing to check is the closure with respect the product, but if $b, b' \in O_l(I)$ then $bb'I \subset bI \subset I$ and the same holds for the right order. They are indeed orders as they are complete $R-$lattices: for each $b \in B$, there exists $k \in K$ such that $kbI \subset I$, as we can think $b$ as a vector with coefficients in

$K = Frac(R)$. Choose $k$, for example, as the product of all denominators of each entry in $b$. Then take a basis of $B$ as a $K-$vector space and notice that both $O_l$ and $O_r$ contain a $K-$basis of $B$.

**Example 1.1.10 (Existence of ideals and orders)** There always exists an ideal, namely the $R-$module generated by a $K-$basis of $B$. Therefore there exist always two orders in $B$, i.e. the left and right order associated with $I$.

**Remark 1.1.11** *By the usual argument we can apply the Zorn's Lemma to chains of orders and considering union of incapsulated orders we can notice that every order is contained in a maximal one.*

**Definition 1.1.12** *Let $I$ be an ideal of $B$ and take $O_l$ and $O_r$ as defined above. We say that $I$ is:*

- *a left ideal for $O_l$;*
- *a right ideal for $O_r$;*
- *a two-sided ideal if $O_l = O_r$;*
- *normal if both $O_l$ and $O_r$ are maximal;*
- *integral if $I \subset O_l \cap O_r$;*
- *principal if there exists $b \in B$ such that $I = O_l b = bO_r$.*

*We define the inverse of $I$, as $I^{-1} = \{b \in B \mid IhI \subset I\}$ and, taken $I$ and $J$ two ideals, we define the product $IJ := \left\{\sum_{l=1}^{n} i_l j_l \;\middle|\; i_l \in I, j_l \in J, n \in \mathbb{N}\right\}$.*

We can characterize either $I^{-1}$ and $IJ$.

**Lemma 1.1.13** *a) The product of two ideals is associative and defines an ideal;*

*b) The inverse of the ideal $I$ is an ideal.*

**Proof:** $a$): The associativity is just the associativity in $B$. Further the product is indeed an ideal: notice that $IJ \subset I \cap J$ and so by remark (1.1.7) and it is a finitely generated $R-$module (as $R$ is Noetherian) and it contains a $K-$basis of $B$ as both $I$, $J$ do it. $b$): notice that exists an element $d \in R^{\times}$ such that $dI \subset O_l \subset d^{-1}I$; the first inclusion holds for each $d \in R^{\times}$ and the second one is equivalent to $dO_l \subset I$. Working with the $R-$basis of both $O_l$ and $I$ we can found easily a $d$ as required. Hence we have that (as $R^{\times}$ is in the centre of $B$) $I\, dO_l\, I = dI\, O_l\, I \subset O_l O_l I = O_l I = I$ then $dO_l \subset I^{-1}$ and so, if $I^{-1}$ is a lattice, it is complete as it contains a $K-$basis of $B$. Moreover we have $I^{-1} = II^{-1}I = dII^{-1}d^{-1}I \subset d^{-1}I\, I^{-1}\, d^{-1}I = d^{-2}I$, i.e. $I^{-1} \subset d^{-2}I$ and hence we deduce that $I^{-1}$ is a finitely generated $R-$module. ■

**Remark 1.1.14** *Let $O$ be an order in $B$ and take the principal ideal $I = Ob$. The left order of $I$ is indeed $O$ and the right one is $O' = b^{-1}Ob$, so $I = bO'$. Now it holds that*

$$I^{-1} = b^{-1}O = O'b^{-1} \quad \text{and} \quad II^{-1} = O, \quad I^{-1}I = O'$$

### 1.1.3 Classes of ideals

**Definition 1.1.15** *Two ideals $I$ and $J$ are* right-equivalent *if $I = Jb$ for a some $b \in B^\times$. Such classes of ideals with left order $O$ are called the* left-classes *for $O$. Analogously we define left-equivalence and right-classes for an order.*

**Lemma 1.1.16 ([V80], Ch.1, Lemme 4.9)** *a) The application $I \longmapsto I^{-1}$ induces a bijection between the left-classes for $O$ and its right-classes.*

*b) Let $J$ be a fixed ideal. The association $I \longmapsto JI$ defines a bijection between the left-classes of ideals on the left with respect to $O_l(I)$ and those on the left with respect to $O_l(J)$.*

**Definition 1.1.17** *The* class number *on an order $O$ is defined as the number of classes of ideals related to $O$. The class number of the quaternion algebra $B$ is defined as the class number of any of its maximal order.*

**Note 1.1.18** The class number of $B$ is well-defined as, taken $O$ and $O'$ two maximal orders of $B$, and defined $I = Ob$ and $I' = O'b'$, for $b, b' \in B^\times$, by Remark (1.1.14) and Lemma (1.1.16), *b)* we obtain the equality of the two class numbers.

**Definition 1.1.19** *Two orders $O$ and $O'$ are said to be of the* same type *if they are conjugate by an inner automorphism of $B$, namely, $O = b^{-1}O'b$ for some $b \in B^\times$.*

Obviously this definition gives an equivalence relation on the set of orders and we can consider classes of types related to a given order $O$: we define the *type number related to $O$* as the number of such classes. We define the *type number of $B$* as the type number of a maximal order. There exists a relation between class number and type number, expressed by the

**Proposition 1.1.20** *Define $h$ and $t$ as the class number of $B$ and the type number of $B$ respectively. If $h$ is finite, then $t \leq h$.*

**Proof:** See Corollaire 4.11, Ch. 1 in [**V80**]. ■

## 1.2 Quaternion algebras over local fields

### 1.2.1 Algebras

Throughout this section $K$ will be a local field and we will use the above notation. There exists a strong characterization of quaternion algebras over $K$ as stated in the next theorem.

**Theorem 1.2.1** *Up to isomorphism, there exists a unique division quaternion algebra over any local field $K \neq \mathbb{C}$.*

**Proof:** See Theoreme 1.1, Ch. 1 in [**V80**]. ■

**Note 1.2.2** As $\mathbb{C}$ is algebraically closed we know, by theorem (1.1.4), that the unique (up to isomorphism) quaternion algebra is $M_2(\mathbb{C})$ which is not a division algebra.

**Remark 1.2.3 (Hasse invariant)** Denote with $Quat(K)$ the set of quaternion algebras over the local field $K$, up to isomorphism. We have hence defined a bijection between $Quat(K)$ and $\{\pm 1\}$,

$$\varepsilon : Quat(K) \longrightarrow \{\pm 1\}$$

We say that $B \in Quat(K)$ has Hasse invariant $\varepsilon(B) = -1$ if and only if $B$ is a division algebra, otherwise we say that it has Hasse invariant $\varepsilon(B) = 1$.

By the above considerations we can restate theorem (1.2.1) as

**Theorem 1.2.4** *If $K$ is a local field we have two cases:*

$$Qaut(K) \cong \begin{cases} \{1\} & \text{if } K = \mathbb{C} \\ \{\pm 1\} & \text{otherwise} \end{cases}$$

Recall that the Hilbert symbol of $(a,b) \in K^2 \smallsetminus \{(0,0)\}$ is defined as

$$(a,b) = \begin{cases} 1 & \text{if } aX^2 + bY^2 - Z^2 = 0 \text{ admits a non-trivial solution in } K^3 \\ -1 & \text{otherwise} \end{cases}$$

The importance of such symbol in the quaternion algebra setting is that, in the case of $Char(K) \neq 2$, it determines completely the Hasse invariant.

**Theorem 1.2.5** *Let $K$ be a local field of characteristic different from 2, then taken $B = \{a,b\}_K$ it holds that $\varepsilon(B) = (a,b)$.*

For the definition and the basic properties of the Hilbert symbol we refer to [**Se73**]. In particular, from such properties one can show that the equality in the above theorem is well-defined.

At this point we must introduce a definition that will be useful for the global case.

**Definition 1.2.6** *We say that a quaternion algebra $B$ is* ramified *(over the local field $K$) if it is a division algebra.*

Just notice that to check if $B$ is ramified over $K$ with $char(K) \neq 2$, it is enough to check its associated Hilbert symbol.

### 1.2.2 Maximal orders

Now we want to focus our attention on the study of the maximal orders in a *division* quaternion algebra $B$ over $K$.

Denote with $v$ a discrete valuation on $K$ and suppose that it is normalized (i.e. $v(K^\times) = \mathbb{Z}$). We have two maps: the reduced norm $n : B^\times \longrightarrow K^\times$ and the valuation $v : K^\times \longrightarrow \mathbb{Z}$ then we can consider the composition

$$w = v \circ n : B^\times \longrightarrow \mathbb{Z} \quad \text{s.t.} \quad b \longmapsto v\left(n(b)\right) = v\left(b\bar{b}\right)$$

We can notice that

- $w(bb') = v(n(bb')) = v(bb'\overline{bb'}) = v(bb'\overline{b'}\,\bar{b}) = v(n(b)n(b')) = w(b) + w(b')$;
- If $b' \neq 0$, $w(b+b') = w((bb'^{-1}+1)b') = w(bb'^{-1}+1)+w(b') \geq min\{w(bb'^{-1}), w(1)\}+w(b') = min\{w(b), w(b')\}$ as $w$ is indeed a valuation when restricted to $K(bb'^{-1})$ (as it is a finite extension of $K$ which is hence algebraic). Obviously the inequality holds (trivially) if $b' = 0$.

Then we proved that $w$ defines a discrete valuation on $B$ (up to setting $w(0) = \infty$). Moreover we have

**Lemma 1.2.7** *The valuation ring of $w$, $O := \{b \in B \mid w(b) \geq 0\}$, is the unique maximal order of $B$.*

**Proof:** See Lemme 1.4, Ch. 1 in [**V80**]. ■

### 1.2.3 Eichler orders for local fields

Let $K$ denotes a non-archimedean local field, $O_K$ its ring of integers and take $B = M_2(K)$. As in [**V80**], Ch. II, §2, one can identify the orders in $B$ as endomorphisms of sublattices just noticing that the identification $M_2(K) \cong End(K^2)$ is indeed natural. Hence, taken two maximal orders $O = End(L)$, $O' = End(M)$ for $L$ and $M$ complete lattices, we can notice that, for each $x, y \in K^\times$, $End(Lx) = O$ and $End(My) = O'$ as we are considering the $K-$endomorphisms. This invariance under multiplication allows us to suppose that $L \subset M$. Then, by the Elementary divisor theorem, there exist two $O_K-$basis $(f_1, f_2)$ and $(f_1\pi^a, f_2\pi^b)$, respectively of $L$ and $M$; here $f_1$ and $f_2$ are elements in $K$, $\pi$ is a uniformizer for $K$ and we can take $a, b \in \mathbb{N}$. One can prove that the integer $|a-b|$ is invariant under the multiplication by $x$ and $y$, then we can define a notion of distance between two maximal orders in $B$. We say that the distance between $O$ and $O'$ is hence $|a - b|$.

**Example 1.2.8** The distance between $M_2(K)$ and $\left(\begin{smallmatrix} O_K & \pi^{-n}O_K \\ \pi^n O_K & O_K \end{smallmatrix}\right)$ is $n$ as we have the two basis (of the associated lattices) $(1, 1)$ and $(1, \pi^n)$. Notice that we understand the elements of $K^2$ as column vectors.

**Definition 1.2.9** *Let $O$ be an Eichler order in $M_2(K)$. We say that it has level $\pi^n O_K = \mathfrak{p}^n$ if it is the intersection of two maximal orders of distance $n$. We denote*

$$O_n = M_2(K) \cap \begin{pmatrix} O_K & \pi^{-n}O_K \\ \pi^n O_K & O_K \end{pmatrix} = \begin{pmatrix} O_K & O_K \\ \pi^n O_K & O_K \end{pmatrix}$$

*the* canonical *Eichler order of level $n$.*

## 1.3 Quaternion algebras over global fields

### 1.3.1 Algebras

In this section fix $K$ to be a global field. We usually refer to $v$ as a place for $K$ and consider $K_v$ the corresponding field. Set $S_\infty$ for the set of archimedean places of $K$ and $S_f$ for the set of the non-archimedean ones. For each $v \in S_f$ we define $N(v) := N_{K_v}(\mathfrak{p}_v) = \#O_v/\mathfrak{p}_v$ the cardinality of the residue class field at $v$ (for $O_v$ and $\mathfrak{p}_v$ the ring of integers and its maximal ideal respectively).

**Definition 1.3.1** *A quaternion algebra $B$ over $K$ is* ramified at the place $v$ *if $B_v := B \otimes_K K_v$ is a division algebra, i.e. $B_v$ is ramified over $K_v$ (as in 1.2.6). In particular, if $K = \mathbb{Q}$, we say that the quaternion algebra is* definite *if it is ramified at infinity.*

**Remark 1.3.2** If $Char(K) \neq 2$ and $B = \{a, b\}_K$, by the characterization of quaternion algebra over local fields, we have that $K$ is ramified at the place $v$ if and only if the Hilbert symbol over $K_v$, $(a, b)_v = -1$. This procedure gives an algorithmic way to compute the ramification places of $B$.

We have the fundamental lemma:

**Lemma 1.3.3** *$B$ is ramified at only finitely many places of $K$.*

**Proof:** Lemme 3.1, Ch III, [**V80**]. ■

**Definition 1.3.4** *The* reduced discriminant *of $B$, $d(B)$, is the product of the finite places at which $B$ is ramified. If $K$ is a number field, the reduced discriminant is an integral ideal in $O_K$. Usually, if $K = \mathbb{Q}$, we identify $d(B)$ with an integer (in the obvious way). Further denote $Ram(B)$ for the set of ramified places for $B$: it is the disjoint union of $Ram_f(B) = Ram(B) \cap S_f$ and $Ram_\infty(B) = Ram(B) \cap S_\infty$.*

### 1.3.2 Classification of quaternion algebras over global fields

One can show, via the analytic construction of certain Zeta-functions (See [**V80**], Ch. III, §3 for details) the following theorem.

**Theorem 1.3.5 (Existence and classification)** *Let $B$ be a quaternion algebra over the global field $K$, then:*

*i) the cardinality of $Ram(B)$ is even;*

*ii) For each finite set $S$, of places of $K$, such that $\#S$ is even, then there exists one and only one quaternion algebra $B$ over $K$, up to isomorphism, such that $Ram(B) = S$.*

**Proof:** See [**V80**], Ch. III, Theoreme 3.1. ■

As a nice corollary of such powerful theorem we have the product formula for the Hilbert symbol. More precisely, if $char(K) \neq 2$, *ii)* is equivalent to the

**Corollary 1.3.6 (Hilbert reciprocity law)** *Let $K$ be a global field of characteristic different from 2. For $a, b \in K^\times$, denote $(a, b)_v$ their Hilbert symbol in $K_v$. Then*

$$\prod_{v \in S} (a, b)_v = 1$$

**Proof:** It is an immediate conseguence of the characterization of the ramification with the Hilbert symbol. ■

### 1.3.3 Maximal and Eichler orders

Analogously to the case of local fields we are interested in the study of orders and (mainly) Eichler orders.

Let $S$ be a non empty finite set of places for $K$ and suppose, if $K$ is a number field, that $S$ contains $S_\infty$. Hence we can consider the ring of $S-$integers $R = R_{(S)} = \cap_{v \notin S} (O_v \cap K)$ which is a Dedekind domain. For example, if $K$ is a number field and $S = S_\infty$, hence $R$ is the ring of integers of $K$. Let $L$ be a $R-$lattice for a quaternion algebra $B$ over $K$. Define $L_v = O_v \otimes_R L$ for each place $v \notin S$ and $L_v = B_v$ for $v \in S$.

**Definition 1.3.7** *For each place $v \notin S$ of $K$ and for each complete $R-$lattice $Y$ in $B$ we call $Y_v$ the* localization *of $Y$ at the place $v$.*
*We say that a property is* local *if it holds for $Y$ if and only if it holds for $Y_v$, for each $v \notin S$.*

**Example 1.3.8 ([V80], *Exemples de proprietes locales*, Ch. IV, §5A)** For a $R-$lattice in $B$, to be

- an order,
- a maximal order,
- an Eichler order, which is the intersection of two maximal ($R-$)orders,
- an ideal,
- an integral ideal

is a local property.

We are hence able to give the significant notion of level of an Eichler order, over a global field.

**Definition 1.3.9** *The* level *of an* $(R-)$*Eichler order is the integral ideal,* $N$*, of* $R$ *such that* $N_v$ *is the level of* $O_v$ *for each* $v \notin S$*. As for the discriminant, if* $K = \mathbb{Q}$*, we can think at* $N$ *as an integer.*

Furthermore one can prove ([**V80**], Ch. II, *corollaire* 5.3) that *an* $(R-)$*Eichler order is maximal if and only if its level is* $R$*. In particular, if* $K = \mathbb{Q}$*, in this case we say that the level is* 1.

**Remark 1.3.10** In the case of $\mathbb{Q}$ as base field, one can show (see [**D03**], Ch.4, §1) that the level of an Eichler order $O$ can be identified with the $R-$module index of $O$ in one of the two maximal orders which realize the definition of Eichler order.

**Definition 1.3.11 (Eichler condition)** *We say that a non empty and finite set of places of* $K$ *verifies the Eichler condition for the quaternion algebra* $B$ *(over* $K$*), if it contains at least one place at which* $B$ *is not ramified.*

## 1.4 Topology on quaternion algebras

Suppose now that $K$ is a topological field. Any quaternion algebra, $B$, over $K$, can be endowed with a topology, namely that induced by the field $K$ via the (4-dimensional) $K-$vector space structure on $B$. In particular, we are interested in $K$ being a finite extension of $\mathbb{Q}_p$, or $K = \mathbb{R}, \mathbb{C}$, i.e. a local field of characteristic zero. In this case $K$ is a locally compact Hausdorff space and so is the quaternion algebra. As the sum, the subtraction and the product of two elements in $B$ are polynomial functions in the coordinates of the elements (viewing $B \cong K^4$ as a $K$ module), one can show that $B$ is a locally compact topological ring and that $B^\times$ is a locally compact topological group with the subspace topology. For $K = \mathbb{Q}_p$ one can prove a stronger result, namely

**Lemma 1.4.1** *Let* $B$ *be a quaternion algebra over* $\mathbb{Q}_p$ *and* $R$ *an order of* $B$*. Then* $R$ *is compact and* $R^\times \subset B^\times$ *is a compact subgroup.*

**Proof:** See [**M89**], Lemma 5.1.1, $i$). ■

Let now $B$ be a quaternion algebra over $\mathbb{Q}$ and denote with $\mathbb{A}$ the ring of adèles for $\mathbb{Q}$. We can consider the tensor product

$$B_{\mathbb{A}} = B \otimes_{\mathbb{Q}} \mathbb{A}$$

and endow it with the topology induced by $\mathbb{A}$ via the identification of $B_{\mathbb{A}}$ with $\mathbb{A}^4$. Again this construction give rise to a topological ring which is locally compact (as $\mathbb{A}$ is locally compact) and we can consider $B_{\mathbb{A}}$ as a subset of the product $\prod_{v \text{ place of } \mathbb{Q}} B_v$, for $B_v = B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ (just recall that $B \otimes_{\mathbb{Q}} -$ is a right-adjoint functor and notice that, as $\mathbb{Q}$ is a field, it is exact). The topological ring $B_{\mathbb{A}}$ is called the *adèlization of* $B$. Considering the subspace topology on $B_{\mathbb{A}}^\times$, it becomes a locally compact topological group, called the *adèlization of* $B^\times$. Again we can identify it as a subset of the product $\prod_{v \text{ place of } \mathbb{Q}} B_v^\times$.

**Remark 1.4.2** One can notice that, with the induced topology, the norm map

$$n : B^\times \longrightarrow K^\times$$

is a continuous homomorphism as it is polynomial in the coordinates. We have the exact sequence (of topological groups)

$$1 \longrightarrow B^1 \longrightarrow B^\times \xrightarrow{n} K^\times$$

for $B^1 = \{b \in B^\times \mid n(b) = 1\}$ with the subspace topology. With more generality, as recalled in [**JL70**], pag. 7, it holds the

**Lemma 1.4.3** *If $B$ is a division algebra over a local field $K$, it happens that*

$$[K^\times : n(B^\times)] = \begin{cases} 1 & \text{if } K \text{ is non-archimedean} \\ 2 & \text{if } K \text{ is } \mathbb{R} \end{cases}$$

Thus we deduce the exactness on the right of the above sequence if $K$ is non-archimedean. We can notice that $B^1$ is closed in $B^\times$ as preimage of $\{1\}$. Suppose that $K$ is a local field of characteristic zero and that $B$ is a division algebra. In this case it can be proved that $B^1$ is compact as pointed out in [**V80**], pag. 81, and that $B^1 = SL(2, K)$ in the split case (and so it is not compact).

**Note 1.4.4** Let $B$ be a quaternion algebra over the local field $K$. We can notice that

$$\begin{aligned} &B^\times = B \smallsetminus \{0\} && \text{if } B \text{ is a division algebra} \\ &B^\times \cong GL_2(K) && \text{if } B \text{ is a split algebra} \end{aligned}$$

so we have

$$\begin{aligned} &B^\times/K^\times = B\smallsetminus\{0\}/K^\times \cong \left(K^4\smallsetminus\{0\}\right)/K^\times \cong \mathbb{P}(K^4) && \text{if } B \text{ is a division algebra} \\ &B^\times/K^\times \cong GL_2(K)/K^\times = PGL_2(K) && \text{if } B \text{ is a split algebra} \end{aligned}$$

By [**We73**], Ch. 2, §1, Corollary 1, we know that $B^\times/K^\times$ is compact if $B$ is a division algebra. On the other hand it is known that $PGL_2(K)$ is not compact. For example, it is not compact as it is not closed in $\mathbb{P}^4_K$. Recall that $GL_2(K)$ admits an open immersion in $K^{4+1}$ (see (2.1)) and since the projection to the quotient is an open map, $PGL_2(K)$ has to be an open subset of $\mathbb{P}^4$. It is well known that the projective space $\mathbb{P}^4$ is Hausdorff, compact and connected, so we deduce immediately that, if $PGL_2(K)$ is compact, it must be a (non-trivial) closed and open subset in $\mathbb{P}^4$. We produced then a contradiction. The above observations lead to the

**Lemma 1.4.5** *Let $K$ be a local field (with $char(K) = 0$) and $B$ a quaternion algebra over $K$. Hence $B^\times/K^\times$ is compact if and only if $B$ is a division algebra.*

## 1.5 Adèlic point of view

Let $\mathbb{A}_K$ be the adèles ring for $K$. Set $X_K$ as either $K$ or a quaternion algebra over $K$, call it $B$ as above. Set $X_\mathbb{A} = X_K \otimes_K \mathbb{A}$ and for each place $v$, $X_v = X_K \otimes K_v$. For this paragraph, just for ease of discussion, set $S = S_\infty$ and so $R_{(S)} = O_K$. The following theorem is the natural generalization of the adèlic theory for global fields to quaternion algebras.

**Theorem 1.5.1** 1) *Adèles: $X_K$ is discrete in $X_\mathbb{A}$ and $X_\mathbb{A}/X_K$ is compact.*

2) *Approximation theorem: for each place $v$, $X_K + X_v$ is dense in $X_\mathbb{A}$.*

3) *Idèles: $X_K^\times$ is discrete in $X_\mathbb{A}^\times$.*

4) *For each place $v$ (only the infinite ones if $K$ is a number field) there exists a compact subset $C$ in $X_\mathbb{A}$ such that $X_\mathbb{A}^\times = \overline{X_K^\times X_v^\times C}$ (i.e. the space is dense in $X_\mathbb{A}$).*

**Proof:** [**V80**], Ch. III, §1, Theoreme 1.4. ■

The adèlic setting allows us to prove a characterization of the ideal classes and the conjugation classes of an Eichler order.

Let $R$ be an Eichler order of level $N$ in $B$. Define $R_\mathbb{A} = \prod_v R \otimes_{O_K} O_v$ with, as in (1.3.3), $R \otimes_{O_K} O_v = B_v$ for $v \in S$ and $R_\mathbb{A}^\times$ its group of units.

**Theorem 1.5.2 (Adèlic dictionary)** 1) *The left-ideals with respect to* $R$ *are in bijection with the set* $R_{\mathbb{A}}^{\times}\backslash B_{\mathbb{A}}^{\times}$. *The association is defined as* $(b_v) \in B_{\mathbb{A}}^{\times} \longmapsto I$ *such that* $I_v = R_v b_v$ *for each place* $v \notin S$. *Moreover the classes of left-ideals for* $R$ *are in bijection with the double coset* $R_{\mathbb{A}}^{\times}\backslash B_{\mathbb{A}}^{\times}/B^{\times}$.

2) *The Eichler orders of level* $N$ *are in bijection with the coset* $N(R_{\mathbb{A}})\backslash B_{\mathbb{A}}^{\times}$ *where* $N(R_{\mathbb{A}})$ *is the normalizer of* $R_{\mathbb{A}}$ *in* $B_{\mathbb{A}}^{\times}$. *The association is defined as* $(b_v) \in B_{\mathbb{A}}^{\times} \longmapsto R'$ *such that* $R'_v = b_v^{-1} R_v b_v$ *for each* $v \notin S$. *Furthermore the types of Eichler orders of level* $N$ *are in bijection with the double coset* $B_K^{\times}\backslash B_{\mathbb{A}}^{\times}/N(R_{\mathbb{A}})$.

**Proof:** See [**V80**], Ch. III, §5B. ■

We have further the following

**Theorem 1.5.3** *The class number of an Eichler order of level* $N$ *is finite. The number of types of such order is finite too.*

**Proof:** See [**V80**], Ch. III, § 5B, Theoreme 5.4 and Corollaire 5.5 for the first assertion. The finiteness of the type number follows immediately from proposition (1.1.20). ■

**Proposition 1.5.4** *If* $S$ *satisfies the Eichler condition for* $B$ *then any two Eichler* $R_{(S)}-$*orders of the same level are conjugate.*

**Proof:** See [**D03**], Ch. 4, Prop. 4.4. ■

**Note 1.5.5** Before ending the paragraph we should mention an useful formulation of the above notions in the case $K = \mathbb{Q}$ and $S = \{\infty\}$. Following [**D03**] we can consider $\hat{\mathbb{Z}}$ the profinite completion of $\mathbb{Z}$ and the ring of finite adèles for $\mathbb{Q}$, $\hat{\mathbb{Q}} = \mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$. Take $R$ an Eichler ($\mathbb{Z}-$)order of level $N$ and set $\hat{R} := R \otimes \hat{\mathbb{Z}}$, $\hat{B} := B \otimes \hat{\mathbb{Q}}$. Then the set of conjugacy classes of Eichler orders of level $N$ is in natural bijection with the double coset $B^{\times}\backslash\hat{B}^{\times}/\hat{R}$. In fact in this case, by definition, $R_{\mathbb{A}} = B_{\infty} \times \prod_p R \otimes_{\mathbb{Z}} \mathbb{Z}_p \cong B_{\infty} \times R \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$ (as $R \otimes_{\mathbb{Z}} -$ is a right adjoint functor and hence it preserves limits) and $\prod_p \mathbb{Z}_p = \hat{\mathbb{Z}}$. Hence

$$R_{\mathbb{A}}^{\times}\backslash B_{\mathbb{A}}^{\times}/B^{\times} \cong \hat{R}^{\times}\backslash\hat{B}^{\times}/B^{\times}$$

since the place at infinity is a direct factor in $\mathbb{A}$. In particular, as stated in [**D03**], every element of the form

$$x\hat{R}x^{-1} \cap B \qquad \text{for } x \in \hat{B}^{\times}$$

is an Eichler ($\mathbb{Z}$-)order of level $N$ and any such order is of this form.

## 1.6 Computing methods and examples

Given $B$ a quaternion algebra over the global field $K$ of characteristic different from 2, we are interested in computing its discriminant and so its ramifications. Just notice that in the case of $K = \mathbb{Q}$, which is the principal field we want to work with, the computation of the Hilbert symbol amounts essentially to compute a Kronecker symbol.

**Remark 1.6.1 ([V80], Ch. II, *Calcul du symbole de Hilbert*)** If $K = \mathbb{Q}_p$ with $p \neq 2$ and $a, b \in \mathbb{Z} \smallsetminus \{0\}$, hence it is possible showing that

$$(a,b)_p = (a,b)_{\mathbb{Q}_p} = \begin{cases} 1 & \text{if } p \nmid a,\ p \nmid b \\ \left(\frac{a}{p}\right) & \text{if } p \nmid a \text{ and } p \| b. \end{cases}$$

The remark does not involve the computation of $(a,b)_2$ but this computation can be indeed avoided using the product formula, in fact. By definition of the Hilbert symbol, $(a,b)^{-1} = (a,b)$ and so

$$(a,b)_2 = (a,b)_\infty \prod_{p \neq 2} (a,b)_p$$

For sake of completeness we recall the properties of the Hilbert symbol and we refer to [**Se73**], chapter III, for the straightforward proofs. Taken $a, b \in K^\times$ (for $K$ a field), we have

- $(ax^2, by^2) = (a,b)$ (invariance under squares);
- $(a,b) = (b,a)$ (symmetry);
- $(a,b)(a,c) = (a,bc)$ (bilinearity);
- $(a, 1-a) = 1$

Focusing our attention to $K = \mathbb{Q}$ we can say a bit more, as stated in the following.

**Note 1.6.2 ([V80], Ch. *III*, § 3, *Exemple*)** Let $\{a,b\}$ be a quaternion algebra over $\mathbb{Q}$. By definition of Hilbert symbol, it is ramified at infinity if and only if $a$ and $b$ are both negative. Its reduced discriminant, $d$, is given by a product of an odd number of primes if $a$ and $b$ are negative and even otherwise. For example

| $\{a,b\}$ | $\{-1,-3\}$ | $\{-2,-5\}$ | $\{-1,-7\}$ | $\{-1,-11\}$ | $\{-2,-13\}$ | $\{-3,-119\}$ | $\{-3,-10\}$ |
|---|---|---|---|---|---|---|---|
| $d$ | 3 | 5 | 7 | 11 | 13 | 17 | 30 |

A fast method to compute examples of such quaternion algebras is provided by using the parity to avoid the study of $(a,b)_2$ and by

$$\begin{aligned} &\text{if } p \equiv -1 \pmod 4 \text{ than } \{1,p\} \text{ has reduced discriminant } p \\ &\text{if } p \equiv \phantom{-}5 \pmod 8 \text{ than } \{1,p\} \text{ has reduced discriminant } p \end{aligned}$$

It is also easy to give rise to quaternion algebras of fixed reduced discriminant: it amounts to determine two integers $a$ and $b$ with given conditions on their local Hilbert symbols. For example

$$\{-1,3\}, \quad d = 6; \qquad \{3,5\}, \quad d = 15; \qquad \{-1,7\}, \quad d = 14;$$

Moreover, it holds that

$$\begin{aligned} &\text{if } p \equiv -1 \pmod 4 \text{ than } \{-1,p\} \text{ has reduced discriminant } 2p \\ &\text{if } p \equiv \phantom{-}5 \pmod 8 \text{ than } \{-2,p\} \text{ has reduced discriminant } 2p. \end{aligned}$$

Furthermore it can be proved that

**Proposition 1.6.3** *The unique (up to isomorphism) definite quaternion algebra over $\mathbb{Q}$ of reduced discriminant $p$ (i.e the unique one ramified exactly at infinity and $p$) is given by the rule*

$$\begin{array}{lll} p = \phantom{-}2 & & \{a,b\} = \{-1,-1\} \\ p \equiv -1 & \pmod 4 & \{a,b\} = \{-1,-p\} \\ p \equiv \phantom{-}5 & \pmod 8 & \{a,b\} = \{-2,-p\} \\ p \equiv \phantom{-}1 & \pmod 8 & \{a,b\} = \{-p,-q\} \end{array}$$

*where $q$ is a prime such that $q \equiv -1 \pmod 4$.*

**Proof:** See [**P80**], Proposition 5.1. ∎

## 1.7 The Class Number and the Eichler mass formula

### 1.7.1 Dedekind Zeta functions: local case

Let $X$ be either a non-archimedean local field $K$ or a quaternion algebra $B$ over $K$ such that $B$ does not contain $\mathbb{R}$. Let $O_K$ be the ring of integers for $K$ and set $k$ as its residue class field. Take $\mathcal{B}$ an order in $X$ containing $O_k$. The definition of order in $K$ is analogous to the definition (1.1.6) given for quaternion algebras where $R = O_k$ and similarly, with the definition (1.1.12) in mind, we can define the notion of ideal and integral ideal for the order $\mathcal{B}$.

Taken hence $I$ an integral ideal in $\mathcal{B}$, we define its norm, as $N_X(I) := \#\mathcal{B}/\mathcal{I}$.

**Definition 1.7.1** *The Dedekind zeta function of $X$ is the complex function of complex variable*

$$\zeta_X(s) = \sum_{I \subset \mathcal{B}} \frac{1}{N_X(I)^s}$$

*where the sum is taken over the integral left (or right) ideal of a maximal order $\mathcal{B}$ in $X$.*

Obviously one can show that the definition is well stated and moreover that it holds the following

**Proposition 1.7.2** *Let $q = \#k$ be the cardinality of the residue class field $k = O_K/\mathfrak{p}_K$. Hence,*

$$\zeta_K(s) = \frac{1}{1 - \frac{1}{q^s}}$$

$$\zeta_B(s) = \begin{cases} \zeta_K(2s) & \text{if } B \text{ is a division algebra} \\ \zeta_K(2s)\zeta_K(2s-1) & \text{if } B = M_2(K) \end{cases}$$

**Proof:** See Proposition 4.2, Ch. II in [**V80**]. ■

### 1.7.2 Dedekind Zeta functions: global case

Let now $K$ be a global field and take $X$ to be either $K$ or a quaternion algebra over $K$. For a place $v$, let $X_v = X \otimes_K K_v$, hence we can state the

**Definition 1.7.3** *The zeta Dedekind function of $X$ is the infinite product of all the zeta Dedekind functions of $X_v$ for all finite places $v$. The infinite product is absolutely convergent whenever the complex variable $s$ has real part greater than 1. Then*

$$\zeta_X(s) = \prod_{v \text{ finite}} \zeta_{X_v}(s) \text{ for } Re(s) > 1$$

By the definition of $\zeta_X$ and proposition (1.7.2), we can obtain, just killing all the contributions of the unramified places, the *multiplication formula*.

**Proposition 1.7.4 (Multiplication formula)** *Let $K$ be a global field and $B$ be a quaternion algebra over $K$. Then it holds the following equality:*

$$\zeta_B\left(\frac{s}{2}\right) = \zeta_K(s)\zeta_K(s-1) \prod_{v \in Ram_f(B)} \left(1 - N(v)^{1-s}\right).$$

### 1.7.3 The Class Number and the Eichler mass formula

We want now to give a couple of useful results for computing the class number and the $w_i$ (which are defined in the theorem below).

**Theorem 1.7.5** *Let $B/K$ be a quaternion algebra over the number field $K$, ramified at each archimedean place of $K$ and $R$ an Eichler order in $B$ of level $\mathfrak{N}$. Let $\mathfrak{D}$ be the reduced discriminant of $B$ and $n$ the class number of $R$. Let $\{I_i\}_{i=1}^n$ a set of representative for the left ideal classes of $R$, with $R_i$ the correspondent right orders. Set $w_i = [R_i^\times : \mathcal{O}_K^\times]$, where $\mathcal{O}_K$ is the ring of integers for $K$. Define $r_1$ as the number of real embeddings of $K$, hence we have*

$$\sum_{i=1}^{n} \frac{1}{w_i} = 2^{1-r_1}|\zeta_K(-1)|h_K N \prod_{\mathfrak{p}|\mathfrak{D}} (N(\mathfrak{p})-1) \prod_{\mathfrak{p}|\mathfrak{N}} \left(\frac{1}{N(\mathfrak{p})}+1\right)$$

*where $\mathfrak{p}$ are prime ideal of $\mathcal{O}_K$, $h_K$ is the class number of the number field $K$ and $\zeta_K$ the zeta function of $K$.*

**Proof:** See [**V80**], Corollaire 2.3. ■

We are interested in the case $K = \mathbb{Q}$. Hence $\mathfrak{D}$ and $\mathfrak{N}$ can be thought as integers and so we refer to them as $D$ and $N$ respectively. The classical results on values of the $\zeta-$Riemann function, provide $\zeta(-1) = -\frac{1}{12}$ and then the above formula is given by

$$\sum_{i=1}^{n} \frac{1}{w_i} = |\zeta(-1)|N \prod_{p|D} (p-1) \prod_{p|N} \left(\frac{1}{p}+1\right) = \frac{N}{12} \prod_{p|D} (p-1) \prod_{p|N} \left(\frac{1}{p}+1\right)$$

There exists a useful result for computing the class number of a given order; for proving it we need to restrict our hypotheses such that the quaternion algebra over $\mathbb{Q}$ is ramified only at one prime $p$ and at infinity. Before we need to define the notion of reduced discriminant for an Eichler order.

**Definition 1.7.6** *Let $R$ be an Eichler order of level $\mathfrak{N}$ in a quaternion algebra of reduced discriminant $\mathfrak{D}$. The* reduced discriminant *of $R$ is $\mathfrak{d} = \mathfrak{N}\mathfrak{D}$.*

**Theorem 1.7.7** *Let $p$ be a prime, $M$ a positive integer prime to $p$, $r$ a non-negative integer and $B$ the quaternion $\mathbb{Q}$-algebra ramified exactly at $p$ and infinity (so its reduced discriminant $D$ equals $p$). Let $R$ be an Eichler order in $B$ of reduced discriminant $d = p^{2r+1}M$ (i.e. of level $N = d/p$). Then its class number, $n$, is given by*

$$n = \frac{d}{12}\left(1-\frac{1}{p}\right)\prod_{q|M}\left(1+\frac{1}{q}\right) + \begin{cases} \frac{1}{4}\left(1-\left(\frac{-4}{p}\right)\right)\prod_{q|M}\left(1+\left(\frac{-4}{q}\right)\right) & \text{if } 4 \nmid d \\ 0 & \text{if } 4|d \end{cases}$$
$$+ \begin{cases} \frac{1}{3}\left(1-\left(\frac{-3}{p}\right)\right)\prod_{q|M}\left(1+\left(\frac{-3}{q}\right)\right) & \text{if } 9 \nmid d \\ 0 & \text{if } 9|d \end{cases}$$

*where $\left(\frac{-}{p}\right)$ is the Kronecker symbol at the prime $p$.*

**Proof:** See [**P80**], Theorem 1.12, observing that its definition of level is our definition of reduced discriminant. ■

In the case of the quaternion algebras in the above theorem we can formulate again the theorem (1.7.5) as the reduced discriminant is $p$. We can moreover consider the case of a maximal Eichler order and so its level has to be 1. Hence

$$\sum_{i=1}^{n} \frac{1}{w_i} = \frac{1}{12}(p-1) = \frac{p-1}{12}$$

**Note 1.7.8** With the above hypothesis, one can prove (See [**G87**], § 1) that $W = \prod_{i=1}^{n} w_i$ is equal to the denominator of the rational number $(p-1)/12$ and that we have the following table:

| $p$ | $W$ | $w_i > 1$ | $n$ |
|---|---|---|---|
| 2 | 12 | 12 | 1 |
| 3 | 6 | 6 | 1 |
| $\equiv 5 \pmod{12}$ | 3 | 3 | $(p+7)/12$ |
| $\equiv 7 \pmod{12}$ | 2 | 2 | $(p+5)/12$ |
| $\equiv 11 \pmod{12}$ | 6 | $3, 2$ | $(p+13)/12$ |
| $\equiv 13 \pmod{12}$ | 1 | | $(p-1)/12$ |

# 2 Algebraic groups and Tamagawa measures

## 2.1 Algebraic groups

### 2.1.1 Basic notions

Let $k$ be an algebraically closed field.

**Definition 2.1.1** *An algebraic group defined over $k$ is an algebraic variety $G$, over $k$, which is a group object in the category of algebraic varieties over $k$. This means that there exist two morphisms of $k-$varieties*

- $m : G \times G \longrightarrow G$ *such that* $(x, y) \longmapsto xy$ *and* • $\iota : G \longrightarrow G$ *such that* $x \longmapsto x^{-1}$,

*satisfying, together with $e : G \longrightarrow Spec(k) = \{*\}$, the usual commutative diagrams encoding the group structure. We say moreover that $G$ is linear if it is an affine variety. Fix $k[G]$ as the ring of coordinates for $G$, i.e. the quotient of $k[X_{i,j}, t]$ by the ideal $I$ generated by all the polynomials defining $G$ as a variety.*

**Example 2.1.2** $GL_n(k)$ is a linear algebraic group as it can be identified with the affine variety in $k^{n^2+1}$ defined by

$$GL_n(k) = \left\{ (x_{i,j}, t) \in k^{n^2} \times k \;\middle|\; det\left( (x_{i,j})_{i,j} \right) \cdot t = 1 \right\}$$

Obviously every closed subgroup of $GL_n(k)$ is a linear algebraic group (as closed subsets define subvarieties) and it holds, as noticed in [**K67**], that *every linear algebraic group over $k$ is a closed subgroup of $GL_n(k)$.* One can notice (see [**Bo66**], §1.5) that a (linear) algebraic group is irreducible if and only if it is connected.

**Remark 2.1.3 (Abelian Varieties)** A connected projective variety which is an algebraic group is an abelian variety. Every abelian variety is an abelian algebraic group. The easy example of an abelian variety (e.g. over $\mathbb{C}$) is an elliptic curve (for details see [**Sil09**], Chapter *III*).

**Definition 2.1.4** *Let $G$ be a linear algebraic group over $k$. Suppose that $k$ is perfect. We say that $G$ is defined over $K$, for $K$ a subfield of $k$, if the equations defining $G$ have coefficients in the field $K$. In this case we say that $G$ is an algebraic group over $K$ (by abuse of notation). As in the case of an algebraically closed field, we can speak about the ring of coordinates, $K[G]$ which is indeed the quotient of $K[X_{i,j}, t]$ by $I \cap K[X_{i,j}, t]$.*

We can talk about algebraic groups in a more convenient manner, namely thinking them as functors. As in the above definition, suppose that $G$ is an algebraic linear group over a field $K$. For any $K-$algebra $B$, we can consider the set $G_B := G \cap GL_n(B)$; it is indeed a group and it can be identified with $\mathrm{Hom}_K(K[G], B)$ (with the usual machinery of the representability of the maximal spectrum). We can see immediately that this construction preserves arrows, meaning that a homomorphism of $K-$algebras $\rho : A \longrightarrow B$ is mapped to a $K-$morphism $\bar{\rho} : G_A \longrightarrow G_B$. Thus we may give the following, equivalent, definition.

**Definition 2.1.5** *An algebraic group over $K$ is a functor from $K-$algebras to groups,*

$$G : \underline{K - Alg} \longrightarrow \underline{Grp}$$

*which is representable by a finitely generated $K-$algebra $K[G]$, such that $\overline{K} \otimes_K K[G]$ has no nilpotent elements for $\overline{K}$ an algebraic closure of $K$.*

We should notice that we required the absence of nilpotent element because we force $G$ to be induced by an algebraic group over an algebraically closed field.

**Notation:** Usually the affine algebraic groups associated with the general linear group and the special linear group are denoted with $GL_n$ and $SL_n$ respectively, with the association on objects as $GL_n(A)$ and $SL_n(A)$.

**Example 2.1.6 (See [Mil15] Ch. 2, §a and [K67], §1)**

- The multiplicative algebraic group $\mathbb{G}_m$ over $K$ is the multiplicative group of $K$ which we can identify with $GL_1(K)$. The corresponding functor

$$\underline{K - Alg} \longrightarrow \underline{Grp} \qquad \text{such that} \qquad R \rightsquigarrow (R^\times, \cdot)$$

is representable (after being composed with the forgetful functor) by $K[T, T^{-1}]$. In fact $\mathrm{Hom}_{K-Alg}\big(K[T, T^{-1}], R\big) \cong R^\times$ as $T$ can be mapped only to an invertible element in $R$ and $T$ determines uniquely the homomorphism. We can notice that, taken $\overline{K}$ an algebraic closure of $K$, $\overline{K} \otimes K[T, T^{-1}] \cong \overline{K}[T, T^{-1}]$ which does not contain any nilpotent element since it is an integral domain.

- The general linear group $GL_n$ is associated with the functor $R \rightsquigarrow (GL_n(R), \cdot)$ and such functor (composed with the forgetful functor) is representable by $K\left[X_{11}, \ldots, X_{nn}, \frac{1}{det(X_{ij})}\right]$, essentially by its realization as an algebraic variety. Again, tensoring with an algebraic closure of $K$, we obtain an integral domain.

### 2.1.2 Lie algebra of an algebraic group

We can associate a Lie algebra to an affine algebraic group (over a field of characteristic zero).

**Theorem 2.1.7 (Cartier)** *Every affine algebraic group over a field of characteristic zero is smooth.*

**Proof:** See Theorem 3.38 in [**Mil15**], noticing that his definition of algebraic group is more general than the one provided here. ■

Let $G$ be an affine algebraic group over $k$, algebraically closed field of characteristic 0. The theorem allows us to consider the tangent space of $G$ at the identity element, which can be identified with the space of $k-$derivation on $k[G]$, commuting with right-translation. Denote it $\mathfrak{g} := Der_k(k[G])$. With the bracket operation defined as $[f, g] := f \circ g - g \circ f$, the couple $(\mathfrak{g}, [\cdot, \cdot])$ defines a Lie-algebra, called the *Lie-algebra associated with the (affine) algebraic group G*.

### 2.1.3 Quaternion algebras

Let now $B$ be a quaternion algebra over a field $K$ (of characteristic zero). We can think to $B^\times$ as an algebraic group over $K$, defined by the association $R \rightsquigarrow (B \otimes_K R)^\times$ for a $K-$algebra $R$. We can start noticing that $\mathrm{Hom}_{K-Alg}\left(K[T, T^{-1}], B \otimes_K -\right) \cong (B \otimes_K -)^\times$ as functors from $\underline{K - Alg}$ to $\underline{Grp}$. It is possible proving that such functor, composed with the forgetful functor, is indeed representable by a finitely generated $K-$algebra (see [**Mil15**], appendix $A$, §y, A.125) but furthermore one can show that this is a particular case of a more general construction, namely the *Weil restriction of scalars*. In particular, this procedure, applied to a linear group $G$, yields to another linear group $Res_{B/K}(G)$ with a description analogous to that above where, as a functor, $Res_{B/K}(G)(A) = G(B \otimes_K A)$ (see e.g. [**Mil15**], Ch. 2 §h). Hence we can identify $B^\times$ with the functor

$$\underline{K - Alg} \longrightarrow \underline{Grp} \qquad \text{such that} \qquad R \rightsquigarrow \left((B \otimes_K R)^\times, \cdot\right).$$

## 2.2 Tamagawa measure

Let $G$ be a finite-dimensional affine algebraic group over a *number field* $K$. Suppose that $G$ is connected, $dim(G) = n$ (as a variety) and take $x \in G$ a point. By theorem (2.1.7) $x$ is regular and hence, let $x_1, \ldots, x_n$ be local coordinates for $G$ at $x$ (i.e. a system of parameters). We can hence consider the Käler differentials of $K[G]$ and take a $n-$differential form $\omega$, on $G$; in a neighbourhood of $x$, $\omega$ can be written as $\omega = f(X)dx_1 \cdots dx_n$ for $f(X)$ a rational function which is defined at $x$. We say that $\omega$ is defined on $K$ if $f$ and the coordinate functions $x_i$ are defined over $K$. Recall that given a morphism $\psi : W \longrightarrow G$ of algebraic varieties, we have the differential form $\psi^*(\omega)$, defined on $W$, via pull back of $\omega$; it is obtained, locally, by composition of $\psi$ with $f$ and the coordinate functions, with the usual rules for the changes of variables.

For each $g \in G$, the left-translation map $\lambda_g : G \longmapsto G$ such that $x \longmapsto gx$, is a morphism and so, we can pull back each differential form $\omega$ on $G$ to another form $\lambda_g^*(\omega)$. Thus we can consider the space of left-invariant differential $n-$forms on $G$, which are also defined over $K$.

**Proposition 2.2.1** *There exists a left-invariant differential form $\omega$, defined over $K$, such that $\omega$ is non-zero. Further, $\omega$ is unique up to a constant in $K^\times$.*

**Proof:** See [**K67**], §3 and [**We82**], §2.2, in particular Theorem 2.2.2 and its corollary. ■

**Example 2.2.2**
- Take $G = \mathbb{G}_a$ the additive algebraic group, i.e. the group associated with the additive group of $K$ (namely the functor "representable" by $K[T]$). In this case $\lambda_g(x) = g + x$ and hence $\lambda_g^*(dx) = d(\lambda_g(x)) = d(g + x) = dx$ as $g \in G$ and where $x$ is the coordinate function. Since $dim(G) = dim(K[T]) = 1$ we found indeed $\omega = dx$.
- If $G = \mathbb{G}_m$, we can notice that $dim(G) = dim(K[T, T^{-1}]) = 1$. Also, $\lambda_g(x) = gx$ and hence taking $\omega = dx/x$ we are done.

- If $G = GL_n$, $dim(G) = dim(K[X_{ij}, det(X_{ij})^{-1}]) = n^2$. $\lambda_g$ is the left multiplication with the matrix $g$ and so, taking $\omega = \prod dx_{ij}/(det(x_{ij})^n)$ we are done (as we have to choose an order for the product and we can use the alternating property).

We would like to determine a measure associated with $\omega$. As $G$ is a connected linear group (i.e. a subgroup of $GL_n$), we can identify it as a closed subset of the affine $m-$dimensional space, for $m = n^2 + 1$. Let $\mathbb{A} = \mathbb{A}_K$ be the ring of adèles for $K$ and consider the adèle group $G_{\mathbb{A}}$, which is the group of points in $\mathbb{A}^m$ satisfying the equations of $G$. $G_{\mathbb{A}}$ is endowed with the topology induced by the product topology of $\mathbb{A}^m$, thus $G_{\mathbb{A}}$ is a locally compact topological group. Furthermore the group $G_{\mathbb{A}}$ can be realized as the restricted product of the groups $G_{K_v}$ with respect to their compact subgroups $G_{O_v}$ for each finite place $v$ in $K$ (where $G_{K_v}$ and $G_{O_v}$ have the obvious definition for $K_v$ the local field with valuation ring $O_v$).

**Remark 2.2.3** *It can be shown that the definition of $G_{\mathbb{A}}$ as restricted product ensures that $G_{\mathbb{A}}$ itself does not depend on the embedding in $\mathbb{A}^m$. The reason is that a change of embedding corresponds to a change of $G_{O_v}$ for only finitely many places $v$.*

$K \hookrightarrow \mathbb{A}$ is discrete and hence also $G_K \hookrightarrow G_{\mathbb{A}}$ is a discrete subgroup. Since $G_{\mathbb{A}}$ is a locally compact topological group we can take a left-invariant Haar measure on it. By the product formula we know that for each non-zero principal adèle $a \in K^{\times}$, $\prod_v |a|_v = 1$ and that for almost all $v$, $|a|_v = 1$. Hence the right-multiplication by elements in $G_K$ does not affect the measure on $G_{\mathbb{A}}$, then we have an induced left-invariant measure on the space $G_{\mathbb{A}}/G_K$.

Now, we have to fix a choice of $\mu_v$, Haar measures on the additive groups $K_v^+$. There are various possible normalizations, but all we require is that

- $\mu_v(O_v) = 1$ for almost all finite places $v$;
- if $\mu = \prod \mu_v$ is the product measure on $\mathbb{A}$, then (the induced measure) $\mu(\mathbb{A}/K) = 1$.

**Example 2.2.4 (of normalization)** One choice of normalization is that in which we take $\mu_v(O_v) = 1$ for all finite place and $c_v$ times the Lebesgue measure for the infinite places, such that those positive real numbers $c_v$ satisfy

$$\prod_{v \in \infty} c_v = 2^{r_2} |disc_{K/\mathbb{Q}}|^{-\frac{1}{2}}$$

for $r_2$ the number of complex embeddings of $K$ and $disc_{K/\mathbb{Q}}$ the discriminant of $K$. In particular, if $K = \mathbb{Q}$, we have $r_2 = 1$, $disc_{\mathbb{Q}} = 1$ and so $c_{\infty} = 1$. This means that $\mu$ on $\mathbb{A}_{\mathbb{Q}}$ can be chosen such that $\mu_p(\mathbb{Z}_p) = 1$ for all $p$ and $\mu_{\infty}$ is the Lebesgue measure on $\mathbb{R}$.

Let $\omega$ be the non-zero left-invariant form on $G$. Suppose that, in a neighbourhood of $x^0 \in G$, it can be written as $\omega = f(x)dx_1 \cdots dx_n$ for $f$ a rational function in $x = (x_i)_i$ and $x_i$ the coordinate functions at $x^0$ (not necessarily zero at it). We can express $f$ as a power series in $t_i = x_i - x_i^0$, with coefficients in $K$ ,

$$f(x) = \sum_{(i)} a_{(i)} (x_i - x_i^0)^{i_1} \cdots (x_n - x_n^0)^{i_n}.$$

**Remark 2.2.5** This can be proved, for example, with the Cohen Structure Theorem. In particular, we can consider the local ring at the regular point $x^0$ and embed it in its completion. Due to the Cohen structure theorem, the latter is indeed a power series ring with coefficient in $K$.

Let $K_v$ be a completion of $K$ and assume that each $x_j^0$ is in $K_v$. Hence $f$ is a power series with coefficients in $K_v$, which converges in some neighbourhood of the origin in $K_v^n$. There exists hence a neighbourhood $U$ of $G_{K_v}$ such that the map $\varphi : x \longmapsto (t_1, \ldots, t_n)$ is a homeomorphism of $U$ onto $U'$. We can suppose that the above power series converges in the chosen neighbourhood of the origin in $K_v^n$. We have the product measure $\mu_v \times \cdots \times \mu_v$ on $K_v^n$ which we denote by $dt_1 \cdots dt_n$. Hence we have a positive measure on $U'$, defined by $|f(t)|_v dt_1 \cdots dt_n$. Via the homeomorphism $\varphi$ we can pull this measure back to $U$ obtaining a positive measure on $U$, namely $\omega_v$.

In particular, taken $g$ to be a measurable (or continuous) real-valued function, compactly supported on $G_{K_v}$, we have

$$\int_U g(x) d\omega_v(x) = \int_{U'} g\left(\varphi^{-1}(t)\right) |f(t)|_v dt_1 \cdots dt_n$$

As in [**We82**], page 14, it can be proved that the measure $\omega_v$ is independent of the choice of the local coordinates $x_i$.

**Definition 2.2.6 (Tamagawa measure)** *If the infinite product*

$$\prod_{v \notin \infty} \omega_v\left(G_{O_v}\right)$$

*converges (absolutely), we define the Tamagawa measure on $G_{\mathbb{A}_K}$ as the product measure*

$$\tau = \prod_{v \notin \infty} \omega_v$$

*Explicitly, if $S$ is a finite set of places containing all the infinite ones, and if, for each $v \in S$, $U_v$ is an open set in $G_{K_v}$ with compact closure, then $\tau$ is the unique Haar measure on $G_{\mathbb{A}_K}$ for which*

$$\tau\left(\prod_{v \in S} U_v \times \prod_{v \notin S} G_{O_v}\right) = \prod_{v \in S} \omega_v\left(U_v\right) \times \prod_{v \notin S} \omega_v\left(G_{O_v}\right)$$

*In the case in which the above infinite product does not converge absolutely, some factors have to be introduced to guarantee the convergence.*

*A family of $\{\lambda_v\}$ of strictly positive real numbers, indexed by the places of $K$, is a set of convergence-factors if the infinite product*

$$\prod_{v \notin \infty} \lambda_v^{-1} \omega_v\left(G_{O_v}\right)$$

*is absolutely convergent. Hence, the Tamagawa measure, relative to the family $\{\lambda_v\}$, is the product*

$$\tau = \prod_{v \notin \infty} \lambda_v^{-1} \omega_v$$

**Remark 2.2.7**
- *In both cases the measure $\tau$ is independent of the choice of the form $\omega$. In fact, as $\omega$ is unique up to constant, replacing $\omega$ with $c\omega$ for a $c \in K^\times$, we have (by construction)*

$$(c\omega)_v = |c|_v \omega_v \qquad \Longrightarrow \qquad \textit{(e.g.)}\ \tau = \prod_v |c|_v \omega_v = \prod_v \omega_v$$

*as $\prod_v |c|_v = 1$ by the product formula.*

- *The Tamagawa measure is, as noticed in the definition, a (canonical) normalization of the Haar measure on $G_{\mathbb{A}_K}$.*

Taken $v$ a place of $K$, we can consider the norm of $v$, $Nv$ (recall that it is defined as the cardinality of the quotient $O_v/\mathfrak{p}_v$ for $\mathfrak{p}_v$ the associated maximal ideal). One can prove the following equalities (see [**K67**]).

- $G = \mathbb{G}_a$: $\quad\omega_v(G_{O_v}) = 1;$
- $G = \mathbb{G}_m$: $\quad\omega_v(G_{O_v}) = 1 - \frac{1}{Nv};$
- $G = GL_m$: $\quad\omega_v(G_{O_v}) = \left(1 - \frac{1}{Nv}\right) \cdots \left(1 - \frac{1}{(Nv)^m}\right);$

As we know that $\zeta_K(s)^{-1} = \prod_v \left(1 - \frac{1}{(Nv)^s}\right)$ and it does not converge at $s = 1$, we can deduce that in the case $G = GL_m$ we have to introduce a set of convergence-factors, for example

$$\lambda_v = 1 - \frac{1}{Nv}$$

**Remark 2.2.8** *We will consider the Tamagawa measure on the algebraic group $B^\times/Z^\times$ over the adèles, namely on $B^\times(\mathbb{A})/\mathbb{A}^\times$.*

# 3 Jacquet–Langlands correspondence

## 3.1 Representations of $GL_2(\mathbb{A})$

### 3.1.1 Cusp forms in the adelic setting

#### 3.1.1.1 Characters

Just fix the following convention: a character on a topological group $X$ is a continuous homomorphism $\omega : A \longrightarrow \mathbb{C}^\times$. We say that a character is *unitary* if its image is contained in $S^1 \hookrightarrow \mathbb{C}^\times$.

**Definition 3.1.1** *A* Hecke character *is a continuous homomorphism $\varepsilon_{\mathbb{A}} : \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ which is trivial on $\mathbb{Q}^\times$. We say that $\varepsilon_{\mathbb{A}}$ is of finite order if the image is a finite group.*

Since $\mathbb{Q}$ has class number 1, we have that $\mathbb{A}^\times = \mathbb{Q}^\times\mathbb{R}_+^\times\hat{\mathbb{Z}}^\times$ and so, taken a Dirichlet character modulo an integer $N$, we can define an associate Hecke character $\varepsilon_{\mathbb{A}}$. Let $a = \alpha x u \in \mathbb{A}^\times$ written with the decomposition of $\mathbb{A}^\times$ and hence set (considering the natural projection $\hat{\mathbb{Z}} \longrightarrow \mathbb{Z}/N\mathbb{Z}$)

$$\varepsilon_{\mathbb{A}}(\alpha x u) = \varepsilon(u^{-1} \pmod N).$$

One can show that every Hecke character of finite order is indeed induced by a Dirichlet character and that every continuous character, $\omega$, on $\mathbb{A}^\times/\mathbb{Q}^\times$ is of the form $\omega(-) = \varepsilon_{\mathbb{A}}(-)|\cdot|^s$ for a certain $s \in \mathbb{C}$, a Dirichlet character $\varepsilon$ and where $|\cdot|$ represents the adelic absolute value. In the end recall that a character $\mu : \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$ is unramified if it is trivial on $\mathbb{Z}_p^\times$.

#### 3.1.1.2 Cusps

Let $G$ be the algebraic group over $\mathbb{Q}$ associated with $GL_2(\mathbb{Q})$ and denote $G_{\mathbb{Q}} = GL_2(\mathbb{Q})$, $G_{\mathbb{A}} = GL_2(\mathbb{A})$, $G_\infty = GL_2(\mathbb{R})$ and $G_f = GL_2(\mathbb{A}_f)$ where $\mathbb{A}_f$ is the ring of finite adèles. Consider $\mathfrak{H}^\pm = \mathbb{C} \smallsetminus \mathbb{R}$ and consider the usual $G_\infty-$action via Moebius transformations on it. Hence define $K_\infty = SO_2(\mathbb{R})\mathbb{R}^\times = Stab_{G_\infty}(i)$. As that action is transitive, we can identify $\mathfrak{H}^\pm$ with the quotient $G_\infty/K_\infty$ and define $j : G_\infty \times \mathfrak{H}^\pm \longrightarrow \mathbb{C}$ such that $j(\gamma, z) = cz + d$ for $\gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)$.

Now consider a function $\varphi : G_{\mathbb{Q}}\backslash G_{\mathbb{A}} \longrightarrow \mathbb{C}$ and $k \in \mathbb{N}$, even. We can consider the following conditions:

(i) $\varphi(gk) = \varphi(g)$ for all $k$ in some open compact subgroup $K$ of $G_f$.;

(ii) $\varphi(gk_\infty) = j(k_\infty, i)^{-k} det(k_\infty)\varphi(g)$ for $k_\infty \in K_\infty$ and $g \in G_{\mathbb{A}}$;

(iii) for all $g \in G_f$, and $\tau = hi \in \mathfrak{H}^\pm$ (for $h \in G_\infty$) the map

$$\begin{aligned} \mathfrak{H}^\pm &\longrightarrow \mathbb{C} \\ \tau = hi &\longmapsto \varphi(gh)j(h,i)^k det(h)^{-1} \end{aligned}$$

is holomorphic;

(iv) $\varphi$ is slowly increasing, which means that for every $c > 0$ and every compact subset of $G_{\mathbb{A}}$, call it $K$, there exist two constants $A$ and $B$ such that

$$\left|\varphi\left(\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) h\right)\right| \leq A|a|^B$$

for all $h \in K$ and $a \in \mathbb{A}^\times$ with $|a| > c$ (where $|a|$ stands for the adèlic absolute value);

(v) $\varphi$ is *cuspidal*, i.e. for (almost) all $g \in G_{\mathbb{A}}$

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

for a non trivial Haar measure $dx$.

We define hence the $G_f-$module

$$S_k := \left\{\varphi : G_{\mathbb{Q}}\backslash G_{\mathbb{A}} \longrightarrow \mathbb{C} \;\middle|\; \varphi \text{ satisfies (i),(ii),(iii),(iv),(v)}\right\}$$

where the action is given by right translation. If $O$ is an open compact subgroup of $G_f$, we define

$$S_k(O) = (S_k)^O = \left\{\varphi \in S_k \;\middle|\; \varphi(go) = \varphi(g),\;\; g \in G_{\mathbb{A}},\;\; o \in O\right\}$$

hence $S_k = \cup_O S_k(O)$ over all open compact subgroups. We can consider two special choices for such $O$. Let $N$ be a positive integer,

$$U_0(N) = \left\{\gamma \in GL_2(\hat{\mathbb{Z}}) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{N\hat{\mathbb{Z}}}\right\}$$
$$U_1(N) = \left\{\gamma \in GL_2(\hat{\mathbb{Z}}) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \pmod{N\hat{\mathbb{Z}}}\right\}$$

then it holds the following

**Theorem 3.1.2** *For an element $\varphi \in S_k(U_1(N))$ we define a function $f_\varphi : \mathfrak{H} \longrightarrow \mathbb{C}$ such that*

$$f_\varphi(hi) = \varphi(h)j(h,i)^k det(h)^{-1} \qquad \textit{for } h \in GL_2(\mathbb{R})^+$$

*Then $f_\varphi$ is an element in $S_k(\Gamma_1(N))$ and the association $\varphi \longmapsto f_\varphi$ defines an isomorphism*

$$S_k(\Gamma_1(N)) \cong S_k(U_1(N))$$

*Furthermore, taken $\varepsilon$ a Dirichlet character modulo $N$ and $\varepsilon_{\mathbb{A}}$ the associated Hecke character, we have that the above isomorphism restricts to*

$$S_k(\Gamma_0(N), \varepsilon) \cong S_k(U_0(N), \varepsilon_{\mathbb{A}}) := \left\{\varphi \in S_k(U_1(N)) \;\middle|\; u\varphi = \varepsilon_{\mathbb{A}}(det(u))\varphi \textit{ for all } u \in U_0(N)\right\}$$

*in particular, $S_k(\Gamma_0(N)) \cong S_k(U_0(N))$.*

**Proof:** See [**Cas73**]. ■

More precisely, one can define Hecke operators on such spaces and so the above isomorphisms become *Hecke-equivariant.* For details we refer to [**DI95**], §11. One can moreover prove that

**Proposition 3.1.3** *Each $\varphi$ as above is a cuspidal automorphic form on $GL_2(\mathbb{A})$ in the sense of the following section (3.1.6.2). In particular, we have determined a correspondence between holomorphic cusp forms and cuspidal automorphic representation of $GL_2(\mathbb{A})$.*

**Proof:** See [**Bu98**], §3.6. ■

### 3.1.2 Representations over $p-$adic fields

Let $p$ be a prime, $G_p = GL_2(\mathbb{Q}_p)$, $K_p = GL_2(\mathbb{Z}_p)$ the maximal compact opensubgroup of $G_p$ and $Z_p$ the center of $G_p$. Take $\pi : G_p \longrightarrow Aut(V)$ be a representation of $G_p$ on a complex vector space $V$.

**Definition 3.1.4** *A representation $\pi : G_p \longrightarrow Aut(V)$ is said to be* admissible *if*

*(i) every $v \in V$ is fixed by some open subgroup of $G_p$ (in this case we say that $\pi$ is* smooth*);*

*(ii) for every open compact subgroup $U$ of $G_p$, the subspace $V^U$ of vectors fixed by $U$ is finite-dimensional.*

It holds the following

**Proposition 3.1.5 ([JL70] Prop. 2.7)** *A finite-dimensional admissible representation is continuous and the only continuous irreducible finite-dimensional representations of $G_p$ are of the form $g \longmapsto \omega(det(g))$ for $\omega$ a character on $\mathbb{Q}_p^\times$.*

In their book, Jacquet and Langlands classified all the irreducible infinite-dimensional admissible representation of $G_p$. Let $\mu_1$ and $\mu_2$ two characters on $\mathbb{Q}_p^\times$ and define

$$\mathcal{C}_{loc.}(\mu_1, \mu_2) = \left\{ \varphi : G_p \longrightarrow \mathbb{C} \;\middle|\; \begin{matrix} \varphi \text{ is locally constant and for each } a_1,a_2 \in \mathbb{Q}_p^\times \\ \varphi\left(\left(\begin{smallmatrix} a_1 & * \\ 0 & a_2 \end{smallmatrix}\right) g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \varphi(g) \end{matrix} \right\}$$

where $|\cdot|$ is the $p-$adic absolute value. We have an obvious action of $G_p$ on $\mathcal{C}_{loc.}(\mu_1, \mu_2)$ via right translation. Denote such representation with $\rho(\mu_1, \mu_2)$. It is possible proving that $\rho(\mu_1, \mu_2)$ is reducible if and only if $\mu := \mu_1\mu_2 = |\cdot|^{\pm 1}$.

**Definition 3.1.6** *Whenever $\rho(\mu_1, \mu_2)$ is irreducible it is called a* principal series *representation.*

Suppose now that the representation is reducible, we have two cases, namely $\mu = |\cdot|$ and $\mu = |\cdot|^{-1}$. If $\mu = |\cdot|^{-1}$ then $\rho(\mu_1, \mu_2)$ has a 1-dimensional subrepresentation and in particular, setting $\omega = \mu_1|\cdot|^{1/2} = \mu_2|\cdot|^{-1/2}$ we have a function $g \longmapsto \omega(det(g))$ which is stable under the action of $G_p$: such map spans hence a 1-dimensional invariant subspace. In the other case, i.e. if $\mu = |\cdot|$, there exists a 1-dimensional quotient of $\rho(\mu_1, \mu_2)$.

**Definition 3.1.7** *In both the above cases, the infinite-dimensional subquotient of $\rho(\mu_1, \mu_2)$ is called the* special *or* Steinberg representation *and it is denoted with $sp(\mu_1, \mu_2)$.*

Let $\pi(\mu_1, \mu_2)$ be the unique irreducible infinite-dimensional subquotient of $\rho(\mu_1, \mu_2)$ (i.e. $\pi(\mu_1, \mu_2)$ is either $\rho(\mu_1, \mu_2)$ or $sp(\mu_1, \mu_2)$). One can characterize such representation with the

**Proposition 3.1.8 ([Go70], §1, Thm. 4.7)** *$\pi(\mu_1, \mu_2)$ is equivalent to $\pi(\mu_1', \mu_2')$ if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$.*

**Definition 3.1.9** *(i) Let $V$ a complex vector space. We define the admissible dual of $V$ as*

$$\check{V} = \left\{ \varphi : V \longrightarrow \mathbb{C} \;\middle|\; \begin{smallmatrix} \varphi \text{ is linear and exists } U, \text{ open an compact} \\ \text{of } G_p \text{ such that } \varphi \text{ is invariant under } U \end{smallmatrix} \right\}$$

*(ii) Each admissible representation of $G_p$ not of the form $\pi(\mu_1, \mu_2)$, are called* supercuspidal. *They are characterized by the property that for all $v \in V$ and functional $\psi \in \check{V}$, the functions $g \longmapsto \psi(\pi(g)v)$, called matrix coefficients, have compact support modulo the center $Z_p$.*

**Remark 3.1.10** Every irreducible admissible representation of $G_p$ defines a character of the centre $Z_p$. In fact, $Z_p \cong \mathbb{Q}_p^\times$ is a totally disconnected locally compact group and so by Schur's lemma (see [**Bu98**], proposition 4.2.4) $Aut_\pi(V) \cong \mathbb{C}^\times$, where $Aut_\pi(V)$ is the space of all (invertible) operators commuting with $\pi$. As the center $Z_p$ has to be mapped to $Aut_\pi(V)$ (because it is abelian), we obtain hence a character $\pi|_{Z_p} : Z_p \cong \mathbb{Q}_p^\times \longrightarrow \mathbb{C}^\times$, called the *central character of* $\pi$.

We are interested in considering particular admissible representation of $G_p$.

**Definition 3.1.11** *Let $(\pi, V)$ be an admissible representation of $G_p$ on which there exists a $G_p-$invariant positive-definite Hermitian form. We call* unitarizible *every representation of this type.*

One can classify completely the irreducible ones, namely they are

1) *Continuous series*: principal series $\pi(\mu_1, \mu_2)$ with $\mu_1$ and $\mu_2$ unitary;

2) *Complementary series*: principal series $\pi(\mu, \overline{\mu}^{-1})$ with $\mu\overline{\mu} = |\cdot|^\sigma$ for a real $\sigma$, $0 < |\sigma| < 1$;

3) *Discrete series*: special or supercuspidal representations with unitary central character.

**Proposition 3.1.12 ([JL70], Lemma 15.2)** *Unitarizible discrete series representations are square integrable i.e. their matrix coefficients are square integrable modulo the centre, explicitly, called $\pi$ the (homomorphism of) representation,*

$$\int_{G_p/Z_p} \Big| \psi(\pi(g)v) \Big|^2 dg < \infty$$

*for $dg$ a non trivial Haar measure on $G_p$.*

**Example 3.1.13 ([DI95] Example 11.2.2)** The unitarizible special representations are those of the form $sp(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ for a unitary character $\chi$.

We need a last definition concerning a condition on an invariant subspace of the representation.

**Definition 3.1.14** *An infinite-dimensional irreducible admissible representation $\pi$ of $G_p$ (on $V$) is said to be unramified if the subspace $V^K = \{v \in V \mid \pi(K_p) \cdot v = v\}$ is 1-dimensional.*

One can prove (see [**DI95**], §11.2) that the unramified representations are the principal series representations $\pi(\mu_1, \mu_2)$ for $\mu_1\mu_2^{-1} \neq |\cdot|^{\pm 1}$ and $\mu_1$, $\mu_2$ unramified characters. Further it can be proved that $V^K$ is spanned by the function $\varphi_0$ on $G_p$ such that, for each $k \in K_p$,

$$\varphi_0 \left( \left( \begin{smallmatrix} a_1 & b \\ 0 & a_2 \end{smallmatrix} \right) k \right) = \mu_1(a_1)\mu_2(a_2) \left| \frac{a_1}{a_2} \right|^{1/2}$$

thus the space $V^K$ is 1-dimensional.

### 3.1.3 Representation over archimedean local fields

#### 3.1.3.1 A brief recall on Lie algebras

For this paragraph we refer mainly to [**Bou89**], I.1.2, I.3.1, I.3.4 and III.3.12.

Let $V$ be a vector space. We have a structure of Lie algebra on $\mathfrak{gl}(V) = (\mathrm{End}(V), [\cdot,\cdot])$ with the bracket given by the commutator, i.e. $[f,g] = f\circ g - g\circ f$. In particular, for each Lie algebra (e.g. either real or complex) $(\mathfrak{g}, [\cdot,\cdot])$, we can define a representation of $\mathfrak{g}$ as the couple $(\rho, V)$ for $\rho$ a homomorphism of Lie algebras and $V$ a vector space such that

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V).$$

We are interested in some Lie algebras associated with the algebraic groups $GL_2(R)$ and $O_2(R) = \{A \in GL_2(R) \mid AA^t = I\}$ for $R = \mathbb{R}, \mathbb{C}$.

- $GL_2(R)$: $\mathfrak{gl}_2(R) = (M_2(R), [\cdot,\cdot])$ with $[\cdot,\cdot]$ the commutator;
- $O_2(R)$: $\mathfrak{so}_2(R) = \left\{M \in \mathfrak{gl}_2(R) \mid M^t = -M\right\}$ with the induced bracket. Note that $\mathfrak{so}_2(R)$, as a set, is the set of antisymmetrical matrices.

**Remark 3.1.15 (Lie algebra of an algebraic group)** In particular, we can associate a Lie algebra to every (linear) algebraic group over the field $K$ of characteristic zero. The procedure amounts *essentially* to consider the tangent space at the unit element, identify it with the space of $K-$derivation and equip it with the commutator bracket. We briefly discussed this construction in 2.1.2 but for details we refer to [**Mil13**], $II$.3.

We should recall the so called adjoint representation of a Lie algebra. Take $(\mathfrak{g}, [,])$ a Lie algebra and consider

$$ad : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}) \quad \text{s.t.} \quad g \longmapsto ad(g) := [h \mapsto [g,h]]\,.$$

It defines a representation of $\mathfrak{g}$ called the adjoint action. We have also another adjoint action, this time of $GL_2$ on $\mathfrak{gl}_2$. It is defined as

$$Ad : GL_2(R) \longrightarrow \mathfrak{gl}(\mathfrak{gl}_2(R)) \quad \text{s.t.} \quad X \longmapsto Ad(X) := \left[h \mapsto XhX^{-1}\right].$$

An other important object we must introduce is the universal enveloping algebra.

**Definition 3.1.16 ([Bou89], Ch. 1, §2, Definition 1)** *Let $\mathfrak{g}$ be a Lie algebra over a field (or ring) $k$ with bracket $[\ ,\ ]$. We can consider the tensor algebra $T$, which is defined as*

$$T = k \oplus \mathfrak{g} \oplus (\mathfrak{g}\otimes\mathfrak{g}) \otimes (\mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}) \oplus \cdots$$

*and the two-sided ideal*

$$J = \left\langle x\otimes y - y\otimes x - [x,y] \;\middle|\; x,y \in \mathfrak{g}\right\rangle.$$

*The associative algebra $T/J$ is called the* universal enveloping algebra *of $\mathfrak{g}$.*

Notably the representation $Ad$ extends to a representation of the universal enveloping algebra of $GL_2$.

#### 3.1.3.2 Representations over $\mathbb{R}$

Fix the notation: $G_\infty = GL_2(\mathbb{R})$, $K_\infty = O_2(\mathbb{R})$ the maximal compact subgroup of $G_\infty$. Let $\mathfrak{g}$ the complexification of the Lie algebra associated with $G_\infty$, i.e. $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$. We recall the

**Definition 3.1.17** *A representation $(\pi, H)$ of a group $G$ on an Hilbert space $(H, \langle\cdot,\cdot\rangle)$, is said to be* unitary *if $\pi(g)$ is unitary for each $g \in G$, i.e. $\langle \pi(g)h, \pi(g)h'\rangle = \langle h, h'\rangle$.*

Let $\pi$ be a unitary representation of $G_\infty$ on a Hilbert space $V$ such that the map $G_\infty \times V \longrightarrow V$ is continuous. Denote $V_0$ for the subspace of $K_\infty$-finite vectors in $V$ i.e. the subspace of the vectors $v \in V$ such that $\pi(K_\infty)v$ is finite-dimensional.

We want to associate to $\pi$ a so called $(\mathfrak{g}, K_\infty)-$module.

**Definition 3.1.18** *A $(\mathfrak{g}, K_\infty)-$module is a complex vector space $V_0$ with actions of $\mathfrak{g}$ and $K_\infty$, such that all vectors in $V_0$ are $K_\infty-$finite and such that*

1) *the two actions are compatible, namely, for $v \in V_0$, $k \in K_\infty$, $X \in \mathfrak{g}$,*
$$k \cdot (X \cdot v) = (Ad(k)X) \cdot (k \cdot v);$$
2) *for $X \in \mathfrak{so}_2(\mathbb{R})$ (i.e. in the Lie algebra associated with $K_\infty$), it holds that*
$$\frac{d}{dt}\left(exp(tX) \cdot v\right)\Big|_{t=0} = X \cdot v.$$

Taken $\pi$ as above we can define a representation of the Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{gl}_2(\mathbb{R})$ (i.e. in the Lie algebra of $G_\infty$) and for $v \in V_0$, it exists the derivative

$$(\star_{d\pi}) \qquad \frac{d}{dt}\pi\left(exp(tX)\right)v\Big|_{t=0} = \lim_{t\to 0}\frac{\pi\left(exp(tX)\right)v - v}{t}$$

and it defines an element of $V_0$. Such derivation defines, by linear extension to $\mathfrak{g}$, an homomorphism of Lie algebras,

$$d\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}_2(V_0) \quad \text{ s.t. } \quad X \longrightarrow \left[v \mapsto \frac{d}{dt}\pi\left(exp(tX)\right)v\Big|_{t=0}\right]$$

namely we have defined a representation of the Lie algebra $\mathfrak{g}$ on $V_0$. We have also a representation (of groups) of $K_\infty$ on $V_0$ induced by the restriction of $\pi$ to that subgroup, i.e. $\pi|_{K_\infty}$. Denote with $\pi_0$ the couple of representations $(d\pi, \pi|_{K_\infty})$. Furthermore we can prove that such couple satisfies the conditions of the above definition. In fact, for example, we can notice that the two first conditions are true:

- every $v \in V_0$ is $K_\infty-$finite, by definition;
- for each $v \in V_0$, $k \in K_\infty$ and $X \in \mathfrak{g}$, it holds that
$$\pi(k)\left(d\pi(X)v\right) = d\pi\left(Ad(k)X\right)\left(\pi(k)v\right)$$
as
$$\begin{aligned} d\pi\left(Ad(k)X\right)\left(\pi(k)v\right) &= \frac{d}{dt}\pi\left(exp(t\ Ad(k)X)\right)\left(\pi(k)v\right)\Big|_{t=0} = \frac{d}{dt}\pi\left(exp(t\ kXk^{-1})\right)\left(\pi(k)v\right)\Big|_{t=0} \\ &= \frac{d}{dt}\pi\left(k\ exp(tX)k^{-1}\right)\left(\pi(k)v\right)\Big|_{t=0} = \pi(k)\frac{d}{dt}\pi\left(exp(tX)\right)(v)\Big|_{t=0} = \\ &= \pi(k)\left(d\pi(X)v\right) \end{aligned}$$
since $\frac{d}{dt}\left(exp(tkXk^{-1})\right)\big|_{t=0} = \frac{d}{dt}\left(k\ exp(tX)k^{-1}\right)\big|_{t=0}$.

It can be indeed proved that also the condition 2) in the above definition holds and so $(\pi_0, V_0)$ is a $(\mathfrak{g}, K_\infty)-$module. We refer to [**Bu98**], chapter $II$, paragraphs §2.2, §2.3 and §2.4, for a proof of it and for a way more general discussion of such theory.

**Definition 3.1.19** *We say that a* $(\mathfrak{g}, K_\infty)-$*module* $V_0$ *is* admissible *if* $\mathrm{Hom}_{K_\infty}(W, V_0)$ *is finite-dimensional, for each* $\rho : K_\infty \longrightarrow GL(W)$.

**Definition 3.1.20** *Taken an admissible* $(\mathfrak{g}, K_\infty)-$*module we say it is* unitarizible *if it is isomorphic to a* $V_0$.

The notion of irreducibility and homomorphism of $(\mathfrak{g}, K_\infty)-$modules are the natural ones.

Recall that each character $\varepsilon : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times$ is of the form $\varepsilon(t) = sgn(t)^m|t|^s$ for $m \in \{0, 1\}$ and $s \in \mathbb{C}$. We say that $\varepsilon$ is the *central character* of a $(\mathfrak{g}, K_\infty)-$module, if $\{\pm 1\} = K_\infty \cap \mathbb{R}^\times$ acts via $sgn(\cdot)^m$ (where the centre of $G_\infty$ is identified with $\mathbb{R}^\times$) and the centre of $\mathfrak{g}$ acts by multiplication by $s$ (and where $Z(\mathfrak{g}) = \mathbb{C}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is identified with $\mathbb{C}$). It can be proved that

**Proposition 3.1.21** *Every irreducible* $(\mathfrak{g}, K_\infty)-$*module admits a central character.*

**Proof:** In [**Wal88**], Lemma 3.3.2 provides a result analogous to the Schur's Lemma for this setting. With the same procedure as in remark (3.1.10) we conclude. ■

**Remark 3.1.22** If $\varepsilon$ is the central character of a unitary representation of $G_\infty$, the induced $(\mathfrak{g}, K_\infty)-$module, $V_0$, has $\varepsilon$ as central character.

Let now $\mu_1$ and $\mu_2$ be two characters of $\mathbb{R}^\times$ and consider the space

$$\mathcal{B}(\mu_1, \mu_2) := \left\{ \varphi : G_\infty \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{c} \varphi \text{ is right } K_\infty\text{-finite and for each } a_1, a_2 \in \mathbb{R}^\times,\ g \in G_\infty \\ \varphi\left(\left(\begin{smallmatrix} a_1 & * \\ 0 & a_2 \end{smallmatrix}\right) g\right) = \mu_1(a_1)\mu_2(a_2)\left|\frac{a_1}{a_2}\right|^{1/2} \varphi(g) \end{array} \right\}$$

where *right* $K_\infty$*-finite* means that the space of functions $Span\{g \mapsto \varphi(gk)\}_{k \in K_\infty}$ is finite-dimensional. The action of $K_\infty$ on $\mathcal{B}(\mu_1, \mu_2)$ is defined by right translation and that of $\mathfrak{g}$ is exactly the one defined above by equation $(\star_{d\pi})$. With those two actions one can prove that $\mathcal{B}(\mu_1, \mu_2)$ is a $(\mathfrak{g}, K_\infty)-$module with central character $\mu_1\mu_2$. Let now $\mu = \mu_1\mu_2^{-1}$. We have the following three cases:

- the $(\mathfrak{g}, K_\infty)-$module, $\mathcal{B}(\mu_1, \mu_2)$ is irreducible unless $\mu(t) = sgn(t)t^n$ for some $n \in \mathbb{Z} \setminus \{0\}$;
- if $\mu(t) = sgn(t)t^n$ with $n > 0$, then $\mathcal{B}(\mu_1, \mu_2)$ contains exactly one proper $(\mathfrak{g}, K_\infty)-$submodule $\mathcal{B}(\mu_1, \mu_2)^s$ which is infinite-dimensional. Instead the quotient $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}(\mu_1, \mu_2)^s$ ha dimension exactly $n$.
- if $\mu(t) = sgn(t)t^n$ with $n < 0$ then $\mathcal{B}(\mu_1, \mu_2)$ contains exactly one proper subgroup $\mathcal{B}(\mu_1, \mu_2)^f$ which dimension is $|n|$. The quotient $\mathcal{B}(\mu_1, \mu_2)/\mathcal{B}(\mu_1, \mu_2)^f$ is instead infinite-dimensional.

Fix $\pi(\mu_1, \mu_2)$ as $\mathcal{B}(\mu_1, \mu_2)$ if it is irreducible, $\mathcal{B}(\mu_1, \mu_2)^f$ otherwise; in both those cases we call $\pi(\mu_1, \mu_2)$ a *principal series for* $G_\infty$ and we call it *limit of discrete series* if $\mu = sgn(t)$. In the second case we fix (and only in this case) $\sigma(\mu_1, \mu_2) = \mathcal{B}(\mu_1, \mu_2)^s$ and we call it a *discrete series.* Furthermore every irreducible $(\mathfrak{g}, K_\infty)-$module is isomorphic to one of such module, for a couple of character $(\mu_1, \mu_2)$. It holds also that $\pi(\mu_1, \mu_2) \cong \pi(\mu_1', \mu_2')$ if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$ and $\sigma(\mu_1, \mu_2) \cong \sigma(\mu_1', \mu_2')$ if and only if $\{\mu_1, \mu_2\} = \{\mu_1', \mu_2'\}$ or $\{sgn(\cdot)\mu_1', sgn(\cdot)\mu_2'\}$.

We can hence characterize the unitary irreducible representations of $G_\infty$, $\pi$ via the induced unitarizible $(\mathfrak{g}, K_\infty)-$module. Explicitly we have

1) *Continuous series*: $\pi(\mu_1, \mu_2)$ principal series with $\mu_1$ and $\mu_2$ unitary;
2) *Complementary series*: $\pi(\mu, \overline{\mu}^{-1})$ principal series with $\mu\overline{\mu} = |\cdot|^\sigma$ for some real number $\sigma$ such that $0 < |\sigma| < 1$;

3) *Discrete series*: $\sigma(\mu_1, \mu_2)$ with unitary central character.

In particular, such representation is uniquely determined up to isomorphism and has $\mu_1\mu_2$ as central character. Analogously to the non-archimedean case we can define the *matrix coefficients* and talk about *square-integrable* representations $\pi$; those ones are indeed the representations associated with unitarizible discrete series.

#### 3.1.3.3 Representation over $\mathbb{C}$

Taken again two characters $\mu_1$ and $\mu_2$ one can give, analogously to the real case, the notion of principal series representation, $\rho(\mu_1, \mu_2)$. The main difference in this case is that there are no discrete series in the complex case. In particular, $\rho(\mu_1, \mu_2)$ is irreducible except when $\mu_1\mu_2^{-1}(x) = x^p\overline{x}^q$ for $p, q \in \mathbb{Z}$ such that $pq > 0$ and if it happens, the corresponding infinite-dimensional quotient is again of the form $\rho(\mu_1', \mu_2')$ for appropriate characters $\mu_1'$ and $\mu_2'$.

For a complete discussion we refer to [**Ge75**], Remark 4.8.

### 3.1.4 The global case

Once developed the local theory we can consider the global one. The main tool is the restricted tensor product of representations.

For each prime $p$, suppose exists an irreducible admissible representation $\pi_p : G_p \to GL(V_p)$. One can notice that $V_p$ is either one or infinite-dimensional as discussed in (3.1.2). Suppose additionally that $\pi_p$ is unramified for all $p$ not in a finite set of places $S$. For each $p \notin S$, choose a non-zero vector $e_p$ in the 1-dimensional $V_p^{K_p}$ for $K_p = GL_2(\mathbb{Z}_p)$. Define the space

$$W = Span\left\{ \bigotimes_p v_p \;\middle|\; v_p = e_p \text{ for all but finitely many } p \right\}$$

which is a $G_f$ module equipped with the componentwise action on each generators and hence extending it linearly on $W$.

**Proposition 3.1.23** *$W$ defines an irreducible representation of $G_f$*

$$G_f \longrightarrow Aut(W)$$

*which is called the restricted tensor product of the $\pi_p$ and it is denoted $\bigotimes_p \pi_p$. Such representation is independent (up to isomorphism) of the choice of the $e_p$. Moreover the $G_f-$module $W$ is admissible, meaning that both the following conditions hold:*

(*a*) *every vector in $W$ is fixed by some open subgroup of $G_f$;*

(*b*) *for every compact open subgroup, $U$, of $G_f$, the subspace of $U-$fixed vectors if finite-dimensional.*

**Proof:** See [**Fl79**], §2, mainly *Example 2* which let us apply the general theory developed in the article, to our particular case. ■

We have now to deal also with the archimedean part. Suppose that there exists an irreducible admissible $(\mathfrak{g}, K_\infty)-$module, $V_\infty$, for $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$ and $K_\infty = O_2(\mathbb{R})$. We can consider

$$V = V_\infty \otimes W$$

which is endowed with a structure of $(\mathfrak{g}, K_\infty) \times G_f-$module, i.e. the two actions are compatible (as the two actions act, roughly speaking, *"componentwise"*). We can notice that $V$ is irreducible, meaning that it has no proper $(\mathfrak{g}, K_\infty) \times G_f-$submodules. It can be proved further that $V$ is admissible in the sense that the two conditions

$(A)$ every vector in $V$ is fixed by some open subgroup of $G_f$;

$(B)$ for every compact open subgroup $U$ of $G_f$, the subspace of $U-$fixed vectors in $V$ defines an admissible $(\mathfrak{g}, K_\infty)-$module;

hold. But we can give a stronger result.

**Theorem 3.1.24** *Every irreducible and admissible $(\mathfrak{g}, K_\infty) \times G_f-$module can be written as a restricted tensor product and the local factors $V_p$ and $V_\infty$ are unique up to isomorphism.*

**Proof:** See [**JL70**], Ch. 9 and [**Fl79**], Theorem 3. ■

If moreover we suppose that each $\pi_p$ is unitarizible, then we are able to choose $e_p$ to be a unitary vector for each $p \notin S$. Repeating the above procedure it is possible showing that the obtained $G_f-$module is equipped with an invariant positive-definite Hermitian form. Considering the completion with respect the induced measure we obtain a $\hat{W}$, which defines a unitary representation of $G_f$. Further, if also $V_\infty$ is unitarizible, the construction determines an admissible unitary representation of $G_{\mathbb{A}}$.

**Remark 3.1.25** *Suppose that $\pi$ is an irreducible admissible representation $\pi : G_f \longrightarrow Aut(W)$. Then, by the above theorem, it is isomorphic to the tensor product $\bigotimes_p \pi_p$ for a collection of local representations $\pi_p : G_p \longrightarrow Aut(V_p)$. One can deduce that $\pi$ admits a central character which is defined as the one induced by the local central characters. In particular, almost all the local representations are unramified.*

### 3.1.5 Note on the arbitrary number field case

Suppose now that $F$ is an arbitrary number field and take $G = GL_2$ as an algebraic group over $F$. As above we denote $G_v = GL_2(F_v)$ for each place of $F$ and for the non-archimedean ones $K_v = GL_2(O_v)$. Again we can construct $G(\mathbb{A}_F)$ (e.g.) as the restricted product of the $G_v$ with respect to the $K_v$. We define

$$K = \prod_{v \text{ finite}} K_v$$

and it is an open and compact subgroup of $G(\mathbb{A}_{F,f})$ and again $G(F) \hookrightarrow G(\mathbb{A}_F)$ is discrete and cocompact. In particular, we have that

- In the *non-archimedean* case the theory is indeed the same as in the case of the $p-$adic fields. The definitions are the obvious generalization to the case of $F_v$ and the results are exactly the same; see for example [**Ge75**], Ch. 4, § B.*The $p-$adic Theory*, mainly definition 4.9, theorems 4.18 and 4.21;

- In the *archimedean* case, again, the theory is analogous to that on $\mathbb{Q}$, with the following modifications (see [**Ku04**]):

$$G_\infty = \prod_{v \text{ arch.}} G(F_v) \cong \underbrace{GL_2(\mathbb{R}) \times \cdots GL_2(\mathbb{R})}_{r_1} \times \underbrace{GL_2(\mathbb{C}) \times \cdots GL_2(\mathbb{C})}_{r_2}$$

$$K_\infty = \prod_{v \text{ arch.}} K_v \cong \underbrace{O_2(\mathbb{R}) \times \cdots O_2(\mathbb{R})}_{r_1} \times \underbrace{U_2(\mathbb{C}) \times \cdots U_2(\mathbb{C})}_{r_2}$$

  for $r_1$ and $r_2$ the number of, respectively, real and complex (conjugated pairs of) embeddings of $F$ and $U_2(\mathbb{C}) = \{A \in M_2(\mathbb{C}) \mid A^*A = AA^* = 1\}$ for $A^*$ the adjoint matrix. Again, one can associate a Lie algebra to $G_\infty$ which center is identified with $Z(\mathfrak{g}) \cong \otimes_v Z(\mathfrak{g}_v)$. Hence one can speak about $(\mathfrak{g}, K_\infty)-$modules in a similar fashion.

- The *global* case is carried out in an analogous manner considering $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_{F,f})-$modules with the same definitions: see e.g. definition 2.4 in [**Ku04**].

### 3.1.6 Cuspidal Automorphic representations of $GL_2$

#### 3.1.6.1 Some analysis

Just start with a brief recall about $L^2-$spaces; for details we refer to [**Ru87**]. Let $(X, \mu)$ be a measurable space with positive measure $\mu$. We can define, for each complex-valued measurable function on $X$, $f$, and for $p \in \mathbb{Z}$, the usual $p-$norm, $||f||_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$, and hence define

$$L^p(X, \mu) = \left\{ f : X \longrightarrow \mathbb{C} \;\middle|\; f \ \mu\text{-measurable}, \ ||f||_p < +\infty \right\}$$

Taken $f, g \in L^p(X, \mu)$, denote $d(f, g) = ||f - g||_p$ for the induced metric on $L^p(X, \mu)$. Notice that $d(f, g) = 0$ if and only if the functions $f$ and $g$ coincide for almost all $x \in X$. We can hence define an equivalence relation on $L^p(X, \mu)$ such that $f \sim g$ if and only if $d(f, g) = 0$. Taken now $F$ and $G$ two equivalence classes we define $d(F, G) = d(f, g)$ for any $f \in F$ and $g \in G$. We can regard $L^p(X, \mu)$ not only as a space of functions but also as a space of these equivalence classes of functions. One can prove easily that $d$ defines a metric and moreover that $(L^p(X, \mu), d)$ is a $\mathbb{C}-$vector space which is a Banach space with respect this metric. Further, defining

$$\langle f, g \rangle = \int_X f(x)\overline{g(x)} d\mu(x)$$

it defines an inner product on $L^2(X, \mu)$ which is then endowed with a structure of Hilbert space (just notice that $\langle f, f \rangle = ||f||_2{}^2$).

Before specializing to our case we have to provide a definition of sum of Hilbert spaces.

Let $(E_n)_n$ be a sequence of Hilbert spaces, endowed with the scalar product $\langle \cdot, \cdot \rangle_n$. Let $E$ be the space

$$E = \left\{ x = (x_1, x_2, \ldots, x_n, \ldots) \in \prod_{n \geq 1} E_n \;\middle|\; \sum_{n \geq 1} ||x_n||_n^2 < +\infty \right\}$$

We can define a structure of vector space on $E$ considering the product by a scalar $\lambda \in \mathbb{C}$ as

$$\lambda \cdot x = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n, \ldots) \Longrightarrow \sum_{n \geq 1} ||\lambda x_n||_n^2 = |\lambda|^2 \sum_{n \geq 1} ||x_n||_n^2 < +\infty$$

We can notice that, for all $n \geq 1$,

$$||x_n + y_n||_n^2 = 2\left(||x_n||_n^2 + ||y_n||_n^2\right)$$

since, by direct computation, it can be proved that

$$||x + y||^2 + ||x - y||^2 = 2\left(||x||^2 + ||y||^2\right)$$

for all $x$ and $y$ in a normed space with norm induced by a (complex) scalar product. Thus the series

$$\sum_{n \geq 1} ||x_n + y_n||_n^2 \leq 2\left(\sum_{n \geq 1} ||x_n||_n^2 + \sum_{n \geq 1} ||y_n||_n^2\right) < +\infty$$

and so we can define the sum on $E$ as the componentwise sum. By straightforward computations it is possible showing that those operations define a structure of vector space on $E$. Further we can notice that, by the well-known Chauchy-Schwarz inequality,

$$|\langle x_n, y_n \rangle_n| \leq ||x_n||_n \cdot ||y_n||_n \leq \frac{1}{2}\left(||x||^2 + ||y||^2\right)$$

hence, if $(x_n)$ and $(y_n)$ are in $E$, we can set

$$\langle x, y\rangle = \sum_{n\geq 1}\langle x_n, y_n\rangle_n$$

since the sum is absolutely convergent. In particular, such map $\langle\cdot,\cdot\rangle : E\times E \longrightarrow \mathbb{C}$ defines a Hermitian form on $E$. Moreover it is a positive definite and nondegenerate form as $\langle x, x\rangle = \sum_{n\geq 1}||x_n||^2$. It remains to show that the space is complete. This is indeed a classical and easy computation with Cauchy sequences and for its proof we refer to [**Diu60**], Ch. $VI$, §4.

**Definition 3.1.26** *The space $(E, \langle\cdot,\cdot\rangle)$ is a Hilbert space called the* Hilbert sum *of the Hilbert spaces $(E_n, \langle\cdot,\cdot\rangle_n)$. Sometimes we refer to it as*

$$\left(E = \hat{\bigoplus}_n E_n, \langle\cdot,\cdot\rangle\right)$$

*to highlight the underlying components.*

**Remark 3.1.27 (Finite sum)** If the sequence $(E_n)_n$ is finite all the constructions become trivial and with the analogous definition of Hermitian form we construct the Hilbert sum of a finite number of Hilbert spaces.

#### 3.1.6.2 Definition of Cuspidal Automorphic representation

Let $K$ be a number field and let $\mu$ be an invariant measure on $G_{\mathbb{A}} := GL_2(\mathbb{A}_K)$; recall that $K \hookrightarrow \mathbb{A}_K$ is discrete and so is $G_K := GL_2(K) \hookrightarrow G_{\mathbb{A}}$. Consider the Hilbert space $L^2(G_K\backslash G_{\mathbb{A}})$ with respect the measure $\mu$; e.g. we can take a (right) Haar measure on $GL_2(\mathbb{A})$ (which is a locally compact topological group) and consider the induced quotient measure which, by abuse of notation, we denote with $\mu$. Define $Z_K = K^\times \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $Z_{\mathbb{A}_K} = \mathbb{A}_K^\times \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ for the center of $G_K$ and $G_{\mathbb{A}}$. As $G_{\mathbb{A}}$ acts unitarily on $L^2(G_K\backslash G_{\mathbb{A}})$ via right translation, we can define the obvious representation of $G_{\mathbb{A}}$,

$$\rho : G_{\mathbb{A}} \longrightarrow Aut\left(L^2(G_K\backslash G_{\mathbb{A}})\right) \quad \text{such that} \quad g \longmapsto [\varphi(x)\mapsto\varphi(xg)]\,.$$

The right translation via matrices in $Z_{\mathbb{A}}$ commutes with the representation $\rho$ (by definition of center) and so it can be proved (see [**Go70**] §3.3) that $L^2(G_K\backslash G_{\mathbb{A}})$ decomposes into the sum of the Hilbert spaces. Each of these spaces is of the form $L^2(G_K\backslash G_{\mathbb{A}}, \omega)$, for each unitary character $\omega$ of $Z_{\mathbb{A}}/Z_K \cong \mathbb{A}^\times/K^\times$,

$$L^2(G_K\backslash G_{\mathbb{A}}, \omega) = \left\{\varphi \in L^2(G_K\backslash G_{\mathbb{A}}/Z_{\mathbb{A}}) \;\middle|\; \varphi(\gamma g z) = \omega(z)\varphi(g) \text{ for } \gamma\in G_K,\, g\in G_{\mathbb{A}},\, z\in Z_{\mathbb{A}}\right\}$$

for the induced quotient measure on $G_K\backslash G_{\mathbb{A}}/Z_{\mathbb{A}} = Z_{\mathbb{A}}G_K\backslash G_{\mathbb{A}}$. For every $\varphi \in L^2(G_K\backslash G_{\mathbb{A}}, \omega)$ and $g\in G_{\mathbb{A}}$ we can define the function

$$\mathbb{A} \ni x \longmapsto \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right) \in \mathbb{C}$$

which is a $K-$invariant function on $\mathbb{A}$ as for $k\in K$, $\left(\begin{smallmatrix}1&k\\0&1\end{smallmatrix}\right) \in G_K$ and $\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&k\\0&1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&a+k\\0&1\end{smallmatrix}\right)$ for $k\in K$ and $a\in\mathbb{A}\smallsetminus K$. By definition of the space $L^2(G_K\backslash G_{\mathbb{A}})$ and that of integrability, for almost all $g\in G_{\mathbb{A}}$, the above function is also absolutely square-integrable over $\mathbb{A}/K$, where $\mathbb{A}/K$ is endowed with a non trivial $Haar$ measure. We would like at this point to encode some *cuspidality* condition. Let

$$L_0^2(G_K\backslash G_{\mathbb{A}}, \omega) := \left\{\varphi \in L^2((G_K\backslash G_{\mathbb{A}}, \omega)) \;\middle|\; \begin{array}{c} G_{\mathbb{A}}\ni g \longmapsto \int_{\mathbb{A}/K}\varphi\left(\left(\begin{smallmatrix}1&y\\0&1\end{smallmatrix}\right)g\right)dy \\ \text{vanishes almost everywhere on } G_{\mathbb{A}}\end{array}\right\}$$

the set of *parabolic* functions. It is possible noticing that $L^2_0(G_K\backslash G_{\mathbb{A}},\omega)$ is a closed subspace of $L^2(G_K\backslash G_{\mathbb{A}},\omega)$ and that it is stable under the action of $G_{\mathbb{A}}$; in fact, taken $g'\in G_{\mathbb{A}}$,

$$\int_{\mathbb{Q}\backslash\mathbb{A}}\varphi\left(\left(\begin{smallmatrix}1&y\\0&1\end{smallmatrix}\right)g\right)dy=\int_{\mathbb{Q}\backslash\mathbb{A}}\varphi\left(\left(\begin{smallmatrix}1&y\\0&1\end{smallmatrix}\right)\left(gg'^{-1}\right)g'\right)dy$$

and denoted with $D$ the set of measure zero on which the function does not vanish, after the action it remains of zero-measure (by definition of Haar mesure) and it is indeed $Dg'^{-1}$. As $L^2_0(G_K\backslash G_{\mathbb{A}},\omega)$ satisfies those properties, it makes sense considering its decomposition into direct sum of Hilbert spaces. In particular, it is possible showing that such decomposition exists (See [**Go70**], §3.5, Theorem 1 and its corollary) and that

$$L^2_0(G_K\backslash G_{\mathbb{A}},\omega)=\hat{\bigoplus}H_\alpha$$

where the sum is taken over a countable set of *minimal* closed irreducible subspaces, stable under the action of $G_{\mathbb{A}}$. Hence we can consider unitary representations $\rho: G_{\mathbb{A}}\longrightarrow GL(H_\alpha)=Aut(H_\alpha)$ such that $g\longmapsto[\varphi\mapsto\varphi\cdot g]$.

**Definition 3.1.28** *The isomorphism classes of those unitary representations are called* cuspidal unitary automorphic representation of $GL_2(\mathbb{A})$ *with central character* $\omega$.

**Theorem 3.1.29 (Multiplicity one)** *Each isomorphism class in the above decomposition occurs with only finite multiplicity. More precisely such multiplicity is one, meaning that if* $H_\alpha\cong H_\beta\Longrightarrow H_\alpha=H_\beta$.

**Proof:** See [**JL70**], §10 and §11. ■

We can restrict our attention to a subspace of $L^2_0(G_K\backslash G_{\mathbb{A}},\omega)$.

**Definition 3.1.30** *With the notation used in this section, let* $\mathcal{A}_0(\omega)$ *be the subspace of* $L^2_0(G_K\backslash G_{\mathbb{A}},\omega)$ *of function* $\varphi$ *satisfying*

*(i)* $\varphi(g)$ *is right* $K-$*finite, for* $K=K_\infty GL_2(\hat{\mathbb{Z}})$ *the maximal compact;*

*(ii)* $\varphi$ *is right* $\mathfrak{z}$*-finite, for* $\mathfrak{z}$ *the center of the enveloping algebra* $\mathfrak{g}$*;*

*(iii)* $\varphi$ *is smooth as a function on* $G_\infty$*;*

*(iv)* $\varphi$ *is* slowly increasing *as in (3.1.1.2).*

*Each element of* $\mathcal{A}_0(\omega)$ *is called a* cuspidal automorphic form on $GL_2(\mathbb{A})$ with central character $\omega$.

**Theorem 3.1.31** *(i)* $\mathcal{A}_0(\omega)$ *is an admissible* $(\mathfrak{g},K_\infty)\times G_f-$*module;*

*(ii)* $\mathcal{A}_0(\omega)$ *is a dense subset in* $L^2_0(G_K\backslash G_{\mathbb{A}},\omega)$*;*

*(iii)* $\mathcal{A}_0(\omega)$ *decomposes as algebraic direct sum of irreducible admissible* $(\mathfrak{g},K_\infty)\times G_f-$*module (which are still unitary representations)*

$$\mathcal{A}_0(\omega)=\bigoplus V_\alpha$$

*with each* $V_\alpha$ *dense in the correspondent* $H_\alpha$.

**Proof:** See [**Ge75**], §5 mainly theorem 5.1. ■

**Theorem 3.1.32 (Strong multiplicity one)** *Let $V_\alpha$ and $V_\beta$ be two constituents of the decomposition at point (iii) of the previous definition. Denote the $G_p-$modules associated with those constituents with $V_{\alpha,p}$ and $V_{\beta,p}$, and suppose that $V_{\alpha,p} \cong V_{\beta,p}$ for all but finitely many primes $p$. Then $V_\alpha = V_\beta$.*

**Proof:** See [**Ge75**], §6 and theorem 5.14 which is proved in it. ■

**Remark 3.1.33** We can focus on a particular quotient, namely $X = \mathbb{R}_+^\times G_\mathbb{Q} \backslash G_\mathbb{A}$, where $R_+^\times$ is identified with the centre $Z_\infty$ of $GL_2(\mathbb{R})^+$. $X$ is endowed with a $G_\mathbb{A}-$invariant measure as the above spaces and the same constructions can be applied to $X$. We obtain hence, $L^2(X)$ and its closed invariant (with respect to the action of $G_\mathbb{A}$ as above) subset $L_0^2(X)$. Again, $L_0^2(X)$ decomposes as a sum of Hilbert spaces $R_\alpha$, where the $R_\alpha$ are closed irreducible subspaces, stable under the action of $G_\mathbb{A}$. We can consider also cuspidal automorphic forms in the same manner as above.

## 3.2 Jacquet–Langlands correspondence

### 3.2.1 Representations of quaternion algebras

Let $B$ be a quaternion algebra over a number field $F$ and consider the group of invertible elements of $B$ as an algebraic group over $F$. For each place of $F$ set $B_v = (B \otimes_F F_v)$ and denote the algebraic group of the invertible elements of $B$ as

$$G' := B^\times \qquad \Longrightarrow \qquad G'_v = G'(F_v) = (B \otimes_F F_v)^\times$$

for each place $v$ of $F$.

For each unramified place of $F$, $v$, we know that there exists an isomorphism

$$\theta_v : B_v \longrightarrow M_2(F_v)$$

and we want to fix such isomorphism. Let hence $\mathcal{B}$ be a maximal order in $B$ and denote by $\mathcal{B}_v$ the induced module in $B_v$. We can choose $\theta_v$ as that isomorphism such that $\theta_v(\mathcal{B}_v) = M_2(O_v)$ (if $v$ is a finite place). Thus we can identify $G'_v$ with $GL_2(F_v)$. We define moreover $K'_v$ as the maximal compact subgroup of $G'_v$ such that $\theta_v(K'_v) = K_v := GL_2(O_v)$ and the correspondent maximal compact for the infinite places. We can hence restate all the theory developed in the matrix case obtaining analogous notions and definitions, up to composing with $\theta_v$.

On the other hand, if $v$ is ramified we know that there exists only one maximal order $\mathcal{B}_v$ in $B_v$, namely $\{x \in B_v \mid |n(x)|_v \leq 1\}$ and then $K'_v = \{x \in G'_v \mid |n(x)|_v = 1\}$ is the maximal compact subgroup of $G'_v$. As in the matrix case, one can establish a relation between representations of $G'_v$ and that of a group algebra (more precisely an algebra of compactly supported locally constant functions or subspaces of smooth functions). In this way one can give an analogous definition of *admissibility* and repeat all the procedure in the matrix case and obtain a classification of the representations of $G'_v$.

**Theorem 3.2.1 (Peter–Weyl)** *Let $K$ be a compact group. Then*

*(i) The matrix coefficients (see below (5.1.1.2)) of finite-dimensional unitary representations of $K$ are dense in the continuous functions $\mathcal{C}(K)$ and hence in $L^p(K)$ for each $1 \leq p < \infty$;*

*(ii) Any irreducible unitary representation of $K$ is finite-dimensional;*

*(iii) Any unitary representation of $K$ decomposes as Hilbert sum of irreducible unitary representations.*

**Proof:** See theorem 2.4.1 in [**Bu98**] and §4, Part $I$, in [**Bu13**]. ■

**Remark 3.2.2** *For $v$ ramified, as $G'_v$ is compact modulo its center (see (1.4.5)), we have that every irreducible unitary admissible representation of $G'_v$ is finite-dimensional.*

Moreover any irreducible continuous representation of $G'_v$, which is finite-dimensional, is admissible (in the usual sense).

With the technique of the tensor product representation, we can think about irreducible representations of $G'_{\mathbb{A}_F} = B^\times(\mathbb{A}_F)$, namely of the form

$$\pi' = \bigotimes_v \pi'_v$$

for local representations $\pi'_v$ of $G'_v$.

#### 3.2.1.1 Cuspidal forms on quaternion algebras

We should start this section defining what is a cusp form on a quaternion algebra. Let

$$X' = Z'_\infty G'_F \backslash G'_{\mathbb{A}_F}$$

for $Z'_\infty$ the centre of $G'_\infty$ (with a description analogous to (3.1.5)). As in the $GL_2$ case, this quotient is equipped with a $G'_{\mathbb{A}_F}-$invariant measure.

**Definition 3.2.3** *A cusp form on $G'_{\mathbb{A}_F}$ is an irreducible unitary representation of $G'_{\mathbb{A}_F}$ which occurs in the natural representation $R'$ of $G'_{\mathbb{A}_F}$ on $L^2(X')$. For natural representation we mean the one determined by right-multiplication by elements of $G'_{\mathbb{A}_F}$.*

Considering the natural representation of $G'_{\mathbb{A}_F}$ we can give the following definition, where $K'$ is a maximal compact in $G'_{\mathbb{A}_F}$ and $\mathfrak{z}'$ is the centre of the universal enveloping algebra of the complexified Lie algebra of $G'_\infty$, where the universal enveloping algebra is defined as in (3.1.16).

**Definition 3.2.4** *An automorphic form on $G'_{\mathbb{A}_F}$ (or for the quaternion algebra $B$) is an element of the space of $K'-$finite and $\mathfrak{z}'-$finite function in $L^2(X')$. Those elements can be, equivalently, defined as the functions $\varphi$ on $G'_{\mathbb{A}_F}$ satisfying*

*(i)* $\varphi(\gamma g) = \varphi(g)$ *for all* $\gamma \in G'_F$;

*(ii)* $\varphi(zg) = \varphi(g)$ *for all* $z \in Z'_\infty$;

*(iii)* $\varphi(g)$ *is right* $K'-$*finite;*

*(iv)* *as a function on* $G'_\infty$, $\varphi$ *is right* $\mathfrak{z}'-$*finite.*

**Remark 3.2.5** We do not have to require the *"slowly increasing"* condition as the underlying space is compact.

In particular, following [**Ge75**], we give the

**Definition 3.2.6** *Let $\pi'$ be a unitary representation of $G'_{\mathbb{A}}$, with central character $\psi$. Then $\pi'$ is a* cuspidal form *of $G'_{\mathbb{A}}$ if it is an irreducible unitary representation of $G'_{\mathbb{A}}$ which occurs in the decomposition of the natural representation (i.e. that induced via right translation) of $G'_{\mathbb{A}}$ on $L^2(X', \psi)$ which is defined as the space of $L^2-$functions, $\varphi$, such that $\varphi(zb) = \psi(z)\varphi(b)$ for $z \in Z(G'_{\mathbb{A}})$.*

### 3.2.2 Fourier transform and measures

#### 3.2.2.1 Schwartz-Bruhat functions and Characters

Before talking about representation we need the definition of the Schwartz-Bruhat functions and some remarks on additive characters. Let $X$ be either a local field, $K$, or a quaternion algebra over $K$, $B$, such that $B \not\supset \mathbb{R}$.

**Definition 3.2.7** *The space of the Schwartz-Bruhat functions on $X$ is*

$$S(X) := \begin{cases} \left\{ f : X \longrightarrow \mathbb{C} \;\middle|\; f \text{ is smooth and rapidly decreasing function} \right\} & \text{if } X \supset \mathbb{R} \\ \\ \left\{ f : X \longrightarrow \mathbb{C} \;\middle|\; f \text{ is locally constant and with compact support} \right\} & \text{otherwise} \end{cases}$$

**Remark 3.2.8** *It is possible to endow $S(X)$ with a topology and we will refer to it as the* standard *topology. As the technical definition is not relevant to our purpose, we refer to* [**We64**], *Ch. 1, paragraph 11, for a precise construction.*

Let $G$ be a locally compact group and take $dg$ a Haar measure on it. For each isomorphism $a$ of $G$, we call $d(ag)$ the Haar measure defined on $G$ by $\int_G f(ag)d(ag) := \int_G f(g)dg$ for each measurable function $f$ on $G$. The proportionality factor of those two measures is $||a|| = d(ag)/dg$ is called the *module of the isomorphism $a$*. We have, for each isomorphisms $a$ and $b$ and each measurable subset $C$,

$$vol(aC) = \int_G \chi_{aC}(g)dg =: \int_G \chi_{aC}(ag)d(ag) = \int_G \chi_C(g)d(ag) = ||a|| \int_G \chi_C(g)dg = ||a||vol(C);$$

$$||ab||vol(C) = vol(abC) = ||a||vol(bC) = ||a|| \cdot ||b||vol(c) \Longrightarrow ||a|| \cdot ||b|| = ||ab||$$

for $\chi_C$ the characteristic (or indicator) function of $C$. We have then

**Definition 3.2.9** *The module of an element $x \in X^\times$, denoted with $||x||_X$ is the module of two isomorphisms of multiplication on the left (or on the right) in $X$. The norm of $x$ is $N(x) = N_X(x) := ||x||_X^{-1}$.*

In particular, one can show by direct computation that, if $|x|$ is the usual module of a real or complex number, we have for $x \in X$

$$||x||_{\mathbb{R}} = |x|; \quad ||x||_{\mathbb{C}} = |x|^2; \quad ||x||_X = N(x)^{-1} = N_X(\mathcal{B}x)^{-1} = (\#\mathcal{B}/\mathcal{B}x)^{-1} \text{ if } X \not\supset \mathbb{R}$$

for $\mathcal{B}$ a maximal order of $X$ (containing $O_K$).

**Remark 3.2.10 (Characters)** The association $x \longmapsto N(x)^s$ defines a character on $X$ and it is unitary if and only if $s \in \mathbb{C}$ is purely imaginary. Furthermore every character on the quaternion algebra $B$ is of the form $\chi_B = \chi_K \circ n$ for $n$ the reduced norm on $B$ and $\chi_K$ a character on $K$.

**Example 3.2.11** $K = \mathbb{Q}_p$, $x \longmapsto exp(2\pi i\langle x\rangle)$ with $\langle x\rangle = ap^{-m}$ the unique rational number in $[0,1] \cap \mathbb{Q}$, such that $x - \langle x\rangle \in \mathbb{Z}_p$ (as $\mathbb{Q}_p \ni x = \sum_{i=n}^{+\infty} a_p p^i$ for $n \in \mathbb{Z}$ and so take $ap^{-m} = \left(\sum_{i=n}^{-1} a_i p^{i+m}\right) p^{-m}$ with $m = \inf\{m \mid 0 < m \leq -n : a_{-m} \neq 0\}$).

#### 3.2.2.2 Self-dual measure

Let $K$ be a local field of characteristic zero and denote $K^+$ for the additive abelian group of $K$. Let $\psi$ be a non-trivial additive character of $K^+$.

**Lemma 3.2.12** *Taken the topological dual character group* $\mathrm{Hom}_{cts}(K^+, \mathbb{C}^\times)$*, we have a topological and algebraic isomorphism*

$$K^+ \longrightarrow \mathrm{Hom}_{cts}(K^+, \mathbb{C}^\times) \qquad s.t. \qquad \eta \longmapsto \psi_\eta$$

*for* $\psi_\eta(\xi) = \psi(\eta\xi)$.

**Proof:** See [**Ta67**] lemma 2.2.1. ■

We can hence give the

**Definition 3.2.13 (Fourier transform)** *For each* $f \in L^1(K^+)$ *we can define the Fourier transform of* $f$ *as*

$$\hat{f}(\eta) = \int_{K^+} f(\xi)\psi(-\eta\xi)d\xi$$

*for* $d\xi$ *a non-trivial Haar measure on* $K^+$.

If $\hat{f} \in L^1(K^+)$ we can consider the Fourier transform of $\hat{f}$ and moreover we have the so-called inversion formula

**Theorem 3.2.14 (Inversion formula)** *For those* $f \in L^1(K^+)$ *such that* $\hat{f} \in L^1(K^+)$ *there exists a constant* $c$ *such that*

$$\hat{\hat{f}}(-\xi) = \int_{K^+} \hat{f}(\xi)\psi(\eta\xi)d\xi = c \cdot f(\xi)$$

**Proof:** See the proof of theorem 2.2.2 in [**Ta67**]. ■

**Remark 3.2.15** As recalled in [**Ku04-1**] one property of the Schwartz-Bruhat functions is that the Fourier transform defines an automorphism of the Schwartz-Bruhat space.

By definition of Haar measure combined with the above theorem, we can normalize the measure such that $c = 1$.

**Definition 3.2.16 (Dual and self-dual measure)** *We define the dual measure* $d^*\xi$ *(of* $d\xi$ *with respect to* $\psi$*) as the Haar measure on* $K^+$ *such that it holds the inversion formula*

$$f(\xi) = \int_{K^+} \hat{f}(\xi)\psi(\eta\xi)d^*\xi$$

*for* $f \in L^1(K^+)$. *We say that the Haar measure on* $K^+$ *is self-dual with respect to* $\psi$ *if*

$$\hat{\hat{f}}(-\xi) = f(\xi)$$

*for each* $f \in L^1(K^+)$ *such that* $\hat{f} \in L^1(K^+)$*, i.e* $d\xi$ *coincides with its dual measure.*

#### 3.2.2.3 Self-dual measure on quaternion algebras

We can generalize the notion of self-dual measure in the setting of quaternion algebras. Let $B$ be a division quaternion algebra over the local field $K$. Attached to such quaternion algebra we have a natural form which is the reduced norm, $n : B \longmapsto K$ such that $n(b) = b\bar{b}$ (see section (1) for details and notations). Recall that we have the reduced trace on $B$, denoted with $t : B \longrightarrow K$ such that $t(b) = b + \bar{b}$.

Let $\tau$ be an additive character on $K$, $\tau : K \longrightarrow \mathbb{C}^\times$ and suppose that it is non-trivial; by paragraph (3.2.2.1) we know that a such $\tau$ exists. By Lemma (1.1.2) we know that $(b, b') \longmapsto \langle b, b' \rangle := t(bb')$ is a non-degenerate bilinear form on $B$ and we can consider the pairing

$$\langle \ , \ \rangle : B \times B \longrightarrow K$$

It is non-degenerative (as $t$ is so) and so we can identify $B$ with its dual, $B^* = Hom_K(B, K)$, via $B \ni x \longmapsto [\langle x, - \rangle : B \to K]$. Via this identification we can define the Fourier transform of a Schwartz-Bruhat function, explicitly, taken $\phi \in S(B)$, we have

$$\hat{\phi}(x) := \int_B \phi(y) \langle x, y \rangle dy$$

for $dy$ an Haar measure on $B$ such that it is normalized with $\hat{\hat{\phi}}(x) = \phi(-x)$; it is called *the self-dual Haar measure with respect to* $\langle \ , \ \rangle$.

**Note 3.2.17** In the case of a quaternion algebra $B$ which does not contain $\mathbb{R}$, we know that $S(B)$ is the space of locally constant function on $B$ with compact support. We can notice immediately that the Fourier transform of a Schwartz-Bruhat function is then a Schwartz-Bruhat function, as predicted by the remark in the previous paragraph.

Let now $\tau$ as chosen above and set $f(b) = \tau(n(b))$.

**Lemma 3.2.18** *There exists a constant $\gamma$, which depends on $\langle \ , \ \rangle$ and $K$, such that, for each $\phi \in S(B)$,*

$$\widehat{(\phi * f)}(x) = \gamma \ f^{-1}(\overline{x}) \hat{\phi}(x)$$

*for $\phi * f$ the convolution of $\phi$ and $f$. Furthermore $\gamma$ is an explicit factor, namely $\gamma = -1$.*

**Proof:** See [**JL70**], Ch. 1, Lemma 1.1 and Lemma 1.2. ■

### 3.2.3 Weil representations

#### 3.2.3.1 Representations associated with bilinear forms

Let $K$ be a local field and $G = GL_2(K)$. In this paragraph we are going to highlight the notion of Weil representation; this object is a representation of $G$ which is associated canonically to a quadratic form defined over $K$. For this purpose, let $B$ be the *"unique"* division quaternion algebra defined over $K$. Let $f$, $\tau$ and $\langle \cdot, \cdot \rangle$ as in paragraph (3.2.2.3) and set $|a|_B := \langle a, a \rangle^2$.

We can now consider the representation of $SL_2(K)$ in $S(B)$ such that

$$r : SL_2(K) \longrightarrow Aut\,(S(B))$$

it is defined by

$$(\star_{Weil}) \qquad \begin{cases} r\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\phi(x) = \tau\left(u\cdot n(x)\right)\phi(x); \\[2ex] r\left(\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix}\right)\phi(x) = |a|_B^{\frac{1}{2}}\phi(ax); \\[2ex] r\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\phi(x) = -\hat{\phi}(\overline{x}); \end{cases}$$

for $u \in K$, $a \in K^\times$ and $\phi \in S(B)$.

**Proposition 3.2.19** *The matrices*

$$\alpha(u) := \begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}, \qquad \beta(a) := \begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} \qquad \textit{and} \qquad s := \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

*for $a \in K^\times$ and $u \in K$, generate the group $SL_2(K)$ with the following relations*

*$(a)$: $s\cdot\beta(a) = \beta(a^{-1})\cdot s$; $\quad (b)$: $s^2 = -\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$; $\quad (c)$: $s\cdot\alpha(a)\cdot s = -\beta(a^{-1})\cdot\alpha(-a)\cdot s\cdot\alpha(-a^{-1})$;*

*together with the condition $\alpha(0) = \beta(1)$.*

**Proof:** See [**JL70**], Ch.1, pag. 7 and [**We64**]. ■

The above proposition guarantees that, if it exists, such representation $r$ is *unique*. In particular, Shalika and Tanaka proved in [**ST69**] that this representation indeed exists. We have hence associated a representation of $SL_2(K)$ to the couple $(B, n)$. During the construction of $r$ we have however choose a character $\tau$ and so it is correct to consider the association $(B, n, \tau) \longmapsto r$. Nevertheless, taken $a \in K^\times$, the representation $r_a$ obtained via $(B, n, \tau(ax))$ is related to $r$ with

$$(\star_a) \qquad r_a(x) = r\left(\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix} x \begin{pmatrix}a^{-1} & 0\\ 0 & 1\end{pmatrix}\right)$$

The above equation is easily verified checking that both sides satisfy the conditions $(\star_{Weil})$. Taken $b \in B^\times$, we can define two operators on $S(B)$ by

$$\lambda(b)\phi(x) := \phi(b^{-1}x) \qquad \text{and} \qquad \rho(b)\phi(x) := \phi(xb)$$

hence, if $a = n(b)$, by the definition of the above operators and the relation between $r_a$ and $r$, it holds that

$$r_a(x)\lambda(b^{-1}) = \lambda(b^{-1})r(x) \qquad \text{and} \qquad r_a(x)\rho(b) = \rho(b)r(x)$$

In particular, $r$ and $r_a$ are equivalent if $a \in n(B^\times)$ and this happens if $K \neq \mathbb{R}$ as in this case $n(B^\times) = K^\times$ (see section (1.4)). Then the association $(B, n) \longmapsto r$ is indeed well-defined (up to equivalence). Moreover, if $n(b) = 1$, by the above conditions, obviously $\rho(b)$ and $\lambda(b)$ commute with $r(x)$.

**Proposition 3.2.20** *If the space $S(B)$ is endowed with the* standard *topology (see 3.2.8) then the representation $r : S_2(K) \longrightarrow Aut(S(B))$ is continuous. Moreover, as $S(B) \subset L^2(B)$, $r$ extends to a unitary representation $r : SL_2(K) \longrightarrow Aut(L^2(B))$.*

**Proof:** See [**Ge75**] Ch. 7, Remark 7.1 and [**ST69**]. ■

We have determined, in a unique way, a representation (more precisely a class of representations) of $SL_2(K)$ associated with the quaternion algebra $B$. Using $r$ we have hence to construct a representation of $GL_2(K)$. For this purpose, define $G_+ = \{A \in GL_2(K) \mid det(A) \in n(B^\times)\}$: by what we have already seen we can deduce that, in our case, $G_+ = GL_2(K)$.

**Lemma 3.2.21** *The representation $r$ of $SL_2(K)$ on $S(V)$ extends to a representation of $GL_2(K)$, which, by abuse of notation we denote again with $r$, characterized by*

$$r : GL_2(K) \longrightarrow Aut\,(S(V)) \text{ such that } r\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi(x) = \phi(xh)$$

*with $a = n(h)$.*

**Proof:** See [**Ge75**], Ch.7, Lemma 7.3. ■

**Definition 3.2.22** *The above representation $r$, of $GL_2(K)$, is called the* Weil representation *associated with the quaternion algebra $B$.*

The main feature of this type of representations is the strict bound between them and representations on the multiplicative group $B^\times$. Let $\pi'$ be a finite-dimensional representation of $G' := B^\times$ on a complex vector space $H$. We can consider the tensor product of $r$ with the trivial representation of $SL_2(K)$ on $H$ obtaining a representation, which we call again $r$, on $S(B) \otimes_{\mathbb{C}} H$.

**Remark 3.2.23** We can think at $S(B) \otimes_{\mathbb{C}} H$ as the space of functions from $B$ to $H$ whose coordinate entries, for a fixed basis of $H$, are Schwartz-Bruhat functions on $B$.

The observation allows us to consider the subspace of $S(B) \otimes_{\mathbb{C}} H$ defined by the condition

$$\phi(xh) = \pi'(h^{-1})\phi(x) \text{ for all } h \in G' \text{ with } n(h) = 1.$$

By the equations $(\star_a)$ we can notice that this subspace is invariant under $r$ as

$$\pi'(h^{-1})r(y)\phi(x) = \rho(h)r(y)\phi(x) = r(y)\rho(h)\phi(x) = r(y)\phi(xh).$$

This invariance gives rise to a subrepresentation called $r_{\pi'}$.

**Remark 3.2.24** The restriction to the centre $Z(B^\times) = K^\times$ of the representation $\pi'$ is such that $\pi'(1) = 1_d$ the identity matrix with $d = dim_{\mathbb{C}}(H)$ and $\pi'(k) = \chi(a)1_d$ for $\chi$ a homomorphism of groups from $K^\times$ to $\mathbb{C}^\times$. This happens as image of an abelian group is an abelian group and so $\pi'(Z(B^\times))$ is contained in $Z(Aut(H)) \cong Z(GL_d(\mathbb{C})) = \mathbb{C}^\times 1_d$. Such $\chi$ is called the central character of $\pi'$.

**Proposition 3.2.25** *$r_{\pi'}$ extends to a representation of $GL_2(K)$ satisfying*

$$r_{\pi'}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right)\phi(x) = |h|_B^{\frac{1}{2}}\pi'(h)\phi(xh)$$

*for $a = n(h)$. Moreover, taken $\chi'_\pi$ the central character of $\pi'$, for each $a \in K^\times$,*

$$r_{\pi'}\left(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right) = \chi'_\pi(a)1_d$$

*Further, if $H$ is an Hilbert space and $\pi'$ is unitary, $r_{\pi'}$ can be extended to a unitary representation of $GL_2(K)$ in $L^2(B,\pi')$. Here $L^2(B,\pi')$ is the closure of $S(B,\pi') = S(B) \otimes_{\mathbb{C}} (\pi', H)$ in the Hilbert space of $L^2-$functions from $B$ to $H$.*

**Proof:** See [**Ge75**], Ch. 7. ■

By abuse of notation let $r_{\pi'}$ be the representation of $GL_2(K)$ in the above proposition. We have the following two fundamental theorems.

**Theorem 3.2.26** *Let $K$ be a non-archimedean local field and take $B$, the unique (up to isomorphism) quaternion division algebra on $K$. Then, for each irreducible unitary representation $\pi'$ of $B^\times$,*

*(I) The representation $r_{\pi'}$ decomposes as the direct sum of $d = dim(\pi')$ mutually equivalent irreducible representations $\pi(\pi')$ of $GL_2(K)$;*

*(II) Each $\pi(\pi')$ is supercuspidal if $d > 1$ and special if $d = 1$.*

*Furthermore,*

*(III) All the supercuspidal and special representation of $GL_2(K)$ are obtained via this construction. Explicitly, the association*

$$\pi' \longmapsto \pi(\pi')$$

*gives a bijection between the set of equivalence classes of finite-dimensional representation of $B^\times$ and the set of equivalence classes of special and supercuspidal representations of $GL_2(K)$.*

*If $K = \mathbb{R}$ and $B$ is the quaternion algebra isomorphic to the Hamilton's quaternions, the association*

$$\pi' \longmapsto \pi(\pi')$$

*defines a bijection between the set of equivalence classes of finite-dimensional representation of $B^\times$ and the set of equivalence classes of discrete series representations of $GL_2(K)$.*

**Proof:** See theorem 7.6 in [**Ge75**], remark 7.7 and following. ■

### 3.2.4 The correspondence

We are now prepared to state the most important correspondence between representations on $GL_2$ and representations on a division quaternion algebra. The notions in section (3.1) generalize to the case of number fields (as noticed in section (3.1.5)) and hence we have the

**Theorem 3.2.27 (Jacquet-Langlands Correspondence)** *Let $B$ be a division quaternion algebra over the number field $K$, $S$ the set of ramified places in $B$ and $G' := D^\times$ (thought as an algebraic group). To each admissible irreducible unitary representation $\pi' = \bigotimes_v \pi'_v$ of $G'_{\mathbb{A}_K}$, let $\pi$ be the representation of $G_{\mathbb{A}_K} = GL_2(\mathbb{A}_K)$ such that*

$$\pi_v \cong \begin{cases} \pi'_v & \text{if } v \notin S \\ \pi_v(\pi'_v) & \text{if } v \in S \end{cases}$$

*where $\pi_v(\pi'_v)$ denotes, as in section (3.2.3.1), the irreducible component of the Weil representation $r(B_v)$ induced by $\pi_v$. Then we have*

*(I) $\pi = \otimes_v \pi_v$ is a cusp form for $G_{\mathbb{A}_K}$ if $\pi'$ is a $d-$dimensional cusp forms for $G'_{\mathbb{A}_K}$, for $d > 1$;*

*(II) the association*

$$\pi' \longmapsto \pi$$

*restricted to the collection of $d-$dimensional, with $d > 1$, cusp forms on $G'_{\mathbb{A}_K}$ is a bijection onto the collection of (all equivalence classes of) cusp form $\bigotimes_v \pi_v$ on $G_{\mathbb{A}_K}$, such that $\pi_v$ is square-integrable for each place $v \in S$.*

**Proof:** See [**Ge75**], Theorem 10.5. ■

## 3.3 Modular forms on quaternion algebras

Considering all the correspondences expressed in the previous sections one can restate the Jacquet–Langlands conjecture in the special case of modular forms on quaternion algebras. We refer to [**BD07**] for this section in which we introduce the basic notions about modular forms on quaternion algebras.

Let $N$ be a positive integer and suppose that such $N$ can be written as the product $N = pN^+N^-$, where $p$ is a prime, $N^-$ is a square-free product of an odd number of primes and $p$, $N^+$ and $N^-$ are relatively prime integers. As stated in the previous sections there exist a unique (up to isomorphism) quaternion algebra over $\mathbb{Q}$, definite and ramified exactly at the primes dividing $N^-$. Call such quaternion algebra $B$ and recall that $B_\infty := B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$, $B_l := B \otimes_{\mathbb{Q}} \mathbb{Q}_l \cong M_2(\mathbb{Q}_l)$ for each prime $l$ which does not divide $N^-$ and $B_l$ is isomorphic to the (unique up to isomorphism) quaternion division algebra over $\mathbb{Q}_l$ if the prime $l$ divides $N^-$.

Let $\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z} \cong \prod_{l \text{ prime}} \mathbb{Z}_l$ be the profinite completion of $\mathbb{Z}$ and define, for any (commutative unitary) ring $A$,

$$\hat{A} = A \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}.$$

With this notation we have $\hat{\mathbb{Q}} = \mathbb{A}_f$ the ring of finite adèles of $\mathbb{Q}$. We can think to $B^\times$ as an algebraic group over $\mathbb{Q}$ (see section (2.1)) defining

$$B^\times(L) = (B \otimes_{\mathbb{Q}} L)^\times$$

for any $\mathbb{Q}-$algebra $L$. Once for all fix the following notation:

$$\hat{B}^\times = B^\times(\hat{\mathbb{Q}}) = B^\times(\mathbb{A}_f) \subset \prod_l B_l^\times$$

for the group of adèlic points of $B^\times$ and, taken $b \in \hat{B}^\times$, denote its $p$-component with $b_p \in B_p^\times$. For each $p$ which does not divide $N^-$ we have an isomorphism of $\mathbb{Q}_p-$algebras

$$\iota_p : B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p \longrightarrow M_2(\mathbb{Q}_p)$$

which restricts to an isomorphism, which we call again $\iota_p$ by abuse of notation, between

$$\iota_p : B_p^\times = (B \otimes_{\mathbb{Q}} \mathbb{Q}_p)^\times \longrightarrow GL_2(\mathbb{Q}_p)$$

Analogously to the arguments showed in section 1.4, we can endow $\hat{B}^\times$ with the topology induced by $\hat{\mathbb{Q}}$, still obtaining a locally compact topological group. Let $\Sigma = \prod_l \Sigma_l$ be a compact open subgroup of $\hat{B}^\times$. Let $A$ be a $\mathbb{Q}_p-$vector space (or sometimes a $\mathbb{Z}_p-$module) such that the semigroup of matrices in $M_2(\mathbb{Z}_p)$ with non-zero determinant acts on the left, linearly, on $A$.

**Definition 3.3.1** *An $A-$valued modular form on $B^\times$ of level $\Sigma$ is a function*

$$\phi : \hat{B}^\times \longrightarrow A$$

*satisfying*

$$\phi(gb\sigma) = \iota_p(\sigma_p^{-1}) \cdot \phi(b),$$

*for all $g \in B^\times$, $b \in \hat{B}^\times$ and $\sigma \in \Sigma$. Let $S(\Sigma; A)$ be the space of such modular forms.*

**Remark 3.3.2** First of all, we should notice that the space $\mathbb{X}_\Sigma$ is finite. In fact, it is compact as $B^\times \backslash \hat{B}^\times$ is compact. By continuity, the image of $\Sigma$, as a subgroup of $B^\times \backslash \hat{B}^\times$, is still open and so it has finite index as the quotient is compact. Hence $\mathbb{X}_\Sigma$ has to be finite.

The second remark is the following. Giving an element of $S(\Sigma; A)$ is equivalent to give the set of its values on a set of representative of the finite double coset space

$$\mathbb{X}_\Sigma := B^\times \backslash \hat{B}^\times / \Sigma.$$

Obviously if $A$ is finite-dimensional so must be $S(\Sigma; A)$.

We consider two cases for the module $A$, namely

- $A = \mathbb{Z}_p$ with the trivial action of the semigroup. The $\mathbb{Z}_p$−valued modular forms of level $\Sigma$ are said to be of *weight* 2 and the $\mathbb{Z}_p$−module of such modular forms denoted by

$$S_2(\Sigma) := S(\Sigma; \mathbb{Z}_p).$$

- Let $\mathcal{P}_k(\mathbb{Q}_p)$ denote the space of homogeneous polynomials in two variables of degree $k-2$ with coefficients in $\mathbb{Q}_p$. It is equipped with a right action of $GL_2(\mathbb{Q}_p)$ with the usual rule

$$(P|_\gamma)(x, y) := P(ax + by, cx + dy), \text{ for } \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

  We take

$$V_k(\mathbb{Q}_p) := \mathrm{Hom}\left(\mathcal{P}_k(qp), \mathbb{Q}_p\right)$$

  with induced left-action

$$(\gamma v)(P) = v(P|_\gamma).$$

  The $V_k(\mathbb{Q}_p)$−valued modular forms of level $\Sigma$ are said to be of *weight* $k$ and the $\mathbb{Q}_p$−module of such modular forms is denoted as

$$S_k(\Sigma) := S(\Sigma; V_k(\mathbb{Q}_p)).$$

We should fix the open compact subgroup $\Sigma$. For this purpose let $R$ be a maximal order of $B$ such that

$$\iota_p(R \otimes_{\mathbb{Z}} \mathbb{Z}_p) = M_2(\mathbb{Z}_p).$$

**Remark 3.3.3** As the quaternion algebra is definite it does not satisfy the Eichler condition and so one can notice, as done in [**BD07**], §2, that the order $R$ is not unique, even up to conjugation.

For each prime $l$ not dividing $N^-$, $R \otimes \mathbb{Z}_p$ is isomorphic to $M_2(\mathbb{Z}_p)$ and we can fix the isomorphism $\iota_l$ as defined above, such that $\iota_l(R \otimes \mathbb{Z}_l) = M_2(\mathbb{Z}_l)$. We define hence

$$\Sigma_0(N^+, N^-) = \prod_l \Sigma_l \quad \text{such that} \quad \Sigma_l = \begin{cases} (R \otimes \mathbb{Z}_p)^\times & \text{if } l | N^- \\ \iota_l^{-1}\left(\Gamma_0(N^+ Z_l)\right) & \text{otherwise} \end{cases}$$

where

$$\Gamma_0(nZ_p) = \left\{ \gamma \in GL_2(\mathbb{Z}_p) \;\middle|\; \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod n\mathbb{Z}_p \right\}.$$

We can consider the spaces

$$S\left(\Sigma_0(pN^+, N^-); A\right)$$

which are endowed with an action of Hecke operators $T_l$ for $l$ not dividing $N$ (e.g. via Brandt matrices as in [**G87**]; see also (4.2.3)). We can give explicit formulas in the case of weight

$k$. In particular, the Hecke operator $T_l$ on $S_k(\Sigma_0(N^+, N^-))$ is defined via the double coset decomposition

$$GL_2(\mathbb{Z}_l) \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} GL_2(\mathbb{Z}_l) = \coprod_{n=0}^{l} \sigma_n(l) GL_2(\mathbb{Z}_l)$$

as

$$(T_l \phi)(b)(P) = \sum_{n=0}^{l} \phi(b\sigma_n(l))(P|_{\sigma_n(l)}).$$

We can also define the Hecke operator $U_p$ characterized by the double coset decomposition

$$\Gamma_0(p\mathbb{Z}_p) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma_0(p\mathbb{Z}_p) = \coprod_{n=0}^{p-1} \sigma_n \Gamma_0(p\mathbb{Z}_p)$$

and so, on $S_k(\Sigma_0(N^+, N^-))$, we have

$$(U_p \phi)(b)(P) = \sum_{n=0}^{p-1} \phi(b\sigma_n)(P|_{\sigma_n}).$$

We are now able to state a new Jacquet–Langlands correspondence. We refer to [**BD07**], §2.3 for the necessary references.

Take $S_k(\Gamma_0(N)) = S_k(\Gamma_0(N), \mathbb{Q}_p)$ the space of classical modular form of weight $k$, level $\Gamma_0(N)$ and coefficient in $\mathbb{Q}_p$ (i.e. $S_k(\Gamma_0(N)) = S_k(\Gamma_0(N), \mathbb{Q}) \otimes \mathbb{Q}_p$). Suppose, as above that $N$ can be written as the product $N = pN^+N^-$ with $p$ prime, $N^-$ is a square-free product of an odd number of primes and $p$, $N^+$ and $N^-$ are coprime. Recall that a modular form in $S_k(\Gamma_0(N))$ is *old at* $N^-$ if it can be written as sum of $g(dz)$ for $g(z) \in S_k(\Gamma_0(M))$ for $M$ not divisible by $N^-$ and $d \in \mathbb{Z} \smallsetminus \{0\}$. We have hence a subspace $S_k^{N^- -old}(\Gamma_0(N))$ of the $N^-$−old forms and we define the orthogonal complement of this space as the space of $N^-$−new form on $\Gamma_0(N)$; denote this latter space with $S_k^{N^- -new}(\Gamma_0(N))$. With the above notation we have the

**Theorem 3.3.4 (Jacquet–Langlands: modular forms)** *There exist Hecke–equivariant isomorphisms*

$$S_k(\Sigma_0(N^+, N^-)) \xrightarrow{\sim} S_k^{N^- -new}(\Gamma_0(N^+N^-)),$$
$$S_k(\Sigma_0(pN^+, N^-)) \xrightarrow{\sim} S_k^{N^- -new}(\Gamma_0(N)).$$

# 4 The Gross–Kudla formula

## 4.1 The Atkin–Lehner involution

Let $N$ be a positive integer and consider the congruence group $\Gamma_0(N)$.

**Definition 4.1.1** *We denote by $w_Q$ any matrices in $GL_2(\mathbb{Z})^+$ such that*

$$w_Q = \begin{pmatrix} Qx & y \\ Nz & Qw \end{pmatrix} \qquad \text{and} \qquad det(w_Q) = Q$$

*for each $Q|N$ and $(Q, N/Q) = 1$ (i.e. $Q||N$). If $q$ is a prime dividing $N$, by abuse of notation we write $w_q$ for $w_{q^\alpha}$ for $q^\alpha||N$ ($\alpha \geq 1$).*

**Remark 4.1.2**
- The condition $(Q, N/Q) = 1$ is redundant as if $(Q, N/Q) = d > 1$, then $Qd|Q^2xw - Nzy = det(w_Q) = Q$, which is impossible.

- One can prove that the product of two matrices $w_{Q_1}$ and $w_{Q_2}$ is a matrix of the form $w_{Q_3}$ with $Q_3$ the least common multiple of $Q_1$ and $Q_2$.

**Lemma 4.1.3 (Lemma 17 [AL87])** *With the notation as in the above definition. Let $f \in S_k(\Gamma_0(N))$, then $f|w_q \in S_k(\Gamma_0(N))$, i.e. the $k-$slash operator with $w_q$ defines an endomorphism of $S_k(\Gamma_0(N))$. Moreover the operator is independent of the choice of $x, y, z$ and $w$ in the definition of the matrix $w_q$.*

Combining the remark and the lemma we deduce that $|w_q$ give rise to an involution on the space of cusp forms.

**Definition 4.1.4** *We define the* Atkin–Lehner involution *as the operator $|w_p$ (or simply $w_p$) for $p$ dividing $N$.*

Furthermore the following theorem holds:

**Theorem 4.1.5** *Let $f(\tau)$ be a newform on $\Gamma_0(N)$ of weight $k$, $p$ a prime dividing $N$ and $w_p$ the corresponding matrix. If $f(\tau) = q + \sum_{n=2}^{+\infty} a_n q^n$, then*

$$f|w_p = \varepsilon_p(f) \cdot f, \text{ where } \varepsilon_p(f) = \pm 1.$$

*Further, if $p$ exactly divides $N$ then $\varepsilon_p(f) = -p^{1-\frac{k}{2}} a_p$*

**Proof:** See [**AL87**], Theorem 3, $(iii)$ (remembering that their weight $k$ is half of our definition of weight). ■

**Remark 4.1.6** *If $k = 2$ then we have that, for each prime $p||N$, $a_p = \pm 1$.*

We can hence notice that the operator $|w_p$ determines a decomposition of $S_k(\Gamma_0(N))^{new}$ in the two eigenspaces corresponding to $\lambda_p = \pm 1$.

## 4.2 The main formula

### 4.2.1 The hypotheses

Let $N \in \mathbb{N}_{\geq 1}$ be a square-free integer and $f$, $g$ and $h$ three cusps forms of weight 2 on $\Gamma_0(N)$. We suppose that $f$, $g$ and $h$ are all normalized eigenforms for the Hecke algebra, and are all newforms of level N. The function $F(z_1, z_2, z_3) = f(z_1)g(z_2)h(z_3)$ is then a newform of weight $(2, 2, 2)$ for $\Gamma_0(N)^3$. Assume also that the Fourier expansions of these newforms are given by

$$f(\tau) = \sum_{n=1}^{+\infty} a_n q^n \qquad g(\tau) = \sum_{n=1}^{+\infty} b_n q^n \qquad h(\tau) = \sum_{n=1}^{+\infty} c_n q^n$$

with $a_1 = b_1 = c_1 = 1$ and, as usual, $q = \exp(2\pi i \tau)$. For a prime $p$ dividing $N$ we define $\varepsilon_p = -a_p b_p c_p$ and we have an involution $u_p = w_p \times w_p \times w_p$ on the space of forms of weight $(2, 2, 2)$ where $w_p$ is the Atkin–Lehner involution on the space of forms on $\Gamma_0(N)$. By theorem (4.1.5) we know that $F|u_p = \varepsilon_p \cdot F$ holds, for all $p|N$, as it holds that $f|w_p = -a_p \cdot f$, $g|w_p = -b_p \cdot g$ and $h|w_p = -c_p \cdot h$. Moreover, since $N$ is square-free, the coefficients $a_p$, $b_p$ and $c_p$ are equal to $\pm 1$. So each $\varepsilon_p$ equals $\pm 1$ as well as $\varepsilon$, which is defined as $\varepsilon = -\prod_{p|N} \varepsilon_p$. For each prime $l$ which does not divide $N$, we can factor the polynomials

$$1 - a_l X + l X^2 = (1 - \alpha_l X)\left(1 - \alpha_l' X\right)$$
$$1 - b_l X + l X^2 = (1 - \beta_l X)\left(1 - \beta_l' X\right)$$
$$1 - c_l X + l X^2 = (1 - \gamma_l X)\left(1 - \gamma_l' X\right)$$

**Definition 4.2.1** *The triple product L-function*

$$L(f \otimes g \otimes h,\ s) = L(F,\ s)$$

*is the function defined by the convergent Euler product*

$$L(F,\ s) = \prod_{l \nmid N} L_l(F,\ s) \cdot \prod_{p \mid N} L_p(F,\ s)$$

*where*

$$\begin{aligned} L_l(F,\ s) &= \left(1 - \alpha_l\beta_l\gamma_l \cdot l^{-s}\right)^{-1} \cdot \left(1 - \alpha_l\beta_l\gamma_l' \cdot l^{-s}\right)^{-1} \cdots \left(1 - \alpha_l'\beta_l'\gamma_l' \cdot l^{-s}\right)^{-1} \\ L_p(F,\ s) &= \left(1 - a_pb_pc_p \cdot p^{-s}\right)^{-2} \cdot \left(1 - a_pb_pc_p \cdot p^{1-s}\right)^{-2} \end{aligned}$$

**Remark 4.2.2** Each $L_l(F,\ s)$ has degree 8 with respect to $l^{-s}$ and each *bad Euler factor* $L_p(F,\ s)$ has degree 3 with respect to $p^{-s}$. Then, by comparison with the $\zeta\left(s - \frac{3}{2}\right)$ (with the usual techniques of analytic number theory) it is possible deducing that the Euler product converges absolutely for $Re(s) > \frac{5}{2}$ i.e. *the triple product L-function is defined in the half plane* $Re(s) > \frac{5}{2}$.

We can define as well the archimedean $L-$factor as (See [**GK92**])

$$L_\infty(F, s) = (2\pi)^{3-4s}\,\Gamma(s)\Gamma(s-1)^3$$

and hence define, for $Re(s) > 5/2$, the function

$$\Lambda(F, s) = L_\infty(F, s)L(F, s)$$

**Proposition 4.2.3** *The function $\Lambda(F, s)$ has an analytic continuation to the whole $s-$plane and satisfies the functional equation*

$$\Lambda(F, s) = \varepsilon \cdot N^{10-5s} \cdot \Lambda(F, s-4)$$

**Proof:** See [**GK92**], Prop. 1.1 and § 7. ■

From now on we assume that the sign in the functional equation is +1 i.e $\prod_{p|N} \varepsilon_p = -1$ which is equivalent to say that $\#\{p \mid p \text{ divides } N,\ \varepsilon_p = -1\}$ is odd.

### 4.2.2 Curves and orders in quaternion algebras

Let $B$ the unique (up to isomorphism) quaternion algebra over $\mathbb{Q}$, ramified at the even set

$$S = \{p \mid p \text{ divides } N,\ \varepsilon_p = -1\} \cup \{\infty\}$$

and let $R$ be an Eichler order in $B$ with reduced discriminant $N$ and level $L = N/D$ where $D$ is the discriminant of $B$ i.e. $D = \prod_{p \in S \setminus \{\infty\}} p$. Such Eichler order is unique up to local conjugation. In particular, for $p \in S$, $R_p = R \otimes \mathbb{Z}_p$ is the unique maximal order in the local division algebra $B_p = B \otimes \mathbb{Q}_p$. Instead, for $p \notin S$, $R_p$ is conjugate to

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \;\middle|\; c \equiv 0 \pmod{N\mathbb{Z}_p} \right\}$$

in the matrix algebra $B_p \cong M_2(\mathbb{Q}_p)$. Let hence $\hat{R} = R \otimes \hat{\mathbb{Z}}$ and $\hat{B} = B \otimes \hat{\mathbb{Z}} = B \otimes_{\mathbb{Q}} \mathbb{A}_f$, where $\mathbb{A}_f$ is the ring of finite Adèles over $\mathbb{Q}$. Let $n$ be the class number of $R$ and let $\{I_1, \ldots, I_n\}$ be a set of ideal representing the ideal group, with $I_1 = R$. For $0 \leq i \leq n$ let $R_i$ the right order of the ideal $I_i$ and define (the setting is that of theorem (1.7.2) with base field $\mathbb{Q}$) the groups $\Gamma_i = R_i^\times/\mathbb{Z}^\times = R_i^\times/(\pm 1)$. It can be proved (See [**G87**], §1) that each $\Gamma_i$ is finite and hence we can define $w_i = \#\Gamma_i$. Recall that these integers are independent of the choice of $R$ as is the choice of the set of representative $\{I_i\}$. Gross proved (See [**G87**], § 3) that to a quaternion algebra $B$ and an Eichler order $R$ of $B$, as above, can be associated a curve $Y$ over $\mathbb{Q}$ of genus zero, endowed with a right action of the group $B^\times/\mathbb{Q}^\times$. Moreover he showed that exist a curve $X$ defined as the double coset space

$$X = \left(\left(\hat{R}^\times \backslash \hat{B}^\times\right) \times Y\right) \Big/ B^\times$$

which is the disjoint union of $n$ curves of genus zero over $\mathbb{Q}$. Indeed (See [**G87**], § 3) $X$ can be written as

$$X \cong \coprod_{i=1}^{n} Y/\Gamma_i.$$

Let $Pic(X)$ be the free abelian group of rank $n$ of isomorphism classes of divisors on $X$. This has, as basis, the elements $\{e_1, \ldots, e_n\}$ where $e_i$ has degree 1 on the component $X_i = Y/\Gamma_i$ and degree 0 on $X_j$ for $j \neq i$. We define $P = Pic(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{i=1}^n \mathbb{Q}e_i$.

### 4.2.3 Brandt matrices and the height pairing

Let $1 \leq l, j \leq n$ fixed. With the notation as above, we define the product $M_{l,j} = I_j^{-1} I_l = \{\sum a_k b_k \mid a_k \in I_j^{-1},\ b_k \in I_l\}$ which is a left ideal in $R_j$ with right order $R_l$ (see lemma (1.1.16)). We set $n(b)$ as the reduced norm of $b \in M_{l,j}$ and $n(M_{l,j})$ as the unique positive rational number such that the quotients $n(b)/n(M_{l,j})$ are integers without common factors (recall that an ideal is a lattice). Define the theta series $\theta_{l,j}$ as

$$\theta_{l,j} = \frac{1}{2w_j} \sum_{b \in M_{l,j}} e^{2\pi i\left(n(b)/n(M_{l,j})\right)\tau} = \sum_{m \geq 0} B_{l,j}(m) q^m \qquad \text{where } q = e^{2\pi i \tau}$$

These functions (on the upper half plane) are modular forms of weight 2 for $\Gamma_0(N)$ (e.g. those are theta functions associated with a lattice). Their Fourier coefficients are given by the entries of the Brandt matrices of degree $m$ i.e.

$$B(m) = (B_{i,j}(m))_{1 \leq i,j \leq n}$$

We can notice that

$$B(0) = \frac{1}{2}\begin{pmatrix} \frac{1}{w_1} & \frac{1}{w_2} & \cdots & \frac{1}{w_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{w_1} & \frac{1}{w_2} & \cdots & \frac{1}{w_n} \end{pmatrix} \qquad B(1) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$$

and for $m \geq 1$, it can be proved that $B(m)$ has non-negative integer entries.

**Note 4.2.4** Pizer showed that the Brandt matrices depend only on the level and not on the choice of the order (See [**P80**], Prop. 2.17).

**Proposition 4.2.5** *For all $m \geq 1$ and $i = 1, \ldots, n$ we define on $Pic(X)$ the Hecke correspondence*

$$t_m(e_i) = \sum_{j=1}^{n} B_{i,j}(m) e_j$$

*i.e. on the basis $\{e_i\}_{i=1}^n$ of $Pic(X)$ we define the action of $t_m$ by multiplication with the transpose $B(m)^{\mathrm{t}}$ of the $m-th$ Brandt matrix.*

**Proof:** See [**G87**], Proposition 4.4. ■

It's possible defining a height pairing $\langle\ ,\ \rangle$ on $Pic(X)$, with values in $\mathbb{Z}$, as follows:

$$\langle e_i, e_j\rangle = 0 \text{ if } i \neq j \text{ and } \langle e_i, e_i\rangle = w_i$$

and then extended bi-additively, namely, if $e = \sum a_i e_i$ and $e' = \sum b_i e_i$ are two divisor classes , then $\langle e, e'\rangle = \sum_{i=1}^{n} a_i b_i w_i$. This pairing is positive definite and moreover the next proposition holds.

**Proposition 4.2.6** *For all classes e and e′ in $Pic(X)$,*

$$\langle t_m(e), e'\rangle = \langle e, t_m(e')\rangle$$

**Proof:** See [**G87**], Proposition 4.6. ■

We can extend all these results to $Pic(X)\otimes\mathbb{Q}$ and it can be proved (See [**GK92**], §10 and [**G87**], §4) that, for each prime $l$ which doesn't divide $N$, the operators $t_l$ commute with each other and are self-adjoint with respect to the height pairing. They may therefore be simultaneously diagonalized on $P_{\mathbb{R}} = Pic(X) \otimes \mathbb{R}$. Further we have the following

**Proposition 4.2.7** *If $f = \sum a_n(f)q^n$ is a cusp form of weight 2 for $\Gamma_0(N)$, there exists a unique line $\langle a_f\rangle$ in $P_{\mathbb{R}}$ such that*

$$t_l(a_f) = a_l(f)\cdot a_f \quad \textit{for all primes } l \not| N$$

**Proof:** See [**GK92**], Proposition 10.2. ■

**Remark 4.2.8** *This proposition establishes a correspondence between cusp forms and line in $P_{\mathbb{R}}$, preserving the set of eigenvalues.*

In terms of our choice for a basis for $Pic(X)$ we can write $a_f = \sum_{i=1}^{n} \lambda_i(f) e_i$ where $\lambda_i(f) \in \mathbb{Q}(f)$ are algebraic and uniquely determined up to a scalar.

### 4.2.4 The formula

In [**GK92**], Gross and Kudla showed that there exists a precise relation between the special value of the $L-$function at the critical point $s = 2$ and all the object we have introduced. More precisely they proved the following

**Theorem 4.2.9 (Corollary 11.3, [GK92])** *With the notation as above, it holds that*

$$L\left(f\otimes g\otimes h, 2\right) = \frac{||\omega_f||^2||\omega_g||^2||\omega_h||^2}{2\pi N 2^t}\frac{\left(\sum_{i=1}^{n} w_i^2\lambda_i(f)\lambda_i(g)\lambda_i(h)\right)^2}{\sum_{i=1}^{n} w_i\lambda_i(f)^2\cdot\sum_{i=1}^{n} w_i\lambda_i(g)^2\cdot\sum_{i=1}^{n} w_i\lambda_i(h)^2}$$

*where $t = \#\{p : p|N\}$ and $\omega_f = 2\pi i f(z_1)dz_1$ (with $\omega_g$ and $\omega_h$ having a similar definition). More precisely we can normalize the Petersson inner product as*

$$\langle f_1, f_2\rangle_{Pet} = 2^3\pi^2 \int\limits_{\Gamma_0(N)\backslash\mathfrak{H}} f_1(z)\overline{f_2(z)}dxdy \quad \textit{with} \quad z = x + iy$$

*and so*

$$||\omega_f||^2 = 2^3\pi^2 \int\limits_{\Gamma_0(N)\backslash\mathfrak{H}} |f(z)|^2 dxdy$$

*with analogous expressions for $||\omega_g||^2$ and $||\omega_h||^2$.*

**Proof:** See [**GK92**]. ■

## 4.3 Examples

We want now discuss the case $g = f$, $h = f$. Let $f$, as before, be a new cuspidal eigenform for $\Gamma_0(N)$ and suppose moreover that $f = \sum a_n q^n$ has integral Fourier coefficients with $a_1(f) = 1$. With the notations introduced in this section we can compute the algebraic part

$$A(F) = \frac{\left(\sum_{i=1}^{n} w_i^2 \lambda_i(f)^3\right)^2}{\left(\sum_{i=1}^{n} w_i \lambda_i(f)^2\right)^3}$$

of the special value of the $L-$function. We focused on the case of conductor $N = 37$ and $N = 15$.

### 4.3.1 Case $N = 37$

In this case the conductor $N = 37$ is prime so $\varepsilon = -\varepsilon_{37}$ and $t = \#\{p \mid p \text{ divides } N\} = \#\{37\} = 1$. Since we require that $\varepsilon = +1$, follows that $\varepsilon_{37} = -1$. By definition of $\varepsilon$ the quaternion algebra over $\mathbb{Q}$, $B$, is ramified at $S = \{\infty\} \cup \{p | \varepsilon_p = -1\} = \{\infty, 37\}$ i.e. its discriminant is $D = 37$. Since $37 \equiv 1 \pmod 4$ and $37 \equiv 5 \pmod 8$, by Proposition (1.6.3), up to isomorphism, $B$ is defined as the quaternion algebra given by $\{a, b\} = \{-2, -37\}$. Let $R$ be an Eichler order in $B$ of reduced discriminant $N$ and so level $L = N/D = 37/37 = 1$ (and hence a maximal one). As $37 \equiv 1 \pmod{12}$, by Note (1.7.8), we deduce that the class number $n = \frac{37-1}{12} = 3$. From Note (1.7.8) we also have that $W = w_1 w_2 w_3 = 1$ (as $\frac{N-1}{12} \in \mathbb{Z}$) hence we deduce that $w_1 = w_2 = w_3 = 1$.

**Remark 4.3.1** One can deduce (in an algorithmic way) the value of each $w_i$ via the structure of $B(0)$ as in §(4.2.3). From [**P80**] we know that

$$B(0) = \frac{1}{2}\begin{pmatrix} 1/w_1 & 1/w_2 & 1/w_3 \\ 1/w_1 & 1/w_2 & 1/w_3 \\ 1/w_1 & 1/w_2 & 1/w_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and so we deduce that each $w_i = 1$.

**Note 4.3.2** Without using $B(0)$ to determine the $w_i$ we have to associate each $w_i$ to the correspondent element of the basis of $Pic(X)$ and hence to the correspondent $\lambda_i(f)$. We can do it using the linear relations between the $w_i$ given by Prop. (4.2.6), i.e. from the fact that each operator $t_m$ is self-adjoint with respect to the height pairing. Anyway in this case, since all the $w_i$ are equal to 1, there is nothing to do.

At this point we can compute a basis of eigenforms for the space $S_2\left(\Gamma_0(37)\right)^{new}$. Moreover we are looking for forms with integer coefficients of their Fourier $q-$expansion. By [**Ste12**] we know that

$$\begin{aligned} f_1 &= q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + 4q^{10} - 5q^{11} + \cdots - q^{37} + O(q^{38}) \\ f_2 &= q + q^3 - 2q^4 - q^7 - 2q^9 + 3q^{11} + \cdots + q^{37} + O(q^{38}) \end{aligned}$$

define a basis of eigenforms for $S_2\left(\Gamma_0(37)\right)^{new}$. Since $1 = -\varepsilon_{37} = a_{37}(f)$ we deduce that $f = \sum a_n(f) q^n = f_2$. We have now to compute the eigenfunction $a_f \in Pic(X) \otimes_{\mathbb{Z}} \mathbb{R}$. By Prop. (4.2.5) and Prop. (4.2.7) we know that it suffices compute the eigenvectors and eigenvalues of the transpose matrix of the Brandt matrix at $p$, where $p$ is prime and does not divide $N$, associated with a maximal Eichler order (and so of level 1). In [**P80**] have been computed the Brandt matrices for an Eichler order of level 1 which are

$$B(2) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \quad B(3) = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 3 & 0 \end{pmatrix} \quad B(5) = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad B(7) = \begin{pmatrix} 2 & 3 & 3 \\ 3 & 2 & 3 \\ 3 & 3 & 2 \end{pmatrix}$$

Computing the diagonalization of $B(2)^{\mathrm{t}} = B(2)$ and $B(3)^{\mathrm{t}} = B(3)$ we obtain

$$P_2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1/2 & 1 \\ 1 & -1/2 & -1 \end{pmatrix} \qquad D_2 = \begin{pmatrix} 3 & & \\ & 0 & \\ & & -2 \end{pmatrix} = P_2^{-1} \cdot B(2) \cdot P_2$$

$$P_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1/2 & 1 \\ 1 & -1/2 & -1 \end{pmatrix} \qquad D_3 = \begin{pmatrix} 4 & & \\ & 1 & \\ & & -3 \end{pmatrix} = P_3^{-1} \cdot B(3) \cdot P_3$$

We can notice that $a_2(f) = 0 = D_2(2,2)$ and $a_3(f) = 1 = D_3(2,2)$. Then, since $a_f$ is, up to scalar, the unique vector (written with respect to the standard basis) such that $B(m)^{\mathrm{t}} \cdot a_f = a_m(f) \cdot a_f$, we can take $a_f$ with integer coefficients. Moreover we can choose $a_f$ as an indivisible element in $Pic(X) \otimes \mathbb{Q}$, i.e., if $a_f = (\lambda_1(f), \lambda_2(f), \lambda_3(f))^{\mathrm{t}}$, this means that $gcd\,(\lambda_1(f), \lambda_2(f), \lambda_3(f)) = 1$. From the uniqueness follows that

$$\langle a_f \rangle = \langle \begin{pmatrix} 1 \\ -1/2 \\ -1/2 \end{pmatrix} \rangle \quad \text{and, up to scalar, we choose} \quad a_f = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

hence $\lambda_1(f) = 2$, $\lambda_2(f) = -1$ and $\lambda_3(f) = -1$.

All the information we have found are collected in the following table.

| $N$ | $\varepsilon_p$ | $n = rk_{\mathbb{Z}}\,(Pic(X))$ | $\lambda_i(f)$ | $w_i$ |
|---|---|---|---|---|
| 37 | $\varepsilon_{37} = -1$ | 3 | 2, 1, 1 | 1, 1, 1 |

Hence we have

$$M_3 = \left( \sum_{i=1}^{3} w_i^2 \lambda_i(f)^3 \right) = 2^3 - 1 - 1 = 6 \qquad M_2 = \left( \sum_{i=1}^{3} w_i \lambda_i(f)^2 \right) = 2^2 + 1 + 1 = 6$$

and so

$$A(F) = \frac{M_3^2}{M_2^3} = \frac{6^2}{6^3} = \frac{1}{6}.$$

### 4.3.2 Case $N = 15$

We have now the case of conductor $N = 15$ which is not a prime number. Hence $\varepsilon = -\varepsilon_3 \varepsilon_5$ and $t = \#\{p \mid p \text{ divides } N\} = 2$. As $\varepsilon = +1$, it follows that either $\varepsilon_3$ or $\varepsilon_5$ can equal $-1$ and so the quaternion algebra over $\mathbb{Q}$, $B$, is ramified at $S = \{\infty\} \cup \{p | \varepsilon_p = -1\}$ with $\#S = 2$. For determining such $p$ we have to look at a basis of eigenfunctions for $S_2(\Gamma_0(15))^{new}$, which, by [**Ste12**], is given by

$$f_1 = q - q^2 - q^3 - q^4 + q^5 + q^6 + 3q^8 + q^9 - q^{10} - 4q^{11} + q^{12} - 2q^{13} - q^{15} + O(q^{16})$$

then, as the space has dimension one, $f = f_1$. Moreover

$$\varepsilon_3 = -a_3(f) = 1 \qquad \text{and} \qquad \varepsilon_5 = -a_5(f) = -1$$

It follows that $B$ is ramified at $S = \{\infty, 5\}$ with discriminant $D = 5$. Since $5 \equiv 1 \pmod 4$ and $5 \equiv 5 \pmod 8$, by Prop. (1.6.3), then, up to isomorphism, $B$ is defined as the quaternion algebra given by $\{a, b\} = \{-2, -5\}$. Let $R$ be an Eichler order in $B$, of reduced discriminant $N$ and so level $L = N/D = 15/5 = 3$. By Theorem (1.7.7) holds that the class number is

$$n = \frac{15}{12}\left(1-\frac{1}{5}\right)\left(1+\frac{1}{3}\right)+\frac{1}{4}\underbrace{\left(1-\left(\frac{-4}{5}\right)\right)}_{=0}\left(1+\left(\frac{-4}{3}\right)\right)+\frac{1}{3}\underbrace{\left(1-\left(\frac{-3}{5}\right)\right)}_{=2}\overbrace{\left(1+\left(\frac{-3}{3}\right)\right)}^{=1}=$$

$$= \frac{15}{12}\cdot\frac{4}{5}\cdot\frac{4}{3}+\frac{2}{3}=\frac{6}{3}=2$$

where $\left(\frac{-}{q}\right)$ is the Kronecker symbol at the prime $q$. By Theorem (1.7.5) we deduce that

$$\sum_{i=1}^{2}\frac{1}{w_i} = |\zeta(-1)|\cdot 3(5-1)\left(1+\frac{1}{3}\right) = \frac{1}{12}\cdot 3\cdot 4\cdot\frac{4}{3} = \frac{4}{3}$$

Since each $w_i$ is a positive integer we deduce that, considering $w_1 \geq w_2$ and putting $M = \frac{4}{3}$,

$$w_1 + w_2 = M\cdot w_1 w_2 \Longrightarrow w_1 = \frac{w_2}{Mw_2 - 1}$$

We notice that $f(x) = \frac{x}{Mx-1}$, with $M > 1$, is strictly decreasing for $x \geq 1$ and has image contained in $\left[\frac{1}{M}, \frac{1}{M-1}\right]$, therefore it is uniquely determined the couple $(w_1, w_2) \in \mathbb{N}^2_{\geq 1}$ with $w_1 \geq w_2$, such that $\sum \frac{1}{w_i} = \frac{4}{3}$; moreover $w_1$ belongs to the set $\left[\frac{3}{4}, 3\right] \cap \mathbb{N}$. It follows that either $w_1 = 3$ and $w_2 = 1$ or $w_1 = 1$ and $w_2 = 3$.

**Remark 4.3.3** Using the structure of $B(0)$, and since in [**P80**] we have

$$B(0) = \frac{1}{2}\begin{pmatrix} 1/w_1 & 1/w_2 \\ 1/w_1 & 1/w_2 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1/3 \\ 1 & 1/3 \end{pmatrix},$$

we can deduce, with a mere computational approach, that $w_1 = 1$ and $w_2 = 3$.

We can now compute the Brandt matrix at $p$, where $p$ is prime and does not divide 15, associated with an Eichler order of level 3. In [**P80**] the matrix at $p = 2$ has already been computed and it is

$$B(2) = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}.$$

We have now to compute the characteristic polynomial of $B(2)^{\mathrm{t}}$

$$P(\lambda) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda+1)(\lambda-3)$$

and, as $a_2(f) = -1$, we need the eigenvector associated with the eigenvalue $-1$, i.e. $v = (x, y)^{\mathrm{t}}$ such that

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} = -\begin{pmatrix} x \\ y \end{pmatrix} \Longrightarrow \begin{cases} 2x + 3y & = -x \\ x & = -y \end{cases}$$

hence, by the uniqueness of the eigenfunction $a_f$ follows that

$$\langle a_f \rangle = \langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle = \langle v \rangle \quad \text{and, up to scalar, we choose} \quad a_f = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

so $\lambda_1(f) = 1$ and $\lambda_2(f) = -1$.

**Note 4.3.4** Differently from the previous example, without using $B(0)$ to determine the $w_i$, we have to associate each $w_i$ to the correspondent $\lambda_i(f)$. We can compute directly

$$\langle B(2)^{\mathrm{t}} \cdot \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle \;=\; \langle\begin{pmatrix}2&3\\1&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle \;=\; \langle\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\rangle \;=\; w_2$$

$$\langle\begin{pmatrix}1\\0\end{pmatrix}, B(2)^{\mathrm{t}} \cdot \begin{pmatrix}0\\1\end{pmatrix}\rangle \;=\; \langle\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}2&3\\1&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}\rangle \;=\; \langle\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}3\\0\end{pmatrix}\rangle \;=\; 3w_1$$

and notice that the two rows are equal by proposition (4.2.6). Therefore it holds that $3w_1 = w_2$ and so $w_1$ and $w_2$ have to be $w_1 = 1$ and $w_2 = 3$.

We can fill in the following table with all the information we have found until this point. Hence we have

| $N$ | $\varepsilon_p$ | $n = rk_{\mathbb{Z}}\,(Pic(X))$ | $\lambda_i(f)$ | $w_i$ |
|---|---|---|---|---|
| 15 | $\varepsilon_3 = +1$, $\varepsilon_5 = -1$ | 2 | 1, $-1$ | 1, 3 |

$$M_3 = \left(\sum_{i=1}^{2} w_i^2 \lambda_i(f)^3\right) = 1 - 3^2 = -8 \qquad M_2 = \left(\sum_{i=1}^{2} w_i \lambda_i(f)^2\right) = 1 + 3 = 4$$

and so

$$A(F) = \frac{M_3^2}{M_2^3} = \frac{(-8)^2}{4^3} = \frac{2^6}{2^6} = 1.$$

# 5 Jacquet's conjecture

## 5.1 Representations

### 5.1.1 Matrix coefficients and contragredient representations

We must recall definition (3.1.9) and develop the notions of admissible dual and matrix coefficient in a more general setting.

#### 5.1.1.1 Contragredient representations

**Definition 5.1.1** *Let $F$ be a non-archimedean local field. If $(\pi, V)$ is an admissible representation of $GL_2(F)$, the contragredient representation of $GL_2(F)$ is $(\hat{\pi}, \hat{V})$ with $\hat{V}$ is the smooth dual of $V$, which is defined as*

$$\hat{V} = \left\{\Lambda : V \longrightarrow \mathbb{C} \;\middle|\; \begin{smallmatrix}\Lambda \text{ is linear and exists } U\text{, open subgroup} \\ \text{of } GL_2(F) \text{ such that } \Lambda \text{ is invariant under } U\end{smallmatrix}\right\}$$

*where for the invariance property we mean that $\Lambda(\pi(u)v) = \Lambda(v)$ for all $u \in U$ and $v \in V$. Its elements are called smooth linear functionals on $V$. The action is defined by*

$$\hat{\pi}(g)\Lambda(v) := \Lambda(\pi(g^{-1})v)$$

*for all $g \in GL_2(F)$, $v \in V$ and $\Lambda \in \hat{V}$.*

In the setting of the definition, one can prove that the contragredient representation is equivalent, in the irreducible case, to the representation $(\pi^*, V)$ such that $\pi^*(g) = \pi({}^{\mathrm{t}}g^{-1})$. Moreover if $\omega$ is the central character of $\pi$, we can define the representation $\omega^{-1} \otimes \pi$ of $GL_2(F)$ on $V$ defined by $\omega^{-1} \otimes \pi(g) = \omega\,(det(g))^{-1}\,\pi(g)$. This representation is equivalent to $\hat{\pi}$.

**Remark 5.1.2 (Twists)** *Let $(\pi, V)$ be an admissible representation of $GL_2(F)$ and $\chi$ a character of $F^\times$. We can define the twist of $\pi$ as the representation of $GL_2(F)$ on $V$ defined by*

$$(\chi \otimes \pi)(g) = \chi(det(g))\,\pi(g)$$

and it is another representation of $GL_2(F)$ on the same space $V$.

**Proposition 5.1.3 ([Bu98], 4.6.1)** *Let $(\pi, V)$ be an unramified representation of $GL_2(F)$, then the contragredient representation is also unramified.*

**Proof:** By the equivalence with $\pi^*$ we can notice that, if it exists a $v \in V$ such that $\pi(K)v = v$ ($K = GL_2(O_F)$) hence, as ${}^{\mathrm{t}}K^{-1} = K$, we are done. ■

Let $F = \mathbb{R}$ or $\mathbb{C}$ and take $K \subset GL_2(F)$ the maximal compact subgroup.

**Definition 5.1.4** *Let $(\pi, V)$ be a representation of $K$ (e.g. the restriction of a representation of $GL_2(F)$). For each $(\rho, V_\rho)$ irreducible finite-dimensional representation of $K$, we can define the $\rho-$*isotypic part *of $(\pi, V)$ as*

$$V(\rho) = \bigoplus_W W$$

*for $W$ varying among all the $K-$submodules of $V$, which are isomorphic to $V_\rho$.*

**Definition 5.1.5** *Let $F = \mathbb{R}$ or $\mathbb{C}$ and (with the usual notation) consider $(\pi, V)$ a $(\mathfrak{g}, K_\infty)-$module for $K \subset GL_2(F)$ the maximal compact subgroup. Let*

$$\hat{V} = \left\{ \Lambda : V \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{c} \Lambda \text{ is linear and it is zero on } V(\rho) \text{ for} \\ \text{almost all irreducible representations } \rho \text{ of } K_\infty \end{array} \right\}$$

*The action $\hat{\pi}$ of $K_\infty$ is given by*

$$\hat{\pi}(k)\Lambda(v) = \Lambda\left(\pi(k^{-1})v\right)$$

*and that of $\mathfrak{g}$ is given by*

$$\hat{\pi}(X)\Lambda(v) = -\Lambda(\pi(X)v).$$

*This $(\mathfrak{g}, K_\infty)-$module is called the* contragredient $(\mathfrak{g}, K_\infty)-$*module of $(\pi, V)$.*

Analogously to the non-archimedean case we have the equivalence between the contragredient $(\mathfrak{g}, K_\infty)-$module and the $(\mathfrak{g}, K_\infty)-$module defined as $(\pi^*, V)$ with $\pi^*(g) = \pi({}^{\mathrm{t}}g^{-1})$.

**Definition 5.1.6** *Let $(\pi, V)$ be an irreducible admissible representation of $GL_2(\mathbb{A})$ and write $\pi = \otimes_v \pi_v$. We define the contragredient representation as $\hat{\pi} = \otimes \hat{\pi}_v$.*

We would like to work with cuspidal representation in particular it holds that

**Proposition 5.1.7** *Let $(\pi, V)$ be an automorphic (unitary) cuspidal representation of $GL_2(\mathbb{A})$, then the controgredient representation $(\hat{\pi}, \hat{V})$ is an automorphic (unitary) cuspidal representation of $GL_2(\mathbb{A})$.*

**Proof:** See proposition 8.9.6 (and proposition 9.5.8) in [**GH11**]. ■

Furthermore we can give two stronger results about unitary representations.

**Proposition 5.1.8** *Let $p$ be a prime and let $(\pi, V)$ be an irreducible smooth representation of $GL_2(\mathbb{Q}_p)$. If $(\pi, V)$ is unitary hence the contragredient representation is unitarizible.*

**Proof:** See proposition 9.1.4 in [**GH11**]. ■

**Proposition 5.1.9** *Let $(\pi, V)$ be an irreducible admissible $(\mathfrak{g}, K_\infty)-$module. If it is unitary hence the contragredient module is unitarizible.*

**Proof:** See proposition 9.4.2 in [**GH11**]. ■

**Definition 5.1.10** *The complex conjugated representation of $(\pi, V)$ (complex representation) is the representation $(\overline{\pi}, \overline{V})$, for $\overline{V}$ the complex conjugated space and $\overline{\pi}$ the representation acting as $\overline{\pi}(g)\phi := \overline{\pi(g)\phi}$.*

**Proposition 5.1.11** *Let $(\pi, V)$ be an (irreducible admissible) unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_\mathbb{Q})$ and write it as $\pi = \pi_{(\mathfrak{g},K_\infty)} \otimes \bigotimes_p \pi_p$. Then:*

- $(a)$ *the contragredient representation is a unitarizible cuspidal automorphic representation;*
- $(b)$ *the complex conjugate representation $(\overline{\pi_{(\mathfrak{g},K_\infty)}} \otimes \bigotimes_p \overline{\pi_p}, \overline{V})$ is equivalent to the contragredient representation.*

**Proof:** $(a)$ is obtained by the above propositions and by proposition 9.5.2 in [**GH11**]. $(b)$ is a particular case of proposition 8.9.6 in [**GH11**]. ■

**Note 5.1.12** Similarly to that pointed out in (3.2.1) we can extend the definition of controgredient representation to representation of quaternion algebras. In particular, we are interested in the generalization of the above proposition having in mind the Ichino's formula (6.1).

#### 5.1.1.2 Matrix coefficients

**Theorem 5.1.13 (Riesz–Fréchet representation theorem)** *Let $H$ be a Hilbert space with Hermitian product $\langle \cdot, \cdot \rangle$. We define the continuous dual of $H$ as*

$$H^* = \left\{ \Lambda : H \longrightarrow \mathbb{C} \;\middle|\; \Lambda \text{ linear and continuous functional} \right\}$$

*Then $\Lambda$ is an element of $H^*$ if and only if there exists a unique $y \in H$ such that*

$$\Lambda(x) = \langle x, y \rangle$$

*for all $x \in H$.*

**Proof:** See [**Ha51**], §17, theorems 1, 2 and 3. ■

Let $(\pi, V)$ be a representation of a topological group $G$. We can consider the space of continuous functionals on $V$ and consider maps of the form

$$g \longmapsto \Lambda(\pi(g)v)$$

for $g \in G$, $\Lambda$ continuous linear operator on $V$ and $v \in V$. We can define the matrix coefficients for a representation $(\pi, V)$ of $GL_2(F)$ ($F$ non-archimedean) as maps of the (above) form

$$g \longmapsto \Lambda(\pi(g)v)$$

for $g \in GL_2(F)$, $\Lambda \in \hat{V}$, $v \in V$. Analogously we define matrix coefficients for $(\mathfrak{g}, K_\infty)-$modules with the contragredient module and, in particular, we can give a global definition in the same manner.

**Note 5.1.14** *The Riesz-Fréchet representation theorem assures that, in the unitary case, all the matrix coefficients are of the form*

$$g \longmapsto \Lambda(\pi(g)v) = \langle \pi(g)v, w \rangle$$

*for a certain $w \in V$ and for $\langle \cdot, \cdot \rangle$ the Hermitian product.*

## 5.2 Triple product $L-$functions

### 5.2.1 On the representations of $GL_2 \times GL_2 \times GL_2$

Let $G$ be the algebraic group $GL_2 \times GL_2 \times GL_2$ over the number field $F$. We consider the representations of $G$ (as a group) as tensor products of three representations of $GL_2$. In particular, the notion of automorphicity and cuspidality are the same introduced in (3.1.6.2), ensuring that all the three components satisfy the conditions. For this reason we usually denote by $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ a representation of $G$ for $\pi_i$ a representation of $GL_2$ for $i = 1, 2, 3$. Since each $\pi_i$ defines a central character, we can speak about the product of the central characters associated with the $\pi_i$.

### 5.2.2 The holomorphic case

Following Harris and Kudla, we can write explicitly the $L-$function associated with three irreducible cuspidal automorphic representations of $GL_2(\mathbb{A}_{\mathbb{Q}})$, i.e. in the particular case $F = \mathbb{Q}$. In the above hypothesis let $\pi_1$, $\pi_2$ and $\pi_3$ be three irreducible cuspidal automorphic representations. We can associate to them three holomorphic cusp forms $f_1$, $f_2$ and $f_3$ of weight $k_1$, $k_2$ and $k_3$ respectively (as consequence of the Jacquet-Langlands correspondence). We can moreover suppose that those forms are normalized newforms of level $N_1$, $N_2$ and $N_3$ respectively and with nebentypus (i.e. character) $\varepsilon_i$, for $i = 1, 2, 3$.

Denoted by $a_i(n)$ the $n-$th coefficient of the Fourier $q-$expansion of $f_i$ we have, for each $i$, the classical Hecke $L-$series

$$L(s, f_i) = \sum_{n=1}^{+\infty} \frac{a_i(n)}{n^s} = \prod_p L(s, \pi_{i,p})$$

where, for each prime $p \not| N_i$,

$$L(s, \pi_{i,p}) = \frac{1}{(1 - a_i(p)p^{-s} + \varepsilon_i(p)p^{k_i-1-2s})} = \frac{1}{\left(1 - \alpha_i^1(p)p^{-s}\right)\left(1 - \alpha_i^2(p)p^{-s}\right)}$$

Let $S_f$ be the set of primes dividing the product $N_1N_2N_3$ and let $S = S_f \cup \{\infty\}$. Hence we can define the local factor

$$L\left(s, (\pi_1 \otimes \pi_2 \otimes \pi_3)_p\right) = \prod_{\eta:\{1,2,3\}\to\{1,2\}} \frac{1}{1 - \alpha_1^{\eta(1)}(p)\alpha_2^{\eta(2)}(p)\alpha_3^{\eta(3)}(p)p^{-s}}$$

then we have the

**Definition 5.2.1** *The restricted $L-$function $L^S(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ is defined as*

$$L^S(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = \prod_{p \notin S} L\left(s, (\pi_1 \otimes \pi_2 \otimes \pi_3)_p\right)$$

**Example 5.2.2 (Gross–Kudla $L-$function)** The $L-$function which appears in the Gross–Kudla formula is indeed the product of the restricted $L$-function with the non-archimedean factors when $p|N_1N_2N_3$ (which is $N^3$ in the notation of the section (4.2) ). In particular, with the notations in (4.2) and denoted $\pi_f$, $\pi_g$ and $\pi_h$ the corresponding irreducible cuspidal automorphic representations, we have that

$$L(s, f \otimes g \otimes h) = \left(\prod_{l|N} \frac{1}{(1 - a_lb_lc_l \cdot l^{-s}) \cdot (1 - a_lb_lc_l \cdot l^{1-s})^2}\right) L^S(s, \pi_f \otimes \pi_g \otimes \pi_h)$$

We can give an explicit description of this extra factor, which we call $\kappa(s) = \kappa(s, N, f, g, h)$. In fact, as notice in (4.2), we have

$$\varepsilon_l = -a_l b_l c_l \in \{\pm 1\}$$

Hence, denoted by $D$ the discriminant of the quaternion algebra as in (4.2) we have

$$\kappa(s) = \prod_{l|N} \frac{1}{(1 - a_l b_l c_l \cdot l^{-s}) \cdot (1 - a_l b_l c_l \cdot l^{1-s})^2} =$$

$$= \left( \prod_{p|D} \frac{1}{(1 - l^{-s}) \cdot (1 - l^{1-s})^2} \right) \cdot \left( \prod_{p|\frac{N}{D}} \frac{1}{(1 + l^{-s}) \cdot (1 + l^{1-s})^2} \right)$$

**Example 5.2.3 (Garrett's triple product $L$−function)** Let $f, \varphi$ and $\psi$ three normalized holomorphic eigen-cuspforms of weight $k$, for the full congruence group $SL_2(\mathbb{Z})$. Then we are in the above case, with trivial characters $\varepsilon_i$ and such that $S_f$ is empty. Namely the Garrett's $L$−function is

$$L_{f,\varphi,\psi}(s) = L^{\{\infty\}}(s, f \otimes \varphi \otimes \psi)$$

Garrett provided an integral representation of this $L$−function in [**Ga87**]. For its functional equation and its properties we refer to [**Ga87**] and [**PSR87**].

Set $\xi_i : \mathbb{A}^\times \longrightarrow \mathbb{C}^\times$ as the central character of $\pi_i$, for each $i$, and consider the product $\omega(x) = \xi_1(x)\xi_2(x)\xi_3(x)$. Define further the *weight* of the $L$−function $L^S(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$,

$$w = k_1 + k_2 + k_3 - 3.$$

**Theorem 5.2.4** *There exists a meromorphic extension of the $L$−function. Moreover, if the product $\omega(x) = ||x||_{\mathbb{A}}^w$ and $\omega$ is odd (for $||\cdot||_{\mathbb{A}}$ the adèlic norm), then there exist*

- *Euler factors for each $p \in S_f$,*

  $$\tilde{L}(s, (\pi_1 \otimes \pi_2 \otimes \pi_3)_p) = \frac{1}{P(p^{-s})}$$

  *for $P(t) \in \mathbb{C}[t]$ and $P(0) = 1$;*

- *Archimedean factor $\Psi(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ (in the holomorphic case as above it can be identified with a product of $\Gamma$−functions and $\zeta$−functions;*

- *$\varepsilon$−factors, $\tilde{\varepsilon}(s, (\pi_1 \otimes \pi_2 \otimes \pi_3)_p)$;*

*such that, defined*

$$\tilde{L}_{fin}(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = L^S(s, \pi_1 \otimes \pi_2 \otimes \pi_3) \cdot \prod_{p \in S_f} \tilde{L}(s, (\pi_1 \otimes \pi_2 \otimes \pi_3)_p)$$

$$\tilde{\varepsilon}(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = \prod_{p \in S_f} \tilde{\varepsilon}(s, (\pi_1 \otimes \pi_2 \otimes \pi_3)_p)$$

*it holds that*

(*a*) *there exists a functional equation*

$$\tilde{L}_{fin}(s, \pi_1 \otimes \pi_2 \otimes \pi_3) = \tilde{\varepsilon}(s, \pi_1 \otimes \pi_2 \otimes \pi_3)\Psi(s, \pi_1 \otimes \pi_2 \otimes \pi_3)\tilde{L}_{fin}(w + 1 - s, \pi_1 \otimes \pi_2 \otimes \pi_3)$$

(*b*) *$\tilde{L}_{fin}(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ has no poles at the center $s = \frac{w+1}{2}$ of the critical strip.*

**Proof:** This theorem sums up (as noticed in [**HK91**], §1) the main results in the article of Piatetski-Shapiro and Rallis, [**PSR87**], namely Theorems 5.2 and 5.3. ■

### 5.2.3 Langlands $L-$functions

The above theorem can be reformulated in order to deal with more general $L-$functions associated with automorphic forms.

At this point we should introduce the so-called *Langlands dual* of an algebraic group. Since the construction and the properties are not of so easy description we will develop exactly what we need and only in two particular cases. For a complete discussion of the Langlands dual we refer to [**Bo79**] and to [**AG91**], §2.

Let $F$ be a number field. The Langland dual, ${}^LG$, of the linear algebraic group $G$ (over $F$) is a semi-direct product

$$ {}^LG := \hat{G} \rtimes \Gamma $$

where $\Gamma := Gal(\overline{F}/F)$ is the absolute Galois group and $\hat{G}$ is a complex linear algebraic group. We have to define a topology on the Langlands dual. In particular, the topology is that on the underlying space as proved by proposition 27 in [**Bou07**], which states that

**Proposition 5.2.5** *Let $L$ and $N$ two topological groups with an action of $L$ on $N$. Consider the map*

$$ N \times L \ni (n, l) \longmapsto l \cdot n \in N $$

*where $\cdot$ is the action of $l$ on $n$. Taken the product topology on the product, suppose that the above map is continuous. Therefore the product topology on the underlying space of $N \rtimes L$ is compatible with the structure of group, making the direct product a topological group.*

**Proof:** See proposition 27 in [**Bou07**], $TG\ III.18$, §2, $N°10$. ■

**Definition 5.2.6** *The topology described by the above proposition is usually referred to as the* semi-direct product topology*.*

Since $\hat{G}$ is a complex algebraic group we have a natural topology on it and the Galois group $\Gamma$ is a profinite group, so it is endowed with the profinite topology. With a precise study of the construction of the Langlands dual, as presented in [**Bo79**], and with proposition (5.2.5) it can be proved that the map

$$ \hat{G} \times \Gamma \ni (g, \gamma) \longmapsto \gamma \cdot g \in \hat{G} $$

is continuous, and then, that the Langlands dual is a topological group with the semi-direct product topology.

**Definition 5.2.7 ([Bo79], §2.6, §8)** *A representation of the Langland dual is a continuous homomorphism $r :\ {}^LG \longrightarrow GL_m(\mathbb{C})$ whose restriction to $\hat{G}$ is a morphism of complex Lie groups.*

*An element of an algebraic linear group $G$, is semi-simple if it is diagonalizable (as a matrix). An element $x = (u, \gamma) \in\ {}^LG$ is semi-simple if its image under any representation $r$ is so.*

**Proposition 5.2.8 (Proposition 2.2, §2.1.2, [Hi00])** *Let $G$ be a profinite group and consider a continuous group representation of it on a finite-dimensional complex space, endowed with the euclidean topology. Hence the representation has finite image.*

**Remark 5.2.9 (§2.6, [Bo79])** Let $r$ be a representation of ${}^LG$. By proposition (5.2.8), $ker(r)$ contains an open subgroup of $\Gamma$, hence it can be proved that $r$ factors through $\hat{G} \rtimes Gal(L/F)$ for $L$ a particular finite Galois extension of $F$. Moreover, by proposition (5.2.5) and since $Gal(L/F)$ is finite (and so endowed with the discrete topology), $\hat{G} \rtimes Gal(L/F)$ has a natural structure of complex algebraic group.

One can hence associate to an automorphic representation of $G(\mathbb{A}_F)$ (with the usual definition), say $\pi = \otimes_v \pi_v$, the set

$$\sigma(\pi) = \left\{ \sigma_v(\pi) = \sigma(\pi_v) \;\middle|\; v \notin S \right\}$$

for $S$ the finite set of places, $S \supset S_\infty$ at which $\pi_v$ is ramified. Here, $\sigma_v(\pi) = \sigma(\pi_v)$ is a conjugacy class of a semi-simple element in ${}^LG$, such that the projection on the Galois group is the Frobenius class at $v$ (for a recall on decomposition groups and Frobenius at a prime see [**SD01**], §5).

Moreover one can prove that such set determines uniquely the representation $\pi$.

We can hence define an automorphic $L-$function associated with an automorphic representation, $\pi$, of $G(\mathbb{A}_F)$ and to a (finite-dimensional) representation of ${}^LG$,

$$r : \, {}^LG \longrightarrow GL_m(\mathbb{C})$$

We associate to this couple of representations the family

$$\left\{ r(\sigma_v(\pi)) \;\middle|\; v \notin S \right\}$$

of semi-simple conjugation classes in $GL_m(\mathbb{C})$, where $S$ is as above. Then we can finally give the

**Definition 5.2.10** *The general (restricted) automorphic $L-$function is defined as the product*

$$L_S(s, \pi, r) = \prod_{v \notin S} det \left( 1 - \frac{r(\sigma_v(\pi))}{(Nv)^s} \right)^{-1}$$

*for $Nv$ the norm of the prime associated with the (finite) place $v$ and $s \in \mathbb{C}$.*

#### 5.2.3.1 Rankin–Selberg $L-$functions

Let $F = \mathbb{Q}$. We can define the so-called *Rankin–Selberg* $L-$functions.

**Definition 5.2.11** *Let $\pi_1$ and $\pi_2$ be two automorphic cuspidal representations of $GL_2(\mathbb{A})$ with unitary central characters and let $S \supset S_\infty$ the set of places at which at least one of the two is ramified. Let $f_1$ and $f_2$ two cusps forms (with characters $\varepsilon_i$, weight $k_i$) corresponding to $\pi_1$ and $\pi_2$ via the Jacquet–Langlands correspondence. For each $p \notin S$ take $\alpha_i(p)$ and $\beta_i(p)$ such that (for $a_i(n)$ the Fourier coefficients)*

$$\left(1 - a_i(p)p^{-s} + \varepsilon_i(p)p^{k_i-1-2s}\right) = \left(1 - \alpha_i(p)p^{-s}\right)\left(1 - \beta_i(p)p^{-s}\right)$$

*The Rankin–Selberg $L-$function of $\pi_1 \otimes \pi_2$ (with $\pi_1 \otimes \pi_2$ viewed as a representation of $GL_2(\mathbb{A}) \times GL_2(\mathbb{A})$) is the Euler product*

$$L_S(s, \pi_1 \otimes \pi_2) := \prod_{p \notin S} L_p(s, \pi_{1,p} \otimes \pi_{2,p})$$

*for*

$$L_p(s, \pi_{1,p} \otimes \pi_{2,p}) := \prod_{i=1}^{2} \prod_{j=1}^{2} \frac{1}{1 - \alpha_i(p)\beta_j(p)p^{-s}}$$

We can realize (and generalize) this function as a Langlands $L-$function. Suppose that $F$ is a generic number field and take $G = GL_2 \times GL_2$ as an algebraic group over $\mathbb{A}_F$. The Langland dual, ${}^LG$, of $G$ is the semi-direct product

$${}^LG := (GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes \Gamma$$

where the absolute Galois group $\Gamma := Gal(\overline{F}/F)$ acts trivially on $\hat{G} := GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$. Let $\pi_1$ and $\pi_2$ be two automorphic (cuspidal unitary) representations of $GL_2(\mathbb{A}_F)$ thus $\pi_1 \otimes \pi_2$ is an automorphic (cuspidal unitary) representation of $G$. Take $r$ a representation of ${}^LG$ such that the restriction to $\hat{G}$ is the natural

$$r : GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \longrightarrow GL_2(\mathbb{C})$$

induced by the tensor product. Thus one can prove that the Rankin–Selberg $L-$function is

$$L_S(s, \pi_1 \otimes \hat{\pi}_2) = L_S(s, \pi_1 \otimes \pi_2, r).$$

**Proposition 5.2.12** *The Rankin–Selberg $L-$function admits a meromorphic continuation to the whole complex plane. Further, it has a pole at $s = 1$ if and only if $\pi_2 \cong \hat{\pi}_1$.*

**Proof:** See [**Bu98**], Prop. 3.8.4, Prop. 3.8.5 and following. ■

#### 5.2.3.2 Adjoint $L-$functions

Let $F = \mathbb{Q}$. Suppose that the Rankin–Selberg $L-$function admits a pole at $s = 1$. Hence, we are in the case

$$L_S(s, \pi \otimes \hat{\pi})$$

for $\pi$ automorphic cuspidal (unitary) representation of $GL_2(\mathbb{A})$. In particular, we can factor it as

$$L_S(s, \pi \otimes \hat{\pi}) = \zeta_{F,S}(s) \cdot L_S(s, \pi, Adj)$$

for $\zeta_{F,S}(s)$the partial Dedekind zeta-function of F as in (1.7.2) up to the factors in $S$, i.e.

$$\zeta_{F,S}(s) = \zeta_F(s) \cdot \prod_{p \in S \cap S_f} \left(1 - \frac{1}{(N(p)^s)}\right)$$

and $L_S(s, \pi, Adj)$ the *adjoint* $L-$function.

In the case $F = \mathbb{Q}$ the decomposition is obvious by the definition of $L_S(s, \pi \otimes \hat{\pi})$; in fact one can prove that the Hecke coefficients associated with the contragredient representation are, with the notations as in the above paragraph, $\alpha(p) := \alpha_2(p) = \beta_1(p)^{-1}$ and $\beta(p) := \beta_2(p) = \alpha_1(p)^{-1}$. Thus we have an explicit expression

$$L_S(s, \pi, Adj) = \prod_{p \notin S} \left(1 - \frac{\alpha(p)\beta(p)^{-1}}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{\alpha(p)^{-1}\beta(p)}{p^s}\right)^{-1}$$

In [**GJ76**] (denoted by $L_2(s, \sigma, \chi)$) and in [**Shi75**] (denoted by D(s) ) it is possible to find a proof that such function is entire.

Let now $G = GL_2$ thought as an algebraic group over $\mathbb{A}_F$ for $F$ a number field. The Langland dual, ${}^L G$, of $G$ is the semi-direct product

$$ {}^L G := GL_2(\mathbb{C}) \rtimes \Gamma $$

where the absolute Galois group $\Gamma := Gal(\overline{F}/F)$ acts trivially on $\hat{G} := GL_2(\mathbb{C})$. Let $\pi$ be an automorphic (cuspidal unitary) representation of $GL_2(\mathbb{A}_F)$ and $\hat{\pi}$ its contagredient. Hence $\pi \otimes \hat{\pi}$ defines an automorphic representation on $GL_2(\mathbb{A}_F) \times GL_2(\mathbb{A}_F)$. We can consider the representation of ${}^L G$, $r = Ad$, the adjoint representation of $GL_2(\mathbb{C})$ on its Lie algebra $Lie(\hat{G}) = \mathfrak{gl}_2$. Notice that considering the representation on the quotient $Lie(\hat{G})/Lie(Z(\hat{G}))$ is the same, in fact the scalar matrices define the trivial automorphism. Recall that it is defined as $Ad(g)(X) = gXg^{-1}$ for $X \in \mathfrak{gl}_2$ and $g \in GL_2(\mathbb{C})$. Hence, the associated Langlands $L-$function is

$$ L_S(s, \pi, Ad) = L_S(s, \pi \times \hat{\pi}) = \zeta_{F,S}(s) \cdot L_S(s, \pi, Adj) $$

**Proposition 5.2.13** *Let $f$ be a normalized new eigen-cuspform of level $\Gamma_1(N)$, weight 2 and character $\varepsilon$. Let $\pi_f$ be the automorphic cuspidal representation associated by the Jacquet–Langlands correspondence. Let $C_\varepsilon$ the conductor of the character, then*

$$ L(1, \pi_f, Adj) = \frac{2^2 \pi^2}{\delta(N) \cdot N \cdot C_\varepsilon \cdot \varphi(N/C_\varepsilon)} \cdot < f, f >_{Pet, \Gamma_1(N)} $$

*for $\varphi$ the Euler function and* $\delta(N) = \begin{cases} 2 & \textit{if } N \leq 2 \\ 1 & \textit{otherwise} \end{cases}$.

**Proof:** See Theorem 5.1 in [**Hi81**] keeping in mind that the notion of weight is half of our weight. Notice also that the definition of $L(s, f, \omega)$ coincides with the one of $L(s, \pi_f, Adj)$ if $f$ is as in the statement and observing that (with the notation in the article) $\alpha_p \beta_p = \varepsilon(p)$ implies $\varepsilon(p)\alpha_p^2 = \alpha_p \beta_p^{-1}$ as in our definition. ■

#### 5.2.3.3 $L-$functions for $Res_{E/F}(GL_2)$

Let $E$ be an étale cubic algebra over the number field $F$, i.e. an algebra of degree 3 over $F$ such that $E \otimes_F \overline{F} \cong \overline{F} \times \overline{F} \times \overline{F}$ for $\overline{F}$ an algebraic closure of $F$. In particular, such algebra can be only of one of the following types:

$$ E \cong \begin{cases} F \times F \times F & \\ F \times F' & \text{for } F' \text{ a field extension of } F \text{ of degree 2,} \\ \text{field extension of } F \text{ of degree 3.} & \end{cases} $$

In particular, we write $\mathbb{A}_E$, by abuse of notation, for $\mathbb{A}_E = \mathbb{A}_F \otimes_F E$ and we can consider $GL_2$ as an algebraic group over $F$. As in (2.1.3), we can take the Weil restriction $Res_{E/F}(GL_2)$. For example, if $E$ is split, so that $E = F \times F \times F$, we have that

$$ Res_{E/F}(GL_2)(F) = GL_2(\mathbb{A}_E) = GL_2(\mathbb{A}_F \otimes_F E) \cong GL_2(F) \times GL_2(F) \times GL_2(F). $$

The Langland dual, ${}^L G$, of $G$ is the semi-direct product

$$ {}^L G := (GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes \Gamma $$

where the absolute Galois group $\Gamma := Gal(\overline{F}/F)$ acts on $\hat{G} := GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ via the homomorphism

$$\rho : \Gamma \longrightarrow S_3$$

for $S_3$ the permutations group over 3 elements. We define $\rho$ as a homomorphism associated with $E/F$ e.g.

- if $E$ is a field, then $\rho$ associates to a homomorphism in $\Gamma$ the permutation of the roots of the minimal polynomial of $E$ over $F$;
- if $E \cong F \times F'$ (with the above notation), then $\rho$ associates to a homomorphism in $\Gamma$ the permutation of the roots of the minimal polynomial of $F'$ over $F$;
- if $E \cong F \times F \times F$, then $\rho$ is the trivial homomorphism.

**Example 5.2.14 (Rankin triple $L-$function)**
(See *Case II examples: Rankin triple $L-$function* in Part II, §1 in [**AG91**])

Let $G = Res_{E/F}(GL_{2,E})$ for $E$ the split cubic étale algebra over $F$ (i.e. $E = F \times F \times F$). Let $\pi_1$, $\pi_2$ and $\pi_3$ a triple of cuspidal automorphic (unitary) representations of $GL_2(F)$ and consider $\Pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ as a representation of $G(F)$. Since the absolute Galois group acts trivially on $\hat{G} = GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$, we can consider the natural (induced by the tensor product) $8-$dimensional representation

$$r : \ {}^L G \longrightarrow GL(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \cong GL_8(\mathbb{C})$$

The associated $L-$function is the Rankin triple product $L-$function. In particular, if $F = \mathbb{Q}$ and $f$, $\varphi$, $\psi$ are the associated cusp forms, then

$$L(s, \Pi, r) = L^S_{f,\varphi,\psi}(s)$$

with the notation as in the previous paragraph. We will usually refer to the complete $L-$function with the notation

$$L(s, \pi_1 \otimes \pi_2 \otimes \pi_3) := L_S(s, \Pi, r) \doteq L^{\{\infty\}}(s, \Pi, r)$$

i.e., up to some local factors, the $L-$function studied in [**Ga87**] and [**PSR87**].

**Remark 5.2.15 (Representations of ${}^L G$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$)** One can determine a representation of ${}^L G$ on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ in a quite *"canonical"* manner. We can choose

$$r : \ {}^L G = (GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})) \rtimes \Gamma \longrightarrow GL(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2) \cong GL_8(\mathbb{C})$$

such that the restriction to $GL_2(\mathbb{C}) \times GL_2(\mathbb{C}) \times GL_2(\mathbb{C})$ is the natural $8-$dimensional representation i.e. that induced by the tensor product, and such that $\Gamma$ acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ via the morphism $\rho$.

## 5.3 The conjecture

The name *conjecture* is misleading, in fact, it is indeed a well-known theorem, proved by M. Harris and S. Kudla in [**HK91**] and [**HK04**].

Let $F$ be a number field and let $\pi_1$, $\pi_2$ and $\pi_3$ be cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$. Let $L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ be the triple product $L-$function associated with this triple of automorphic representations. Suppose that the product of the central characters of the $\pi_i$ is trivial. Let $\{B^\times_\alpha \mid \alpha \in A\}$ (for $A$ a certain set of indexes) denote the set of all multiplicative group of quaternion algebras (up to isomorphism) which are ramified only at places where the representations $\pi_1$, $\pi_2$ and $\pi_3$ are all discrete series. For each $B^\times_\alpha$ we denote by $\pi_i^{B_\alpha}$, for $i = 1, 2, 3$, the automorphic representations of $B^\times_\alpha$ which are associated with each $\pi_i$ via the Jacquet–Langlands correspondence. Then

**Theorem 5.3.1 (Harris and Kudla)** *$L(s, \pi_1 \otimes \pi_2 \otimes \pi_3)$ vanishes at the center of symmetry of its functional equation if and only if, for all $\alpha \in A$ and all $f_{i,\alpha} \in \pi_i^{B_\alpha}$, the integral*

$$I\left(f_{1,\alpha}, f_{2,\alpha}, f_{3,\alpha}\right) = \int_{\mathbb{A}_F^\times B_\alpha(F)^\times \backslash B_\alpha(\mathbb{A}_F)^\times} f_{1,\alpha}(x) f_{2,\alpha}(x) f_{3,\alpha}(x) d^\times x$$

*vanishes, where $d^\times x$ is the Tamagawa measure on $B_\alpha(F)^\times \backslash B_\alpha(\mathbb{A}_F)^\times$.*

# 6 Ichino's formula

## 6.1 The formula

We are now able to state the so-called Ichino's formula. It is strictly related to the Jacquet's conjecture as it relates the central value of the triple product $L-$function with the global trilinear forms

$$I(\phi_1, \phi_2, \phi_3) = \int_{\mathbb{A}_F^\times B^\times(F) \backslash B^\times(\mathbb{A}_F)} \phi_1(x)\phi_2(x)\phi_3(x) \ \ d^\times x$$

defined as above.

Let $F$ be a number field, $E$ an étale cubic algebra over $F$ and $B$ a quaternion algebra over $F$. Let $\Pi$ be an irreducible cuspidal unitary automorphic representation of $GL_2(\mathbb{A}_E) = GL_2(\mathbb{A}_F \otimes_F E)$ such that the central character of $\Pi$ is trivial on $\mathbb{A}_F^\times$ . By the definition of $\mathbb{A}_E$ and the nature of $E$, we can work on each factor and suppose that there exists a Jacquet–Langlands lift for $\Pi$ to $\Pi^B$, irreducible unitary cuspidal automorphic representation of $B^\times(\mathbb{A}_E)$. We can take an element

$$I \in \mathrm{Hom}_{B^\times(\mathbb{A}_F)\times B^\times(\mathbb{A}_F)}\left(\Pi^B \otimes \widehat{\Pi}^B, \mathbb{C}\right)$$

for $\widehat{\Pi}^B$ the contragredient representation of $\Pi^B$, defined as the double integral

$$I\left(\phi \otimes \phi'\right) = \int_{\mathbb{A}_F^\times B^\times(F) \backslash B^\times(\mathbb{A}_F)} \int_{\mathbb{A}_F^\times B^\times(F) \backslash B^\times(\mathbb{A}_F)} \phi(x)\phi'(x') \ \ d^\times x \, d^\times x'$$

where $\phi \in \Pi^B$, $\phi' \in \widehat{\Pi}^B$, and $d^\times x$ and $d^\times x'$ are the Tamagawa measures on $\mathbb{A}_F^\times \backslash B^\times(\mathbb{A}_F)$ (and then thought as measures on the quotient). Notice that it makes sense considering the above integral as we have shown in paragraph (5.1.1.1).

**Note 6.1.1** We denoted both the trilinear form in the Jacquet conjecture and the bilinear form on $\Pi^B \times \widehat{\Pi}^B$ with the letter $I$. We must mention that this choice should not produce any confusion as those two objects are strictly related. Especially, we have already seen that we can consider representation $\Pi$ induced by a triplet of representations and we will see in (6.2.1) that the bilinear form is a product of two trilinear forms, every time we consider elementary tensors in $\Pi^B$.

**Remark 6.1.2 (see [I08], Remark 1.2)** Prasad proved that, if $F$ is a local non-archimedean field and $E$ is a cubic étale algebra over $F$, hence

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{B^\times(\mathbb{A}_F)}\left(\Pi^B, \mathbb{C}\right)\right) \leq 1$$

in [**Pr90**] (see theorems 1.1, 1.2, 1.3 and 1.4) for $E$ the split algebra and in [**Pr92**] (see theorems A, B and $C$) for the other two cases. An analogous result has been proved by Loke in [**Lo01**] in the case of an archimedean local field. Thus, by definition of restricted tensor product representations, we can deduce that it holds also in the global case. By definition of tensor product (i.e. considering the equivalence between bilinear maps on the product and linear maps on the tensor) we deduce that

$$\dim_{\mathbb{C}}\left(\mathrm{Hom}_{B^\times(\mathbb{A}_F)\times B^\times(\mathbb{A}_F)}\left(\Pi^B\otimes\widehat{\Pi}^B,\mathbb{C}\right)\right)\leq 1.$$

We define at this point the $B^\times(\mathbb{A}_E)-$invariant pairing between $\Pi^B$ and $\widehat{\Pi}^B$ defined by

$$\langle\phi,\phi'\rangle=\int_{\mathbb{A}_E^\times B^\times(E)\backslash B^\times(\mathbb{A}_E)}\phi(h)\;\phi'(h)\;dh$$

for $\phi\in\Pi^B$, $\phi'\in\widehat{\Pi}^B$ and $dh$ is the Tamagawa measure on $\mathbb{A}_E^\times\backslash B^\times(\mathbb{A}_E)$. Similarly, for each place $v$, we can choose

(*i*) a $B^\times(E_v)-$invariant pairing $\langle\cdot,\cdot\rangle_v$ between $\Pi_v^B$ and $\widehat{\Pi}_v^B$ such that $\langle\phi_v,\phi'_v\rangle=1$ for almost all $v$ and for $\phi=\otimes_v\phi_v\in\Pi^B$ and $\phi'=\otimes_v\phi'_v\in\widehat{\Pi}_v^B$;

(*ii*) a Haar measure $d^\times x_v$ on $F_v^\times\backslash B^\times(F_v)$ such that $\mathrm{vol}\left(O_v^\times\backslash GL_2(O_v),d^\times x_v\right)=1$ for almost all $v$. By definition of the Tamagawa measure (2.2.6) there exists hence a constant $C$ such that

$$d^\times x=C\prod_v d^\times x_v$$

is the Tamagawa measure on $\mathbb{A}_F^\times\backslash B^\times(\mathbb{A}_F)$.

With (*i*) and (*ii*), for all $v$ we define an element

$$I_v\in\mathrm{Hom}_{B^\times(F_v)\times B^\times(F_v)}\left(\Pi_v^B\otimes\widehat{\Pi}_v^B,\mathbb{C}\right)$$

as

$$I_v\left(\phi_v\otimes\phi'_v\right)=\frac{\zeta_{F_v}(2)}{\zeta_{E_v}(2)}\cdot\frac{L_v(1,\Pi_v,Adj)}{L_v(1/2,\Pi_v,r)}\int_{F_v^\times\backslash B^\times(F_v)}\langle\Pi_v^B(x_v)\phi_v,\phi'_v\rangle_v\;\;d^\times x_v$$

for $\phi_v\in\Pi_v^B$, $\phi'_v\in\widehat{\Pi}_v^B$. Here $L_v(s,\Pi_v,r)$ is the local factor of the Rankin triple product $L-$function as in section (5.2.3.3) and $L_v(1,\Pi_v,Adj)$ is the local factor of the adjoint $L-$function.

**Note 6.1.3 (Dedekind zeta functions for étale cubic algebras)** We have already defined the notion of Dedekind zeta function for a number field and for a quaternion algebra in section (1.7.2) and we can define the zeta function for a cubic étale algebra over a number field. In particular, it is consistent with the decomposition of the algebra as product of number fields, as it is defined as

$$\zeta_E=\begin{cases}\zeta_F^3 & \text{if } E=F\times F\times F\\ \zeta_F\cdot\zeta_{F'} & \text{if } E=F\times F',\text{ for } F'/F \text{ quadratic}\\ \text{the usual Dedekind zeta function} & \text{if } E \text{ is a cubic extension of } F\end{cases}$$

We denote $\zeta_{E_v}$ for the local component at $v$.

Combining all the introduced objects, Ichino managed to state the following remarkable result.

**Theorem 6.1.4 (Ichino's formula)** *Under the hypotheses*

$$\langle\phi,\phi'\rangle=\prod_v\langle\phi_v,\phi'_v\rangle_v\qquad\textit{and}\qquad d^\times x=\prod_v d^\times x_v$$

*we have the equality, as elements in* $Hom_{B^\times(\mathbb{A}_F)\times B^\times(\mathbb{A}_F)}\left(\Pi^B\otimes\widehat{\Pi}^B,\mathbb{C}\right)$,

$$I=\frac{1}{2^c}\cdot\frac{\zeta_E(2)}{\zeta_F(2)}\cdot\frac{L(1/2,\Pi,r)}{L(1,\Pi,Adj)}\cdot\prod_v I_v$$

*for*

$$c=\begin{cases}3 & \text{if } E=F\times F\times F,\\ 2 & \text{if } E=F\times F', \text{ for } F' \text{ a quadratic extension of } F,\\ 1 & \text{if } E \text{ is a cubic extension of } F.\end{cases}$$

**Proposition 6.1.5 (Remark 1.3 in [I08])** *With the above notations and without the hypotheses in the previous theorem we have*

$$\frac{I(\phi\otimes\phi')}{\langle\phi,\phi'\rangle}=\frac{1}{2^c}\cdot\frac{\zeta_E(2)}{\zeta_F(2)}\cdot\frac{L(1/2,\Pi,r)}{L(1,\Pi,Adj)}\cdot\prod_v\frac{I_v(\phi_v\otimes\phi'_v)}{\langle\phi_v,\phi'_v\rangle_v}$$

*for* $\phi=\otimes_v\phi_v\in\Pi^B$ *and* $\phi'=\otimes_v\phi'_v\in\widehat{\Pi}^B_v$ *such that* $\langle\phi,\phi'\rangle\neq 0$.

## 6.2 The split case over $\mathbb{Q}$

We restrict now our attention to the case $F=\mathbb{Q}$ and $E=\mathbb{Q}\times\mathbb{Q}\times\mathbb{Q}$ and we will consider $\Pi$ formed of three cuspidal automorphic unitary irreducible representations of $GL_2(\mathbb{A}_\mathbb{Q})$ with trivial product of the central characters. Notice that this is equivalent to the condition imposed by Ichino that the central character of $\Pi$, as a representation of $GL_2(\mathbb{A}_E)$, is trivial on $\mathbb{A}_F^\times$. Again, $B$ is a quaternion algebra over $\mathbb{Q}$ with reduced discriminant $D$ and we denote $S:=Ram(B)$.

### 6.2.1 Global trilinear forms

We start considering the two objects

$$(A)\qquad I\left(\phi\otimes\phi'\right)=\int_{\mathbb{A}_\mathbb{Q}^\times B^\times(\mathbb{Q})\backslash B^\times(\mathbb{A}_\mathbb{Q})}\int_{\mathbb{A}_\mathbb{Q}^\times B^\times(\mathbb{Q})\backslash B^\times(\mathbb{A}_\mathbb{Q})}\phi(x)\phi'(x')\ d^\times x\ d^\times x'$$

$$(B)\qquad \langle\phi,\phi'\rangle=\int_{\mathbb{A}_E^\times B^\times(E)\backslash B^\times(\mathbb{A}_E)}\phi(h)\ \phi'(h)\ dh$$

We begin considering the object $(A)$ and we can notice immediately that $I$ is absolutely convergent (since the underlying space is compact) so, by Fubini–Tonelli theorem we can write

$$I\left(\phi\otimes\phi'\right)=\int_{\mathbb{A}_\mathbb{Q}^\times B^\times(\mathbb{Q})\backslash B^\times(\mathbb{A}_\mathbb{Q})}\phi(x)\ d^\times x\cdot\int_{\mathbb{A}_\mathbb{Q}^\times B^\times(\mathbb{Q})\backslash B^\times(\mathbb{A}_\mathbb{Q})}\phi'(x')\ d^\times x'$$

Now on *suppose that the quaternion algebra $B$ is definite* and take $R$ to be an Eichler order of level $N$ in $B$, with level prime to the reduced discriminant of $B$. Let $\hat{B}^\times=B^\times(\hat{\mathbb{Q}})$ for $\hat{\mathbb{Q}}$ the finite adèles of $\mathbb{Q}$ and denote $\hat{R}:=R\otimes\hat{\mathbb{Z}}$. Consider $d^\times x$ the Tamagawa measure on $\mathbb{A}^\times\backslash B^\times(\mathbb{A})$ hence the Tamagawa number of $\mathbb{A}^\times\backslash B^\times(\mathbb{A})$, i.e. its volume, is

$$\mathrm{vol}\left(\mathbb{A}^\times\backslash B^\times(\mathbb{A}),d^\times x\right)=2$$

(see for example [**V80**], page 71, Theoreme 2.3).

**Proposition 6.2.1 (§4.1, [Hs17])** *There exists a (positive rational) number,* $\mathrm{vol}(\hat{R}^\times)$, *such that, for every* $f\in L^1\left(B^\times\backslash B^\times(\mathbb{A})/B_\infty^\times\hat{R}^\times\right)$, *it holds*

$$\int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})}f(x)\ d^\times x=\mathrm{vol}(\hat{R}^\times)\sum_{\bar{x}\in B^\times\backslash\hat{B}^\times/\hat{R}^\times}f(x)\cdot(\#\Gamma_x)^{-1}$$

*for* $\bar{x}$ *the double coset* $\bar{x}=B^\times x\hat{R}^\times$, *and* $\Gamma_x:=\left(B^\times\cap x\hat{R}^\times x^{-1}\right)\mathbb{Q}^\times/\mathbb{Q}^\times$.

**Proof:** Firs of all just notice that $B^\times\backslash B^\times(\mathbb{A})/B_\infty^\times\hat{R}^\times \cong B^\times\backslash\hat{B}^\times/\hat{R}^\times$ as the place at infinity is in direct product with the finite adèles. As $\mathbb{Q}$ has class number 1, we have the decomposition $\mathbb{A}^\times = \mathbb{R}^\times\mathbb{Q}^\times\hat{\mathbb{Z}}^\times$ so $\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A}) = B^\times\backslash B^\times(\mathbb{A})/\mathbb{R}^\times\mathbb{Q}^\times\hat{\mathbb{Z}}^\times$. As $\hat{R}$ contains a copy of $\hat{\mathbb{Z}}$ and $B_\infty^\times$ contains $\mathbb{R}^\times\mathbb{Q}^\times$, we can realize $B^\times\backslash B^\times(\mathbb{A})/B_\infty^\times\hat{R}^\times$ as a quotient of $\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})$. Now it is obvious that the integration of $f$ gives a sum of values of $f$ times the volume of each coset of $B^\times\backslash B^\times(\mathbb{A})/B_\infty^\times\hat{R}^\times$. The existence of the constant is now true since the number of cosets is finite by (1.5.5). ■

**Note 6.2.2 (§4.1, [Hs17])** We can determine the constant in the above proposition. The first step is noticing that

$$B^\times \cap x\hat{R}^\times x^{-1} = \left(B \cap x\hat{R}x^{-1}\right)^\times$$

as $\left(x\hat{R}x^{-1}\right)^\times = x\hat{R}^\times x^{-1}$. Hence, by Note (1.5.5) we know that $B^\times \cap x\hat{R}^\times x^{-1}$ are the Eichler orders of level $N$ in $B$ and $n = \#\left(B^\times\backslash\hat{B}^\times/\hat{R}^\times\right)$ is the class number of $R$. With the notation as in theorem (1.7.5) we have that (chosen a set of representatives) $\#\Gamma_{x_i} = 2w_i$. We can consider $f(x) = 1$ constant function (it is a $L^1-$function as it is continuous on a compact space) hence

$$\begin{aligned}2 = \mathrm{vol}\left(\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A}), d^\times x\right) &= \int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})} d^\times x = \\ &= \mathrm{vol}(\hat{R}^\times) \sum_{\bar{x}\in B^\times\backslash\hat{B}^\times/\hat{R}^\times} \frac{1}{\#\Gamma_x} = \frac{\mathrm{vol}(\hat{R}^\times)}{2}\sum_{i=1}^n \frac{1}{w_i}\end{aligned}$$

thus, by the Eichler mass formula (1.7.5), we have

$$\begin{aligned}\mathrm{vol}(\hat{R}^\times) = 4\left(\sum_{i=1}^n \frac{1}{w_i}\right)^{-1} &= 4\left(\frac{N}{12}\prod_{p\mid D}(p-1)\prod_{p\mid N}\left(\frac{1}{p}+1\right)\right)^{-1} = \\ &= \frac{48}{N}\prod_{p\mid D}\frac{1}{(p-1)}\prod_{p\mid N}\left(\frac{p}{p+1}\right)\end{aligned}$$

for $D$ the reduced discriminant of $B$.

Consider now $(B)$. We are in the split case, so the Tamagawa measure $dh$ is the product of the three Tamagawa measures, one on each component of

$$\mathbb{A}_E^\times B^\times(E)\backslash B^\times(\mathbb{A}_E) = \left(\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})\right) \times \left(\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})\right) \times \left(\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})\right).$$

The Fubini–Tonelli theorem guarantees the truth of the expression

$$\langle\phi,\phi'\rangle = \int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})} \phi_1(x)\,\phi_1'(x)\,d^\times x \int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})} \phi_2(y)\,\phi_2'(y)\,d^\times y \int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})} \phi_3(z)\,\phi_3'(z)\,d^\times z$$

for $\phi = \phi_1\otimes\phi_2\otimes\phi_3 \in \Pi^B = \pi_1^B\otimes\pi_2^B\otimes\pi_3^B$ and $\phi' = \phi_1'\otimes\phi_2'\otimes\phi_3' \in \widehat{\Pi}^B = \hat{\pi}_1^B\otimes\hat{\pi}_2^B\otimes\hat{\pi}_3^B$. With proposition (6.2.1) we can express those integrals as a product of finite sums, under the restriction of choosing elements in $L^1\left(B^\times\backslash B^\times(\mathbb{A})/B_\infty^\times\hat{R}^\times\right)$.

### 6.2.2 Local trilinear forms

After having considered the global trilinear forms we have to deal with the local trilinear forms. In the split case, they are of the form

$$I_v\left(\phi_v \otimes \phi'_v\right) = \frac{1}{\zeta^2_{F_v}(2)} \cdot \frac{L_v(1, \Pi_v, Adj)}{L_v(1/2, \Pi_v, r)} \int_{F_v^\times \backslash B^\times(F_v)} \langle \Pi_v^B(x_v)\phi_v, \phi'_v\rangle_v \ d^\times x_v$$

with the notations as in section (6.1). By lemma (1.4.5) we deduce that:

- $(a)$ if $v \in Ram(B)$ then $F_v^\times \backslash B^\times(F_v)$ is compact;
- $(b)$ if $v \notin Ram(B)$ then $F_v^\times \backslash B^\times(F_v)$ is not compact.

Thus, in the case $(a)$ the integral

$$\int_{F_v^\times \backslash B^\times(F_v)} \langle \Pi_v^B(x_v)\phi_v, \phi'_v\rangle_v \ d^\times x_v$$

is absolutely convergent. In particular, Ichino recall that, under our hypotheses, i.e. that $\Pi$ is an irreducible unitary cuspidal automorphic representation, one can apply the result of Shalika and Kim [**KS02**], combined with lemma 2.1 in [**I08**], to show the absolute convergence. We have thus sketched the proof of the following lemma.

**Lemma 6.2.3** *If $\Pi$ is an irreducible unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_E)$, the integral*

$$\int_{F_v^\times \backslash B^\times(F_v)} \langle \Pi_v^B(x_v)\phi_v, \phi'_v\rangle_v \ d^\times x_v$$

*is absolutely convergent for each place $v$, in both cases $(a)$ and $(b)$.*

This result allows us to compute the above local integral in the unramified cases (and so in almost all cases) if applied with a second lemma, namely

**Lemma 6.2.4** *Let $F_v$ be non-archimedean, $B$ split at $v$ and assume that $\Pi_v = \Pi_v^B$ is unramified. Let $\Phi$ be the matrix coefficient of $\Pi$ (cfr. §(5.1.1.2)) such that*

- *$\Phi(1) = 1$;*
- *$\Phi(k_1 g k_2) = \Phi(g)$ for $k_2$, $k_2 \in GL_2(O_E)$ and $g \in GL_2(E)$.*

*Take $d^\times x_v$ the Haar measure on $F_v^\times \backslash B^\times(F_v)$ such that $vol(O_v^\times \backslash GL_2(O_v), d^\times x_v) = 1$. Then the integral*

$$\int_{F_v^\times \backslash GL_2(F_v)} \Phi(x_v) \ d^\times x_v = \zeta^2_{F_v} \cdot \frac{L_v(1/2, \Pi_v, r)}{L_v(1, \Pi_v, Adj)}$$

*if it is absolutely integrable.*

**Proof:** See [**I08**], Lemma 2.2. ■

The only step left consist in proving that $\Phi(x_v) = \langle \Pi_v^B(x_v)\phi_v, \phi'_v\rangle_v$ satisfies the hypotheses of the lemma, but we have

- $\Phi(1) = \langle \Pi_v^B(1)\phi_v, \phi'_v\rangle_v = \langle \phi_v, \phi'_v\rangle_v = 1$ for a suitable choice of the pairing $\langle\cdot,\cdot\rangle_v$;
- by the invariance of the pairing and since $\phi$ and $\phi'$ are $GL_2(O_E)-$invariant (see section (3.1.4)),

$$\begin{aligned}\Phi(k_1 g k_2) = \langle \Pi_v^B(k_1 g k_2)\phi_v, \phi'_v\rangle_v = \langle \Pi_v^B(g)\Pi_v^B(k_2)\phi_v, \Pi_v^B(k_1^{-1})\phi'_v\rangle_v = \\ = \langle \Pi_v^B(g)\phi_v, \phi'_v\rangle_v = \Phi(g).\end{aligned}$$

**Note 6.2.5** We can always suppose that $\langle \phi_v, \phi'_v \rangle_v = 1$ exactly at those places $v$ at which $B$ is split and, at the same time, $\Pi_v$ is unramified. Hence we have that

$$\frac{I_v(\phi_v \otimes \phi'_v)}{\langle \phi_v, \phi'_v \rangle_v} = 1$$

and so the infinite product in the Ichino's formula becomes indeed a finite product

$$\prod_v \frac{I_v(\phi_v \otimes \phi'_v)}{\langle \phi_v, \phi'_v \rangle_v} = \prod_{v \in Ram(B)} \frac{I_v(\phi_v \otimes \phi'_v)}{\langle \phi_v, \phi'_v \rangle_v}.$$

### 6.2.3 Normalization of the measure

The Ichino's formula comes with the constant $C$ and to make it explicit we have to fix such constant. We consider the normalization chosen by Ichino in [**I08**], §5.

Let $F$ be a number field and let $\psi = \otimes_v \psi_v$ be a non-trivial additive character of $\mathbb{A}_F/F$. For each place $v$ we choose the Haar measure on $F_v^\times$

$$d^\times z_v = \zeta_{F_v}(1)|z_v|_{F_v}^{-1} dz_v$$

for $dz_v$ the self-dual (additive) Haar measure on $F_v$ with respect to $\psi_v$. We take hence the Haar measure on $B^\times(F_v)$ as

$$d^\times x_v = \zeta_{F_v}(1)\zeta_{F_v}(2)|n(x_v)|_{F_v}^{-2} dx_v$$

for $n$ the reduced norm of $B(F_v)$ and $dx_v$ is the self-dual Haar measure on $B(F_v)$ with respect to the pairing $\psi_v(t(x\bar{y}))$ (for $t$ the reduced trace of $B(F_v)$). We need also the measure on the quotient space, namely the Haar measure on $F_v^\times \backslash B^\times(F_v)$ defined as the quotient of $d^\times x_v$ by $d^\times z_v$ and we follow Ichino denoting it again with $d^\times x_v$ (by abuse of notation). With this choice of measures we have that, considered $d^\times x$ the Tamagawa measure on $\mathbb{A}_F^\times \backslash B^\times(\mathbb{A}_F)$, as stated in [**I08**],

$$d^\times x = C \cdot \prod_v d^\times x_v \qquad \text{with} \qquad C = \zeta_F(2)^{-1}.$$

## 6.3 Back to the Gross–Kudla formula

We are almost ready to deduce how the Gross–Kudla formula is hidden behind the Ichino's formula. We are in the case $F = \mathbb{Q}$, $E = \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ and $\Pi = \pi_f \otimes \pi_g \otimes \pi_h$ for $f$, $g$ and $h$ normalized new eigenforms of level $\Gamma_0(N)$ ($N$ square-free) and weight 2. Furthermore we take $B$ the definite quaternion algebra over $\mathbb{Q}$, ramified at $\{p : \varepsilon_p = -1\}$, with the notation as in section (4.2). Put $S = Ram(B)$ and suppose that $\Pi$ is unramified outside $S$.

We can start with three observations:

- In our case the adjoint $L-$function factors (see [**Co16**], §3.1) as the product

$$L(s, \Pi, Adj) = L(s, \pi_f, Adj)L(s, \pi_g, Adj)L(s, \pi_h, Adj);$$

- The Gross-Kudla $L-$function is a slight modification of the one in Ichino's formula. In particular, the variable $s$ has been shifted by a factor $+\frac{3}{2}$ with respect that of Ichino and, as noticed in example (5.2.2), it differs by a multiplicative factor. For ease of writing we denote $\kappa(s) = \kappa(s, N, f, g, h)$ the factor, so it holds

$$\underbrace{L\left(s + \frac{3}{2}, f \otimes g \otimes h\right)}_{\text{Gross–Kudla}} = \kappa\left(s + \frac{3}{2}\right) \underbrace{L(s, \Pi, r)}_{\text{Ichino}};$$

- As $S$ contains $\{\infty\}$ we have to deal with the local integral at infinity and hence we need the factors of $\zeta$, $L(s,\Pi,Adj)$ and $L(s,\Pi,r)$ at infinity. The former two objects are defined as
$$\zeta_\infty(s) := \pi^{-\frac{s}{2}}\Gamma(s) = \pi^{-\frac{s}{2}}\int_0^\infty e^{-t}t^s\frac{dt}{t}$$
and, as in [**Co16**], §3.1, for $\pi_f$ cuspidal irreducible unitary representation of $GL_2(\mathbb{A}_\mathbb{Q})$ associated with $f \in S_k(\Gamma_0(N))$,
$$L_\infty(s,\pi_f,Adj) := 2\,(2\pi)^{-(s+k-1)}\,\Gamma(s+k-1)\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right).$$
Plugging in the needed values of $s$ in the above formulas and taking $k=2$ we obtain
$$\zeta_\infty(2) = \frac{1}{\pi} \qquad \text{and} \qquad L_\infty(1,\pi_f,Adj) = \frac{\Gamma(k)\Gamma(1)}{2^{k-1}\pi^{k+1}} = \frac{1}{2\pi^3}.$$
In the end, we must define $L_\infty(s,\Pi,r)$ and so we put, following [**GK92**],
$$L_\infty(s,\Pi,r) = L_\infty\left(s+\frac{3}{2}, f\otimes g\otimes h\right) = (2\pi)^{-3-4s}\,\Gamma\left(s+\frac{3}{2}\right)\Gamma\left(s+\frac{1}{2}\right)^3.$$
In particular, at the critical value we have
$$L_\infty\left(1/2,\Pi,r\right) = L_\infty\left(2, f\otimes g\otimes h\right) = (2\pi)^{-5}\,\Gamma\left(2\right)\Gamma\left(1\right)^3 = \frac{1}{2^5\pi^5}.$$

We are now able to recover great part of the Gross–Kudla formula.

**6.3.0.1 Global integrals:** We must choose the two functions $\phi \in \Pi^B$ and $\phi' \in \widehat{\Pi}^B$. Let $R$ be an Eichler order in $B$ with reduced discriminant $N$ and level $L = N/D$ where $D$ is the discriminant of $B$. By proposition (4.2.7) we know there exist (essentially) unique vectors $a_f$, $a_g$ and $a_h$ in $Pic\,(X)\otimes_\mathbb{Z}\mathbb{R}$ (where $X$ is defined in section (4.2.2)), with the usual notation we write $a_f = \sum_{i=1}^n \lambda_i(f)e_i$, $a_g = \sum_{i=1}^n \lambda_i(g)e_i$ and $a_h = \sum_{i=1}^n \lambda_i(h)e_i$, with $n$ the class number of $R$ and $\{e_i\}$ the standard basis of $Pic(X)\otimes_\mathbb{Z}\mathbb{R}$. Define, for $f$ and analogously for $g$ and $h$,
$$\Phi_f : B^\times\backslash B^\times(\mathbb{A})/B_\infty^\times\hat{R}^\times \cong B^\times\backslash\hat{B}^\times/\hat{R}^\times = \coprod_{i=1}^n B^\times x_i\hat{R}^\times \longrightarrow \mathbb{C}$$
$$\bar{x}_i \longmapsto \Phi_f\left(\bar{x}_i\right) = \frac{w_i}{\sqrt{\mathrm{vol}(\hat{R}^\times)}}\lambda_i(f) \in \mathbb{R}$$
as in proposition (6.2.1) and the following observation. We can thus take $\phi_f$ as the map induced by $\Phi_f$ to $\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})$ and hence define $\phi = \phi_f\otimes\phi_g\otimes\phi_h$. By the characterization of the contragredient representation, in the unitary case, as the complex conjugate representation, we take $\phi' = \overline{\phi}$. With this choice of $\phi$ and $\phi'$, and considering both proposition (6.2.1) and paragraph (6.2.1), we have
$$\frac{I(\phi\otimes\overline{\phi})}{\langle\phi,\overline{\phi}\rangle} =$$
$$= \frac{\int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})}\phi_f(x)\phi_g(x)\phi_h(x)\;d^\times x\cdot\int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})}\overline{\phi_f(y)\phi_g(y)\phi_h(y)}\;d^\times y}{\int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})}|\phi_f(x)\,|^2\,d^\times x\int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})}|\phi_g(y)\,|^2\,d^\times y\int_{\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})}|\phi_h(z)\,|^2\,d^\times z} =$$
$$= \frac{\left(\mathrm{vol}(\hat{R}^\times)\sum_{i=1}^n\frac{1}{w_i}\frac{wi^3}{\mathrm{vol}(\hat{R}^\times)^{3/2}}\lambda_i(f)\lambda_i(g)\lambda_i(h)\right)^2}{\left(\sum_{i=1}^n w_i\lambda_i(f)^2\right)\left(\sum_{i=1}^n w_i\lambda_i(g)^2\right)\left(\sum_{i=1}^n w_i\lambda_i(h)^2\right)} =$$
$$= \frac{1}{\mathrm{vol}(\hat{R}^\times)}\cdot\frac{\left(\sum_{i=1}^n w_i^2\lambda_i(f)\lambda_i(g)\lambda_i(h)\right)^2}{\left(\sum_{i=1}^n w_i\lambda_i(f)^2\right)\left(\sum_{i=1}^n w_i\lambda_i(g)^2\right)\left(\sum_{i=1}^n w_i\lambda_i(h)^2\right)}.$$

We deduce that, called $A(F)$ the algebraic part of the Gross–Kudla formula,

$$\frac{I(\phi\otimes\overline{\phi})}{\langle\phi,\overline{\phi}\rangle}=\frac{A(F)}{\mathrm{vol}(\hat{R}^\times)}=\frac{N}{D\cdot 2^4\cdot 3}\prod_{p|D}(p-1)\prod_{p|\frac{N}{D}}\left(1+\frac{1}{p}\right)\cdot A(F)$$

**Remark 6.3.1** As $B^\times\backslash B^\times(\mathbb{A})/B^\times_\infty\hat{R}^\times$ and $\mathbb{A}^\times B^\times\backslash B^\times(\mathbb{A})$ are compact we can notice that step functions, like $\phi_f$, are indeed absolutely integrable and also square integrable.

**6.3.0.2 Adjoint $L-$factors:** By proposition (5.2.13) we know that it holds, as $f$ has trivial character,

$$L(1,\pi_f,Adj)=\frac{2^2\pi^2}{\delta(N)\cdot N\cdot\varphi(N)}\cdot\langle f,f\rangle_{Pet,\Gamma_1(N)}$$

for $\varphi$ the Euler function, $\delta(N)=2$ if $N\leq 2$ and $\delta(N)=1$ otherwise. By definition of Petersson inner product we know that

$$\langle f,f\rangle_{Pet,\Gamma_1(N)}=\underbrace{[\Gamma_1(N):\Gamma_0(N)]}_{=\varphi(n)}\langle f,f\rangle_{Pet,\Gamma_0(N)}$$

thus, recalling that do not exist modular new eigenforms of weight 2 and level $\Gamma_0(N)$ for $N\leq 10$ (see, for example, [**Ste12**] or [**M89**], §2.5 for a way more theoretical approach),

$$L(1,\pi_f,Adj)=\frac{2^2\pi^2}{N}\cdot\langle f,f\rangle_{Pet,\Gamma_0(N)}.$$

By the decomposition of the triple adjoint $L-$function we have

$$\begin{aligned}L(1,\Pi,Adj)&=L(1,\pi_f,Adj)L(1,\pi_g,Adj)L(1,\pi_h,Adj)=\\&=\frac{2^6\pi^6}{N^3}\cdot\langle f,f\rangle_{Pet,\Gamma_0(N)}\langle g,g\rangle_{Pet,\Gamma_0(N)}\langle h,h\rangle_{Pet,\Gamma_0(N)}=\\&=\frac{1}{2^3N^3}\cdot\langle f,f\rangle_{Pet}\langle g,g\rangle_{Pet}\langle h,h\rangle_{Pet}\end{aligned}$$

where the last equality comes from the choice of normalization in (4.2).

**6.3.0.3 A first formula:** We can put together the above formulas obtaining

$$\begin{aligned}&L\left(2,f\otimes g\otimes h\right)\cdot\prod_{v\in Ram(B)}\frac{I_v(\phi_v\otimes\phi'_v)}{\langle\phi_v,\phi'_v\rangle_v}=\\&=\kappa(2)\frac{2^3}{\zeta(2)}\frac{N}{D\cdot 2^4\cdot 3}\prod_{p|D}(p-1)\prod_{p|\frac{N}{D}}\left(1+\frac{1}{p}\right)\cdot A(F)\frac{\langle f,f\rangle_{Pet}\langle g,g\rangle_{Pet}\langle h,h\rangle_{Pet}}{2^3N^3}=\\&=\kappa(2)\prod_{p|D}(p-1)\prod_{p|\frac{N}{D}}\left(1+\frac{1}{p}\right)\cdot A(F)\frac{\langle f,f\rangle_{Pet}\langle g,g\rangle_{Pet}\langle h,h\rangle_{Pet}}{D\pi^2 2^3N^2}.\end{aligned}$$

Thus, as in (5.2.2) we have

$$\kappa(2)=\prod_{p|D}\frac{p^2}{(p-1)^2}\frac{p^2}{p^2-1}\prod_{p|\frac{N}{D}}\frac{p^2}{(p+1)^2}\frac{p^2}{p^2+1}$$

so

$$\kappa(2)\prod_{p|D}(p-1)\prod_{p|\frac{N}{D}}\left(1+\frac{1}{p}\right)\frac{1}{D}=$$

$$=\prod_{p|D}\frac{p^2}{(p-1)^2}\frac{p^2}{p^2-1}\frac{(p-1)}{p}\prod_{p|\frac{N}{D}}\frac{p^2}{(p+1)^2}\frac{p^2}{p^2+1}\frac{p+1}{p}=$$

$$=\prod_{p|D}\frac{p}{p-1}\frac{p^2}{p^2-1}\prod_{p|\frac{N}{D}}\frac{p}{p+1}\frac{p^2}{p^2+1}.$$

**6.3.0.4 The local integrals (at infinity):** We must now deal with the factor at infinity. For this purpose we should investigate the nature of the local component of our $\phi$ as chosen in (6.3.0.1). By definition we can immediately notice that $\phi_\infty$ is the constant function (as the global function is left $B_\infty^\times-$invariant) and it is non-zero (again by definition). Hence, we can follow the argument in section 4.9 in [**Hs17**] for determining the value of the local integral. In [**Hs17**] the component at infinity is indeed chosen to be $\phi_\infty = 1$ (with the notation of the paper, it holds in fact, that $\kappa_i = 0$ for $i = 1, 2, 3$) but the normalization of the local factor by the division on the pairing, guarantees that the computations can be carried on in the same manner. Moreover it is possible noticing that the representation space is 1-dimensional in that case. Also our choice of the *"contragredient"* component is consistent with that of Hsieh and nevertheless we can take a pairing analogous to that considered in §4 of [**Hs17**]. Then we have

$$\frac{I_\infty(\phi_\infty\otimes\overline{\phi}_\infty)}{\langle\phi_\infty,\overline{\phi}_\infty\rangle_\infty}=\frac{L_\infty(1,\Pi_\infty^B,Adj)}{\zeta_\infty(2)^2L_\infty(1/2,\Pi_\infty,r)}\int_{\mathbb{R}^\times\backslash B^\times(\mathbb{R})}\frac{\langle\Pi_\infty^B(x_\infty)\phi_\infty,\overline{\phi}_\infty\rangle_\infty}{\langle\phi_\infty,\overline{\phi}_\infty\rangle_\infty}dx_\infty=\frac{1}{2^2\pi^2}$$

for $dx_\infty$ the Lebesgue measure induced on the quotient. Hsieh managed to obtain such result considering some polynomial representations. In particular, he combined the values of the $L-$functions and of the $\zeta-$function at infinity with his lemma 4.11 in which it is computed explicitly the main part of the above integral. In our case the lemma becomes way more easy as the values $\kappa_i$ and $\kappa_i^*$ in [**Hs17**] are all zero.

**6.3.0.5 The local integrals (at the bad primes):** It remains to compute the local factor at the bad primes, namely those indexed by (rational) primes dividing the level of the modular forms. First of all, we must provide the following structure lemma.

**Lemma 6.3.2** *Let $f$ be a cuspidal holomorphic modular form of level $\Gamma_1(N)$, with $N$ square-free. Hence the cuspidal automorphic representation of $GL_2(\mathbb{A})$ associated with $f$ is such that the local components $\pi_p$ are special representations for each $p$ dividing $N$.*

**Proof (Sketch):** One can prove the lemma characterizing the local representations with the notion of *conductor*. In particular, we refer to [**DI95**], §11.2 and [**Ge75**], Remark 4.25, for definitions and a complete list of conductors. It can be proved that in the square-free case, the conductor forces the above local components to be special representations. We refer also to [**DI95**], Examples 11.5.3 and 11.5.4. ■

Theorem (3.2.26) assures that the local representations $\Pi_p^B$, for $p|N$, are one-dimensional. Hence the local component $\phi_p$ is unique up to a scalar multiple then we can repeat all the observations made in the above paragraph. The definitions of the two $L-$functions are given (as we have already noticed) up to local bad factors. In particular, we can complete them with

$$L_p(s,\pi_p,Adj):=\zeta_p(s+1)\qquad\text{and}\qquad L_p(s,\Pi_p,r):=\zeta_p\left(s+\frac{1}{2}\right)^2\zeta_p\left(s+\frac{3}{2}\right).$$

As noticed in [**Hs17**], §4, we can suppose that the central character of $\Pi_p^B$ is trivial whenever evaluated at $p$ and so, following Hsieh, we can prove that

$$\frac{I_p(\phi_p \otimes \overline{\phi}_p)}{\langle \phi_p, \overline{\phi}_p \rangle_p} = \frac{2}{\zeta_p(1)^2} = \frac{2p^2}{(p-1)^2}$$

for each $p$ dividing $N$ and which is in the set of ramification for $B$ (otherwise the above factor equals 1 by our previous computations).

**6.3.0.6 Conclusion:** We are almost done and it remains only to give some conclusive remarks.

First of all, combining all the information of the above paragraphs we can recover the Gross–Kudla formula, in particular, the local integrals at the bad primes provide the factor $2^t$ in the formula. In fact $t = \#\{p : \varepsilon_p = -1\} = \#\{p : p|D\}$ for $D$ the discriminant of $B$.

Secondly, we must emphasize that the constructions and procedures in [**Hs17**] are more general as they are meant to deal with families of modular forms with varying weights and levels.

In the end, we should mention that the Gross–Kudla formula has been recovered up to some explicit factors and that this is due to the choice of the measures in [**Hs17**]. A deeper analysis of that measures should guarantee our claim.

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