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The Construction of Moduli Spaces and Geometric Invariant Theory

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Declaration of Authorship

I, DINAMO DJOUNVOUNA, declare that this thesis titled, "The Construction of Moduli Spaces and Geometric Invariant Theory" and the work presented in it are my own. I confirm that:

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Signed: Dinamo Djounvouna

Date:July 24, 2017

"Do not worry about your difficulties in Mathematics. I can assure you mine are still greater."

Albert Einstein

Stellenbosh University
Università Degli Studi di Padova

Abstract

Faculty of Sciences
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The Construction of Moduli Spaces and Geometric Invariant Theory

by DINAMO DJOUNVOUNA

In algebraic geometry, classification is a key question. When studying geometric objects, it is desirable to classify them according to different criteria in order to be able to distinguish the equivalent classes in this category. Moduli problems are essentially classification problems. Given a collection of geometric objects, we want to classify them up to a notion of equivalence that we are given. Moduli spaces arise as spaces of solutions of geometric classification problems, and they may carry more geometric structures than the objects we are classifying. The construction of moduli spaces is important in algebraic geometry and difficult in general. To any moduli problem $\mathcal{M}$, corresponds a moduli functor, and the study of the classification problem reduces to that of the representability of that functor. On the other hand, moduli spaces may arise as the quotient of a variety by a group action. Quotients of schemes by reductive groups arise in many situations. Many moduli spaces may be constructed by expressing them as quotients. Geometric Invariant Theory (GIT) gives a way of performing this task in reasonably general circumstances.

The aim of this thesis is to construct moduli spaces both in the functorial point of view and as quotients.

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Contents

List of Abbreviations

$\mathrm{Spec}\,\mathrm{k}$	Spectrum of k
$\mathrm{Aut}(G)$	The group of automorphisms of G
GL_n	**G**eneral **L**inear group
$\dim V$	The dimension of V
$\mathrm{Card}(A)$	The cardinality of A
Set	The category of sets
$\mathrm{Hom}_{\mathcal{C}}(A, B)$	The set of all morphisms from A to B in the category $\mathcal{C}$
Fam	Functor of families of objects in some category $\mathcal{C}$
Fam_g	Functor of families of nonsingular projective curves of genus g
$\mathrm{F}(S)$	Families of objects parametrized by S
id	The identity map
$\mathrm{PSh}(\mathcal{A})$	The category of presheaves on $\mathcal{A}$
Sch	The category of schemes
Sch/S	The category of schemes over S
Var	The category of all varieties
Diff	The category of differentiable manifolds
$\mathrm{Obj}(\mathcal{A})$	The set of objects of the category $\mathcal{A}$
$\mathrm{F}(d, n)$	The Grassmannian functor
$\mathrm{G}(d, n)$	The Grassmannian of degree d in an n-dimensional vector space
$\mathbb{A}^n$	The n-dimensional affine space
$\mathbb{P}^n$	The n-dimensional projective space

Chapter 1

Introduction and Summary

1.1 Introduction

In algebraic geometry, one of the significant fields of research is Moduli Theory. Moduli theory is the study of the way in which objects in algebraic geometry (or in other areas of mathematics) vary in families and is fundamental to an understanding of the objects themselves. The theory goes back at least to Riemann in the mid-nineteenth century, but moduli spaces were first rigorously constructed in the 1960s by Mumford and others. The theory has continued to develop since then, perhaps most notably with the infusion of ideas from physics after 1980. Frequently, one would like to parametrize a class of objects of certain fixed type, that is to find an algebro-geometric object (such as a variety, a scheme, a manifold or an algebraic stack) whose underlying set of points corresponds bijectively to the equivalence classes of the objects we want to parametrize.

Then, a moduli space appears to be a solution to the classification problem. If a moduli space M exists, then the bijection between the set of equivalence classes and the underlying set of points $|M|$ of M allows us to transport any structure on $|M|$ to the set of equivalence classes, so that the latter can be seen as a geometric object.

1.2 Motivation

As Brian Osserman [23] said in his introduction,

> The idea of a moduli space is that its points correspond to a geometric object of interest. The underlying set of the space is thus determined, but not its geometric structure. In most cases, one can describe a geometric structure in an ad hoc manner, but it is not immediately obvious how to rigorously formulate the idea that the resulting object is "the" moduli space for the problem. This is the problem addressed by the theory of representable functors, with the key idea being that we specify not only the objects of interest, but also what families of these objects should look like.

As above, moduli spaces arise in nature as geometric solutions to classification problems in algebraic geometry. More precisely, given a collection $\mathcal{A}$ of objects (such as vector bundles, varieties, algebraic manifolds, subschemes, linear subspaces of dimension d of an n-dimensional vector space V, morphisms, elliptic curves), we would like to parametrize them with respect to an equivalence relation that we are free to choose. Moduli spaces tell us which objects can be considered as the same and how the objects vary continuously in families. We need the notion of families of objects in order to assign a topology on the moduli space.

A simple motivating example is the following. Given an algebraically closed field k and a positive integer n, one may consider the set of all lines in k^{n+1} passing through the origin. The set $\mathcal{A}$ of all lines passing through the origin is in fact the set of equivalence classes of all lines in k^{n+1} with respect to the colinearity (which defines an equivalence relation on the set of all lines). From basic algebraic geometry, it is known that the set $\mathcal{A}$ is in fact the projective space $\mathbb{P}^n$. Moreover, $\mathbb{P}^n$ is not just a set; it is a topological space endowed with the Zariski topology. This gives an algebro-geometric object.

We would like to do the same construction with the collection of all linear subspaces of dimension d of k^{n+1}; that is we would like to endow the collection of d-dimensional linear subspaces of k^{n+1} with an algebro-geometric structure (for instance a structure of projective variety or scheme). Our task will be to prove that it is always possible to endow this collection with a structure of variety. The geometric object obtained is called the Grassmannian of degree d. The Grassmannian is a fundamental object of study in algebraic geometry. This object plays an important role in the construction of moduli spaces, and found also application in security network channel. They are not only important in constructing other interesting algebraic object but have a rich structure (structure of variety, scheme) and are of interest in their own right.

Another important object that motivates the development of Moduli Theory was the construction of an algebro-geometric object parametrizing Riemann surfaces. This study started with Riemann, and he was the first to use the term moduli space for the object parametrizing a collection of objects.

1.3 Objectives

The objectives of this thesis are to:

(1) Give a rigorous formulation of the notion of moduli problem and moduli space;

(2) Construct a fine moduli space (the Grassmannian) for the moduli problem of d-dimensional linear subspaces of an n-dimensional vector space V;

(3) Construct a coarse moduli space for the moduli problem of projective curves of genus g;

(4) Give a relationship between the notions of coarse moduli spaces and categorical quotients;

(5) Study the moduli problem of projective hypersurfaces using the techniques of Geometric Invariant Theory due to Mumford.

1.4 Structure of this thesis

This thesis is divided into two parts:

- Moduli problems and the representability of the moduli functor (Chapters 2, 3 and 4),
- Moduli spaces and Geometric Invariant Theory (Chapters 5, 6 and 7).

The first part of this thesis is devoted to a rigorous formulation of the main objects of concern. In Chapter 2, following the ideas of [1, Chapter 4], we define the main notions of this thesis, such as family of objects, moduli functors, fine moduli spaces, universal families, coarse moduli spaces and tautological families. Chapter 3 is devoted to the construction of a concrete example of a fine moduli space, namely the Grassmannian $\mathrm{G(d, n)}$ which parametrizes all d-dimensional vector subspaces of an n-dimensional vector space V over a field k. Following the ideas of [10, Chapter 8], [23, Appendix A] and [6, Chapter VI], we proved that the moduli problem of d-dimensional linear subspaces of an n-dimensional k-vector space V admits a fine moduli space called the Grassmannian. The notion of the Grassmannian is then the generalization of the projective space. However, most of the moduli problems that arise in nature do not admit a fine moduli space. An example of such moduli problem is that consisting of the parametrization of nonsingular projective curves of a fixed genus g up to isomorphism. The goal of Chapter 4 is to show that the moduli problem of nonsingular projective curves does not admit a fine moduli space, but does, however, admit a coarse moduli space.

The second part of the thesis is devoted to the construction of moduli spaces as quotient varieties. The Geometric Invariant Theory, developed by Mumford [4], is one of the oldest methods dealing with the construction of moduli spaces. This method provides a useful tool to compute moduli spaces as quotient varieties. The aim of Chapter 5 is to describe the necessary tools for the construction of coarse moduli spaces as quotients. We first recall the different notions of quotients, such as categorical quotient, geometric quotient and good quotient following the exposition of [21], [4], [5], and [19]. We then give a relationship between the notion of a coarse moduli space and a categorical quotient. Constructing quotient leads to the study of stable objects for the group action; and stable elements form an open subset. A coarse moduli space can then be realized as a quotient of the set of stable or semistable elements by a group action. Therefore, to construct a moduli space we need to compute stable or semistable elements for the group action. The computation of stable or semistable elements of a group action is sometimes not an easy task. A powerful criterion for the description of stable or semistable elements is given by the Hilbert-Mumford numerical criterion. This is the goal of Chapter 6. In Chapter 7, following the ideas of [21], [5], [19] and [13], we apply those techniques to the moduli problem of hypersurfaces that consists of classifying hypersurfaces of fixed degree d in a projective space $\mathbb{P}^n_{\mathrm{k}}$ up to the action of the automorphism group PGL_{n+1}.

1.5 Background

Given a set $\mathcal{M}$ of objects a certain set of axioms defined on them, along with a rule $\sim_{\mathcal{M}}$ for saying when two objects can be identified (equivalence relation), one looks for a variety that classifies these objects. In this context, to classify objects means to find a variety whose closed points are in bijective correspondence with the classes of objects we want to parametrize. As we want our moduli space to be assigned with a topological structure, the first step should be to formulate a concrete notion of a family of these objects, and from that notion we define a moduli functor that encodes the information on the objects of our study. A moduli problem consists of two things: a class of objects of concern along with an equivalence relation defined on our objects, and a notion of family of objects. The study of a moduli problem reduces to that of its associated moduli functor, and for this reason we frequently identify a moduli problem with its moduli functor. Then, a natural question that arises immediately when studying a moduli problem is: to what extent, is the moduli functor representable? An affirmative answer to this question leads to the notion of fine moduli spaces. A variety representing a moduli functor is called a fine moduli space and it does really parametrize the objects of concern. This representing variety comes with a universal family corresponding to the identity morphism id_M. Most moduli functors fail to be representable. However, even if the moduli functor is not representable, one can still parametrize the objects we are concerned with by looking for a weaker notion rather than the representability. By requiring only a natural transformation which satisfies the additional conditions of

(1) being bijective when restricted to closed points, and

(2) of being universal among morphism from an arbitrary object to M,

we obtain the notion of a coarse moduli space, and it gives again a parametrization of our geometric objects of concern. The notion of a coarse moduli space is the second best possible notion of a moduli space.

Chapter 2

Moduli Problems and Moduli Spaces

2.1 Overview of this chapter

Suppose we are given a set $\mathcal{M}$ of objects possessing some fixed algebro-geometric structure in Geometry, along with an equivalence relation $\sim_{\mathcal{M}}$ on $\mathcal{M}$ that we are free to choose. Then we may consider the set of equivalence classes $\mathcal{M}/\sim_{\mathcal{M}}$. This defines a set. But we can get on $\mathcal{M}/\sim_{\mathcal{M}}$ more than a set-theoretic structure. Our goal is to describe $\mathcal{M}/\sim_{\mathcal{M}}$ algebro-geometrically. More precisely, it may be possible to put an algebro-geometric structure on the set $\mathcal{M}/\sim_{\mathcal{M}}$. A first fundamental question a moduli problem addresses is to find an object M such that there is a natural bijective correspondence between the underlying set $|M|$ and the set $\mathcal{M}/\sim_{\mathcal{M}}$. If there is such a bijective correspondence, from this bijection we may then induce the structure on $|M|$ to $\mathcal{M}/\sim_{\mathcal{M}}$. For this, we start by choosing carefully a category $\mathcal{C}$ called a parameter category, in such a way that its elements are sets possessing that algebro-geometric structure and its morphisms are morphisms of the underlying sets that are compatible with the algebro-geometric structure we want to put on $\mathcal{M}/\sim_{\mathcal{M}}$. After choosing the parameter category for our moduli problem as in 2.2.9, we look for an object $M \in \mathcal{C}$ that represents the moduli functor. If we are able to find such an object M, we say that M is a fine moduli space. The notion of a fine moduli space appears then to be the best solution possible of the moduli problem. However, most of moduli problems of interest do not admit fine moduli spaces (for instance the moduli problem $\mathcal{M}_g$ consisting to classify complex nonsingular projective curves of fixed genus g is not representable, as shown in Chapter 4), and we can still get a solution by relaxing the conditions on the moduli functor. By requiring only for a natural transformation with some extra conditions and the universality of the moduli functor, we obtain the notion of a coarse moduli space which is the best approximation of the notion of a fine moduli space that we can get. A coarse moduli space still classifies our objects.

As we want to understand the geometry of the elements of $\mathcal{M}$, the next step will be naturally to relate the structure on $\mathcal{M}/\sim_{\mathcal{M}}$ to the structure already on the elements of $\mathcal{M}$ in a natural way. This is done by using the notion of a family of elements of $\mathcal{M}/\sim_{\mathcal{M}}$. So, the second fundamental question to be answered while studying a moduli problem is to investigate the ways in which the properties of families influence the structure on M.

2.2 Families of objects

The notion of a family of objects is fundamental in Moduli Theory and it is impossible to define a moduli problem without giving a clear notion of families of objects we are interested in. In fact, the notion of families allows us to assign a topology on the moduli space. In this section, we define families of objects following the exposition of [1, Chapter 4]. For this, we start by determining the notion of a classification problem in 2.2.2. Then, in 2.2.9 we fix the requirements for the category in which we would like to find a solution for our moduli problem. Having defined the parameter category, we then formulate the notion of families of objects of $\mathcal{M}$ in 2.2.11 as the values taken by a functor called classifying functor or moduli functor on base objects in the parameter category.

2.2.1 Definition of a classification problem

A *classification problem* is a problem of the following form: given a set of mathematical objects together with a notion of equivalence relation, find out which object we have up to this equivalence relation. In order words, given a set $\mathcal{M}$ of objects and an equivalence relation $\sim$ defined on $\mathcal{M}$, a classification problem consists of describing the set of equivalence classes $\mathcal{M}/\sim$ algebro-geometrically (that is to assign an algebro-geometric structure to $\mathcal{M}/\sim$).

Assume that we are given a collection $\mathcal{M}$ of algebro-geometric objects of certain fixed type, together with a notion of an equivalence relation $\sim_{\mathcal{M}}$ on these objects. Let $\mathcal{M}/\sim_{\mathcal{M}}$ be the set of equivalence classes. Our ultimate goal is to classify these objects according to different criteria. There are two types of classification problems: the set-theoretical and the structural classification problems.

Definition 2.2.2. A *set-theoretic classification problem* over $\mathcal{M}$ consists of finding some discrete invariants of $\mathcal{M}$. A discrete invariant is a function $f : \mathcal{M}/\sim_{\mathcal{M}} \longrightarrow \mathbb{Z}$ that partitions $\mathcal{M}/\sim_{\mathcal{M}}$.

Example 2.2.3. Suppose that we are given the set $\mathcal{M}_1$ of all elliptic curves. We define the equivalence relation on $\mathcal{M}_1$ as follows: two elliptic curves are equivalent if and only if there exists an isomorphism between them. We know that the j-invariant function classifies them. So, the j-invariant function is a solution to the set-theoretic classification problem of elliptic curves.

Example 2.2.4. Let $\mathcal{M}$ be the set of all finite dimensional vector spaces over some fixed field k, and let say that two vector spaces are equivalent if there exists an isomorphism between them. As two finite dimensional k-vector spaces V and W are isomorphic if and only if they have the same dimension, it follows that these objects are classified by their dimensions. So, the discrete invariant is given by the function

$$\begin{aligned} f : \mathcal{M}/\sim_{\mathcal{M}} &\longrightarrow \mathbb{N} \\ [V] &\longmapsto \dim V. \end{aligned}$$

Example 2.2.5. Let $\mathcal{M}$ be the collection of all finite subsets of $\mathbb{R}$. We define the equivalence relation $\sim_{\mathcal{M}}$ on $\mathcal{M}$ as follows. Two subsets are equivalent if there exists a bijection between them. But, we know that two finite sets are in bijection if and only if they have the same cardinality. So, as in the previous example, we define a map

$$\begin{array}{rcl} f : \mathcal{M}/\sim_{\mathcal{M}} & \longrightarrow & \mathbb{N} \\ [A] & \longmapsto & \mathrm{Card}(A). \end{array}$$

On the other side, one usually would like to endow $\mathcal{M}/\sim_{\mathcal{M}}$ with a geometric structure in order to understand the geometry of the objects we are interested in. This is known as a structural classification problem.

Definition 2.2.6. A *structural classification problem* consists of finding a variety M whose closed points are in bijective correspondence with the equivalence classes $\mathcal{M}/\sim_{\mathcal{M}}$. When the structure on $\mathcal{M}/\sim_{\mathcal{M}}$ is naturally compatible with that on the elements of $\mathcal{M}$, we say that the structural classification problem is well formulated.

Example 2.2.7. Let us find all possible plane conics. First, a plane conic is defined by the zero set $\{(x, y) \in \mathbb{R} : h(x, y) = 0\}$, where h is some polynomial of degree 2, $h(x, y) = ax^2 + by^2 + cxy + dx + ey + f$. Two conics are equivalent if they have the same points. All conics are determined by variation of the coefficients a, b, c, d, e, f. So, there are six parameters to characterize all conics. But, for any non-zero scalar λ, the parameters (a, b, c, d, e, f) and $(\lambda a, \lambda b, \lambda c, \lambda d, \lambda e, \lambda f)$ determine the same conic. This means that the parameter space is given by the set

$$\{(a, b, c, d, e, f) \in \mathbb{R}^6 : (a, b, c) \neq (0, 0, 0)\}/\sim_{\mathcal{M}}$$

with the equivalence relation being defined by identifying the pairs (a, b, c, d, e, f) and $(\lambda a, \lambda b, \lambda c, \lambda d, \lambda e, \lambda f)$ for any non-zero scalar λ.

2.2.8 How to put an additional geometric structure on $\mathcal{M}/\sim_{\mathcal{M}}$

Suppose we are given a set $\mathcal{M}$ of algebro-geometric objects along with an equivalence relation $\sim_{\mathcal{M}}$ on $\mathcal{M}$, and that we want to endow the set of equivalence classes $\mathcal{M}/\sim_{\mathcal{M}}$ with some fixed algebro-geometric structure. The first thing to do is to organize our search in the category $\mathcal{C}$ whose objects are sets possessing that algebro-geometric structure and whose morphisms are those on the underlying sets preserving that algebro-geometric structure. The category $\mathcal{C}$ is called the *parameter category* for the classification problem. Then we try to find an object M in the category $\mathcal{C}$ for which the elements of the underlying set of M, denoted $|M|$, are naturally in bijective correspondence with $\mathcal{M}/\sim_{\mathcal{M}}$. If such an object M exists, then the bijection between the underlying set $|M|$ and $\mathcal{M}/\sim_{\mathcal{M}}$ allows us to transport the structure on $|M|$ to $\mathcal{M}/\sim_{\mathcal{M}}$.

2.2.9 Requirements for the parameter category

Suppose that we want to put an algebro-geometric structure on the set of equivalence classes $\mathcal{M}/\sim_{\mathcal{M}}$. The choice of the parameter category is fundamental. The parameter category $\mathcal{C}$ must be chosen in such a way that the following conditions are satisfied [1]:

(1) $\mathrm{Obj}(\mathcal{C}) = \{$sets possessing the given algebro-geometric structure$\}$,

and the morphisms in $\mathcal{C}$ are maps on the underlying sets which are compatible with that structure, that is to say, those morphisms of the underlying sets that preserve the given algebro-geometric structure. Moreover, if $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are morphisms in $\mathcal{C}$ and $g \circ f$ is the composition, then the corresponding map of the underlying $|g \circ f|$ is the composition $|g| \circ |f|$ of the underlying maps of sets.

(2) $\mathcal{C}$ is a locally small category, that is, for all $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a set.

(3) Let

$$\begin{array}{rcl} \varphi : \mathcal{C} & \longrightarrow & \mathrm{Set} \\ A & \longmapsto & |A| \\ (A \xrightarrow{f} B) & \longmapsto & (|A| \xrightarrow{|f|} |B|) \end{array}$$

be the functor from the category $\mathcal{C}$ to the category Set of sets, which to each object A in $\mathcal{C}$, associates the underlying set $|A|$ of A, and to each morphism $f : A \longrightarrow B$ in $\mathcal{C}$, associates the associated set-theoretic map $|f| : |A| \longrightarrow |B|$. Then for each $A, B \in \mathcal{C}$, the induced map by the above functor

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(A, B) & \longrightarrow & \mathrm{Hom}_{\mathrm{Set}}(|A|, |B|) \\ f & \longmapsto & |f| \end{array}$$

must be injective.

(4) There exists an object P in $\mathcal{C}$ such that $|P| = \{\mathrm{pt}\}$ is a single point and there exists a canonical identification $\mathrm{Hom}_{\mathcal{C}}(P, M) \cong |M|$ for each $M \in \mathcal{C}$ in the following sense: if $f : M \longrightarrow N$ is a morphism in $\mathcal{C}$, then $|f| : |M| \longrightarrow |N|$ is defined by the natural map

$$\begin{array}{rcl} \mathrm{Hom}(P, M) & \longrightarrow & \mathrm{Hom}(P, N) \\ \psi & \longmapsto & f \circ \psi. \end{array}$$

A point P above is called a base point.

Example 2.2.10. The category of schemes over an algebraically closed field k obviously satisfies the requirements above. In this case, the base object is just $\mathrm{Spec}\mathrm{k}$ and, $|M|$ will denote the set of closed points of M.

Next, we would like to relate in a natural way the given algebro-geometric structure to the structure of the elements of $\mathcal{M}$. This is done by using the notion of families of objects that we introduce now.

2.2.11 Families of objects

Given a classification problem $(\mathcal{M}, \sim_{\mathcal{M}})$, and a parameter category $\mathcal{C}$ satisfying the conditions 2.2.9 above, we want to endow the set $\mathcal{M}/\sim_{\mathcal{M}}$ of equivalence classes with an algebro-geometric structure in a natural way. This leads to the notion of families of objects. For instance, many objects, like elliptic curves, that are usually defined over fields, one can extend the notion to an elliptic curve over a scheme B. Such an object can be thought of as a family of elliptic curves, one elliptic curve over each point in the underlying set of B.

Definition 2.2.12. [1, Definition 4.2.3] A *functor of families of objects* for the classification problem $(\mathcal{M}, \sim_{\mathcal{M}})$ is a contravariant functor $\mathrm{Fam} : \mathcal{C} \longrightarrow \mathrm{Set}$ such that:

(a) $\mathrm{Fam}(P) = \mathcal{M}/\sim_{\mathcal{M}}$, for each base point P of $\mathcal{C}$ as defined above,

(b) For each $B \in \mathcal{C}$, we can define an equivalence relation $\sim_B$ on the set $\mathrm{Fam}(B)$ which restricts to the initial equivalence relation $\sim_{\mathcal{M}}$ when B is a base point P,

(c) For any morphism $f : A \longrightarrow B$ in $\mathcal{C}$, the induced morphism

$$\mathrm{Fam}(f) : \mathrm{Fam}(B) \longrightarrow \mathrm{Fam}(A)$$

transforms $\sim_B$-equivalent elements in $\mathrm{Fam}(B)$ into $\sim_A$-equivalent elements in $\mathrm{Fam}(A)$: if $T, T' \in \mathrm{Fam}(B)$ such that $T \sim_B T'$, then $\mathrm{Fam}(f)(T) \sim_A \mathrm{Fam}(f)(T')$.

Definition 2.2.13. [1, Remark 4.2.4] Let B be an object in $\mathcal{C}$. A *family of objects of $\mathcal{M}$ parametrized by B* (or family of $\mathcal{M}$ over B) is an element of the set $\mathrm{Fam}(B)$.

Hence, equivalence relation of families of objetcs satisfies:

(1) A family parametrized by a base point (a single point) is an object of $\mathcal{M}/\sim$.

(2) The equvalence class of two families X and X' paramtrized respectively by base points P and P' is the equivalence class of the two base points P and P'.

Notation 2.2.14. The map $\mathrm{Fam}(f)$ in 2.2.12 is frequently denoted by f^*, and for each family T over B we say that the family $f^*(T)$ is obtained by base change along f, and we sometimes call it the *pullback family of T along f*. In this case we write $f^*T = A \times_B T = T_A$. One of the most important cases that we will use several times is when A is just a single point b, that is to say, if we have $\{b\} \hookrightarrow B$ an inclusion of a point, then T_b is the fiber of the family over the point $b \in B$.

2.2.15 The classification functor

As our ultimate goal is to classify the elements of $\mathcal{M}$ up to the equivalence relation $\sim_{\mathcal{M}}$, and by the hypotheses of the functor of families Fam, the assigment

$$\begin{array}{rcl} \mathrm{F} : \mathcal{C}^{\mathrm{opp}} & \longrightarrow & \mathrm{Set} \\ B & \longmapsto & \mathrm{Fam}(B)/\sim_B \\ (f : A \longrightarrow B) & \longmapsto & [f^*] : \mathrm{Fam}(B)/\sim_B \longrightarrow \mathrm{Fam}(A)/\sim_A \\ & & T/\sim_B \longmapsto f^*(T)/\sim_A \end{array}$$

defines a unique functor called the *classification functor* or the *moduli functor* associated to the moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$. By the use of the axioms of the functor Fam defined in 2.2.12, it follows that the moduli functor associated with a moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$ satisfies the following conditions:

(1) For each object $A \in \mathcal{C}$, there exists a set $\mathrm{F}(A)$ of equivalent families of objects of $\mathcal{C}$ parametrized by A, with respect to some equivalence relation $\sim_A$ on $\mathrm{Fam}(A)$ that restricts to the initial relation $\sim_{\mathcal{M}}$ on $\mathcal{M}$ if A is taken to be a base point,

(2) For every morphism $f : A \longrightarrow B$ in the category $\mathcal{C}$, there is a pullback map (or a base change map) $\mathrm{F}(f) := [f^*] : \mathrm{F}(B) \longrightarrow \mathrm{F}(A)$ which transforms equivalent families over B into equivalent families over A.

The pullback map $[f^*] : \mathrm{F}(B) \longrightarrow \mathrm{F}(A)$ satisfies the following:

(i) $\mathrm{F}(P) = \mathcal{M}/\sim_{\mathcal{M}}$, for each base point P in $\mathcal{C}$. This means that a family parametrized by a base point P is an object of $\mathcal{M}/\sim_{\mathcal{M}}$. The equvalence class of two families X and X' parametrized respectively by the base points P and P' is the equivalence class of P and P'.

(ii) For any morphism $\phi : S' \longrightarrow S$ and for any family X parametrized by S, there is an induced family $[\phi^* X]$ parametrized by S'.

(iii) If $f = \mathrm{id}_A : A \longrightarrow A$ is the identity map, then $[\mathrm{id}_A^* \pi] = \pi$ for every $\pi \in \mathrm{F}(A)$.

(iv) The equivalence relation on families is compatible with the pullback. In order words, if $f : A \longrightarrow B$ is a morphism in $\mathcal{C}$ and $\pi, \pi' \in \mathrm{F}(B)$ are two equivalent families over the same base B; that is, $\pi \sim_B \pi'$, then the base change families $[f^*\pi]$ and $[f^*\pi']$ are equivalent families over A.

(v) If $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are morphisms in $\mathcal{C}$ and π is a family over A, then $[(g \circ f)^*\pi] = [f^*][g^*]\pi$.

The associated moduli functor defined above is fundamental for the study of the given moduli problem, and we will see that the study of a classification problem reduces to that of the representability of its associated classification functor. Before to take further the study of a classification, we need to define the notion of the representability of a functor.

2.3 Representable functors

The notion of a representable functor that we define below is fundamental in the study of any classification problem, and in particular for the notion of moduli spaces that will be defined in the next section. The aim of this section is to provide the necessary tools of representable functors that we will use to study the classification problem in general, and the moduli problem in particular.

The notion of representable functors is the modern language for the classification problem. In fact, given any classification problem $(\mathcal{M}, \sim_{\mathcal{M}})$ there is an associated classification functor F as in 2.2.15 such that $\mathrm{F}(P) = \mathcal{M}/\sim_{\mathcal{M}}$, for any base point P. Since we are looking for an object M in the parameter category $\mathcal{C}$ such that the elements of the set $|M|$ are canonically in bijective correspondence with the equivalence classes $\mathcal{M}/\sim_{\mathcal{M}}$, it will be enough to look for an element M in the category $\mathcal{C}$ whose functor of points $\mathrm{h}_M = \mathrm{Hom}_{\mathcal{C}}(-, M)$ is isomorphic to the functor F. This is because if there exists such an isomorphism $\mathrm{F} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, M)$, by restricting to a base point P we obtain that $\mathcal{M}/\sim_{\mathcal{M}} = \mathrm{F}(P) \cong \mathrm{Hom}(P, M)$. But, by the fourth assumption on the parameter category $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(P, M) \cong |M|$. By the Yoneda's lemma, the functor $\mathrm{Hom}_{\mathcal{C}}(-, M)$ does really determine uniquely the object M.

2.3.1 The functor of points

We can embed any (locally small) category $\mathcal{C}$ into the category $\mathrm{Funct}(\mathcal{C}^{opp}, \mathrm{Set})$ of presheaves on $\mathcal{C}$. This embedding allows viewing an object of $\mathcal{C}$ as a functor. There are many advantages viewing an object as a functor. First, it is much easier to describe the product of two objects by their respective representing functors rather than those objects themselves. Second, this way of viewing objects as functors is useful in moduli theory. In fact, in the moduli theory, one usually seeks for a variety that parametrizes a class of some objects of a given structure. It seems to be easier to look for a functor that represents such an object rather than that object itself. Another advantage of this embedding is that we can gain a lot of information about the geometry of an object by studying its representing functor. As we are mostly interested here in formulating a notion of moduli spaces, this justifies the motivation of this subsection.

Definition 2.3.2. [11] Let Z be a scheme. The *Z-valued points* of a scheme X are defined to be maps $Z \longrightarrow X$, and denoted $X(Z)$. For a ring A, the A-valued points of a scheme X are defined to be the $(\mathrm{Spec} A)$-valued points of the scheme X, and denoted $X(A)$. The most common case of this is when A is a field.

In general to understand the structure of an object X in some good category $\mathcal{A}$ (such as locally small categories, small categories), it is sufficient to know all the morphisms from any object Y to X. We recall that a category $\mathcal{A}$ is *locally small* if $\mathrm{Hom}_{\mathcal{A}}(Y, X)$ is a set for each $X, Y \in \mathcal{A}$. And a category $\mathcal{A}$ is *small* if it is locally small and its collection of objects $\mathrm{Obj}(\mathcal{A})$ is a set.

The hypothesis 2.2.9 on the parameter category $\mathcal{C}$ shows that it is actually a small category. This simplifies a lot of difficulties.

Definition 2.3.3. [6] Let X be an object in a locally small category $\mathcal{A}$. The *functor of points* of X is the contravariant functor

$$\begin{aligned} \mathrm{h}_X := \mathrm{Hom}_{\mathcal{A}}(-, X) : \mathcal{A} &\longrightarrow \mathrm{Set} \\ Y &\longmapsto \mathrm{Hom}_{\mathcal{A}}(Y, X) \\ f &\longmapsto f_* := - \circ f \end{aligned}$$

from the category $\mathcal{A}$ to the category of sets defined by assigning to every object Y in $\mathcal{A}$ the set $\mathrm{Hom}_{\mathcal{A}}(Y, X)$ of morphisms in $\mathcal{A}$ from Y to X; and that assigns to every morphism $f : Y \longrightarrow Z$ in $\mathcal{A}$ the induced morphism

$$\begin{aligned} \mathrm{h}_X(f) := f_* : \mathrm{Hom}_{\mathcal{A}}(Z, X) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(Y, X) \\ \alpha &\longmapsto \alpha \circ f \end{aligned}$$

Furthermore, a morphism $f : X \longrightarrow Y$ in $\mathcal{A}$ induces a natural transformation of functors:

$$\mathrm{h}_f : \mathrm{h}_X \longrightarrow \mathrm{h}_Y$$

given by:

$$\begin{array}{rcl} \mathrm{h}_{f,Z} := \mathrm{h}_f(Z) : \mathrm{h}_X(Z) = \mathrm{Hom}_{\mathcal{A}}(Z,X) & \longrightarrow & \mathrm{h}_Y(Z) = \mathrm{Hom}_{\mathcal{A}}(Z,Y) \\ \alpha & \longmapsto & f \circ \alpha. \end{array}$$

Definition 2.3.4. Let $\mathcal{A}$ be a locally small category. We recall that a presheaf with values in $\mathcal{A}$ is a functor $\mathrm{F} : \mathcal{A}^{\mathrm{opp}} \longrightarrow \mathrm{Set}$. The presheaves with values in $\mathcal{A}$ form a category, denoted $\mathrm{Psh}(\mathcal{A}) = \mathrm{Funct}(\mathcal{A}^{\mathrm{opp}}, \mathrm{Set})$ whose morphisms are given by natural transformations. With these notations, we obtain a functor $\mathrm{h} : \mathcal{A} \longrightarrow \mathrm{PSh}(\mathcal{A})$ given by:

$$\begin{array}{rcl} \mathrm{h} : \mathcal{A} & \longrightarrow & \mathrm{Psh}(\mathcal{A}) \\ X & \longmapsto & \mathrm{h}_X \\ (X \xrightarrow{f} Y) & \longmapsto & (\mathrm{h}_X \xrightarrow{\mathrm{h}_f} \mathrm{h}_Y). \end{array}$$

Remark 2.3.5. The category $\mathrm{Psh}(\mathcal{A})$ of presheaves on $\mathcal{A}$ is not in general locally small. However, it is always locally small if the category $\mathcal{A}$ is small.

This is the first step to construct a functor that represents the scheme X. The next step should be naturally to show that this way of representing a scheme really does determine that scheme. The latter follows easily from the use of Yoneda's lemma. Yoneda's lemma says, roughly speaking, that the condition of representing a given functor uniquely determines an object of a category.

2.3.6 Yoneda's lemma

Lemma 2.3.7 (Yoneda's Lemma)**.** [6, Lemma VI-1] Let $\mathcal{A}$ be a locally small category. Then for any $X \in \mathcal{A}$ and any presheaf $\mathcal{F} \in \mathrm{Psh}(\mathcal{A})$

(a) There is a bijection

$$\begin{array}{rcl} \{\text{natural transformations } \eta : \mathrm{h}_X \longrightarrow \mathcal{F}\} & \longleftrightarrow & \mathcal{F}(X) \\ \eta & \longmapsto & \eta_X(\mathrm{id}_X) \end{array}$$

that assigns to every natural transformation η the value $\eta_X(\mathrm{id}_X)$.

(b) If the functors $\mathrm{Hom}_{\mathcal{A}}(-,X)$ and $\mathrm{Hom}_{\mathcal{A}}(-,X')$ from $\mathcal{A}$ to the category of sets are isomorphic, then X and X' are isomorphic. More generally, the maps of functors from $\mathrm{Hom}_{\mathcal{A}}(-,X)$ to $\mathrm{Hom}_{\mathcal{A}}(-,X')$ are the same as maps from X to X'; that is, the functor

$$\begin{array}{rcl} \mathrm{h} : \mathcal{A} & \longrightarrow & \mathrm{Funct}(\mathcal{A}^{\mathrm{opp}}, \mathrm{Set}) = \mathrm{Psh}(\mathcal{A}) \\ X & \longmapsto & \mathrm{h}_X \end{array}$$

is an equivalence of $\mathcal{A}$ with a full subcategory of the category of functors.

Definition 2.3.8. Let $\mathcal{A}$ be a locally small category. A presheaf $\mathcal{F} \in \mathrm{Psh}(\mathcal{A})$ is *representable* if there exists an object $X \in \mathcal{A}$ and an isomorphism of functors

$$\Psi : \mathcal{F} \longrightarrow \mathrm{Hom}_{\mathcal{A}}(-,X).$$

Therefore, $\mathcal{F}(Y) \simeq \mathrm{Hom}_{\mathcal{A}}(Y, X)$ functorially in each $Y \in \mathcal{A}$. We say that the pair (X, Ψ) represents $\mathcal{F}$. A use of the Yoneda Lemma shows that the pair (X, Ψ) is uniquely determined. An example of a representable functor is given by the functor of points $\mathrm{Hom}_{\mathcal{A}}(-, X)$ for each $X \in \mathcal{A}$, and in this case the representative is just the pair $(X, \mathrm{id}_{\mathrm{Hom}_{\mathcal{A}}(-,X)})$.

The representability of functors is a useful property in general, and in particular for functors representing moduli problems, it allows us to gain more information on the structure of the objects to be classified.

2.4 Moduli problems

In this section, we introduce the main concepts of our study: the notion of moduli problem and moduli space. A moduli problem is essentially a classification, and a moduli space should be ideally a solution to a moduli problem.

2.4.1 Definition of a moduli problem

Suppose we are given an object C of a parameter category $\mathcal{C}$ as defined in 2.2.9, a classification problem $(\mathcal{M}, \sim_{\mathcal{M}})$ where the morphisms and the objects to be classified are defined intrinsically in terms of the given object C, and a functor Fam of families of objects of $\mathcal{M}$ defined over the elements of the parameter category $\mathcal{C}$. Assume in this subsection that a family of object $X \in \mathrm{F}(S)$ is always given by a morphism $X \longrightarrow S$ satisfying some extra conditions. In fact, in all the thesis, our families will be always given by a morphism of the form above.

The following two questions are essential while studying a classification problem.

Question 2.4.2. Does there exist an object M in the parameter category $\mathcal{C}$ and a bijective map between the underlying set of points $|M|$ and the set of equivalence classes $\mathcal{M}/\sim_{\mathcal{M}}$?

Question 2.4.3. What influence do the properties of families have on the structure of the object M found in 2.4.2?

These two questions lead to the fundamental notion of moduli problems. Note that there exist many ways to formulate moduli problems. Here, our moduli problems are defined as follows.

Definition 2.4.4. [1, p. 149] A moduli problem is a classification problem that addresses the questions 2.4.2 and 2.4.3 above. More concretely, to formulate a moduli problem consists of specifying the following data:

(1) an object C in a category $\mathcal{C}$,

(2) a classificaton problem $(\mathcal{M}, \sim_{\mathcal{M}})$ in which the objects of $\mathcal{M}$ are defined with respect to C,

(3) a functor Fam of families of objects as in 2.2.12.

Roughly speaking, a moduli problem consists of objects, families of objects, and equivalence relation of families as defined in 2.2.15.

Suppose we have successfully found an object M that answers the question 2.4.2, that is, there exists an object $M \in \mathcal{C}$ and a bijective map $\varphi : \mathcal{M}/\sim_{\mathcal{M}} \longrightarrow |M|$. From this bijection, we can transport the structure on $|M|$ onto $\mathcal{M}/\sim_{\mathcal{M}}$, which makes $\mathcal{M}/\sim_{\mathcal{M}}$ into an algebro-geometric object. Moreover, we know from Yoneda's lemma that the structure of M is uniquely determined by its functor of points $\mathrm{Hom}(-, M)$. Therefore, the investigation of the ways in which the families of objects parametrized by elements of $\mathcal{M}$ influence the structure of the object M can be reduced to the investigation of the ways the properties of families of objects parametrized by objects of $\mathcal{C}$ influence the morphisms of the form $X \longrightarrow M$, where X is an arbitrary element of $\mathcal{M}$. The investigation of the influence of the properties of families of objects of $\mathcal{M}$ consists of the following questions [1, p. 149].

Question 2.4.5. Suppose we are given an object $Y \in \mathcal{C}$, a family $X \in \mathrm{Fam}(M)$ and a morphism $f : Y \longrightarrow M$ in the category $\mathcal{C}$. So we have a diagram of the form:

$$\begin{array}{ccc} & & X \\ & & \big\downarrow{\scriptstyle \phi} \\ Y & \xrightarrow{f} & M \end{array}$$

By change of basis, we can complete this diagram into a commutative diagram

$$\begin{array}{ccc} f^*X & \xrightarrow{p_X} & X \\ \big\downarrow{\scriptstyle p_Y} & & \big\downarrow{\scriptstyle \phi} \\ Y & \xrightarrow{f} & M \end{array}$$

That is, we obtain an induced family $f^*X \longrightarrow Y$ over the fixed object Y by pulling back the family $\phi : X \longrightarrow M$ parametrized by M along the morphism f.

Conversely, a natural question one may ask is the following: suppose we are given a family $\phi : F \longrightarrow Y$ over Y. Does there exist a morphism $f_F : Y \longrightarrow M$ and a family $X_F \longrightarrow M$ such that its pullback along f_F gives back the initial family F parametrized by the object Y? That is, $f_F^*(X_F) = F$? In other words, given a family $\phi : F \longrightarrow Y$ over Y, can we complete it into a commutative diagram of the form below?

$$\begin{array}{ccc} f_F^*(X_F) = F & \xrightarrow{p_{X_F}} & X_F \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \exists \pi} \\ Y & \xrightarrow{\exists f_F} & M. \end{array}$$

Now we consider all the families over the **fixed** object Y and we want to know if there exists a family as above which does not depend on a particular family parametrized by Y. More precisely, we may ask the following question.

Question 2.4.6. Does there exist a single family $\pi : X \longrightarrow M$ over M which gives an affirmative answer to the question raised in 2.4.5 for any family $F \longrightarrow Y$ over the fixed object Y?

$$\begin{array}{ccc} f^*X = F & \xrightarrow{p_X} & X \\ \downarrow{\scriptstyle \forall\phi} & & \downarrow{\scriptstyle \exists!\pi} \\ Y & \xrightarrow{\exists f} & M. \end{array}$$

We may still ask the same question as above if instead we let Y **vary** over $\mathcal{C}$, and consider all the morphisms over each object $Y \in \mathcal{C}$.

Question 2.4.7. Does there exist a single family $X \longrightarrow M$ over M which gives an affirmative answer to the question raised in 2.4.5 for any object $Y \in \mathcal{C}$?

$$\begin{array}{ccc} f^*X = F & \xrightarrow{p_X} & X \\ \downarrow{\scriptstyle \forall\phi} & & \downarrow{\scriptstyle \exists!\pi} \\ \forall Y & \xrightarrow{\exists f} & M. \end{array}$$

An affirmative answer to Question 2.4.7 implies respectively an affirmative answer Question 2.4.5 and 2.4.6, so that it will be enough to affirmatively answer Question 2.4.7.

Remark 2.4.8. Before we start analysing these questions, let us give a meaning to the requirement that the structure of an object of the category $\mathcal{C}$ on the set of equivalence relation $\mathcal{M}/\sim_{\mathcal{M}}$ should relate in a natural way and reflect the properties of families of objects of $\mathcal{M}$ over elements of $\mathcal{C}$. For this, suppose that we are able to affirmatively give an answer to the question 2.4.7. Then, for each $Y \in \mathcal{C}$, we can consider all the families over Y. This completely determines $\mathrm{Fam}(Y)$, the set of families of objects of $\mathcal{M}$ parametrized by Y. And for each such family X/Y in $\mathrm{Fam}(Y)$, we then obtain a morphism $\phi_X : Y \longrightarrow M$ as follows. For each closed point $y \in Y$, consider the inclusion map $i_y : \{y\} \hookrightarrow Y$. Then by the properties of families of objects, the pullback $(i_y)^*X$ of the family X parametrized by Y along the morphism i_y gives rise to a family parametrized by the base point $\{y\}$. But a family parametrized by a base point is an object of $\mathcal{M}/\sim_{\mathcal{M}}$. Thus we obtain an element of $\mathcal{M}/\sim_{\mathcal{M}}$, and this element can be viewed as an element of M via the bijection between $\mathcal{M}/\sim_{\mathcal{M}}$ and M. This determines a morphism

$$\begin{array}{rcl} \phi_X : Y & \longrightarrow & M. \\ y & \longmapsto & (i_y)^*X \end{array}$$

Hence for any $Y \in \mathcal{C}$, we obtain a map

$$\begin{array}{rcl} \mathrm{Fam}(Y) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(Y, M). \\ X/Y & \longmapsto & \phi_X \end{array}$$

Then, for each Y in $\mathcal{C}$, $\mathrm{Fam}(Y)$ determines $\mathrm{Hom}_{\mathcal{C}}(Y, M)$, and this in a functorial way. Therefore, any property related to the functor Fam should translate into that of $\mathrm{Hom}_{\mathcal{C}}(-, M)$.

2.4.9 Examples of moduli problems

The goal of this subsection is to illustrate the notions of moduli problems by concrete examples. The first example we consider here is the moduli problem of complete varieties.

Example 2.4.10 (Moduli of complete varieties)**.** Let us formulate a moduli problem for complete varieties as follows.

(1) $\mathcal{M} = \{\text{all complete varieties}\}$.

(2) Two complete varieties X and Y are equivalent if they are isomorphic. This defines an equivalence relation on $\mathcal{M}$.

(3) We consider the parameter category $\mathcal{C}$ to be the category Var of all varieties.

(4) A family of complete varieties parametrized by a variety S is a pair (X, f), where X is a variety and $f : X \longrightarrow S$ is a flat morphism whose geometric fibers $X_s = f^{-1}(s)$ are complete varieties.

(5) Two families (X, f) and (X', f') of complete varieties are equivalent if there exists an isomorphism $h : X \longrightarrow X'$ of varieties over S, that is $f = f' \circ h$.

(6) For each variety $S \in \mathcal{C}$, let $\mathrm{Fam}(S)$ be the set of families of complete varieties parametrized by S. Suppose that $h : S' \longrightarrow S$ is a morphism of varieties, and (X, f) is a family of complete varieties parametrized by S. Then the pullback family along the morphism h is the fiber product $S' \times_S X$, which is a family parametrized by S'. This defines a functor

$$\begin{array}{rcl} \mathrm{F} : \mathrm{Var}^{\mathrm{opp}} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \mathrm{Fam}(S)/\sim_S \\ (h : S' \to S) & \longmapsto & [h^*] : \mathrm{Fam}(S)/\sim_S \longrightarrow \mathrm{Fam}(S')/\sim_S \\ & & X/\sim_S \longmapsto [S' \times_S X]/_{\sim_{S'}} \end{array}$$

This clearly defines a moduli problem. This example is just an overview of Chapter 4.

Example 2.4.11 (Moduli problem of vector bundles)**.** Let X be a fixed variety. We can formulate a moduli problem for vector bundles over X as follows.

(1) $\mathcal{M} = \{\text{all vector bundles over } X\}$.

(2) Two vector bundles $p : E \longrightarrow X$ and $p' : E' \longrightarrow X$ are equivalent if they are isomorphic. This again defines an equivalence relation on $\mathcal{M}$.

(3) We consider the parameter category $\mathcal{C}$ to be the category Var of all varieties.

(4) A family of vector bundles over X parametrized by a variety S is a vector bundle $p : E \longrightarrow S \times X$ over $S \times X$.

(5) Two families $p : E_1 \longrightarrow S \times X$ and $p : E_2 \longrightarrow S \times X$ of vector bundles over X parametrized by S are equivalent if there exists a line bundle L over S such that $E_1 \cong E_2 \otimes p_S^* L$.

(6) For each variety $S \in \mathcal{C}$, let $\mathrm{Fam}(S)$ be the set of families of vector bundles over X parametrized by S. Suppose that $h : S' \longrightarrow S$ is a morphism of varieties, and $p : E \longrightarrow S \times X$ is a family of vector bundles parametrized by S. Then the pullback family along the morphism h is the family $(h \times \mathrm{id}_X)^* E \longrightarrow S' \times X$

$$\begin{array}{ccc} (h \times \mathrm{id}_X)^* E & \xrightarrow{p_X} & E \\ \big\downarrow{\scriptstyle p_{S \times X}} & & \big\downarrow{\scriptstyle p} \\ S' \times X & \xrightarrow{h \times \mathrm{id}_X} & S \times X \end{array}$$

So the notion of families of vector bundles over X pametrized by S is compatible with pullback. Note that the restriction $p_s : E_s \longrightarrow \{s\} \times X \cong X$ is again a vector bundle over X.

Therefore, the associated moduli functor is given by:

$$\begin{array}{rcl} \mathrm{F} : \mathrm{Var}^{\mathrm{opp}} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \mathrm{Fam}(S)/\sim_S \\ (h : S' \to S) & \longmapsto & [h^*] : \mathrm{Fam}(S)/\sim_S \longrightarrow \mathrm{Fam}(S')/\sim_S \\ & & [E \xrightarrow{p} S \times X]/\sim_S \longmapsto [(h \times \mathrm{id}_X)^* E \xrightarrow{p'} S' \times X]/_{\sim_{S'}} \end{array}$$

Example 2.4.12 (Quadruple of points in $\mathbb{P}^1$)**.** [15, p.5-19] The main goal of this example is to illustrate the steps to formulate a moduli problem. For this, we will describe each object in the definition of a moduli problem in the sense of Definition 2.4.4. Let

$$\mathcal{M} = \{x = (x_1, x_2, x_3, x_4) \in (\mathbb{P}^1)^4 : x_i \neq x_j, \text{ for } i \neq j\}$$

be the collection of all quadruples (x_1, x_2, x_3, x_4) on the projective line $\mathbb{P}^1$. By quadruple, we always mean a set of four distinct ordered points in $\mathbb{P}^1$. We recognize here that our objects (quadruples in $\mathbb{P}^1$) are defined in terms of $C = \mathbb{P}^1$. We would like to classify these quadruples up to automorphisms of $\mathbb{P}^1$. So, two quadruples (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) are equivalent if there exists an automorphism $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that $f(x_i) = y_i$, for $i = 1, 2, 3, 4$. We would like to put a structure of an algebraic variety over $\mathbb{C}$ on the set of equivalence classes $\mathcal{M}/\sim_{\mathcal{M}}$. So, we consider our parameter category $\mathcal{C}$ to be the category of algebraic varieties over $\mathbb{C}$. The next step is to define a notion of families of quadruples of points parametrized by each base variety $B \in \mathcal{C}$. Let $B \in \mathcal{C}$ be an algebraic variety over $\mathbb{C}$. We define a family of quadruples parametrized by B to be the projection map

$$\begin{array}{rcl} \pi : B \times \mathbb{P}^1 & \longrightarrow & B \\ (x, p) & \longmapsto & x \end{array}$$

with four disjoint sections (that is, their images are disjoint sets)

$$\begin{array}{rcl} \sigma_i : B & \longrightarrow & B \times \mathbb{P}^1 \\ x & \longmapsto & (x, p_i) \end{array}$$

of π (that is $\pi \circ \sigma_i = \mathrm{id}_B$), for $i = 1, 2, 3, 4$. We will denote a family of quadruples of points over a base variety B by the tuple $(B, \pi, (\sigma_1, \sigma_2, \sigma_3, \sigma_4))$. Also, it is clear that these sections σ_i of π are completely determined by the data of four distinct points $(p_1, p_2, p_3, p_4) \in (\mathbb{P}^1)^4$. So, we identify a family of quadruples parametrized by a base variety B to the data of a projection $\pi : B \times \mathbb{P}^1 \longrightarrow B$ and four distinct points (p_1, p_2, p_3, p_4) on the projective space $\mathbb{P}^1$. For this reason, we may sometimes just write a family over B just by $(B, \pi, (p_1, p_2, p_3, p_4))$.

Now, we should verify that the fiber over each base point $b \in B$ is exactly a quadruple of points in $\mathbb{P}^1$. For each base point $b \in B$, the fiber of π over b is exactly $\{b\} \times \mathbb{P}^1$ (which is a copy of the projective space $\mathbb{P}^1$), and for this point b the four sections σ_i give four distinct points on $\mathbb{P}^1$; so an element of $\mathcal{C}$.

Let $f : A \longrightarrow B$ be a morphism in $\mathcal{C}$ and let $(B, \pi, (\sigma_1, \sigma_2, \sigma_3, \sigma_4))$ be a family of quadruples over $B \in \mathcal{C}$. The pullback of π along f is clearly the projection

$$\begin{aligned} \pi_A : A \times \mathbb{P}^1 &\longrightarrow A \\ (a, p) &\longmapsto a \end{aligned}$$

with sections

$$\begin{aligned} \sigma_i : A &\longrightarrow A \times \mathbb{P}^1 \\ a &\longmapsto (a, p_i) \end{aligned}$$

So, our definition of families of quadruples over a base variety is compatible with taking pullback.

We define an equivalence relation $\sim_B$ on families of quadruples over the same base variety B as follows. Two families of quadruples $(B, \pi, (\sigma_1, \sigma_2, \sigma_3, \sigma_4), (p_1, p_2, p_3, p_4))$ and $(B, \pi', (\sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4), (q_1, q_2, q_3, q_4))$ parametrized by the same base variety $B \in \mathcal{C}$ are equivalent if there exists an automorphism $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ such that $f(p_i) = q_i, i = 1, 2, 3, 4$. So, the functor of families of objects of $\mathcal{M}$ over objects of $\mathcal{C}$ is defined by:

$$\begin{aligned} \mathrm{Fam} : \mathcal{C}^{\mathrm{opp}} &\longrightarrow \mathrm{Set} \\ B &\longmapsto \mathrm{Fam}(B) = \{\pi : B \times \mathbb{P}^1 \longrightarrow B,\ \sigma_i : B \longrightarrow B \times \mathbb{P}^1,\ i = 1, 2, 3, 4\} \end{aligned}$$

Therefore, the moduli functor associated with the moduli problem of quadruples in $\mathbb{P}^1$ is given by:

$$\begin{aligned} \mathrm{F} : \mathcal{C}^{\mathrm{opp}} &\longrightarrow \mathrm{Set} \\ B &\longmapsto \mathrm{Fam}(B)/ \sim_B . \end{aligned}$$

We claim that the moduli functor F above is representable and we will prove this fact later. For this, we will prove it step by step as we are going on using the notion of universal family that we will introduce soon. The variety representing this moduli problem will be called a fine moduli space, a notion that we introduce now.

2.5 Fine moduli spaces

The happiest situation while studying a moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$ is when the associated moduli functor F defined in 2.2.15, is representable in the sense of Definition 2.3.8, say represented by an object M in $\mathcal{C}$.

2.5.1 Definition of a fine moduli space

Definition 2.5.2. Let $\mathrm{F} : (\mathcal{C})^{\mathrm{opp}} \longrightarrow \mathrm{Set}$ be the moduli functor associated with a moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$. A *fine moduli space* for the moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$ is pair (M, η) representing the moduli functor F in the sense of Definition 2.3.8. In other words, (M, η) is a fine moduli space for a moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$ if there is an isomorphism of functors $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M = \mathrm{Hom}_{\mathcal{C}}(-, M)$. If this happens, for any object S in $\mathcal{C}$, we get a bijection

$$\eta_S : \mathrm{F}(S) := \{\text{families over } S\}/\sim_S \longrightarrow \mathrm{h}_M(S) := \{\text{morphisms } \ S \longrightarrow M\}. \tag{2.1}$$

In particular, if $S = \{\mathrm{pt}\}$ is a base point, then the underlying set of points $|M|$ of M is in bijection with the set $\mathcal{M}/\sim_{\mathcal{M}}$. Indeed, on the left-hand side of (2.1) we have $\mathrm{F}(\{\mathrm{pt}\})$ which is the set of equivalence classes of the elements of $\mathcal{M}$, and on the right-hand side we have $\mathrm{h}_M(\{\mathrm{pt}\}) = \mathrm{Hom}_{\mathcal{C}}(\{\mathrm{pt}\}, M) \cong |M|$. This proves that M does really parameterize the elements of $\mathcal{M}$. The bijection

$$\eta_{\{\mathrm{pt}\}} : \mathcal{M}/\sim_{\mathcal{M}} \quad \longrightarrow \quad |M|$$

allows us to transport any structure on the object M to the set of equivalence classes.

Furthermore, these bijections η_S are compatible with morphisms $f : T \longrightarrow S$, in the sense that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{F}(S) & \xrightarrow{\eta_S} & \mathrm{h}_M(S) \\ \big\downarrow{\scriptstyle \mathrm{F}(f)=f^*} & & \big\downarrow{\scriptstyle \mathrm{h}_M(f)} \\ \mathrm{F}(T) & \xrightarrow{\eta_T} & \mathrm{h}_M(T). \end{array}$$

In particular, $\eta_M : \mathrm{F}(M) \longrightarrow \mathrm{h}_M(M) = \mathrm{Hom}_{\mathcal{C}}(M, M)$ is an isomorphism, and since $\mathrm{id}_M \in \mathrm{Hom}_{\mathcal{C}}(M, M)$ there is a unique family $\mathcal{U}/M$ parametrized by M, corresponding to the identity map id_M. The family $\mathcal{U}$ will be called the universal family on the fine moduli space M.

Proposition 2.5.3. Suppose that a moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$ admits a fine moduli space (M, η). Then for each family X/B of objects of $\mathcal{M}$ parametrized by each $B \in \mathcal{C}$, there exists a morphism $\varphi_X : |B| \longrightarrow M$, called the morphism associated to the family X/B (or the corresponding morphism to the family X/B).

Proof. The proof of this proposition is the same as the construction given in 2.4.8. Let us recall again how this is constructed.

Assume that (M, η) is a fine moduli space for the moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$. Let X/B be a family parametrized by $B \in \mathcal{C}$. For each closed point $b \in B$, consider the inclusion map $i_b : \{b\} \hookrightarrow B$, which is a morphism. Then we can take the pullback of the family X along the morphism i_b to obtain a family $(i_b)^*X$ parametrized by the single point $\{b\}$. But a family parametrized by a single point is an element of $\mathcal{M}/\sim$, so that the family $(i_b)^*X$ is an element of $\mathcal{M}/\sim$. By the bijection between M and $\mathcal{M}/\sim$, we can view the object $(i_b)^*X$ as an element of M. This determines a morphism

$$\begin{array}{rcl}\phi_X : |B| & \longrightarrow & M \\ b & \longmapsto & (i_b)^*X\end{array}$$

as desired. □

When a fine moduli space M exists, there is a special family $\mathcal{U}$ parametrized by M with a map $\mathcal{U} \longrightarrow M$ such that the fiber over each point $m \in M$ is precisely the object in the equivalence class parametrized by the single point m. Such a family $\mathcal{U}$ satisfies a universal property.

2.5.4 Universal families and moduli spaces

Let $(\mathcal{M}, \sim_{\mathcal{M}})$ be a moduli problem as in 2.4.1. We would like to find a family $\mathcal{U}/M$ of objects of $\mathcal{M}$ parametrized by an object $M \in \mathcal{C}$ that affirmatively answers Question 2.4.7. If such a familiy exists, it is called a universal family for the moduli space M.

Definition 2.5.5. [1, Definition 4.5.3.3.1] A *universal family* for the moduli $(\mathcal{M}, \sim_{\mathcal{M}})$ is a family $\mathcal{U}/M$ of objects of $\mathcal{M}$ parametrized by an object $M \in \mathcal{C}$ which satisfies the following universal property: for any family Z/T parametrized by each T in $\mathcal{C}$, there exists a unique morphism $\phi_Z : T \longrightarrow M$ such that the corresponding map

$$\begin{array}{rcl}\phi_Z^* : \mathrm{F}(M) & \longrightarrow & \mathrm{F}(T) \\ X/\sim_M & \longmapsto & \phi_Z^*X/\sim_T\end{array}$$

sends the equivalence class $[\mathcal{U}/M]$ of the family $\mathcal{U}/M$ to the equivalence class of the initial family $Z \in \mathrm{Fam}(T)$ (that is, $\phi_Z^*\mathcal{U}/\sim_T = Z/\sim_T$). So, if we have a universal family, then the questions 2.4.5, 2.4.6, 2.4.7 will all be affirmatively answered.

Now we would like to relate the notion of fine moduli spaces with that of universal families. We would like to show that the family $\mathcal{U}/M$ corresponding to the identity morphism id_M is a universal family. The next theorem states that this is actually a universal family.

Theorem 2.5.6. Let (M, η) be a fine moduli space for the moduli problem $\mathrm{F} : (\mathcal{C})^{opp} \longrightarrow \mathrm{Set}$, and let $\mathcal{U}/M$ be the family of objects of $\mathcal{M}$ parametrized by M, corresponding to the identity morphism id_M. Then $\mathcal{U}/M$ is a universal family.

Conversely, if $\mathcal{U}/M$ is a universal family parametrized by an arbitrary object $M \in \mathcal{C}$, then M is a fine moduli space.

Proof. Let X/S be any family of objects of $\mathcal{M}$ parametrized by $S \in \mathcal{C}$, and let $\phi_X : S \longrightarrow M$ be the corresponding morphism via the isomorphism

$$\eta(S) : \mathcal{F}(S) \longrightarrow \mathrm{h}_M(S).$$

Since $\mathcal{U}$ is a family over M, we can take its pullback along the morphism ϕ_X to obtain a family $\phi_X^*\mathcal{U}$ over S. We claim that $\phi_X^*\mathcal{U} = X$.

Let us prove the claim above. Since M is a fine moduli space, we have a natural transformation $\eta : \mathcal{F} \longrightarrow \mathrm{h}_M$ of contravariant functors such that the maps $\eta(S) : \mathcal{F}(S) \longrightarrow \mathrm{h}_M(S)$ are bijective for each $S \in \mathcal{C}$. By the naturality of η, the diagram

$$\begin{array}{ccc} \mathrm{F}(M) & \xrightarrow{\eta(M)} & \mathrm{Hom}_{\mathcal{C}}(M, M) \\ \Big\downarrow{\scriptstyle \mathcal{F}(\phi_X)=\phi_X^*} & & \Big\downarrow{\scriptstyle \mathrm{h}_M(\phi_X)=-\circ\phi_X} \\ \mathrm{F}(S) & \xrightarrow{\eta(S)} & \mathrm{Hom}_{\mathcal{C}}(S, M). \end{array}$$

is commutative. That is, $\eta(S) \circ \phi_X^* = \mathrm{h}_M(\phi_X) \circ \eta(M)$. So, we obtain that

$$\eta(S) \circ \phi_X^*(\mathcal{U}) = \mathrm{h}_M(\phi_X) \circ \eta(M)(\mathcal{U}).$$

But $\mathrm{h}_M(\phi_X) \circ \eta(M)(\mathcal{U}) = \mathrm{h}_M(\phi_X)(\mathrm{id}_M) = \mathrm{id}_M \circ \phi_X = \phi_X$. On the other hand, $(\eta(S) \circ \phi_X^*)(\mathcal{U}) = \eta(S)(\phi_X^*(\mathcal{U}))$. Hence, $\eta(S)(\phi_X^*(\mathcal{U})) = \phi_X$. Also, $\eta(S)(X) = \phi_X$ since ϕ_X is the morphism corresponding to the family X/S via the morphism $\eta(S) : \mathcal{F}(S) \longrightarrow \mathrm{h}_M(S)$. It follows that $\eta(S)(\phi_X^*(\mathcal{U})) = \eta(S)(X)$. As $\eta(S)$ is bijective, we deduce that $\phi_X^*(\mathcal{U}) = X$ as desired.

For the converse, assume that $\mathcal{U}/M$ is a universal family. Then, by definition, for any family X/S parametrized by each S in $\mathcal{C}$, there exists a unique morphism $\phi_X : S \longrightarrow M$ such that the corresponding map

$$\begin{array}{rcl} \phi_X^* : \mathrm{F}(M) & \longrightarrow & \mathrm{F}(S) \\ Z/\sim_M & \longrightarrow & \phi_X^* Z/\sim_S \end{array}$$

sends the equivalence class $[\mathcal{U}/M]$ of the family $\mathcal{U}/M$ to the equivalence class of the initial family $X \in \mathrm{Fam}(S)$ (that is, $\phi_X^*\mathcal{U}/\sim_S = X/\sim_S$). Then we consider the functor $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ defined for every $S \in \mathcal{C}$ by:

$$\begin{array}{rcl} \eta(S) : \mathrm{F}(S) & \longrightarrow & \mathrm{h}_M(S), \\ X/S & \longmapsto & \phi_X \end{array}$$

where ϕ_X is the unique morphism above. By the uniqueness of ϕ_X, we deduce that $\eta(S)$ is injective. For the surjectivity, let $\phi \in \mathrm{h}_M(S)$. Taking the pullback of the universal family $\mathcal{U}$ along the morphism ϕ gives rise to a family $\phi^*\mathcal{U}$ parametrized by S. □

Theorem 2.5.7. Let (M, η) be a fine moduli space for the moduli problem $\mathrm{F} : (\mathcal{C})^{opp} \longrightarrow \mathrm{Set}$, and let X/B be any family of objects of $\mathcal{M}$ parametrized by an object $B \in \mathcal{C}$. Then, there exists a unique morphism $f : B \longrightarrow M$ in $\mathcal{C}$ such that $X = f^*\mathcal{U} = B \times_M \mathcal{U}$.

$$\begin{array}{ccc} f^*\mathcal{U} & \xrightarrow{\mathrm{pr}_{\mathcal{U}}} & \mathcal{U} \\ \Big\downarrow{\scriptstyle \mathrm{pr}_B} & & \Big\downarrow{\scriptstyle u} \\ B & \xrightarrow{\exists! f} & M. \end{array}$$

Furthermore, M is unique up to unique isomorphism and parametrizes all the equivalence classes of the moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$.

Proof. Let $\varphi : B \longrightarrow B'$ be a morphism in $\mathcal{C}$. Since F is represented by M, the diagram

$$\begin{array}{ccc} \mathrm{F}(B') & \xrightarrow{\eta_{B'}} & \mathrm{Hom}_{\mathcal{C}}(B', M) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle -\circ\varphi} \\ \mathrm{F}(B) & \xrightarrow{\eta_B} & \mathrm{Hom}_{\mathcal{C}}(B, M). \end{array}$$

commutes, with η_B and $\eta_{B'}$ being bijective. Taking $B = M$, we have $\mathrm{F}(M) \cong \mathrm{Hom}_{\mathcal{C}}(M, M)$ via η_M. By definition, $\mathrm{F}(M)$ is the set of equivalence classes of families of objects of $\mathcal{M}$ parametrized by M. Since the identity morphism $\mathrm{id}_M \in \mathrm{Hom}_{\mathcal{C}}(M, M)$, it follows from the isomorphism η_M that id_M corresponds to the universal family $\mathcal{U}$.

Let us consider an arbitrary family $\pi : X \longrightarrow B$ over B, that is $X \in \mathrm{F}(B)$. We want to find a morphism $f : B \longrightarrow M$ such that $X = f^*\mathcal{U}$. Since $\eta_B : \mathrm{F}(B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(B, M)$ is an isomorphism, then X corresponds to a unique morphism $f : B \longrightarrow M$, that is $\eta_B(\pi) = f$. Then, the diagram

$$\begin{array}{ccc} \mathrm{F}(M) & \xrightarrow{\eta_M} & \mathrm{Hom}_{\mathcal{C}}(M, M) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle -\circ f} \\ \mathrm{F}(B) & \xrightarrow{\eta_B} & \mathrm{Hom}_{\mathcal{C}}(B, M), \end{array}$$

$$\begin{array}{ccc} \mathcal{U} & \longmapsto & \mathrm{id}_M \\ \big\downarrow & & \big\downarrow \\ X & \longmapsto & f \end{array}$$

commutes so that $\eta_B(f^*(\mathcal{U})) = (-\circ f)(\eta_M(\mathcal{U})) = (-\circ f)(\mathrm{id}_M) = \mathrm{id}_M \circ f = f$. Therefore, from $\eta_B(\pi) = f = \eta_B(f^*(\mathcal{U}))$, it follows that $f^*(\mathcal{U}) = \pi$. Hence $f^*(\mathcal{U}) = X$ as desired.

For the unicity of M, assume that there is another pair (η', M') with the same property as (η, M). We have a universal family $\mathcal{U}' \longrightarrow M'$ by definition of M'. For this family, there exists unique morphism $f : M' \longrightarrow M$ in $\mathcal{C}$ such that $\mathcal{U}' = f^*(\mathcal{U})$. Similarly, for the universal family $\mathcal{U} \longrightarrow M$, there exists a unique morphism $g : M \longrightarrow M'$ such that $\mathcal{U} = g^*(\mathcal{U}')$. Hence, $\mathcal{U} = g^* f^*(\mathcal{U}) = (f \circ g)^*\mathcal{U}$ and $\mathcal{U}' = f^* g^*(\mathcal{U}') = (g \circ f)^*\mathcal{U}'$. By uniqueness, it follows that $f \circ g = \mathrm{id}_M$ and $g \circ f = \mathrm{id}'_M$, that is $M \cong M'$ as desired.

The last statement of the theorem has already been proven after Definition 2.5.2. □

Remark 2.5.8. The theorem above tells us that any fine moduli space comes with a universal family that answers the questions 2.4.5, 2.4.6 and 2.4.7, so that it makes sense to say that the notion of fine moduli spaces is the best possible we can expect from a moduli problem.

Proposition 2.5.9. [1] Let $\mathrm{F} : \mathcal{C}^{\mathrm{opp}} \longrightarrow \mathrm{Set}$ be the moduli functor associated with a moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$. Suppose that F admits a fine moduli space (M, η). Let $\mathcal{U}/M$ be a universal family associated with (M, η). Then

1. The fiber $\mathcal{U}_m$ of $\mathcal{U}$ over each $m \in M$ is in the same $\sim_{\mathcal{M}}$-equivalence class as represented by m itself.

2. If X/B and Y/B are two families of objects of $\mathcal{M}$ parametrized by the same base object $B \in \mathcal{C}$ and are such that $[X_b] \sim_{\mathcal{M}} [Y_b]$ for all $b \in B$, then the families X/B and Y/B are equivalent.

Therefore, there is an equivalence between fine moduli space and the associated universal family. It is sometimes useful to define a fine moduli space through the notion of univesal family as follows.

Definition 2.5.10. Let $(\mathcal{M}, \sim_{\mathcal{M}})$ be a moduli problem as in 2.4.4. A fine moduli space for $(\mathcal{M}, \sim_{\mathcal{M}})$ is a pair (U, M), where U is a universal family and M is the space representing the associated moduli functor F.

Example 2.5.11. Let k be an algebraically closed field and let V be a vector space of dimension n over k. So, we may identify V with k^n. We would like to parametrize the 1-dimensional linear subspaces of V, that is, all lines through the origin. We consider the class $\mathcal{M}$ of all lines through the origin. We know that each nonzero vector $v \in V$ determines a unique line L; namely $L = \langle v \rangle \subset \mathrm{k}^n$. Moreover, two nonzero vectors v and v' determine the same line if they are colinear, that is if there exists a nonzero constant c such that $v = cv'$. Hence, we may define an equivalence relation on $\mathcal{M}$ by saying that two lines $L = \langle v \rangle$ and $L' = \langle v' \rangle$ are equivalent if $v = cv'$ for some nonzero constant c. Therefore, the isomorphism class of lines through the origin is the quotient $(V - \{0\})/\mathrm{k}^*$, called the projective space, and denoted by $\mathbb{P}^1_{\mathrm{k}}$. We claim that there exists a universal family.

2.5.12 Conclusion

If a moduli problem $\mathrm{F} : (\mathcal{C})^{\mathrm{opp}} \longrightarrow \mathrm{Set}$ admits a fine moduli space M, then M parameterizes all the equivalence classes of the objects of our study. Furthermore, there is a 1-to-1 correspondence between the equivalence classes of families over a fixed base object B and morphisms $B \longrightarrow M$ in $\mathcal{C}$. Therefore we are to understand the geometry of families of the moduli problem F through the geometry of the moduli space M.

2.6 Pathological behaviour

Most of the moduli problems fail to admit fine moduli spaces. The following two pathologies may prevent a moduli problem from admitting a fine moduli space:

(1) The jump phenomenon: moduli may jump in families in the sense of Definition 2.6.1.

(2) The moduli problem of our study may be unbounded. This means there is no family $X \longrightarrow B$ over an object B which parameterizes all the objects we are interested in. This pathology can be solved by reducing to stable objects that are well behaved.

Definition 2.6.1. Let $\mathcal{C} = \mathrm{Sch}/\mathrm{k}$ be the category of schemes over a field k. A *jump phenomenon* for a moduli problem $(\mathcal{M}; \sim_{\mathcal{M}})$ is a family $X \longrightarrow S$ of elements of $\mathcal{M}$ parametrized by an integral scheme S of dimension at least one, of finite type over k, such that all fibers X_s for $s \in S$ are isomorphic except for one fiber X_{s_0} which is of different type.

Proposition 2.6.2. Let $(\mathcal{M}; \sim_{\mathcal{M}})$ be a moduli problem admitting a jump phenomenon $X \longrightarrow S$. Then there does not exists a fine moduli space for this moduli problem.

Proof. By contradiction, assume that there is a fine moduli space for that moduli problem. Then for the family $X \longrightarrow S$, there corresponds a unique morphism $f : S \longrightarrow M$. The morphism f maps the point s_0 to a point of M and the other closed points to another point of M which must be different from $f(s_0)$. But this is not possible for a morphism of schemes, so a fine moduli space for F fails to exist. Therefore a jump phenomenon prevents the existence of a fine moduli space. □

Example 2.6.3. Let us give a first example that exhibits the jump phenomena. Consider the family $E_t : y^2 = x^3 + t^2x + t^3$ of algebraic curves parameterised by the t-line. Then for any $t \neq 0$ we get smooth elliptic curves all with the same j-invariant (see Definition 4.2.7)

$$j(E_t) = 12^3 \frac{4t^6}{4t^6 + 27t^6} = \frac{12^3 \times 4}{31},$$

which is independent of t; and hence all isomorphic. But for $t = 0$ we get the singular cubic $E_0 : y^2 = x^3$. This is typically a jump phenomena, so the cusp curve cannot belong to a class having a fine moduli space.

Remark 2.6.4. The example above is constructed from the following observation. To any elliptic curves of the form $E : y^2 = x^3 + Ax + B$, are associated the quantities: the discriminant $\Delta = -16(4A^3 + 27B^2)$ and the j-invariant $j = -1728\frac{(4A)^3}{\Delta}$. This suggests that in order to produce a family of elliptic curves with jump phenomena, it suffices to choose the coefficients A and B such that Δ is proportional to A^3, that is $\Delta = uA^3$, for some constant u. In this case, we have: $4A^3 + 27B^2 = uA^3$. Thus $B^2 = vA^3$ for some constant v. In particular, for $A = t^2$ and $u = 1$, we get the family of elliptic curves in the example above.

Example 2.6.5 (Jump phenomenon)**.** Let $\mathcal{M}$ be the collection of all pairs (V, T), consisting of a k-vector space V of dimension n and an endomorphism $T \in \mathrm{End}(V)$. We define an equivalence relation on $\mathcal{M}$ as follows: two pairs (V, φ) and (W, ψ) are $\sim_{\mathcal{M}}$-equivalent if there exists an isomorphism $\mathrm{h} : V \longrightarrow W$ such that $\mathrm{h} \circ \varphi = \psi \circ \mathrm{h}$.

Define a family of objects of $\mathcal{M}$ over a k-vector space S to be a rank n vector bundle X over S with an endomorphism $\varphi : X \longrightarrow X$. Two families (X, φ) and (Y, ψ) are isomorphic under $\sim_S$ if there is an isomorphism $\mathrm{h} : X \longrightarrow Y$ such that $\mathrm{h} \circ \varphi = \psi \circ \mathrm{h}$. Let $\mathcal{C}$ be the category of k-vector spaces. It is clear that the category $\mathcal{C}$ satisfies the conditions 2.2.9, so that the data above define a moduli problem as in 2.4.4. Let $\mathrm{F} : \mathcal{C}^{\mathrm{opp}} \longrightarrow \mathrm{Set}$ be the moduli functor associated.

Let $n \geq 2$. It is always possible to construct families with jump phenomenon. Let us construct families which exhibit the jump phenomenon for $n = 2$. Consider the family over the affine line $\mathbb{A}^1$ given by $(X = \mathcal{O}_{\mathbb{A}^1}^{\oplus 2}, \varphi)$ and

$$\varphi_s \;=\; \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

for $s \in \mathbb{A}^1$. We claim that this family exhibits a jump phenomenon. Indeed, for any $s, t \neq 0$, the matrices φ_t and φ_s are similar so that $\varphi_t \sim \varphi_s$. But, φ_0 and φ_1 are not similar so that $\varphi_0 \nsim \varphi_1$ since these matrices have distinct Jordan normal forms. Therefore this family exhibits a jump phenomenon as desired, so that the associated moduli functor F s not representable.

Example 2.6.6 (Example of unbounded moduli problem)**.** The moduli problem of complete varieties given in Example 2.4.10 is unbounded since there can be no variety that parametrizes all such varieties of all dimensions.

Example 2.6.7 (Non representability of the moduli problem of vector bundles)**.** Let r be a fixed non negative integer. Let $\mathcal{M}$ be the collection of all vector bundles of rank r over smooth mainifolds. We define the equivalence relation on $\mathcal{M}$ to be that given by the isomorphy of rank r vector bundle over manifolds. Let $\mathcal{C} := \mathrm{Diff}$ denote the category of differentiable manifolds. The associated moduli functor is defined as:

$$\begin{aligned} \mathrm{F} : \mathrm{Diff}^{\mathrm{opp}} &\longrightarrow \mathrm{Set} \\ B &\longmapsto \{\text{isomorphism classes of vector bundles of rank } r \text{ over } B\}. \end{aligned}$$

We claim that the functor F is not representable by a smooth manifold. By contradiction, assume that F is represented by a smooth manifold M. By Theorem 2.5.7, for any family $\pi : X \longrightarrow B$ over B, there exists a unique morphism of smooth manifolds $f : B \longrightarrow M$ such that $X = f^*\mathcal{U} = B \times_M \mathcal{U}$, where $\mathcal{U}$ is the universal family corresponding to id_M.

$$\begin{array}{ccc} X = f^*\mathcal{U} & \xrightarrow{\mathrm{p}_{\mathcal{U}}} & \mathcal{U} \\ \Big\downarrow{\scriptstyle \mathrm{p}_B} & & \Big\downarrow{\scriptstyle u} \\ B & \xrightarrow{f} & M. \end{array}$$

Let us prove first that a vector bundle $X \longrightarrow B$ is trivial if and only the corresponding morphism $f : B \longrightarrow M$ is trivial. Let $\{U_i\}_i$ be a trivializing open cover for $\mathcal{U}$. As f is continuous, each $f^{-1}(U_i)$ is an open subset of B so that $\{f^{-1}(U_i)\}_i$ is a trivializing open cover for $f^*(\mathcal{U})$. If the morphism f is constant, say $f = c$ for some constant c, then for each i,

$$f^{-1}(U_i) = \begin{cases} B & \text{if } c \in U_i \\ \emptyset & \text{otherwise}. \end{cases}$$

This means that B itself is a trivializing cover, that is, the morphism $X \longrightarrow B$ is trivial. Conversely, suppose that

$$f^{-1}(U_i) = \begin{cases} B & \text{if } c \in U_i \\ \emptyset & \text{otherwise}, \end{cases}$$

for all i. We claim that f is constant. Otherwise, we can find two distinct points $a, b \in f(B)$. As manifolds are Kolmogorov spaces, then at least one of the points a and b has a neighborhood not containing the other. Without loss of generality, suppose that there exists a neighborhood U of a such that $b \notin U$. As $a, b \in f(B)$, we can write $a = f(a')$ and $b = f(b')$ for some $a', b' \in B$. Since $f(a') = a \in U$, then $a' \in f^{-1}(U)$, so that $f^{-1}(U) \neq \emptyset$. On the other hand, since $f(b') = b \notin U$, then $b' \notin f^{-1}(U)$ so that $f^{-1}(U) \neq B$. But then $f^{-1}(U)$ is neither $\emptyset$, nor B, which is a contradiction.

Let $X \longrightarrow B$ be a trivial vector bundle of rank r. By definition of a trivial vector bundle, all rank k trivial vector bundles over the same base B are isomorphic to the bundle $p_B : B \times \mathrm{k}^r \longrightarrow B$, where p_B is the canonical projection over B. By Theorem 2.5.7, $p_B : B \times \mathrm{k}^r \longrightarrow B$ corresponds to a unique morphism of smooth manifolds $B \longrightarrow M$ such that $X = g^*\mathcal{U}$. From what we showed previously, the morphism $g : B \longrightarrow M$ must be constant. That is, there exists a unique $x \in M$ such that $g(b) = x$ for all $b \in B$. Thus $g : B \longrightarrow \{x\}$. Therefore, any rank r trivial vector bundle over a smooth manifold B is given as a pullback along the constant morphism $B \longrightarrow \{x\}$, for some unique $x \in M$.

Let us consider the case of non-trivial vector bundles. Let $\pi : X \longrightarrow B$ be a rank r non-trivial vector bundle over a smooth manifold B with trivializing open cover $\{U_i\}_i$. By the same arguments as above, there exists a unique morphism of smooth manifolds $f : B \longrightarrow M$ such that $X = f^*(\mathcal{U})$. As the restriction morphism $E_{U_i} \longrightarrow U_i$ is trivial for each i, it follows from the paragraph that the restriction of f to each U_i must be the constant morphism $f_{U_i} : U_i \longrightarrow \{x\}$. It follows that f itself is the constant morphism $f : B \longrightarrow \{x\}$, which contradicts the fact that the vector bundle X was chosen non-trivial. This shows that the moduli functor cannot be represented by a smooth manifold M.

What goes wrong? The problem is the following. Since each vector space has non-trivial automorphisms, even if our vector bundle X is trivial on each U_i, it is possible to glue these trivial bundles in a non-trivial vector bundle. In general, moduli problems classifying objects with non-trivial automorphism fail to admit fine moduli space. This is the case of complex smooth projective curves of a fixed genus g.

As proved above, most moduli problems do not admit fine moduli spaces. Usually one can deal with this problem either by rigidifying the problem, or by enlarging the category of schemes to the category of stacks, or by looking for a weaker solution. Here, we will be interested in the latter case. Before introducing the notion of coarse moduli spaces, let us say few words on the first two solutions.

Rigidifying the problem

Since automorphisms of the objects we are parametrizing are what prevent the existence of a fine moduli space, it is necessary to find a way to eliminate objects with non-trivial automorphisms. Rigidifying a moduli problem consists of changing the moduli problem by asking objects and morphisms to satisfy some additional properties in such a way that the objects obtained with these additional properties no longer have non-trivial automorphisms. For instance, an idea is to decorate the objects with some extra structure so that only the identity automorphism preserves the extra structure. This means the new moduli problem consists of classifying a certain subcategory of the objects of interest. The new classfication problem is called the rigidified problem. Then we can construct a fine moduli space for the rigidified problem. This gives a solution to our new classification problem, but one drawback of this solution is that the fine moduli space obtained does not parameterize the objects we were primary interested in, but a certain subclass. However, it gives an idea of how we can study the moduli problem in its generality. This can be achieved by taking a quotient via an appropriate group action.

Algebraic stacks

Another possible attempt is the look for a solution in a category larger than that of schemes. In 1969, Pierre Deligne and David Mumford introduced algebraic stacks as generalisations of algebraic spaces in general. In particular, algebraic stacks generalise algebraic varieties and schemes. The moduli problem may not be representable in the category of schemes, but it may be representable in the category of stacks. This is probably the most satisfying solution, but in this thesis, we will be concerned with the third solution which consists of finding the best approximation possible of the solution.

2.7 Coarse moduli spaces.

Fine moduli spaces are desirable, but they do not always exist and are frequently difficult to construct, so it is sometimes useful to consider a weaker notion which approximates the best the notion of fine moduli spaces. The concept we are interested in here is that of coarse moduli spaces. The main idea is to require for weaker conditions rather than of the representability of the moduli functor in such a way that the moduli problem remains unchanged. The first condition is to require only for a natural transformation $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ for some object M in $\mathcal{C}$. If this is the case, for every $B \in \mathrm{Sch}$ and for every family $\pi : X \longrightarrow B$ over B, from the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{F}(B) & \xrightarrow{\eta_B} & \mathrm{Hom}_{\mathcal{C}}(B, M) \\ \Big\downarrow{\scriptstyle \mathrm{F}(\pi)} & & \Big\downarrow{\scriptstyle -\circ\pi} \\ \mathrm{F}(X) & \xrightarrow{\eta_X} & \mathrm{Hom}_{\mathcal{C}}(X, M). \end{array}$$

we obtain that $f = \eta_B(\pi) \in \mathrm{Hom}_{\mathcal{C}}(B, M)$. Hence there still exists a morphism $f = \eta_B(\pi) : B \longrightarrow M$ as in Theorem 2.5.7. Furthermore, these morphisms are still natural in the sense that if, $\pi' : \pi^*X = X \times_B B' \longrightarrow B'$ is the base change along the morphism $\varphi : B' \longrightarrow B$, then $f' = \eta(\pi') = \eta(\pi) \circ \varphi$. However, asking only for a natural transformation is far from determining M. In fact, if we have a solution (M, η), then for any morphism $\mu : M \longrightarrow M'$ we get another solution $(M', \mu \circ \eta)$. In particular, if we take $M' = \{\mathrm{pt}\}$ a single point and $\eta(\pi)$ to be the unique morphism $B \longrightarrow \{\mathrm{pt}\}$, then we get a new solution; which implies that each point was a solution of our moduli problem. This is a pathological solution.

Then we are tempted to require that the underlying set of points of M should correspond bijectively to the objects we are interested in. Even with this requirement, it does not fix the geometric structure on the space M: in fact, as long as the morphism $\mu : M \longrightarrow M'$ is bijective on base points, we have the freedom to compose with μ as above to get a new solution. This pathological behaviour may be removed by asking that the natural transformation $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ be universal among such natural transformation.

2.7.1 Definition of a coarse moduli space

Definition 2.7.2. An object $M \in \mathcal{C}$ is a *coarse moduli space* for a moduli functor F if there exists a natural transformation of functors $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ which is universal among such natural transformations, and which is bijective on the level of closed points. More concretely, M is a coarse moduli space for F if η satisfies the following two properties:

(a) $\eta_P : \mathrm{F}(P) \longrightarrow \mathrm{h}_M(P)$ is bijective, for any base point P.

(b) For any scheme N and natural transformation $\nu : \mathrm{F} \longrightarrow \mathrm{h}_N$, there exists a unique morphism of schemes $f : M \longrightarrow N$ such that $\nu = \mathrm{h}_f \circ \eta$, where $\mathrm{h}_f : \mathrm{h}_M \longrightarrow h_N$ is the corresponding natural transformation of presheaves.

Remark 2.7.3. Given a functor $\mathrm{F} : \mathcal{C}^{\mathrm{opp}} \longrightarrow \mathrm{Set}$, we say that M is a best approximation of F if there exists a natural transformation $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ which is universal among such natural transformations. So a coarse moduli space is a best approximation of F satisfying the assertion (a) of Definition 2.7.2.

Proposition 2.7.4. A coarse moduli space for a moduli problem $\mathcal{M}$, if it exists, is unique up to unique isomorphism.

Proof. Suppose that (M, η) and (M', η') are two coarse moduli spaces for $\mathcal{M}$. Since (M, η) is a coarse moduli space for $\mathcal{M}$, by taking $N = M'$ and $\nu = \eta'$ then Property (b) implies that there is a unique morphism $f : M \longrightarrow M'$ such that $\eta' = \mathrm{h}_f \circ \eta$. Similarly, since (M', η') is a coarse moduli space for $\mathcal{M}$, by taking $N = M$ and $\nu = \eta$ then Property (b) implies that there is a unique morphism $f' : M' \longrightarrow M$ such that $\eta = \mathrm{h}_{f'} \circ \eta'$. Hence $\eta = \mathrm{h}_{f'} \circ \eta' = \mathrm{h}_{f'} \circ \mathrm{h}_f \circ \eta$. We can also write η as $\mathrm{id}_M \circ \eta$. So by the uniqueness of $\mathrm{h}_{f'} \circ \mathrm{h}_f$ and the Yoneda's Lemma 2.3.7 we get $f' \circ f = \mathrm{id}_M$. Similarly, $f \circ f' = \mathrm{id}_{M'}$. □

Example 2.7.5. A fine moduli space is a coarse moduli space. In fact, the assertion (a) of Definition 2.7.2 is easily satisfied since η is an isomorphism. For the second, it suffices to take $\mathrm{h}_f = \nu \circ \eta^{-1}$.

2.7.6 Tautological family

When a moduli problem F does not admit a fine moduli space, we lose the notion of a universal family. However, there may exists a weaker notion of family in the case of coarse moduli space, namely the notion of tautological family that we define below.

Definition 2.7.7. [1, Definition 4.8.1.1.1] Let $(\mathcal{M}, \sim_{\mathcal{M}})$ be a moduli problem and $\mathrm{F} : \mathcal{C} \longrightarrow \mathrm{Set}$ be the associated moduli functor. A *tautological family* for the moduli functor F is a pair (X, β), where $X \in \mathrm{Fam}(T)$ for some $T \in \mathcal{C}$, and $\beta : \mathcal{M}/\sim_{\mathcal{M}} \longrightarrow |T|$ is a set theoretic map satisfying:

(i) The map β is a bijection of sets, that is for any base-point-object $P \in \mathcal{C}$

$$\beta : \mathrm{F}(P) = \mathcal{M}/\sim_{\mathcal{M}} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(P, T) = |T|$$

is a bijection identifying a $\sim_{\mathcal{M}}$-equivalence class of objects of $\mathcal{M}$ with the underlying points of T;

(ii) For each point $t \in |T|$ (which we may think of as a morphism $\varphi_t \in \mathrm{Hom}_{\mathcal{C}}(P, T)$ by the identification $\mathrm{Hom}_{\mathcal{C}}(P, T) = |T|$), the equivalence class $X_t/\sim_{\mathcal{M}}$ of the fiber X_t of X over t (that is the equivalence class $\varphi_t^*(X) \in \mathrm{F}(P) = \mathcal{M}/\sim_{\mathcal{M}}$) is equal to that determined by β, that is $\beta^{-1}(t)$.

Example 2.7.8. Let $(\mathcal{M}_0, \sim_{\mathcal{M}_0})$ be the moduli problem that consists of classifying nonsingular projective curves of genus 0 over k, up to isomorphism. We may consider the parameter category to be the category of schemes over k. Then $\mathcal{M}_0$ has a single element, namely $\mathbb{P}^1$. One shows that $M = \mathrm{Spec}\mathrm{k}$ is a coarse moduli space, and the trivial family $\mathbb{P}^1 \longrightarrow \mathrm{Spec}\mathrm{k}$ becomes a tautological family (see Proposition 4.1.9). On the other hand, we will see in §4.2 that the j-line is a coarse moduli scheme for curves of genus 1, but that it has no universal family (a counterexample is provided by Example 2.6.3).

Remark 2.7.9. A universal family of objects of $\mathcal{M}$ satisfies the assertion of Definition 2.7.7, so that any universal family of objects of $\mathcal{M}$ is a tautological family. It is clear that if a taulogical family does not exist, then a universal family does not exist because the notion of a universal family is stronger than the notion of a tautological family. Therefore, to show for instance that a moduli problem does not admit a universal family (or a fine moduli space), it is sufficient to show it does not admit a tautological family. We will use this observation in Chapter 4 to show that the moduli problem of nonsingular porjective curves of genus g does not admit a fine moduli space.

Proposition 2.7.10. [12, Proposition 23.1, p.152] If the functor F associated with a moduli problem $(\mathcal{M}, \sim_{\mathcal{M}})$ is representable by an object $M \in \mathcal{C}$, then M is also a coarse moduli space for F, and the universal family $\mathcal{U}/M$ associated with M is a tautological family.

Now, we may ask the following question: what are the conditions that a coarse moduli space should satisfy to be a fine moduli space? The following proposition gives an answer to this question.

Proposition 2.7.11. Let (M, η) be a coarse moduli space for a moduli problem $\mathcal{M}$. Then (M, η) is a fine moduli space if and only if

(1) there exists a family $\mathcal{U} \longrightarrow M$ over M such that $\eta_M(\mathcal{U}) = \mathrm{id}_M$,

(2) for families $\mathcal{F} \longrightarrow S$ and $\mathcal{G} \longrightarrow S$ parametrized by the same base object S, we have $\mathcal{F} \sim_S \mathcal{G} \Longleftrightarrow \eta_S(\mathcal{F}) = \eta_S(\mathcal{G})$.

Proof. In fact, condition (1) corresponds to the surjectivity of $\eta(T)$ for all $T \in \mathcal{C}$ and condition (2) corresponds to the injectivity of $\eta(T)$ for all $T \in \mathcal{C}$. □

Definition 2.7.12. [12, p.153] Suppose $\mathcal{C}$ is the category of schemes. Let X/S be a family parametrized by S. We say that X/S is *trivial* if it is obtained by base extension from the family $Y/\{\mathrm{pt}\}$ consisting of one element of $\mathcal{M}$ over a point.

If S is a scheme of finite type over k, we say that the family $X \longrightarrow S$ is *fiberwise trivial* or *isotrivial* if all the fibers X_s are isomorphic for all closed points $s \in S$.

Lemma 2.7.13. Suppose $\mathcal{M}$ admits a fine moduli space M. Then every fiberwise trivial family is trivial.

Proof. This is because any family X/S is obtained by pulling back the universal family at Speck. □

For instance, we can use this property to prove that the one-point coarse moduli space for complex nonsingular projectives curves of genus 0 is not a fine moduli space.

Conclusion: Coarse moduli spaces parameterise moduli spaces, but the requirements for coarse moduli spaces are not sufficient enough to guarantee the existence of a universal family as in the case of fine moduli spaces. Moreover, we do not have an equivalent version of Theorem 2.5.7 in the case of coarse moduli spaces.

After defining the fundamental notions of fine moduli space, coarse moduli space and their related properties, we woud like to study concrete example of moduli problems illustrations the notions above. The first example of moduli space that we will study is the Grassmannian. According to V. Lakshmibai and Justin Brown [16]:

> "In algebraic geometry, Grassmannian varieties form an important fundamental class of projective varieties. In terms of importance, they are second only to projective spaces; in fact, a projective space itself is a certain Grassmannian".

They can be realized as moduli spaces of the moduli problem of the classification of linear subspaces. This gives our first example of fine moduli space. The aim of the next chapter is to prove that the moduli functor corresponding to this moduli problem is representable.

Chapter 3

The Representability of the Grassmannian Functor

Let k be an algebraically closed field and V a vector space over k. Let $0 < d \leq n$ be integers. In this chapter, we study the classification problem of linear subspaces of an n-dimensional k-vector space V.

To begin with, the projective space $\mathbb{P}^n(\mathrm{k})$ parametrizes 1-dimensional linear subspaces of k^{n+1}. In other words, it is defined as the isomorphism class (with respect to colinearity) of all lines in k^{n+1} passing through the origin. This parametrization can be generalised to the collection of all linear subspaces of an arbitrary fixed dimension d of a fixed vector space V of dimension n. The space parametrizing d-dimensional subspaces of V is known as Grassmannian of degree d in V, and it is an important object of study in Algebraic Geometry. The construction of Grassmannians can be used to construct other moduli spaces such as Hilbert schemes. The goal of this chapter is to formulate a moduli problem for d-dimensional vector subspaces of V and to prove that such moduli problem admits a fine moduli space. We will first formulate our moduli problem in the context of schemes. We will define the families of d-dimensional subspaces parametrized by a scheme to be a vector bundle with some extra conditions. In order to be able to do so, we will need to make the connection between locally free sheaves and vector bundles. From this, we will freely switch between schemes and vector bundles.

The main references for this chapter are [10, Chapter 8], [23, Appendix A] and [6, Chapter VI].

3.1 Moduli problem of d-dimensional linear subspaces

Let $\mathrm{Grass}(d, n)$ denote the class of all d-dimensional linear subspaces of an n-dimensional k-vector space V. That is,

$$\mathrm{Grass}(d, n) = \{W \subset V : W \text{ is a linear subspace of } V,\ \dim W = d\}.$$

We define the equivalence relation on $\mathrm{Grass}(d, n)$ to be the equality of subspaces of V. It is clear that the set of equivalence classes is again $\mathrm{Grass}(d, n)$. Our aim is to put a structure of variety or scheme on $\mathrm{Grass}(d, n)$. So, we choose our parameter category $\mathcal{C}$ to be the category of k-schemes, that is $\mathcal{C} = \mathrm{Sch}/\mathrm{k}$. Now we would like to define a notion of families of d-dimensional linear subspaces of V parametrized by schemes $S \in \mathrm{Sch}/\mathrm{k}$. Such families should pullback on closed points to a d-dimensional vector bundle, so that the adapted notion to define such families is through vector bundles over S. For this, we need first to make the connection between locally free sheaves and vector bundles.

3.1.1 Connection between locally free sheaves and vector bundles

This subsection is introduced only to give a connection between locally free sheaves and vector bundles. This is useful in the sense that it will imply that locally free sheaves are generalizations of vector spaces. We will use this connection to build up the family of d-dimensional vector subspaces of V over a base scheme S.

Definition 3.1.2. A sheaf of modules $\mathcal{F}$ over a scheme X is said to be *free* if it is isomorphic to a direct sum of copies of $\mathcal{O}_X$. The number of copies of $\mathcal{O}_X$ is called the rank of the free sheaf $\mathcal{F}$.

Definition 3.1.3. A sheaf of modules on a topological space X is said to be *locally free* if there exists a cover of X by open subsets such that $\mathcal{F}\mid_U$ is a free $\mathcal{O}_X\mid_U$-module. It follows immediately that a locally free module is quasi-coherent, and it is coherent if it has finite rank. In the case where the topological space X is connected, the rank is the same on each component.

Proposition 3.1.4. [11] The coherent sheaf associated with a finitely generated module M is locally free if and only if it is a projective module. In other words, if there exists N such that $M \oplus N$ is free.

Definition 3.1.5. [21, p. 10] Let X and E be varieties and $\pi : E \longrightarrow X$ be a morphism of varieties. A *trivialisation* of π over an open subset U of X, is an isomorphism

$$\psi : U \times F \longrightarrow \pi^{-1}(U),$$

for some variety F such that $\pi \circ \psi = \pi_U$. The morphism π is said to be locally trivial if X can be covered by open subsets over each of which π admits a trivialisation.

Definition 3.1.6. [21, p. 10] A *vector bundle* of rank n over X consists of a variety E, a morphism $\pi : E \longrightarrow X$, and a structure of n-dimensional vector space over each fiber $E_x = \pi^{-1}(x)$, such that for all $x \in X$,

(i) the vector structure on E_x is compatible with the structure of variety induced from that of E;

(ii) there exists a neighbourhood U of x and a trivialisation

$$\psi : U \times \mathrm{k}^n \longrightarrow \pi^{-1}(U)$$

of π over U such that the map

$$\begin{array}{ccc} \mathrm{k}^n & \longrightarrow & E_y \\ v & \longmapsto & \psi(y, v) \end{array}$$

is linear for all $y \in U$.

A vector bundle of rank 1 is called a *line bundle.*

Example 3.1.7. Take $E = X \times \mathrm{k}^n$, and give $E_x = \{x\} \times \mathrm{k}^n$ the obvious structure of vector space. The axioms of the definition of a vector bundle are easily satisfied. This bundle is denoted by I_n.

Definition 3.1.8. [21, p. 11] Let (E_1, φ_1) and (E_2, φ_2) be two vector bundles over X. A homomorphism from (E_1, φ_1) to (E_2, φ_2) is a morphism $h : E_1 \longrightarrow E_2$ such that

(i) $\pi_2 \circ h = \pi_1$;

(ii) for all $x \in X$, h restricts to a linear map from E_{1x} to E_{2x}.

We define in the usual way the term isomorphism, endomorphism, automorphism, and so on.

Definition 3.1.9. [21, p. 11] A *trivial vector bundle* is a vector bundle E which isomorphic to the bundle I_n.

A section of a vector bundle (E, π) over X is a morphism $\sigma : X \longrightarrow E$ such that $\pi \circ \sigma = \mathrm{id}_X$.

Example 3.1.10. Let X be a variety. Then for each integer n, and for $E = X \times \mathbb{A}^n$ with the projection morphism $\pi : E = X \times \mathbb{A}^n \longrightarrow X$ is a vector bundle called the *trivial vector bundle.*

Example 3.1.11 (Example of locally free sheaves)**.** Let d be an integer. Then we define the locally free sheaf, denoted $\mathcal{O}_{\mathbb{P}^n}(d)$, of rank 1 on the projective space $\mathbb{P}^n$ to be the sheaf that associates to each open set U, the set

$$\{f/g : \ f, g \text{ homogeneous functions, } g \text{ is non-zero on } U \text{ and } \deg f - \deg g = d\,\}.$$

It is locally free since by restricting to a standard copy of affine space, we have that $U \subset \mathbb{A}^n$ is associated to rational functions f/g defined everywhere on U. So we get precisely $\mathcal{O}_{\mathbb{P}^n}|_{\mathbb{A}^n}$.

Theorem 3.1.12 (The correspondence between vector bundles and locally free sheaves)**.** [27, Section 13.1] There exists a bijective correspondence between locally free sheaves and vector bundles.

Proof. Let $\pi : E \longrightarrow X$ be a vector bundle. We associate a locally free sheaf as follows. To any open set $U \subset X$, define

$$\mathcal{O}_X(U) = \{s : U \longrightarrow E : \pi \circ s \text{ is the identity on } U\}.$$

We claim that this sheaf is locally trivial, and the open sets U can be taken to be those for which $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^n$. Indeed, on these sets, we are looking at functions $U_i \to U_i \times \mathbb{A}^n$ such that the composition with the projection gives the identity. That is, we are looking at morphisms $U_i \to \mathbb{A}^n = \mathbb{A}^1 \times \ldots \times \mathbb{A}^1$, which is simply the list of n regular functions on U_i. Therefore, any vector bundle is a locally free sheaf. In particular, every vector space is a locally free sheaf so that locally free sheaves are generalisations of finite dimensional k-vector spaces.

Conversely, let $\mathcal{F}$ be a locally free sheaf on X of rank n. Then we can find an open cover $\{U_i\}_i$ of X such that $\mathcal{F}\mid_{U_i}$ is locally free for each U_i. As the varieties are quasi-compact, we may choose the cover to be finite. Hence, we have isomorphisms $g_i : \mathcal{F}\mid_{U_i} \to \mathcal{O}^n_{U_i}$ and $g_j : \mathcal{F}\mid_{U_j} \to \mathcal{O}^n_{U_j}$. If we restrict each of these isomorphisms to the intersection $U_{ij} = U_i \cap U_j$, we obtain two different isomorphisms $g_{ij} : \mathcal{F}\mid_{U_{ij}} \longrightarrow \mathcal{O}^n_{U_{ij}}$, with $g_{ij} = g_j g_i^{-1}$ is an automorphism $\mathcal{F}|_{U_{ij}}$. With the isomorphism $\mathcal{O}^n_{U_{ij}}$, we may identify this with an $n \times n$ square matrix of regular functions defined on U_{ij}.

The next step consists of gluing the pieces of X together. Let $U_i \times \mathbb{A}^n$ and $U_j \times \mathbb{A}^n$ be open sets. We can identify them along their intersection U_{ij} by the map

$$\begin{aligned} U_{ij} \times \mathbb{A}^n &\longrightarrow U_{ij} \times U_{ij} \\ (x, v) &\longmapsto (x, g_{ij}(v)). \end{aligned}$$

And we perform this for all i, j, and we call this object E. It comes with a morphism $E \longrightarrow X$ by forgetting the vector coordinate on each point. Hence the fibers are copies of the affine space $\mathbb{A}^n$ which sends $(x, v) \in U_{ij} \times \mathbb{A}^n$ to $(x, g_{ij}(v))$. By construction, around each point there is a neighborhood on which the space is $U \times \mathbb{A}^n$. It remains to shows that this is really a vector bundle. It is enough to show that it is a variety. Certainly, it has an open cover by affine varieties by construction. Moreover, this cover is finite. The result follows easily from the fact that the composition map $g_{ij} \circ g_{jk} \circ g_{ki}$ is the identity map. This establishes the desired correspondence. □

3.1.13 Families of d-dimensional vector subspaces over S

Now we come back to the construction of $\mathrm{Fam}(d, n)$, the functor of families of d-dimensional subspaces of V. For any scheme S, we would like to define the set $\mathrm{Fam}(d, n)(S)$ of families of d-dimensional vector subspaces of V parametrized by S. That is, $\mathrm{Fam}(d, n)(S)$ has to satisfy the following properties:

(P_1) For every morphism of schemes $f : T \longrightarrow S$, the set $\mathrm{Fam}(d, n)(T)$ is obtained by pulling back $\mathrm{Fam}(d, n)(S)$ via the morphism f. In other words $\mathrm{Fam}(d, n)(T) = f^*(\mathrm{Fam}(d, n)(S)) := \mathrm{Fam}(d, n)(S) \times_S T$.

(P_2) For every field k, $\mathrm{Fam(d, n)(Spec k)}$ (which we may simply denote by $\mathrm{Fam(d, n)(k)}$) is the set of all d-dimensional vector subspaces of the vector space k^n.

Now we have to understand the construction of the Grassmannian families over an arbitrary scheme S. This means that we have to find the correct meaning of a d-dimensional subspace while defining $\mathrm{Fam}(d, n)(S)$. The property (P_2) to be satisfied by $\mathrm{Fam}(d, n)(S)$ suggests that the natural way to define a notion of a family of d-dimensional subvector spaces of V is in terms of subbundles of the trivial vector bundle over S. Therefore, we have the following definition.

Definition 3.1.14. A *family of d-dimensional subvector spaces of V parametrized by a variety* S is a subbundle of the trivial vector bundle $\mathbb{A}^d_{\mathrm{k}} \times S$.

Furthermore, the equivalence in Theorem 3.1.12 between locally free sheaves and vector bundles allows us to translate the notion of families $\mathrm{G(d, n)}$ defined with respect to subbundles in terms of locally free sheaves. These two formulations are equivalent and the latter is sometimes easier to handle. So we will define our notion of families of d-dimensional subvector spaces of V parametrized by schemes instead of varieties. From Theorem 3.1.12, we may rephrase the definition of vector a bundle in terms of locally free sheaves as follows.

Definition 3.1.15. [27, Subsection 13.1.5] A *vector bundle of rank* d over a scheme S is a locally free sheaf $\mathcal{F} \longrightarrow S$ over S of rank d.

Definition 3.1.16. [23, Definition A.3.1] Let $\mathcal{F} \longrightarrow S$ be a vector bundle over S of rank d, and let $0 \leq r \leq d$. A rank r subbundle of $\mathcal{F}$ is a subsheaf $\mathcal{G}$ of $\mathcal{F}$ which is locally free of rank r and such that the quotient $\mathcal{F}/\mathcal{G}$ is also locally free (which is necessarily of rank $d-r$).

Remark 3.1.17. The condition that $\mathcal{F}/\mathcal{G}$ is locally free actually implies that the subsheaf $\mathcal{G}$ is locally free, and the converse is not true in general. The definition is formuated in this way because we want our families of d-dimensional subspaces to be invariant under pullback along morphisms in order to satisfy (P_2). The condition that $\mathcal{F}$ is locally free is made in order to ensure that any family should be invariant under pullback along morphisms. Furthermore, we would like that injective maps of locally free sheaves correspond to injective maps of vector bundles. The additional condition that the quotient $\mathcal{F}/\mathcal{G}$ is also locally free is to ensure that injective maps on sheaves correspond to injective maps on the corresponding locally free sheaves.

Proposition 3.1.18. [2, Lemma 1.2] Let $\mathcal{E}$ be a locally free sheaf and $\mathcal{F}$ be a subsheaf of $\mathcal{E}$. Then the following are equivalent:

(i) For each $x \in S$, the map restricted to the fibers $\mathcal{F}|_x \longrightarrow \mathcal{E}|_x$ is injective,

(ii) $\mathcal{F}$ is a subbundle of $\mathcal{E}$, that is both $\mathcal{F}$ and $\mathcal{E}/\mathcal{F}$ are locally free.

The equivalence relation on families is given by equality. That is, two families of d-dimensional subvector spaces are equivalent if and only if they are equal. Therefore, we have a well formulated notion of families of d-dimensional linear subspaces over each scheme S, so that we can define the moduli functor associated to this classification problem as follows.

3.1.19 Grassmannian Functor

From the previous subsection, we can define the functor of families of linear vector subspaces of V as follows:

$$\begin{aligned} \mathrm{Fam}(d,n) : (\mathrm{Sch/k})^{\mathrm{opp}} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \mathrm{Fam}(d,n)(S). \end{aligned}$$

Hence the moduli functor associated to this moduli problem, that we call the Grassmannian functor, is defined by:

$$\begin{aligned} \mathrm{F}(d,n) : (\mathrm{Sch/k})^{\mathrm{opp}} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \mathrm{Fam}(d,n)(S)/\sim_S, \end{aligned}$$

and for each morphism $f : T \longrightarrow S$ we associate the pullback map

$$\mathrm{F}(d,n)(f) : \mathrm{F}(d,n)(S) \longrightarrow \mathrm{F}(d,n)(T)$$

which sends an equivalence class of families over S to equivalence class $f^*\mathcal{F}/\sim_T$ of families over T obtained by pulling back $\mathcal{F}$ along f.

More precisely, $\mathrm{Fam}(d,n)(S)/\sim_S$ is the set of equivalence classes of rank d subbundles of the trivial bundle $\mathcal{O}_S^{\oplus n}$.

Now, we turn our attention to the study of the representability of the Grassmannian functor $\mathrm{F}(d,n)$. We want to find a scheme that represents the above Grassmannian functor. We may construct this only for affine schemes $S = \mathrm{Spec}A$ because of the gluability property of schemes. In fact, schemes are defined by gluing together affine schemes. We proceed by first constructing the Grassmannian scheme in the affine case. For an arbitrary scheme S, we can cover it by affine open schemes, say $\{U_\alpha\}$. Then we construct the Grassmannians $\{\mathrm{F}(d,n)(U_\alpha)\}$, and glue them to obtain the Grassmannian scheme over the scheme S. So we may assume that S is affine.

The main point is to prove the following result.

Theorem 3.1.20. There exists a scheme $\mathrm{G(d,n)} \in \mathrm{Sch/k}$ which represents $\mathrm{F}(d,n)$.

The strategy to prove that $\mathrm{F}(d,n)$ is representable is to show that it is covered by representable subfunctors. If this happens, we then glue the local solutions of the covers and then obtain a scheme which will represent $\mathrm{F}(d,n)$. The following result gives a justification to this stragy.

Theorem 3.1.21. [10, Theorem 8.9] Let $F : (\mathrm{Sch}/S)^{\mathrm{opp}} \longrightarrow \mathrm{Set}$ be a functor that satisfies:

(a) F is a sheaf for the Zariski topology,

(b) F has a Zariski open covering $(f_i : F_i \longrightarrow F)_{i\in I}$ by representable functors F_i.

Then F is representable.

Proof. See [10, Theorem 8.9] or [23] for the proof. □

So, to prove that the Grassmannian functor is representable, it is enough to prove that it is a Zariski sheaf and that it has a Zariski open covering by representable subfunctors. This is done through Lemma 3.1.26, Lemma 3.1.27 and Proposition 3.1.28.

Definition 3.1.22. [11, Definition, p.64] Let X be a topological space and $\mathscr{F}$ be a sheaf of groups, rings, modules, etc. A subsheaf of $\mathscr{F}$ is a sheaf $\mathscr{G}$ of groups, rings, modules etc., such that for every open subset U of X, $\mathscr{G}(U)$ is a subgroup, subring, submodules etc., and the restriction maps of the sheaf $\mathscr{G}$ are induced by those of $\mathscr{F}$.

Definition 3.1.23. [10, Definition 8.5] Let $f : F \longrightarrow G$ be a natural transformation in $\mathrm{PSh(Sch)}$. We say that f is representable if for any $X \in \mathrm{Sch}$ and any $g : \mathrm{h}_X \longrightarrow G$, the fiber product functor $F \times_G \mathrm{h}_X$ is representable. Note that the fiber product $F \times_G \mathrm{h}_X$ is defined to be the functor that assigns to each scheme S the object $F(S) \times_{G(S)} \mathrm{h}_X(S)$.

Definition 3.1.24. Let $F, G : \mathrm{Sch} \longrightarrow \mathrm{Set}$ be functors. An open immersion of F into G is a natural transformation $F \longrightarrow G$ such that for every $X \in \mathrm{Sch}$, the fiber product $\mathrm{h}_X \times_G F$ is isomorphic to h_Y, for some $Y \in \mathrm{Sch}$, and a map $Y \longrightarrow X$ given by $\mathrm{h}_X \times_G F \longrightarrow \mathrm{h}_X$ is an immersion.

Definition 3.1.25. Let G be a functor. An open covering of G is a collection $\{F_i \longrightarrow G, i \in I\}$ of natural transformations such that each natural transformation $F_i \longrightarrow G$ is an open immersion and for every field K, the map

$$\bigcup_{i\in I} F_i(\mathrm{Spec}K) \longrightarrow G(\mathrm{Spec}K)$$

is surjective.

Lemma 3.1.26. [23, Lemma A.3.5] The Grassmannian functor $\mathrm{F}(d,n)$ is a Zariski sheaf.

Proof. The proof follows from the definition of a sheaf. In fact, a subsheaf $\mathcal{F}$ is determined in a unique way by an open cover $\{U_\alpha\}_\alpha$ of subsheaves of $\mathcal{F}$ which agree on the intersections. □

Lemma 3.1.27. The Grassmannian functor $\mathrm{F}(d,n)$ admits an open covering.

Proof. Let us produce a collection of open subfunctors that cover the Grassmannian functor $\mathrm{F}(d,n)$. Let $\{e_1,\dots,e_n\}$ be a basis of the k-vector space V. Let $I \subseteq \{1,\dots,n\}$ containing $n-d$ elements. The inclusion $I \hookrightarrow \{1,\dots,n\}$ induces an homomorphism of S-bundle $\mathcal{O}_S^{\oplus I} \longrightarrow \mathcal{O}_S^{\oplus n}$. By S-bundle, we mean a vector bundle over S. We define a subfunctor $\mathrm{F}_I(d,n)$ of $\mathrm{F}(d,n)$ by setting:

$$\mathrm{F}_I(d,n)(S) := \{\mathcal{U} \in \mathrm{F}(d,n)(S) :\ \mathcal{O}_S^{\oplus I} \hookrightarrow \mathcal{O}_S^{\oplus n} \twoheadrightarrow \mathcal{O}_S^{\oplus n}/\mathcal{U} \quad \text{is an isomorphism of } S\text{-bundles}\}.$$

Or equivalently,

$$\mathrm{F}_I(d,n)(S) := \{\mathcal{U} \in \mathrm{F}(d,n)(S) :\ 0 \to \mathcal{O}_S^{\oplus I} \to \mathcal{O}_S^{\oplus n} \to \mathcal{O}_S^{\oplus n}/\mathcal{U} \to 0 \ \text{ exact seq. of } S\text{-bundles}\}.$$

In other words, $\mathrm{F}_I(d,n)(S)$ consists of those families $\mathcal{U} \in \mathrm{F}(d,n)(S)$ that are rank-d direct summands of $\mathcal{O}_S^{\oplus I}$, that is, such that $\mathcal{U} \oplus \mathcal{O}_S^{\oplus I} = \mathcal{O}_S^{\oplus n}$ and $\mathcal{U} \cap \mathcal{O}_S^{\oplus I} = \emptyset$. The inclusion $\mathrm{F}_I(d,n)(S) \longrightarrow \mathrm{F}(d,n)(S)$ induces a morphism of functors between $\mathrm{F}_I(d,n)$ and $\mathrm{F}(d,n)$, say

$$\iota^I : \mathrm{F}_I(d,n) \longrightarrow \mathrm{F}(d,n).$$

We claim that the functors $\mathrm{F}_I(d,n)$, where the I's are the subsets of $\{1,\dots,n\}$ with $n-d$ elements, are open subfunctors of $\mathrm{F}(d,n)$ and that they cover the Grassmannian functor $\mathrm{F}(d,n)$.

Let us show that $\iota^I : \mathrm{F}_I(d,n) \longrightarrow \mathrm{F}(d,n)$ is an open immersion. That means, for any scheme X, we would like to show that $\mathrm{h}_X \times_{\mathrm{F}(d,n)} \mathrm{F}_I(d,n)$, the fiber product of $\mathrm{h}_X \longrightarrow \mathrm{F}(d,n)$ along the natural transformation $\iota^I : \mathrm{F}_I(d,n) \longrightarrow \mathrm{F}(d,n)$, is isomorphic to h_Y, for some scheme Y, and that the morphism $Y \longrightarrow X$ given by $\mathrm{h}_X \times_{\mathrm{F}(d,n)} \mathrm{F}_I(d,n) \longrightarrow \mathrm{h}_X$ (in the Yoneda's lemma) is an open immersion. Let $g : \mathrm{h}_X \longrightarrow \mathrm{F}(d,n)$ be a natural transformation. Then, by the Yoneda's lemma, the data of a natural transformation $\mathrm{h}_X \longrightarrow \mathrm{F}(d,n)$ is equivalent to the data of an element $[\mathcal{U} \subset \mathcal{O}_X^{\oplus n}]$ of $\mathrm{F}(d,n)(X)$. Now we have to compute the fiber product $\mathrm{h}_X \times_{\mathrm{F}(d,n)} \mathrm{F}_I(d,n)$. Again, by the Yoneda's lemma, a map from h_Y to $\mathrm{h}_X \times_{\mathrm{F}(d,n)} \mathrm{F}_I(d,n)$ is equivalent to an element of $\mathrm{h}_X(Y) \times_{\mathrm{F}(d,n)(Y)} \mathrm{F}_I(d,n)(Y)$. But an element of $\mathrm{h}_X(Y) \times_{\mathrm{F}(d,n)(Y)} \mathrm{F}_I(d,n)(Y)$ is a pair $(\phi : Y \longrightarrow X, [F_Y \subset \mathcal{O}_Y^{\oplus n}])$ such that two elements obtained from ϕ and $[F_Y \subset \mathcal{O}_Y^{\oplus n}]$ are the same. Therefore,

$$\begin{aligned} \mathrm{h}_X \times_{\mathrm{F}(d,n)} \mathrm{F}_I(d,n)(Y) &= \{(\phi, \mathcal{U}) \in \mathrm{h}_X(Y) \times \mathrm{F_I(d,n)(Y)} \mid \phi^*\mathcal{U} \in \mathrm{F_I(d,n)(Y)}\} \\ &= \left\{ \begin{matrix} (\phi,\mathcal{U}) \in \mathrm{h}_X(Y) \times \mathrm{F}_I(d,n)(Y) \mid \mathcal{O}_Y^{\oplus I} \hookrightarrow \mathcal{O}_Y^{\oplus n} \twoheadrightarrow \mathcal{O}_Y^{\oplus n}/\phi^* F \\ \text{is an isomorphism} \end{matrix} \right\}. \end{aligned}$$

Now we consider the projection $\mathcal{U} \longrightarrow \mathcal{O}_X^{\oplus I}$, and define

$$U^I := \{x \in X : \mathcal{U}|_x \longrightarrow \mathcal{O}_X^{\oplus I}|_x \ \text{ is an isomorphism.}\}$$

Then U^I is open, and hence

$$\begin{aligned} \mathrm{h}_X \times_{\mathrm{F}(d,n)} \mathrm{F}_I(d,n)(Y) &= \{\phi : Y \longrightarrow X : \phi(Y) \subset X\} \\ &= \{\phi : Y \longrightarrow U^I : \phi(Y) \subset X\}. \end{aligned}$$

This shows that $\mathrm{h}_X \times_{\mathrm{F}(d,n)} \mathrm{F}_I(d,n)$ is representable and represented by U^I.

Moreover, the collection $\mathrm{F}_I(d,n)$ covers the Grassmannian functor $\mathrm{F}(d,n)$. Let K be a field and $h : \bigcup_I \mathrm{F}_I(d,n)(\mathrm{Spec}K) \longrightarrow \mathrm{F}(d,n)(\mathrm{Spec}K)$ a map. We want to show that h is surjective. By definition, $\mathrm{F}(d,n)(\mathrm{SpecK})$ is the set of all d-dimensional subspaces of V. The map h sends an $(n-d)$-dimensional subspace W to its dual $W^\vee$. Now take a d-dimensional subspace W in $\mathrm{F}(d,n)(\mathrm{Spec}\,K)$. Then its dual $W^\vee$ is an $(n-d)$-dimensional K-vector subspace of K^n, so belongs to some $\mathrm{F}_I(d,n)(\mathrm{Spec}\,K)$. But $W^{\vee\vee} = W$, so that $h(W^\vee) = W$. Hence h is surjective, and this shows that the collection $\{\mathrm{F_I(d,n)} \longrightarrow \mathrm{F(d,n)} : \mathrm{I} \subset \{1,\ldots,\mathrm{n}\},\ |\mathrm{I}| = \mathrm{n-d}\}$ is a an open cover of the Grassmannian functor (d,n). □

Proposition 3.1.28. The functor $\mathrm{F}_I(d,n)$ is represented by the affine variety $\mathbb{A}^{d(n-d)}$.

Proof. Let us prove that $\mathrm{F}_I(d,n)$ is isomorphic to $\mathrm{Hom}(-, \mathbb{A}^{d(n-d)})$. We follow the proof of [10, Lemma 8.13 (2)]. Let $X \in \mathrm{Sch}$ and $\mathcal{U} \in \mathrm{F_I}(d,n)(X)$ be a representative of an element of $\mathrm{F}_I(d,n)(X)$. By definition of $\mathrm{F}_I(d,n)(X)$, we have an isomorphism $w : \mathcal{O}_X^{\oplus I} \longrightarrow \mathcal{O}_X^{\oplus n}/\mathcal{U}$. Let $u_\mathcal{U}$ be the composition

$$u_\mathcal{U} : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{O}_X^{\oplus n}/\mathcal{U} \longrightarrow \mathcal{O}_X^{\oplus I},$$

where the first arrow stands for the projection $u^I : \mathcal{O}_X^{\oplus n} \twoheadrightarrow \mathcal{O}_X^{\oplus n}/\mathcal{U}$ and the second is the inverse of w, $w^{-1} : \mathcal{O}_X^{\oplus n}/\mathcal{U} \longrightarrow \mathcal{O}_X^I$. Then the $\mathrm{Ker}(u_\mathcal{U}) = \mathcal{U}$ and $u_\mathcal{U} \circ u^I = \mathrm{id}_{\mathcal{O}_X^I}$.

Conversely, if $u : \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{O}_X^{\oplus I}$ is a homomorphism such that $u \circ u^I = \mathrm{id}_{\mathcal{O}_X^{\oplus I}}$, then we get $\mathrm{Ker}(u) \in \mathrm{F_I(d,n)(X)}$. This defines a bijective map

$$\begin{aligned} \mathcal{M}(X) := \{u \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus n}, \mathcal{O}_X^{\oplus I}) : u \circ u^I = \mathrm{id}_{\mathcal{O}_X^{\oplus I}}\} &\longrightarrow \mathrm{F}_I(d,n)(X) \\ u &\longmapsto \mathrm{Ker}(u), \end{aligned}$$

which is functorial in X. Hence, we obtain an isomorphism of functor $\mathcal{M} \longrightarrow \mathrm{F}_I(d,n)$.

Let $J := \{1, \dots, n\} \setminus I$. The map

$$\begin{array}{rcl} \mathcal{M}(X) & \longrightarrow & \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus J}, \mathcal{O}_X^{\oplus I}) = \Gamma(X, \mathcal{O}_X)^{\oplus J \times I} \cong h_{\mathbb{A}^{d(n-d)}}(X) \\ u & \longmapsto & u\,|_{\mathcal{O}_X^{\oplus J}} \end{array}$$

is bijective and functorial in X, so an isomorphism $F \longrightarrow h_{\mathbb{A}^{d(n-d)}}$. □

3.2 The Grassmannian as orbit space

Let V be a vector space of dimension $n+1$ over an algebraically closed feld k. Let $\mathbb{P}(V) = \mathbb{P}^n$ denote the projective space of all hyperplanes of V. Let $\mathrm{G(d,n)}$ be the Grassmannian of d-dimensional vector subspaces of $\mathbb{P}^n$ passing through the origin. We may identify this with the set of $(n-d)$-dimensional quotient of V. In this section, we would like to construct the Grassmannian scheme as a quotient using the techniques developed in Chapter 5. In order to be able to use GIT, we need to embed $\mathrm{G(d,n)}$ into a projective variety.

Now we would like to give a structure of variety on $\mathrm{G(d,n)}$. For this, we need a coordinate system for subspaces. It will be enough to cover it by affine charts and determine the patchings in the intersections. Let us consider a system of coordinates $x_0, x_1, \dots, x_n$ of $\mathbb{P}^n$. Let $\Lambda \in \mathrm{G(d,n)}$ and let $e_0, \dots, e_d$ be a basis of Λ. Then any e_i can be uniquely written as:

$$e_i = \sum_{j=0}^{n} a_{ij} x_j.$$

This defines a matrix $A_\Lambda = (a_{ij})_{0 \le i \le d, 0 \le j \le n}$. Therefore, any $\Lambda \in \mathrm{G(d,n)}$ can be identified with a $(d+1) \times (n+1)$ matrix

$$A_\Lambda = \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d0} & a_{d1} & \cdots & a_{dn} \end{pmatrix} \tag{3.1}$$

called the Plücker matrix of Λ. Of course this representation depends to the chosen basis for Λ and the system of coordinates of $\mathbb{P}^n$. By changing the basis of Λ, the matrix (3.1) will change by multiplying on the left by the non-degenerate $(d+1) \times (d+1)$ matrix of order $d+1$ corresponding to the change of the basis of Λ. Let $A_{\Lambda i_0 \dots i_d}$ denote the matrix of A with columns $i_0, \dots, i_d$, and let $p_{i_0 \dots i_d} = \det A_{\Lambda i_0 \dots i_d}$ denote the corresponding minor. Since the rank of A_Λ is $d+1$, there exist some $0 \le i_0 < \cdots < i_d \le n$ such that $p_{i_0 \dots i_d} \neq 0$. If we suppose for instance that the minor corresponding to the first $d+1$ columns is not zero, then $(A_{\Lambda i_0 \dots i_d})^{-1} A_\Lambda$ has the form

$$\tilde{A}_\Lambda = \begin{pmatrix} \cdots & * & 1 & * & \cdots & 0 & * \cdots & 0 & * \cdots & * \cdots \\ \cdots & * & 0 & * & \cdots & 1 & * \cdots & 0 & * \cdots & * \cdots \\ & & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \\ \cdots & * & 0 & * & \cdots & 0 & * \cdots & 1 & * \cdots & * \cdots \end{pmatrix}, \tag{3.2}$$

where, for $0 \leq j \leq d$, the i_j-th column is the canonical vector with 1 in position j and the others entries are zero. The other entries of the matrix B are arbitrary. Since multiplication on the left by a $(d+1) \times (d+1)$ invertible matrix does not change the row space, this shows that $\mathrm{G(d,n)}$ contains an open affine subset, of dimension $(d+1)(n-d)$.

Let

$$U_{i_0,\dots,i_d} = \{\Lambda \in \mathrm{G(d,n)} : \Lambda \text{ is represented by a matrix of the form (3.2)}\}.$$

More precisely, $U_{i_0,\dots,i_d}$ denotes the affine open subset of $\mathrm{G(d,n)}$ corresponding to linear subspaces that do not meet the $(n-d-1)$-dimensional subspace defined by the equation $x_{i_0} = \cdots = x_{i_d} = 0$. Equivalently, $U_{i_0,\dots,i_d}$ is the affine open subset of $\mathrm{G(d,n)}$ corresponding to subspaces such that the maximal minor of the Plücker matrix (3.1) obtained when considering the columns $i_0,\dots,i_d$ is not zero. It follows that $\mathrm{G(d,n)}$ is covered by $\binom{n+1}{d+1}$ charts $U_{i_0,\dots,i_d}$. Each subspace $U_{i_0,\dots,i_d}$ is a row space of a unique matrix of the form (3.2).

The chart given above parametrizes $(d+1)$-dimensional subspaces that surject onto the subspace $\mathrm{Span}\{i_0,\dots,i_d\}$ under projection along the complementary coordinate subspace. We can in particular identify $U_{i_0,\dots,i_d}$ with $\mathrm{k}^{(d+1)(n-d)}$.

3.2.1 The Plücker embedding

Let $U \in \mathrm{G(d,n)}$, and $\{u_1,\dots,u_d\}$ be a basis for U. We define a map

$$\begin{aligned} p : \mathrm{G(d,n)} &\longrightarrow \mathbb{P}(\textstyle\bigwedge^d \mathrm{k}^n) \\ U &\longmapsto [u_1 \wedge \cdots \wedge u_d]. \end{aligned}$$

The map p is well defined. In fact, if $\{u'_1,\dots,u'_d\}$ is another basis of U, then

$$u_1 \wedge \cdots \wedge u_d = \det(C) u'_1 \wedge \cdots \wedge u'_d,$$

where C is the base change matrix, which is a $d \times d$ invertible matrix so that $\det C \neq 0$. It follows that $[u_1 \wedge \cdots \wedge u_d] = [u'_1 \wedge \cdots \wedge u'_d]$. Hence, p is well defined.

The goal is to show that the Grassmannian $\mathrm{G(d,n)}$ can be embedded into the projective space $\mathbb{P}(\bigwedge^d \mathrm{k}^n)$. In order to achieve this goal, we have to show that the map p above is injective and its image is the set of totally decomposible multivectors of $\bigwedge^d \mathrm{k}^n$. For this, we need the following.

Lemma 3.2.2. Let $\{u_1,\dots,u_d\}$ and $\{u'_1,\dots,u'_d\}$ be two linearly independent systems of k^n. Then $u_1 \wedge \cdots \wedge u_d$ and $u'_1 \wedge \cdots \wedge u'_d$ are linearly dependent in $\bigwedge^d \mathrm{k}^n$ if and only if $\mathrm{span}\{u_1,\dots,u_d\} = \mathrm{span}\{u'_1,\dots,u'_d\}$.

Proof. First, notice that $u_1 \wedge \cdots \wedge u_d = 0$ if and only if $u_1,\dots,u_d$ are linearly dependent.

Now, assume that $u_1 \wedge \cdots \wedge u_d$ and $u'_1 \wedge \cdots \wedge u'_d$ are linearly dependent in $\bigwedge^d \mathrm{k}^n$), and let us show that $\mathrm{span}\{u_1,\dots,u_d\} = \mathrm{span}\{u'_1,\dots,u'_d\}$. By contradiction, assume that $\mathrm{span}\{u_1,\dots,u_d\} \neq \mathrm{span}\{u'_1,\dots,u'_d\}$. Then there exists at least one index $l \in \{1,\dots,d\}$ such that $u_l \notin \mathrm{span}\{u'_1,\dots,u'_d\}$. Without loss of generality, we may assume that $l = 1$. Then $u_1, u'_1,\dots,u'_d$ are linearly independent. Therefore $u_1 \wedge u'_1 \wedge \cdots \wedge u'_d \neq 0$ by the observation above. Since $u_1 \wedge \cdots \wedge u_d$ and $u'_1 \wedge \cdots \wedge u'_d$ are linearly dependent in $\bigwedge^d \mathrm{k}^n$, there exists a scalar $\lambda \in \mathrm{k}$ such that $u'_1 \wedge \cdots \wedge u'_d = \lambda u_1 \wedge \cdots \wedge u_d$. It follows that $0 \neq u_1 \wedge u'_1 \wedge \cdots \wedge u'_d = \lambda u_1 \wedge u_1 \wedge \cdots \wedge u_d = 0$, which is a contradiction. Therefore, $\operatorname{span}\{u_1, \ldots, u_d\} = \operatorname{span}\{u'_1, \ldots, u'_d\}$.

Conversely, assume that $\operatorname{span}\{u_1, \ldots, u_d\} = \operatorname{span}\{u'_1, \ldots, u'_d\}$, and let us show that $u_1 \wedge \cdots \wedge u_d$ and $u'_1 \wedge \cdots \wedge u'_d$ are linearly dependent in $\bigwedge^d \mathrm{k}^n$. Since $\{u_1, \ldots, u_d\}$ and $\{u'_1, \ldots, u'_d\}$ span the same linear space, it follows from linear algebra that the basis $\{u_1, \ldots, u_d\}$ can be obtained from the basis $\{u'_1, \ldots, u'_d\}$ by a finite sequence of elementary operations of the form $u'_i \mapsto u'_i + \lambda u'_j$ and $u'_i \mapsto \lambda u'_j$ for $\lambda \in \mathrm{k}$ and $i \neq j$. It is enough to show that these operations change the wedge product only by scalar multiplication. Indeed,

$$\begin{aligned} u'_1 \wedge \cdots \wedge u'_{i-1} \wedge (u'_i + \lambda u'_j) \wedge \cdots \wedge u'_d &= u'_1 \wedge \cdots \wedge u'_d + \lambda u'_1 \wedge \ldots u'_j \wedge \ldots u'_j \wedge \cdots \wedge u'_d \\ &= u'_1 \wedge \cdots \wedge u'_d \end{aligned}$$

and

$$u'_1 \wedge \cdots \wedge u'_{i-1} \wedge (\lambda u'_i) \wedge \cdots \wedge u'_d = \lambda u'_1 \wedge \cdots \wedge u'_d.$$

Now, using the multilinearity of the wedge product and the fact that for any permutation σ of the set $\{1, \ldots, d\}$, $u'_1 \wedge \cdots \wedge u'_d = \operatorname{sign}(\sigma) u'_{\sigma(1)} \wedge \cdots \wedge u'_{\sigma(d)}$, we obtain that $u_1 \wedge \cdots \wedge u_d = \lambda u'_1 \wedge \cdots \wedge u'_d$, for some $\lambda \in \mathrm{k}$. ☐

Lemma 3.2.3. The map

$$\begin{aligned} p : \mathrm{G(d,n)} &\longrightarrow \mathbb{P}(\bigwedge^d \mathrm{k}^n) \\ U &\longmapsto [u_1 \wedge \cdots \wedge u_d]. \end{aligned}$$

is injective and its image is the set of totally decomposible multivectors of $\bigwedge^d \mathrm{k}^n$.

Proof. Let us first show that p is injective. Let U and U' be two elements of $\mathrm{G(d,n)}$. Let $\{u_1, \ldots, u_d\}$ and $\{u'_1, \ldots, u'_d\}$ be respectively a basis of U and U'. If $p(U) = p(U')$, that is, $[u_1 \wedge \cdots \wedge u_d] = [u'_1 \wedge \cdots \wedge u'_d]$, then there exists a nonzero scalar λ such that $u_1 \wedge \cdots \wedge u_d = \lambda u'_1 \wedge \cdots \wedge u'_d$. By Lemma 3.2.2, $\operatorname{span}\{u_1, \ldots, u_d\} = \operatorname{span}\{u'_1, \ldots, u'_d\}$; that is, $U = U'$ as desired.

The image of p is the set of totally decomposable multivectors of $\bigwedge^d \mathrm{k}^n$. To see this, let $L_\omega = \{u \in U : \omega \wedge u = 0\}$, for $\omega \in \bigwedge^d \mathrm{k}^n$. L_ω is a linear subspace of V. For $\omega = u_1 \wedge \cdots \wedge u_d$, we have that $L_\omega = W$ is a d-dimensional linear subspace of k^n (see Lemma 3.2.4), that is, $L_\omega \in \mathrm{G(d,n)}$. The map $\omega \longmapsto L_\omega$ is the inverse of p on its image. ☐

Lemma 3.2.4. [8, Lemma 8.11] For any nonzero $\omega \in \bigwedge^d \mathrm{k}^n$, the space L_ω defined above has dimension at most d, and it is d-dimensional if and only if ω is totally decomposable.

Therefore, we may identify the the Grassmannian $\mathrm{G(d,n)}$ with the set of decomposable multivectors of $\bigwedge^d \mathrm{k}^n$. The map p is the so-called Plücker embedding of $\mathrm{G(d,n)}$. The next step is to show that the set of decomposable multivectors is a closed subset of $\mathbb{P}(\bigwedge^d \mathrm{k}^n)$.

Theorem 3.2.5. [8, Corollary 8.13] The Grassmannian variety $\mathrm{G(d,n)}$ is a projective variety, embedded in a closed subset of $\mathbb{P}(\bigwedge^d \mathrm{k}^n)$ under the Plücker embedding.

3.2.6 Grassmannians as quotients of $\mathrm{M}_{d,n}(\mathrm{k})$ by the action of GL_d

Let $\mathrm{M}_{d,n}(\mathrm{k})$ be the set of $d \times n$ matrices with coefficients in k. Let GL_d acts on $\mathrm{M}_{d,n}(\mathrm{k})$ by left multiplication. Now we can identify $\mathrm{M}_{d,n}(\mathrm{k})/\mathrm{GL}_d$ with $\mathrm{G(d,n)}$ as follows. Let $[A] \in \mathrm{M}_{d,n}(\mathrm{k})/\mathrm{GL}_d$. Then there exists a unique matrix C such that CA is of form (3.2). The matrix C determines a unique d-dimensional vector space V. Conversely, any d-dimensional vector subspace Λ can be identified with a matrix A_Λ, and $(A_{\Lambda i_0 \dots i_d})^{-1} A_\Lambda$ has the form (3.2). Therefore, $\mathrm{G(d,n)} = \mathrm{M}_{d,n}(\mathrm{k})/\mathrm{GL}_d$.

3.3 Grassmannian and flag varieties

Let k be an algebraically closed field of arbitrary characteristic. Let $G = \mathrm{SL}_n(\mathrm{k})$ and let $T(n,\mathrm{k})$ be the subgroup of upper triangular matrices in G.

Definition 3.3.1. A Borel subgroup of G is a subgroup of G which is maximal among the connected solvable subgroups. For dimension reasons, Borel subgroups always exist. Equivalently, a Borel subgroup of G is a subgroup B conjugate to the subgroup of upper triangular matrices $T(n,\mathrm{k})$. In other words, $B = gT(n,\mathrm{k})g^{-1}$ for some $g \in G$. In particular, $T(n,\mathrm{k})$ is a Borel subgroup.

Theorem 3.3.2. [3, Theorem 11.1] Let B be a Borel subgroup of G. Then all Borel subgroups are conjugate to B, and G/B is a projective variety.

Definition 3.3.3. A parabolic subgroup of G is a closed subgroup of G containing a Borel subgroup. Alternatively, a parabolic subgroup P of G is a closed subgroup such that G/P is a complete variety.

Suppose B is a Borel supgroup of G. For convenience, B may be considered as a subgroup of upper triangular matrices. The maximal parabolic subgroups of G that contain this choice of B are of the form P_d, with $1 \le d \le n-1$, where

$$P_d = \{A = \begin{pmatrix} * & * \\ & \\ 0_{(n-d)\times d} & * \end{pmatrix} \in G : \det A = 1\}.$$

In general, for a multiindex $\underline{d} = (d_1, \dots, d_r)$ with $1 \le d_1 < d_2 \dots < d_r \le n$, we define

$$P_{\underline{d}} = \bigcap_{i=0}^{r} P_{d_i}.$$

As $P_{\underline{d}}$ is a finite intersection of closed subgroups, it is closed. Since B is contained in each P_{d_i}, it follows that B is contained in $P_{\underline{d}}$ so that $P_{\underline{d}}$ is a parabolic subgroup of G.

Definition 3.3.4. Let V be an n-dimensional vector space and $0 < n_1 < \cdots < n_r < n$. A flag in V corresponding to $0 < n_1 < \cdots < n_r < n$ is a sequence of subspaces

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V,$$

such that $\dim V_i = n_i$, for each i. Then, the corresponding partial flag variety is defined by:

$$F(V; n_1, \ldots, n_r) = \{\text{All partial flag varieties in } V\}.$$

Example 3.3.5. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of k^n, and $V_i = \langle e_1, \ldots, e_i \rangle$ the subspace generated by $e_1, \ldots, e_i$. Then $0 \subset V_1 \subset \cdots \subset V_n = \mathrm{k}^n$ is a flag, called the standard flag.

The flag variety of the form $F(V; 1, 2, \ldots, n)$ is called full flag. A flag variety has the stucture of a projective variety.

Proposition 3.3.6. [17]. Any flag variety $F(V; n_1, \ldots, n_r)$ is a projective variety.

The group GL_n acts on $F(V; n_1, \ldots, n_r)$ as follows: for any $A \in \mathrm{GL}_n$ and for any flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$; we associate the flag $0 = A(V_0) \subset A(V_1) \subset \cdots \subset A(V_n) = A(V)$.

Proposition 3.3.7. [17] The group $\mathrm{GL}_n(V)$ acts transitively on the flag variety $F(V; n_1, \ldots, n_r)$.

Proposition 3.3.8. [17] Let P be a parabolic subgroup of $\mathrm{GL}_n()$. Then P is the stabilizer of some flag.

Proof. Since P is parabolic, it contains some Borel subgroup. Without loss of generality, we may assume that $P \subset T(n,)$. Then the elements of P have block matrix form with blocks of size m_i, $i = 1, \ldots, r$. Consider the sequence $n_i = \sum_{j=1}^{i} m_j$. Since the $\mathrm{GL}_n(V)$ -action of $F(V; d_1, \ldots, d_r)$ is transitive, there exists a basis and an element $\{V_i\} \in F(V; d_1, \ldots, d_r)$ such that P is the stabilizer of $\{V_i\}$. □

$F(V; d_1, \ldots, d_r)$ can be identified with $\mathrm{GL}_n(V)/B_n$, where $B_n := T(n, \mathrm{k})$ is a Borel subgroup of upper triangular matrices, as follows. By Proposition 3.3.7, $\mathrm{GL}_n(V)$ acts transitively on $F(V; d_1, \ldots, d_r)$, and B_n is the stabilizer of the standard flag. Hence, $F(V; d_1, \ldots, d_r)$ can be identified with $\mathrm{GL}_n(V)/B_n$.

Now we can use the connection between parabolic subgroups P and flags to give a description of $F(V; d_1, \ldots, d_r)$. Let P be a parabolic subgroup of $\mathrm{GL}_n(V)$. Then P is the stabilizer of some flag $\{V_i\} \in F(V; d_1, \ldots, d_r)$. Moreover, $F(V; d_1, \ldots, d_r)$ is the orbit of $\{V_i\}$ under $\mathrm{GL}_n(V)$ so that $F(V; d_1, \ldots, d_r) = \mathrm{GL}_n(V)/P$. As a special case, for $r = 1$ and $d = d_1$, $F(V; d)$ is the set of d dimensional subspaces of V, which is the Grasmannian $\mathrm{G(d, n)}$. Therefore, $\mathrm{G(d, n)} = \mathrm{GL}_n(V)/P$. This gives another description of the Grassmannian in terms of flag varieties and parabolic subgroups of $\mathrm{GL}_n(V)$.

Chapter 4

Moduli Spaces of Curves

To any Riemann surface we can attach an invariant g, called genus, that classifies isomorphism classes of Riemann surfaces. For a compact Riemann surface X, its genus is defined to be $g := \dim_{\mathbb{C}} \Omega(X)$. One of the first moduli problems people studied is to classify curves of genus g up to isomorphism. We would like to construct a space M_g whose geometric points are in one-to-one correspondence with the isomorphism classes of nonsingular projective curves of genus g. The purpose of this chapter is to study the moduli problem of nonsingular projective curves of a fixed genus g. This is an example of moduli problem which does not admit a fine moduli space. The obstruction to the representability of the moduli functor associated with this moduli problem comes from the nontriviality of the automorphism group of curves. However, it admits a coarse moduli space. For this, our method is that described in Chapter 2: we will find a parameter category in which we study the representability of the moduli functor associated with this problem of moduli. We divide this moduli into three cases: the genus 0 case, the genus 1 case, and the case where $g \geq 2$. The case of curves of genus 1 with one marked point forms what we call the moduli problem of elliptic curves.

4.1 The Moduli $\mathscr{M}_0$ of curves of genus 0

Let $\mathscr{M}_0$ denote the class of all *nonsingular projective curves of genus* 0 *over an algebraically closed field* k. The isompurhy of curves of genus 0 defines an equivalence relation $\sim_{\mathscr{M}_0}$ on $\mathscr{M}_0$. Now we consider the set $\mathscr{M}_0/\sim_{\mathscr{M}_0}$ of isomorphism classes of nonsingular projective curves of genus 0. We want to parametrize the set $\mathscr{M}_0/\sim_{\mathscr{M}_0}$ by a scheme or a variety M_0 whose closed points are in bijective correspondence with the set of equivalence classes $\mathscr{M}_0/\sim_{\mathscr{M}_0}$. But we know by the uniformization theorem of compact Riemann surfaces [9, Theorem 2.1, p.82] that any nonsingular projective curve of genus 0 is isomorphic to the projective line $\mathbb{P}^1_{\mathrm{k}}$. Therefore, the set of equivalence classes $\mathscr{M}_0/\sim_{\mathscr{M}_0}$ is just a single point $\{\mathbb{P}^1_{\mathrm{k}}\}$. This means, a moduli space (fine or coarse moduli space), if it exists, should be a scheme with just one closed point. This suggests that Speck is the natural candidate for this moduli problem. We may consider our parameter category to be the category of schemes over k.

Definition 4.1.1. Let S be a scheme over k. A *family of nonsingular projective curves of genus* 0 *parametrized by* S is a scheme X, smooth and projective over S whose geometric fibers are nonsingular projective curves of genus 0. This is to say that for each $s \in S$, if we take the fiber X_s and extend the base scheme to the algebraic closure

$\overline{\kappa(s)}$ of the residue field $\kappa(s)$, then the new curve $X_{\overline{s}} = X_s \times_{\kappa(s)} \overline{\kappa(s)}$ is a nonsingular projective curve of genus 0.

Let

$$\mathrm{Fam}_0(S) := \{X \in \mathrm{Sch}/\mathrm{k} : X \text{ is smooth, projective over } \mathrm{k} \text{ and } X_s \in \mathscr{M}_0, \forall s \in S\}$$

be the set of families of nonsingular projective curves of genus 0 parametrized by S. For any morphism $f : T \longrightarrow S$ of schemes over k, we associate the pullback map

$$\begin{array}{rcl} f^* : \mathrm{Fam}_0(S) & \longrightarrow & \mathrm{Fam}_0(T) \\ X/S & \longmapsto & (X \times_S T)/T. \end{array}$$

This defines a functor Fam_0, called *the functor of families of nonsingular projective curves of genus* 0

$$\begin{array}{rcl} \mathrm{Fam}_0 : \mathrm{Sch}/\mathrm{k} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \mathrm{Fam}_0(S) := \left\{ \begin{array}{c} \text{Families of nonsingular projective} \\ \text{curves of genus } 0 \text{ parametrized by } S \end{array} \right\} \\ f & \longmapsto & f^*. \end{array}$$

Now two families X/S and Y/S parametrized by the same base scheme S are isomorphic if there exists an isomorphism $\varphi : X \longrightarrow Y$. This defines an equivalence relation, denoted $\sim_S$, over $\mathrm{Fam}_0(S)$. From this, we deduce the moduli functor of nonsingular projective curves of genus 0

$$\begin{array}{rcl} \mathrm{F}_0 : \mathrm{Sch}/\mathrm{k} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \mathrm{Fam}_0(S)/\sim_S := \left\{ \begin{array}{c} \text{Isomorphism classes of families of nonsingular} \\ \text{projective curves of genus } 0 \text{ parametrized by } S \end{array} \right\} \\ f & \longmapsto & [f^*], \end{array}$$

where $[f^*]$ is the map defined by:

$$\begin{array}{rcl} [f^*] : \mathrm{Fam}_0(S)/\sim_S & \longrightarrow & \mathrm{Fam}_0(T)/\sim_T \\ {[X/S]} & \longmapsto & [(X \times_S T)/T] \end{array}$$

with $[X/S]$ being the $\sim_S$-equivalence class of the family X/S over S and $[(X \times_S T)/T]$ being the $\sim_T$-equivalence class of the family $(X \times_S T)/T$ over T.

Now, we would like to know if there is a scheme representing the moduli functor F_0. We will prove that this functor is not representable, but admits a coarse moduli space. To prove that the moduli functor F_0 is not representable, we will use the notion of ruled surfaces that we introduce now.

4.1.2 Ruled surfaces

We introduce this notion here to prove that in fact the moduli functor F_0 is not representable. We recall that an algebraic surface is an algebraic variety of dimension two. Here, by a *surface*, we mean a nonsingular projective curve over an algebraically closed field k [11, Chapter V, p. 357].

Definition 4.1.3. [11, Chapter V, p. 369] A *ruled surface* or a *geometrically ruled surface* is a surface X, endowed with a surjective morphism $\pi : X \longrightarrow C$ (where C is a nonsingular projective curve), such that for each $c \in C$, the fiber X_c is isomorphic to the projective space $\mathbb{P}^1$, and such that the morphism π admits a section $\sigma : C \longrightarrow X$.

Example 4.1.4. Let C be a nonsingular projective curve. Then $C \times \mathbb{P}^1$ is a ruled surface. In fact, we choose the morphism $\pi : C \times \mathbb{P}^1 \longrightarrow C$ to be the first projection, which is surjective. And for each $y \in C$, $C_y = \{y\} \times \mathbb{P}^1 \cong \mathbb{P}^1$. Moreover, the morphism $\sigma : C \longrightarrow C \times \mathbb{P}^1$ defined by sending any $c \in C$ to the pair $(c, [1, 0])$ is a section of π. This proves that for any nonsingular projective curve C, the variety $C \times \mathbb{P}^1$ is a ruled surface.

Proposition 4.1.5. [11, Chapter V, Proposition 2.2] Let $\pi : X \longrightarrow C$ be a ruled surface. Then there is a locally free sheaf $\mathscr{E}$ such that X is isomorphic to the projective space bundle $\mathbb{P}(\mathscr{E})$ (as defined in [11, Chapter II, p.162]) over C.

4.1.6 Non-existence of a fine moduli space for $\mathscr{M}_0$

Proposition 4.1.7. The moduli functor F_0 of nonsingular projective curves of genus 0 is not representable.

Proof. To prove this, it suffices to show that the moduli functor F_0 is not a Zariski sheaf. But we know that if such a fine moduli space existed, it should be the one point scheme $\operatorname{Spec} k$. So we need to show that $\operatorname{Spec} k$ is not a fine moduli space for the moduli problem of nonsingular projective curves of genus 0. Let $\pi : X \longrightarrow C$ be a ruled surface. By Proposition 4.1.5, X is isomorphic to $\mathbb{P}(\mathscr{E})$, for some vector bundle $\mathscr{E}$ of rank 2 over C. This implies that the curve C can be covered by open set $\{U_i\}_{i \in I}$ such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{P}^1$, that is X is trivial over U_i. But there exist many ruled surfaces that are not trivial. By definition of a ruled surface, we see that a ruled surface $\pi : X \longrightarrow C$ is in particular a family of curves of genus 0 parametrized by C. It follows that the moduli functor F_0 is not a Zariski-sheaf; this is because the structure of a ruled surface X on the curve C is not determined by knowing its structure locally on the curve C. Therefore, the moduli functor F_0 cannot admit a fine moduli space as desired. □

4.1.8 Existence of a coarse moduli space for F_0

As proved above, the moduli problem of nonsingular projective curves of genus 0 does not admit a fine moduli space. However, the situation is not too bad. In fact, we will prove that the moduli functor F_0 has a coarse moduli space as defined in 2.7.2.

Proposition 4.1.9. [12, Proposition 25.1] The affine scheme $\operatorname{Spec} k$ is a coarse moduli space for the moduli problem $\mathscr{M}_0$, and it admits a tautological family.

Proof. We have to construct a natural transformation $\eta : \mathrm{F}_0 \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/k}(-, \operatorname{Spec} k)$ satisfying the assertions of Definition 2.7.2.

(a) For any natural transformation $\eta : \mathrm{F}_0 \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/k}(-, \operatorname{Spec} k)$, we have that

$$\eta(\operatorname{Spec} k) : \mathrm{F}_0(\operatorname{Spec} k) \cong \{\mathbb{P}^1_k\} \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/k}(\operatorname{Spec} k, \operatorname{Spec} k) \cong \operatorname{Spec} k$$

which is always bijective since this is just a set theoretic map between two sets with a single element each. So, it suffices to construct η satisfying the universal property of a coarse moduli space.

(b) Let X/S be a family of nonsingular projective curves of genus 0 over k parametrized by S. Then we define a morphism $e : S \longrightarrow \operatorname{Spec}\mathrm{k}$ such that for each closed point $s \in S$, $f(s)$ is the unique point of $\operatorname{Spec}\mathrm{k}$. We then obtain a morphism of functors $\varphi : \mathrm{F}_0 \longrightarrow \mathrm{h}_M$ such that for all $S \in \mathrm{Sch}/\mathrm{k}$, $\varphi(S) : \mathrm{F}_0(S) \longrightarrow \mathrm{h}_M(S)$ is the morphism sending a family X/S to the unique morphism $e : S \longrightarrow M = \operatorname{Spec}\mathrm{k}$.

Now we show that the second assertion of Definition 2.7.2 is satisfied, that is to show that the morphism of functors $\varphi : \mathrm{F}_0 \longrightarrow \mathrm{h}_M$ is universal among such morphisms. For this, let $\nu : \mathrm{F}_0 \longrightarrow \mathrm{h}_N$ be another morphism of functors with the same properties as $\varphi : \mathrm{F}_0 \longrightarrow \mathrm{h}_M$. We want to find a unique morphism $f : M \longrightarrow N$ of schemes over k such that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{F}_0 & \xrightarrow{\eta} & \mathrm{h}_M \\ & {\scriptstyle\nu}\searrow & \downarrow{\scriptstyle \mathrm{h}_f} \\ & & \mathrm{h}_N \end{array}$$

where h_f is the morphism induced from f.

Since $M = \operatorname{Spec}\mathrm{k} \in \mathrm{Sch}/\mathrm{k}$, the morphism $\nu(M) : \mathrm{F}_0(M) \longrightarrow \mathrm{h}_N(M)$ sends any family X/M to a unique morphism $e : M \longrightarrow N$ (this is because the functor $\nu : \mathrm{F}_0 \longrightarrow \mathrm{h}_N$ is assumed to have the same properties as the functor η; in particular $\nu(\operatorname{Spec}\mathrm{k}) : \mathrm{F}_0(\operatorname{Spec}\mathrm{k}) \longrightarrow \mathrm{h}_N(\operatorname{Spec}\mathrm{k})$ send any family X/S to a unique morphism $e : M = \operatorname{Spec}\mathrm{k} \longrightarrow N$). In particular, $\nu(M)$ sends the family $\mathbb{P}^1_{\mathrm{k}}/M$ to a unique morphism of schemes $f : M \longrightarrow N$. It remains only to prove that the diagram above is commutative. For this, we will need the result of Lemma 4.1.10 that we state below.

Now, we consider a family $X/S \in \mathrm{F}_0(S)$. Assume first that S is a scheme of finite type over k. Then the fiber X_s of any closed point $s \in S$ is $\mathbb{P}^1_{\mathrm{k}}$, so that all the fibers over closed points of S are isomorphic to $\mathbb{P}^1_{\mathrm{k}}$. So any closed point s of S must go to the same point $n_0 \in N$, where n_0 is the image of the single point of $M = \operatorname{Spec}\mathrm{k}$ by the morphism $f : M \longrightarrow N$. Now we consider the reduced point n_0 as a closed subscheme of N, and want to show that the morphism $f : M \longrightarrow N$ factors through n_0. But from Lemma 4.1.10, it follows that the restriction of any $X \in \mathrm{F}_0(S)$ to any Artinian closed subscheme is trivial, and so, f factors through the reduced scheme $M = \operatorname{Spec}\mathrm{k}$.

Now we consider the case where S is not of finite type over k. The proof is similar to by first case by taking base extensions to the geometric of S and Artinian ring over them. This shows again that the associated morphism $S \longrightarrow N$ factors through the reduced scheme $n_0 \in N$. Therefore $f : M \longrightarrow N$ factors through N. This completes the proof of the proposition.

□

Lemma 4.1.10. [12, Lemma 25.2] Let A be an Artinian ring with residue field κ algebraically closed. Then any $X \in \mathrm{F}_0(\operatorname{Spec}\mathrm{A})$ is trivial, and is isomorphic to $\mathbb{P}^1_{\operatorname{Spec}\mathrm{A}}$.

4.2 The moduli $\mathscr{M}_1$ of curves of genus 1

Nonsingular projective curves of genus 1 over an algebraically closed field k with one marked point are also known as elliptic curves over k. So, by an elliptic curve over an algebraically closed field k, we always mean a nonsingular projective curve C of genus 1 over k together with a rational point $P \in C$. For simplicity, we assume that the algebraically closed field k is of characteristic diffferent from 2 and 3.

4.2.1 Formulation of the moduli $\mathscr{M}_1$

In this section, we are concerned with the moduli problem of elliptic curves, that consists of classifying elliptic curves up to isomorphism. So $\mathscr{M}_1$ denotes the collection of all elliptic curves over a fixed algebraically closed field k. We are interested in the way the elliptic curves vary in family. For this, we would like to find a variety or a scheme over k such that its closed points are in one-to-one correspondence with the isomorphism classes of elliptic curves. So we may consider our parameter category to be the category Sch/k of schemes over k.

Definition 4.2.2. [12, §26, p.169] Let S be a scheme over k. A *family of elliptic curves over* k *parametrized by* S is a morphism $\varphi : X \longrightarrow S$ which is flat and whose geometric fibers are elliptic curves over k, along with a section $\sigma : S \longrightarrow X$. If we consider S to be the affine scheme $S = \mathrm{Spec}\mathrm{k}$, then an elliptic curve over S (or just over k) is a nonsingular projective curve E of genus 1 along with a rational point $P \in E$.

Now we define an equivalence relation on families of elliptic curves as follows. Let $(\varphi : X \longrightarrow S, \sigma : S \longrightarrow X)$ and $(\psi : Y \longrightarrow S, \delta : S \longrightarrow Y)$ be two families of elliptic curves over the same base scheme $S \in \mathrm{Sch}/\mathrm{k}$. We say that these two families are equivalent if there exists an isomorphism $\eta : X \longrightarrow Y$ making the following diagrams commutative:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Y \\ & {\scriptstyle\varphi}\searrow & \downarrow{\scriptstyle\psi} \\ & & S \end{array} \qquad \begin{array}{ccc} S & \xrightarrow{\sigma} & X \\ & {\scriptstyle\delta}\searrow & \downarrow{\scriptstyle\eta} \\ & & Y \end{array}$$

This defines an equivalence relation on families of elliptic curves over k parametrized by S. We denote this equivalence relation by $\sim_S$.

Let

$$\mathrm{Fam}_1(S) := \{\text{Families of elliptic curves over } \mathrm{k} \text{ parametrized by } S.\}$$

Let $f : S \longrightarrow T$ be a morphism in Sch/k. For any family $(\varphi : X \longrightarrow T, \sigma : T \longrightarrow X)$ of elliptic curves over k parametrized by T, we define a new family of elliptic curves over k parametrized by S, $(f^*\varphi : X \times_T S \longrightarrow S, f^*\sigma : S \longrightarrow X \times_T S)$, where $f^*\varphi$ is the second projection and $f^*\sigma$ is the map sending each element $s \in S$ to the pair (X_s, s). This defines a set theoretic map

$$\begin{array}{rcl} f^* : \mathrm{Fam}_1(T) & \longrightarrow & \mathrm{Fam}_1(S) \\ (\varphi : X \longrightarrow T, \sigma : T \longrightarrow X) & \longmapsto & (f^*\varphi : X \times_T S \longrightarrow S, f^*\sigma : S \longrightarrow X \times_T S) \end{array}$$

Hence we define the functor of families of elliptic curves as follows:

$$\begin{aligned} \mathrm{Fam}_1 : \mathrm{Sch}/\mathrm{k} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \{\text{Families of elliptic curves over k parametrized by } S\} \\ (f : S \longrightarrow T &\longmapsto f^* : \mathrm{Fam}_1(T) \longrightarrow \mathrm{Fam}_1(S) \end{aligned}$$

Therefore the moduli functor associated with the moduli problem of elliptic curves over k, denoted F_1, is defined by:

$$\begin{aligned} \mathrm{F}_1 : \mathrm{Sch}/\mathrm{k} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \mathrm{Fam}_1(S)/\sim_S := \left\{ \begin{matrix} \text{Isomorphism classes of families of elliptic} \\ \text{curves over k parametrized by } S \end{matrix} \right\} \\ f &\longmapsto [f^*] \end{aligned}$$

where $[f^*]$ is the map induced from f^* by sending an isomorphism class of families of elliptic curves over k parametrized by T to the isomorphism class of the families of elliptic over k parametrized by S obtained by base extension along f.

$$\begin{aligned} [f^*] : \mathrm{Fam}_1(T)/\sim_T &\longrightarrow \mathrm{Fam}_1(S) \sim_S \\ \left(X \xrightarrow{\varphi} T,\ T \xrightarrow{\sigma} X\right)/\sim_T &\longmapsto \left(X \times_T S \xrightarrow{f^*\varphi} S,\ S \xrightarrow{f^*\sigma} X \times_T S\right)/\sim_S \end{aligned}$$

is well-defined, that is isomorphic elliptic curves are mapped under f to isomorphic elliptic curves.

4.2.3 Non-existence of a fine moduli space of elliptic curves

Proposition 4.2.4. The moduli functor F_1 of elliptic curves is not representable.

The proof of this proposition is based on that given in [12, Proposition 26.2] and we added some extra details for clarity.

Proof. By contradiction, assume that F_1 is representable, say by a pair (φ, M_1). Then it follows from Lemma 2.7.13 that every fiberwise trivial family is trivial.

Let us consider the family of elliptic curves $X_t \longrightarrow S = \mathbb{C}^*$ defined by:

$$y^2 = x^3 + t,\ t \neq 0.$$

Take X to be the zero set of the polynomial $y^2 - x^3 - t$, i.e; $X_t = \{(x, y, t) : y^2 - x^3 - t = 0\}$, and $\pi : X \longrightarrow S$ is the map induced by the projection $(x, y, t) \longmapsto t$. For all $t \neq 0$, the curves X_t have the same j-invariant (as defined in the next section), which implies that this family is isotrivial (that is, all fibers are isomorphic) [11]. Another way to see this consists to writing down the isomorphism $\mu : (x, y, t) \longmapsto (\lambda^{-2}x, \lambda^{-3}y, 1)$, $\lambda^6 = t$, between the fibers X_t and X_1. We claim that this family is not isomorphic to a trivial family, that is; is not isomorphic to $S \times X_1$. Indeed, the isomorphism μ trivializes X over the pullback along the map $\mathrm{k}^* \longrightarrow \mathrm{k}^*$, $\lambda \longmapsto \lambda^6$. Let μ_6 denote the group of 6-th roots of unity. Define an action of μ_6 on $X_1 \times \mathrm{k}^*$ by setting:

$$\begin{array}{rcl}\sigma : \mu_6 \times (X_1 \times \mathrm{k}^*) & \longrightarrow & X \\ (\zeta, (x, y, \lambda)) & \longmapsto & (\zeta^{-2}x, \zeta^{-3}y, \zeta\lambda)\end{array}$$

Hence, X can be characterized by a quotient of $X_1 \times \mathrm{k}^*$ by this action. Since this action is determined by X, its non-triviality proves that this action is determined by the variety X, its nontriviality shows that the variety X is not a trivial family. This contradicts Lemma 2.7.13. Therefore, the moduli functor F_1 associated with the moduli problem $\mathscr{M}_1$ of elliptic curves over k is not representable. □

4.2.5 The j-invariant of an elliptic curve

The first step is to find an invariant that characterizes elliptic curves over k. The most interesting invariant that characterizes elliptic curves is called the j-invariant. Let k be an algebraically closed field, and X be an elliptic curve over k.

Weierstrass equations and Legendre forms

Definition 4.2.6. [26, §III.1, p.42] Every elliptic curve E can be described by an equation of the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \text{ with } a_i \in \mathrm{k}, \tag{4.1}$$

called the Weierstass equation of E. We assume further that the characteristic of k is different from 2. By making the substitution

$$y \longmapsto \frac{1}{2}(y - a_1x - a_3),$$

we obtain an equation of the form

$$E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6, \tag{4.2}$$

where the new coefficients b_2, b_4 and b_6 are given by:

$$b_2 = a_1^2 + 4a_2, \ b_4 = 2a_4 + a_1a_3, \ b_6 = a_3^2 + 4a_6.$$

We also define the following quantities:

$$\begin{array}{rcl} b_8 & = & a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2 \\ c_4 & = & b_2^2 - 24b_4 \\ c_6 & = & -b_2^3 + 36b_2b_4 - 216b_6 \\ \Delta & = & -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6. \end{array}$$

Definition 4.2.7. The j-invariant of the elliptic curve E is the quantity $j(E) = \dfrac{c_4^3}{\Delta}$. This quantity is invariant under isomorphism class of elliptic curves and it does not depend on a particular equation chosen [26, p.44]. The j-invariant plays an important role in the classification of elliptic curves. The quantity Δ is called the discriminant of E.

If we assume further that the charcteristic of k is not 2, then any elliptic curve has a Weierstrass equation of the form:

$$y^2 = x^3 + Ax + B. \tag{4.3}$$

In this case, the discriminant is $\Delta = -16\,(4A^3 + 27B^2)$ and $j = -1728\dfrac{(4A)^3}{\Delta}$.

Theorem 4.2.8. [11, Theorem 4.1, p.317] Let k be an algebraically closed field whose characteristic is not 2. Then:

(a) For any elliptic curve E defined over k, the j-invariant depends only on E;

(b) Two elliptic curves E and E' are isomorphic if and only if they have the same j-invariant;

(c) For any $j_0 \in \mathrm{k}$, there exists an elliptic curve E defined over k such that $j(E) = j_0$.

Corollary 4.2.9. There exists a one-to-one correspondence between the set of isomorphism classes of elliptic curves defined over k and the elements of k.

Proof. The map which, to each isomorphism class of elliptic $[E]$ over k assigns the j-invariant $j(E)$ is clearly a one-to-one correspondence thanks to Theorem 4.2.8. □

Definition 4.2.10. [26, p.49] We say that a Weierstrass equation is in Legendre form if it can be described by an equation:

$$y^2 = x(x-1)(x-\lambda) \tag{4.4}$$

Proposition 4.2.11. [26, Proposition 1.7, p.49] Let k be an algebraically closed field of characteristic different from 2.

(a) Every elliptic curve over k is isomorphic to an elliptic curve in Legendre form

$$E_\lambda : y^2 = x(x-1)(x-\lambda)$$

(b) The j-invariant of the elliptic curve in the Legendre form $E_\lambda : y^2 = x(x-1)(x-\lambda)$ is

$$j(E_\lambda) \quad = \quad 2^8\frac{(\lambda^2-\lambda+1)^3}{\lambda^2\,(\lambda-1)^2}. \tag{4.5}$$

Lemma 4.2.12. Let $E_\lambda : y^2 = x(x-1)(x-\lambda)$ and $E'_\lambda : y^2 = x(x-1)(x-\lambda')$ be two elliptic curves over k given by Legendre forms. Then E_λ and E'_λ are isomorphic if and only if

$$\lambda' \in \left\{\lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{\lambda-1}{\lambda}\right\}.$$

Proof. For the proof, we refer to the proof of the assertion (c) of [26, Proposition 1.7, p. 49] □

4.2.13 Existence of a coarse moduli space of elliptic curves

We now use the notion of the j-invariant associated with each elliptic curve to prove that the j-line $\mathbb{A}^1_j$ is a coarse moduli space for the moduli problem of elliptic curves over an algebraically closed field k. Another fundamental tool to achieve this goal is the use of very ample divisors.

Proposition 4.2.14. [11, Proposition 3.1 (b), page 307] Let D be a divisor on a curve X. Then D is very ample if and only if for any $P, Q \in X$, $\dim |D - P - Q| = \dim |D| - 2$.

Proposition 4.2.15. Let D be a divisor of degree $d \geq 2g+1$ on C. Then $\phi : C \longrightarrow \mathbb{P}^{d-g}$ is a closed embedding.

Proof. This follows by the use of Riemann-Roch theorem. Indeed, applying Riemann-Roch theorem, we have $\ell(D) = d - g + 1$, $\ell(D-p) = d - g$, $\ell(D-p-q) = d - 1 - g$. This is because $\ell(D - K_C) = \ell(K_C - D + p) = \ell(K_C - D + p + q) = 0$. □

Corollary 4.2.16. [11, Corollary 3.2 (b), page 307] Let D be a divisor on a curve X of genus g. If $\deg D \geq 2g+1$, then D is very ample.

Example 4.2.17. [11, Example 3.3.3, page 309] Let X be an elliptic curve, that is, $g = 1$. Then for any divisor D od degree 3, we have that $3 = \deg D = 2g + 1$, so that D is very ample by Corollary 4.2.16. Therefore any elliptic curve can be embedded in $\mathbb{P}^{2g} = \mathbb{P}^2$.

Proposition 4.2.18. The moduli functor F_1 of elliptic curves admits a coarse moduli space and it is given by the j-line $\mathbb{A}^1_j$.

Proof. (a) Let us define a natural transformation $\eta : \mathrm{F}_1 \longrightarrow \mathrm{Hom}_{\mathrm{Sch/k}}(-, \mathbb{A}^1_j)$ satisfying the properties of Definition 2.7.2. More precisely, we want to define a morphism

$$\eta_S : \mathrm{F}_1(S) \longrightarrow \mathrm{Hom}_{\mathrm{Sch/k}}(S, \mathbb{A}^1_j)$$

such that for each family $(\varphi : X \longrightarrow S,\ \sigma : S \longrightarrow X) \in \mathrm{F}_1(S)$ of elliptic curves over k, simply denoted by X/S, the restriction of $\eta_S(X/S) : S \longrightarrow \mathbb{A}^1_j$ to the set of closed points of S is the set theoretic map:

$$\begin{array}{rcl} |\eta_S(X/S)| : |S| & \longrightarrow & \mathbb{A}^1_j \\ s & \longmapsto & j(X_s) \end{array}$$

where $j(X_s)$ denotes the j-invariant of the fiber X_s (which is an elliptic curve over k) of X over the closed point s.

Let $\{U_i = \mathrm{Spec} A_i, i \in I\}$ be an affine open covering of S. Then for each $i \in I$, the divisor 3σ is of degree greater or equal than 3, so that it is very ample. Hence, we obtain an embedding $X_{A_i} \hookrightarrow \mathbb{P}^2_{A_i}$ as in Proposition 4.2.14. Assuming that the characteristic of k is different from 2 and 3, by rational operations over the ring A_i (see [11, Chapter IV, §4]), our elliptic curve can be described by an equation of the form:

$$y^2 = x^3 + ax + b, \text{ with } a, b \in A_i.$$

In this case the j-invariant of our elliptic curve is given by:

$$j = 1728 \frac{4a^3}{(4a^3 + 27b^2)}.$$

Hence, we define a morphism $\psi_i : \mathrm{Spec} A_i \longrightarrow \mathbb{A}^1_j$ for each $i \in I$. By the property of sheaves, we can glue these morphisms ψ_i together to obtain a morphism $\psi : S \longrightarrow \mathbb{A}^1_j$ which assigns to each closed point $s \in S$ the j-invariant of the fiber over s, X_s (which is an elliptic curve over k).

So this defines a natural transformation $\eta : \mathrm{F}_1 \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/\mathrm{k}}(-, \mathbb{A}^1_j)$ with the desired properties. It remains only to show that η satisfies the assertions of Definition 2.7.2.

(b) Let us show that $\eta(\mathrm{Spec} \mathrm{k})$ is a set theoretic bijection between the set of isomorphism classes of elliptic curves over k and the elements of the j-line $\mathbb{A}^1_j$. This means, we want to show that the set theoretic map

$$\begin{aligned} J(\mathrm{Spec}\mathrm{k}) : \mathrm{F}_1(\mathrm{Spec}\mathrm{k}) \cong \mathscr{M}_1/\sim_{\mathscr{M}_1} &\longrightarrow \mathrm{Hom}_{\mathrm{Sch}/\mathrm{k}}(\mathrm{Spec}\mathrm{k}, \mathbb{A}^1_j) \cong \mathbb{A}^1_j \\ [E] &\longmapsto j(E) \end{aligned}$$

is bijective. But this is just Corollary 4.2.9.

(c) We show that the j-line is universal among morphisms of functors of the form $\mathrm{F}_1 \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/\mathrm{k}}(-, N)$ possessing the property (a). So, let $\nu : \mathrm{F}_1 \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/\mathrm{k}}(-, N)$ be another natural transformation possessing the property (a). We want to find a unique morphism $f : \mathbb{A}^1_j \longrightarrow N$ in Sch/k such that $\nu = h_f \circ \eta$, where $h_f : \mathrm{Hom}_{\mathrm{Sch}/\mathrm{k}}(-, \mathbb{A}^1_j) \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/\mathrm{k}}(-, N)$ is the corresponding natural transformation of presheaves. Since the morphism of functors $\nu : \mathrm{F}_1 \longrightarrow \mathrm{Hom}_{\mathrm{Sch}/\mathrm{k}}(-, N)$ satisfies the property (a), this means that for every family X/S of elliptic curves over k parametrized by S, there exists a morphism $S \longrightarrow N$. In particular, considering the family given by the Legendre form $y^2 = x(x-1)(x-\lambda)$ parametrized by the affine scheme $S = \mathrm{Spec} B$, where $B = \mathrm{k}[\lambda, \lambda^{-1}, (1-\lambda)^{-1}]$, there exists a morphism $\psi : \mathrm{Spec} B \longrightarrow N$. Furthermore, this morphism is compatible with the action of the group G of order 6 acting on the λ-line consisting of the substitutions

$$\left\{ \lambda, \frac{1}{\lambda}, 1-\lambda, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}, \frac{\lambda-1}{\lambda} \right\}.$$

Then by Lemma 4.2.12, the fibers of the transported family X' are fiberwise isomorphic to the fibers of the family X. Thus the morphism ψ is compatible with the action of G. Hence the morphism ψ factors through $\mathrm{Spec} B^G$, where B^G denotes the fixed ring of B by the action of G. All that remains is to identify B^G with $\mathrm{k}[j]$. It is clear that $j \in B^G$. Considering the function fields $\mathrm{k}(j) \subseteq \mathrm{k}\left(B^G\right) \subseteq \mathrm{k}(B)$, the latter is of order 6 over the two former, that is: $\left[\mathrm{k}(B) : \mathrm{k}\left(B^G\right)\right] = 6$ and $[\mathrm{k}(B) : \mathrm{k}(j)] = 6$. By the tower property, we have

$$6 = [\mathrm{k}(B) : \mathrm{k}(j)] = \left[\mathrm{k}(B) : \mathrm{k}\left(B^G\right)\right] \left[\mathrm{k}\left(B^G\right) : \mathrm{k}(j)\right] = 6 \left[\mathrm{k}\left(B^G\right) : \mathrm{k}(j)\right].$$

It follows that $\left[\mathrm{k}\left(B^G\right) : \mathrm{k}(j)\right] = 1$. Hence $\mathrm{k}(j) = \mathrm{k}\left(B^G\right)$.

Next, we claim that B is integral over $\mathrm{k}[j]$. To prove the claim, it sufficies to show that the generators λ, λ^{-1} and $(\lambda-1)^{-1}$ of B are integral over $\mathrm{k}[j]$. First, for λ, it follows from Equation (4.5) that

$$\lambda^2(\lambda-1)^2 j = 256(\lambda^2-\lambda+1)^3. \tag{4.6}$$

So λ is a root of the polynomial $X^2(X-1)^2 j - 256(X^2-X+1)^3$, which is a polynomial with coefficient in $\mathrm{k}[j]$. This shows that λ is integral over $\mathrm{k}[j]$.

To show that λ^{-1} and $(\lambda-1)^{-1}$ are integral over $\mathrm{k}[j]$, it is enough to show that they are roots of the polynomial $H(X) = X^2(X-1)^2 j - 256(X^2-X+1)^3$. We have:

$$\begin{aligned} H\left(\lambda^{-1}\right) &= \frac{1}{\lambda^2}\left(\frac{1}{\lambda}-1\right)^2 j - 256\left(\frac{1}{\lambda^2}-\frac{1}{\lambda}+1\right) \\ &= \frac{1}{\lambda^4}(1-\lambda)^2 j - \frac{256}{\lambda^6}\left(\lambda^2-\lambda+1\right)^3 \\ &= \lambda^{-6}\left(\lambda^2(\lambda-1)^2 j - 256(\lambda^2-\lambda+1)^3\right) \\ &= \lambda^{-6}H(\lambda) = 0. \end{aligned}$$

Similarly, we have:

$$\begin{aligned} H\left((\lambda-1)^{-1}\right) &= \frac{1}{(1-\lambda)^2}\left(\frac{1}{1-\lambda}-1\right)^2 j - 256\left(\frac{1}{(1-\lambda)^2}-\frac{1}{1-\lambda}+1\right) \\ &= \frac{1}{(1-\lambda)^4}\lambda^2 j - \frac{256}{(1-\lambda)^6}\left((1-\lambda)^2-(1-\lambda)+1\right)^3 \\ &= (1-\lambda)^{-6}\left(\lambda^2(\lambda-1)^2 j - 256(\lambda^2-\lambda+1)^3\right) \\ &= (1-\lambda)^{-6}H(\lambda) = 0. \end{aligned}$$

This shows that λ^{-1} and $(\lambda-1)^{-1}$ are integral over $\mathrm{k}[j]$ too. Hence B is integral over $\mathrm{k}[j]$. In paticular, B^G is integral over $\mathrm{k}[j]$ since $B^G \subset B$. But these two rings have the same quotient fileds and $\mathrm{k}[j]$ is integrally closed, so $\mathrm{k}[j] = B^G$.

Thus we obtain a morphism $\mathbb{A}^1_j \longrightarrow N$, so $\mathbb{A}^1_j$ has the desired property.

□

4.2.19 Non-existence of a tautological family for $\mathbb{A}^1_j$

The main object of this subsection is to show that the existence of a coarse moduli space does not necessarily imply that of a tautological family. For this, we would like to show that the j-line does not admit a tautological family even if it is a coarse moduli space. By contradiction, assume that the j-line $\mathbb{A}^1_j$ has a tautological family. Let X/S be a family of elliptic curves over k and $s_0 \in S$ such that the corresponding fiber E_{s_0} has j-invariant $j(E_{s_0}) = 0$. The family X/S can be represented by an equation of the form $y^2 = x^3 + ax + b$ with $a, b \in A$ in an affine neighbourhood $\operatorname{Spec} A$ of s_0. At the point s_0, the j-invariant is thus given by $j = 12^3\dfrac{4a^3}{4a^3+27b^2}$. Hence a belongs to the maximal ideal m of the ring A at the point s_0. It follows that $j \in \mathrm{m}^3$, so that the morphism $S \longrightarrow \mathbb{A}^1_j$ is ramified at the point s_0, that is the j-function associated with the family cannot have simple zeroes. This implies in particular that the j-line $\mathbb{A}^1_j$ cannot be a tautological family.

4.3 The moduli problem $\mathscr{M}_g$, with $g \geq 2$

To conclude this chapter, let us briefly formulate in a similar way the moduli problem for nonsingular projective curves of genus $g \geq 2$. We fix an algebraically closed field k and an integer $g \geq 2$. Let

$$\mathscr{M}_g = \{ \text{ all nonsingular projective curves over k, of genus } g\}.$$

In this section, for the sake of simplicity, by a curve of genus g we will always mean a nonsingular projective curve over k of genus g. We consider on $\mathscr{M}_g$ the equivalence relation defined by isomorphy: two nonsingular projective curves over k of genus g are equivalent if and ony if they are isomorphic. As usual, let us denote this equivalence relation by $\sim_{\mathscr{M}_g}$. Now we consider the set $\mathscr{M}_g/\sim_{\mathscr{M}_g}$, and we would like to find a variety M_g that parametrizes the isomorphism classes $\mathscr{M}_g/\sim_{\mathscr{M}_g}$ of nonsingular projective curves of genus g, and also describe the families of such curves over arbitrary schemes. So we consider our parameter category to be the category Sch/k, of schemes over k.

Definition 4.3.1. Let S be a scheme over k. A *family of curves of genus over S* is a morphism $\varphi : X \longrightarrow S$ which is flat, proper over S and whose geometric fibers are curves of genus g.

Now we define an equivalence relation on families of curves of genus g over the same base scheme S as follows. Let $\varphi : X \longrightarrow S$ and $\psi : Y \longrightarrow S$ be two families of curves of genus g over the same base scheme $S \in \mathrm{Sch}/\mathrm{k}$. We say that these two families are equivalent if there exists an isomorphism $\eta : X \longrightarrow Y$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\eta} & Y \\ & \searrow^{\varphi} & \downarrow{\psi} \\ & & S \end{array}$$

This defines an equivalence relation on families of curves of genus g over S. We denote this equivalence relation by $\sim_S$.

Let

$$\mathrm{Fam}_g(S) := \{\text{Families of curves of genus } g \text{ over } S\}$$

Let $f : S \longrightarrow T$ be a morphism of schemes over k. For any family $\varphi : X \longrightarrow T$ of curves of genus g over T, we define a family $f^*\varphi : X \times_T S \longrightarrow S$ of curves of genus g over S, where $f^*\varphi$ is the second projection. We get a set theoretic map

$$\begin{aligned} f^* : \mathrm{Fam}_g(T) &\longrightarrow \mathrm{Fam}_g(S) \\ (X \xrightarrow{\varphi} T) &\longmapsto (X \times_T S \xrightarrow{f^*\varphi} S) \end{aligned}$$

Moreover, this assignment is functorial with respect to families X/T of curves of genus g over T. Hence we define the functor of families of curves of genus g by:

$$\begin{aligned} \mathrm{Fam}_g : \mathrm{Sch/k} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \{\text{Families of curves of genus } g \text{ over } S\} \\ (S \xrightarrow{f} T) &\longmapsto (\mathrm{Fam}_g(T) \xrightarrow{f^*} \mathrm{Fam}_g(S)) \end{aligned}$$

Therefore the moduli functor associated with the moduli problem of curves of genus g, denoted F_g, is defined by:

$$\begin{aligned} \mathrm{F}_g : \mathrm{Sch/k} &\longrightarrow \mathrm{Set} \\ S &\longmapsto \mathrm{Fam}_g/\sim_S := \{\text{Isomorphism classes of families of curves of genus } g \text{ over } S\} \\ f &\longmapsto [f^*] \end{aligned}$$

where $[f^*]$ is the map induced from f^* by sending an isomorphism class of families of curves of genus g over T to the isomorphism class of the families of curves of genus g over S obtained by base extension along f. That is:

$$\begin{aligned} [f^*] : \mathrm{Fam}_g(T)/\sim_T &\longrightarrow \mathrm{Fam}_g(S)/\sim_S \\ \left(X \xrightarrow{\varphi} T\right)/\sim_T &\longmapsto \left(X \times_T S \xrightarrow{f^*\varphi} S\right)/\sim_S \end{aligned}$$

where $\left(X \xrightarrow{\varphi} T\right)/\sim_T$ denotes the $\sim_T$-equivalence class of the family $\varphi : X \longrightarrow T$.

4.3.2 Nontriviality of the automorphism group and non-existence of a fine moduli space

In this section , we want to show that, in general, the nontriviality of the automorphism group of certain curves prevents the existence of a fine moduli space. So, we have to show that it is always possible to construct a non-trivial family of curves of genus g which is isotrivial, that is, a family $X \longrightarrow S$ such that all the fibers X_s, $s \in S$ are isomorphic and the family $X \longrightarrow S$ is not trivial. This is shown is the following lemma.

Lemma 4.3.3. Let C be a curve of genus g with nontrivial automorphism group, and let $G \subseteq \mathrm{Aut}(C)$ be a finite and nontrivial group. Then there exists a nonconstant isotrivial (or fiberwise trivial) family $X \longrightarrow S$ of family of curves of genus g parametrized by S.

Proof. Let us construct a non-constant family $X \longrightarrow S$ such that each fiber X_s, $s \in S$, is isomorphic to the curve C. Consider a scheme S', with a free G-action $\sigma : G \times S' \longrightarrow S'$ on S'. Let $S = S'/G$ be the quotient group scheme of S' by the G-action. Let $X' = S' \times C$. The map

$$\begin{aligned} \sigma' : G \times (S' \times C) &\longrightarrow S' \times C \\ (\varphi, (s, c)) &\longmapsto (\sigma(s, c), \varphi(c)) \end{aligned}$$

defines an action of G on $X' = S' \times C$. Now we want to construct a family over S. Since the finite group G acts freely on the quotient group $X' = S' \times C$, it follows that $X'/G \cong S'/G \times C/G$. Then the first projection $X'/G \cong S'/G \times C/G \longrightarrow S = S'/G$ is a family of curves of genus g parametrized by S. Moreover, the fibers are still isomorphic to X, but the curve C will not in general be isomorphic to $C \times S$. □

Theorem 4.3.4. The moduli functor F_g associated with the moduli problem $\mathscr{M}_g$ of nonsingular projective curves over k of genus g is not representable.

Proof. Let us show that $\mathscr{M}_g$ fails to have a fine moduli space. By contradiction, assume that there exists a variety M_g and an isomorphism $\eta : \mathrm{F}_g \longrightarrow \mathrm{Hom}(-, M_g)$. By Lemma 4.3.3, the moduli problem $\mathscr{M}_g$ possesses a nontrivial family $X \longrightarrow S$ which is isotrivial. Then the map $\eta(S) : \mathrm{F}_g(S) \longrightarrow \mathrm{Hom}(S, M_g)$ sends the nontrivial isotrivial family $X \longrightarrow S$ to a morphism $f : S \longrightarrow M_g$. But the morphism f sends each closed point $s \in S$ to the point corresponding to the isomorphsm class of the fiber X_s of X over s. Since all the fibers are isomorphic, it follows that $f(t) = f(s)$, for all $s, t \in S$. This means that $f(S)$ is a single point. Therefore the family $X \longrightarrow S$ should be isomorphic to the trivial product family.

Now we take the pull-back of the universal family $C_g \longrightarrow M_g$ along f, and then we must have this pull-back map must be equivalent to $\pi : X \longrightarrow S$. But $f^*C_g = C_g \times_{M_g} B = X_s \times S$. This means $X_s \times S$ is equivalent to $\pi : X \longrightarrow S$, which contradicts the assumption that $\pi : X \longrightarrow S$ is not equivalent to $X_s \times S$ for any $s \in S$. □

However, Mumford has proved that the moduli problem $\mathscr{M}_g$ admits a coarse moduli space M_g [4, Theorem 5.1.1]. Deligne and Mumford showed in [24] that the coarse moduli space M_g is a quasi-projective variety and has dimension $3g - 3$ over k. An algebraic proof is given by Fulton [7].

To close this chapter, we summarize without proof what we said in the following theorem. We refer to the references above for the proof.

Theorem 4.3.5. [12, Theorem 27.1, p.178] The moduli functor F_g associated with the moduli problem $\mathscr{M}_g$ of nonsingular projective curves of genus g over k admits a coarse moduli space M_g.

Proof. For the existence of M_g, we refer to [4, Theorem 5.1.1]. The irreducibility of M_g is proved in [24]. □

By Theorem 4.3.5, we have a natural transformation $\eta : \mathrm{F}_g \longrightarrow \mathrm{Hom}_{\mathrm{Sch/k}}(-, M_g)$ satisfying the assertions of 2.7.2. This implies the following:

(a) The map $\eta(\mathrm{Speck}) : \mathrm{F}_g(\mathrm{Speck}) \cong \mathscr{M}_g/\sim_{\mathscr{M}_g} \longrightarrow \mathrm{Hom}_{\mathrm{Sch/k}}(\mathrm{Speck}, M_g) \cong |M_g|$ is bijective. In other words, the closed points of M_g are in bijective correspondence with the isomorphism classes of nonsingular projective curves of genus g over k.

(b) For any family X/S of curves of genus g over S, there is a morphism $f : S \longrightarrow M_g$. This morphism sends any point s in S into the point of M_g corresponding by (a) to the isomorphism class of the fiber X_s of X over s.

4.3.6 Conclusion

We have seen that the moduli problem $\mathscr{M}_g$ of curves of fixed genus $g \in \mathbb{Z}_+$ over an algebraically closed field k does not admit a fine moduli problem. One reason for the failure of the existence of a fine moduli space is the presence of families of curves which are isotrivial, but not trivial. But it has been shown that in any case there always exists a coarse moduli space. Moreover, there does not even exist a tautological family, except for the case $g = 0$.

It is still not an easy task to construct moduli space. One of the oldest methods dealing with the construction of moduli spaces is to realize them a quotient varieties using the Geometric Invariant Theory developed by Mumford [4].

Chapter 5

Geometric Invariant Theory

In this chapter, we will describe a method of constructing quotients in algebraic geometry, and then relate this construction to that of coarse moduli spaces. In fact, moduli problems can be reduced to the existence of nice quotients of the action of an algebraic group scheme G on a scheme X. The natural context for the construction is that of schemes over a fixed base field k. We will start by defining the notions of algebraic groups, their actions and related notions of quotients such as categorical quotients, good quotients geometric quotients. In order to construct categorical quotients, one fundamental question we should answer: is the k-algebra of regular functions $\mathcal{O}(X)$ finitely generated? An affirmative answer to this question is given for the action reductive groups by Nagata's Theorem. We then relate the construction of coarse moduli spaces to forming categorical quotients. The main texbooks for this chapter are [21], [4], [5], and [19].

5.1 Affine algebraic groups

In this section, we introduce the notions of algebraic groups needed for Geometric Invariant Theory. Roughly speaking, a *group scheme over* k is a group in the category of schemes over k. It is called an *affine group scheme* over k if the underlying scheme is affine.

Since the category of all affine subschemes is a full subcategory of all schemes [10], then to give an affine group scheme over k is the same as giving a group in the category of affine schemes over k.

An *abelian variety* is a smooth projective algebraic group. By Chevalley's theorem [18, Theorem 10.25], every algebraic group is an extension of an abelian variety by an affine algebraic group whose underlying scheme is affine. This allows us to restrict our study to that of affine algebraic groups.

Definition 5.1.1. An *algebraic group* over k is a scheme G over k along with morphisms

(1) identity element: $1 : \mathrm{Speck} \longrightarrow G$; (this is just choosing a point of G, we denote this point by e)

(2) group law: $m_G : G \times G \longrightarrow G,\ (g,h) \longmapsto g \cdot h$;

(3) group inversion: $i : G \longrightarrow G,\ g \longmapsto g^{-1}$

satisfying the usual axioms of group:

(a) $m_G\left(g, m_G(h,l)\right) = m_G\left(m_G(g,h), l\right)$,

(b) $m_G(g,e) = m_G(e,g) = g$,

(c) $m_G(g, i(g)) = m_G(i(g), g) = e$.

We will denote an algebraic group G by the pair (G, m_G). If the underlying scheme G is affine, G is called an *affine algebraic group*. This definition can be generalized to a group object in any category. In particular, we say that G is a group variety to mean G is a group in the category of all varieties.

Definition 5.1.2. Let (G, m_G) and (H, m_H) be two algebraic groups over k. A *homomorphism of algebraic groups* from G to H is a morphism of schemes $f : G \longrightarrow H$ such that

$$f \circ m_G = m_H \circ (f \times f).$$

Definition 5.1.3. An algebraic subgroup of an algebraic group (G, m_G) over k is an algebraic group (H, m_H) over k such that H is a k-subscheme of G and the inclusion map is a homomorphism of algebraic groups. We say that an algebraic group G' is an *algebraic quotient* of G if there is a homomorphism of algebraic groups $f : G \longrightarrow G'$ which is flat and surjective.

Example 5.1.4 (Some classical examples of Group Schemes)**.** 1) The additive group $\mathbb{G}_a = \mathrm{Speck}[t]$ over k with the co-group operations

$$m^*(t) = t \otimes 1 + 1 \otimes t, \quad i^*(t) = -t \text{ and } e^*(t) = 0$$

is an algebraic group whose underlying variety is $\mathbb{A}^1$.

2) The multiplicative group $\mathbb{G}_m = \mathrm{Spec}\,\mathrm{k}[t, t^{-1}]$ over k with the co-group operations

$$m^*(t) = t \otimes t, \quad i^*(t) = t^{-1} \text{ and } e^*(t) = 1.$$

is an algebraic group whose underlying variety is $\mathbb{A} - \{0\}$.

3) The General Linear group $\mathrm{GL}_n = \{A \in \mathrm{M}_n(\mathrm{k}) : \det A \neq 0\}$ over k is an open subvariety of $\mathbb{A}^n$. GL_n with the co-group operations:

$$m^*(x_{ij}) = \sum_{l=1}^{n} x_{il} \otimes x_{lj} \text{ and } i^*(x_{ij}) = (x_{ij})^{-1}_{ij}$$

4) Every smooth curve of degree 3 in the projective plane $\mathbb{P}^2$ has the structure of an algebraic group [11, Proposition IV.4.8]. These elliptic curves yield examples of projective, and hence non-affine, algebraic groups.

Definition 5.1.5. A *linear algebraic group* is a subgroup of GL_n which is defined by polynomial equations. Any linear algebraic subgroup is automatically affine. The converse is also true: any affine algebraic group is linear ([13, Theorem 3.9]).

The important examples of linear algebraic groups for our purpose are groups GL_n, SL_n and PGL_n. The Special Linear group $\mathrm{SL}_n := \{g \in \mathrm{M}_n : \det g = 1\}$ is a closed subgroup of GL_n, and PGL_n is isomorphic to a closed subgroup of GL_n.

An action G over k determines a group-valued functor on the category of finitely generated k-algebras; it assigns to any finitely generated k-algebra R given by $R \longmapsto G(R) = \mathrm{Hom}(R, G)$. In the same way, to every k-vector space V, we can associate a group-valued functor $\mathrm{GL}(V)$ that assigns to any finitely generated k-algebra R, the group of R-linear automorphisms $\mathrm{Aut}_R(V \otimes_{\mathrm{k}} R)$. If V is finite-dimensional, then $\mathrm{GL}(V)$ is an affine algebraic group.

Definition 5.1.6. A *linear representation* of an algebraic group G on a vector space V over k is a homomorphism of group valued functors $\rho : G \longrightarrow \mathrm{GL}(V)$. If V is finite-dimensional, this is equivalent to a homomorphism of algebraic groups $\rho : G \longrightarrow \mathrm{GL}(V)$, which we call a *finite dimensional linear representation* of G. Another equivalent terminology for linear representation is rational representation.

5.2 Group actions and basic results

In this section, we recall the notion of group action and describe the action of the k-algebra of regular functions on X.

5.2.1 Group action

Definition 5.2.2. An (algebraic) *action* of an affine algebraic group (G, m_G) on a scheme X is a morphism $\sigma : G \times X \longrightarrow X$ such that the composition $\sigma \circ (\mathrm{id} \otimes e) : X \cong X \times \mathrm{Speck} \longrightarrow X \times G \longrightarrow X$ is the identity and the maps $\sigma \circ (\mathrm{id} \times m_G), \sigma \circ (\sigma \times \mathrm{id}) : X \times G \times G \to X$ are equal. In other words, we have:

$$\sigma(e, x) = x \text{ and } \sigma(g, \sigma(h, x)) = \sigma(m_G(g, h), x) \quad \text{for all } (g, h) \in G \times G \text{ and } x \in X.$$

Definition 5.2.3. A *representation* of an affine group scheme $G = \mathrm{Spec}A$ is a k-vector space V endowed with a linear map $\mu : V \to V \otimes A$ which satisfies the dual relations for those of an action. A vector $v \in V$ is called *invariant* if $\mu(v) = v \otimes 1$, and a subspace $U \subset V$ is called a *subrepresentation* if $\mu(U) \subset U \otimes A$.

Example 5.2.4. Let $V = \mathbb{A}^2$. The following are examples of group actions.

1) Let G be the multiplicative group $\mathbb{G}_m$. The map $(t, (x, y)) \longmapsto (tx, ty)$ is an action of G on V.

2) Let G be the multiplicative group $\mathbb{G}_m$. Then G acts on V via the map $(t, (x, y)) \longmapsto (tx, t^{-1}y)$.

3) Consider the additive group $G = \mathbb{G}_a$. Then G acts on V via the map $(t, (x, y)) \longmapsto (x + ty, y)$.

4) $G = \mathbb{G}_m \times \mathbb{G}_a$ acts on V via the map $((t, s), (x, y)) \to (x + sy, ty)$.

5) For simplicity, suppose that k is an algebraically closed field with characteristic prime to n. Let z be a primitive n-th root of unity. Then $G = \mu_n$ acts on V via the action $(a, (x, y)) \to (z^a x, z^a y)$.

Definition 5.2.5. Let $\sigma : G \times X \longrightarrow X$ and $\pi : G \times Y \longrightarrow Y$ be two actions as above. Then a morphism $f : X \longrightarrow Y$ is *G-equivariant* (or G-morphism) if:

$$f(\sigma(g,x)) = \pi(g, f(x)),$$

for all $g \in G$, $x \in X$. For convenience we denote $\sigma(g,x)$ by $g.x$, so that the previous relation becomes

$$f(g \cdot x) = g \cdot f(x)$$

In particular, when G acts trivially on Y (that is, $g \cdot y = y$ for all $y \in Y$ and all $g \in G$), we say that the morphism f is *G-invariant*. If this happens, we obtain:

$$f(g \cdot x) = f(x).$$

for all $g \in G$ and $x \in X$, which is equivalent to say that the morphism f is constant on each orbit.

5.2.6 The k-algebra $\mathcal{O}(X)$ of regular functions

Let X be a scheme over k, and $\mathcal{O}(X)$ the k-algebra of regular functions $f : X \longrightarrow \mathrm{k}$. For an affine algebraic variety, this is isomorphic to the coordinate algebra. If G and X are both affine, an action of G on X, say $\sigma : G \times X \longrightarrow X$ is equivalent to a coaction homomorphism

$$\sigma^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(G) \times \mathcal{O}(X).$$

This defines a unique action of G on $\mathcal{O}(X)$ given by:

$$(g \cdot f)(x) := f(g^{-1} \cdot x) \text{ for all } x \in X \text{ and } g \in G. \tag{5.1}$$

Definition 5.2.7. Let V be a k-vector space and G an affine algebraic group. An action of G on V is a map

$$\begin{array}{rcl} \sigma : (\text{Affine k-algrebra}) & \longrightarrow & \{\text{Actions on } V \otimes_{\mathrm{k}} R\} \\ R & \longmapsto & \sigma_R \end{array}$$

where $\sigma_R : \mathrm{Hom}(R,G) \times (V \otimes_{\mathrm{k}} R) \longrightarrow V \otimes_{\mathrm{k}} R$ is an action of $\mathrm{Hom}(R,G)$ on $V \otimes_{\mathrm{k}} R$ such that $\sigma_R(g,-)$ is a morphism of R-modules functorial in R. We define in the same way an action of G on k-algebra A, and in this case $\sigma_R(g,-)$ is a morphism of R-algebras. We say that an action of G on a k-algebra A is *rational* if every element of A is contained in a finite dimensional G-invariant linear subspace of A.

Lemma 5.2.8. [22, Lemma 3.1] Let G be an affine algebraic group acting on an affine scheme X. Then

(1) For any finite dimensional vector subspace W of $\mathcal{O}(G)$, there is a finite dimensional G-invariant vector subspace V of $\mathcal{O}(X)$ containing W.

(2) In any case, any $f \in \mathcal{O}(X)$ is contained in a finite dimensional G-invariant subspace of $\mathcal{O}(X)$.

(3) If W is a finite dimensional G-invariant subspace of $\mathcal{O}(G)$, then the action of G on W is given by a rational action.

Since the coordinates ring $\mathcal{O}(X)$ is finitely generated, it follows by the lemma above that this action is rational.

Proof. (1) Let W be a finite dimensional vector subspace of $\mathcal{O}(G)$, say $\dim W = r$. Consider a basis $\{f_1, \ldots, f_r\}$ of W. Let $W' = \mathrm{span}\,\{g \cdot f_i \mid i = 1, \ldots, r,\ g \in G\}$. Certainly, W' contains W. So, it is enough to show that W' is invariant under the action of G and it is finite dimensional. But W' is clearly G-invariant by construction. It remains only to show that W' is finite dimensional.

Let us now show that W' is finite dimensional. The action $\sigma : G \times X \longrightarrow X$ of G on X induces a coaction $\sigma^* : \mathcal{O}(X) \longrightarrow \mathcal{O}(G) \times \mathcal{O}(X)$. Then for all $i = 1, \ldots, r$ and for all $g \in G$, we have:

$$\sigma^*(f_i) = \sum_{j=1}^{n_i} a_{ij} \otimes b_{ij}, \text{ with } a_{ij} \in \mathcal{O}(G) \text{ and } b_{ij} \in \mathcal{O}(X).$$

It follows that, for all $g \in G$,

$$g \cdot f_i = \sum_{j=1}^{n_i} a_{ij}(g) b_{ij}.$$

Let $W'' = \mathrm{span}\,\{b_{ij} \mid i = 1, \ldots, r,\ j = 1, \ldots, n_i\}$. W'' is certainly finite dimensional. Moreover, for all $i = 1, \ldots, r$ and for all $g \in G$, we have that $g \cdot f_i = \sum_{j=1}^{n_i} a_{ij}(g) b_{ij} \in W''$. This proves that $W' \subseteq W''$. Therefore W' is finite dimensional as desired.

(2) The second assertion follows from the first assertion by taking $W = \mathrm{span}\{f\}$.

(3) Let us show that the action of W is given by a rational action. Let $\{f_1, \ldots, f_r\}$ be a basis of W. Then we have:

$$\sigma^*(f_i) = \sum_{j=1}^{n_i} a_{ij} \otimes b_{ij}, \text{ with } a_{ij} \in \mathcal{O}(G) \text{ and } b_{ij} \in \mathcal{O}(X).$$

This gives rise to a morphism we have:

$$a = (a_{ij}) : G \longrightarrow M(n).$$

As every element of G is invertible, this morphism has to factor through the linear group $\mathrm{Gl}(n)$ since.

□

Definition 5.2.9. Let $\sigma : G \times X \longrightarrow X$ be an action of an affine algebraic group G on a scheme X.

For a k-point $x \in X$, the stabilizer G_x of x is defined to be the fiber product of $\sigma_x : G \longrightarrow X$ and $x : \operatorname{Spec} \mathrm{k} \longrightarrow X$. In other words, G_x is the closed subgroup

$$G_x := \{g \in G : g \cdot x = x\}.$$

The orbit of x is the set $G \cdot x = \{g \cdot x : g \in G\}$; that is, $G.x$ is image of the induced morphism $\sigma_x = \sigma(-, x) : G(\mathrm{k}) \longrightarrow X(\mathrm{k})$ given by $g \longmapsto g \cdot x$. If all the G-orbits are closed subsets of X, we say that the G-action is closed. The orbit space is the set of all G-orbits of X, denoted $X/G := \{G \cdot x : x \in X\}$.

Since G_x is the preimage of a closed subscheme of X under σ_x, then it is a closed subscheme of G. What can we say about orbits? This is the aim of the following proposition.

Proposition 5.2.10. [13, Proposition 3.15] Let G be an affine algebraic group acting on a scheme X. If x is a closed point of X, then G_x is a locally closed subset of X. Hence G_x can be identified with the corresponding reduced locally closed subscheme.

Furthermore, the boundary $\overline{G \cdot x} - G \cdot x$ of the orbit $G.x$ is a union of orbits of strictly smaller dimension. In particular, each orbit closure contains a closed orbit of minimal dimension.

Proposition 5.2.11. [21, Lemma 3.7] Consider an affine algebraic group G acting on a scheme X. For any k-point $x \in X(\mathrm{k})$, we have

$$\dim(G) = \dim(G_x) + \dim(G.x).$$

Moreover, the dimension of the stabilizer subgroup viewed as a function $X \longrightarrow \mathbb{N}$ is upper semi-continuous; that is, for every $n \in \mathbb{N}$, the set

$$\{x \in X : \dim G_x \geq n\}$$

is closed in X, and the dimension of an orbit subgroup is lower semi-continuous, that is the set

$$\{x \in X : \dim(G \cdot x) \leq n\}$$

is closed for every $n \in \mathbb{N}$.

5.3 Affine quotients

Given a group action on an affine scheme X, we would like to construct a quotient, which will be ideally the set theoretical space of orbits under G, and will have a structure of an affine scheme. Unfortunately, the orbit space may not always admit a structure of a scheme. We will introduce various alternatives corresponding to different notions of moduli space that we described above. In fact, there exist several notions of a quotient, namely: categorical quotient, geometric quotient, and good quotient.

The first notion is the weakest and dictated by the functorial point of view.

5.3.1 Categorical quotient

The notion of categorical quotient is fundamental in the construction of coarse moduli spaces. We will prove in the next section that any coarse moduli space is in fact a categorical quotient and any categorical quotient is a coarse moduli space if and only if it is an orbit space. This motivates this subsection.

Definition 5.3.2. A *categorical quotient* for the action of an affine algebraic group G on a scheme X, is a scheme Y endowed with a G-invariant morphism $\varphi : X \longrightarrow Y$ satisfying the following universal property: for every G-invariant morphism $f : X \longrightarrow Z$, there exists a unique morphism $h : Y \longrightarrow Z$ such that $f = \varphi \circ h$. In categorical language, it just means that the diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{m_G} & X \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle \varphi} \\ X & \xrightarrow[\varphi]{} & Y \end{array}$$

is a pushout diagram in the category of schemes over $\Bbbk$. As objects defined by a universal property are unique, it follows that a categorical quotient, if it exists, is unique up to a unique isomorphism.

Definition 5.3.3. An *orbit space* is a categorical quotient (Y, φ) such that the preimage of each $\Bbbk$-point in Y is a single orbit.

Let $\varphi : X \longrightarrow Y$ be a categorical quotient of X by the action of an affine algebraic group G on X. By definition, φ is constant on each orbit $G \cdot x$, and as φ is continuous (as all morphisms of schemes), it follows that φ is constant on orbit closures. Hence, a categorical quotient is an orbit space only if the action of G on X is closed (that is the G-orbits are closed).

The categorical quotient has good functorial properties in the sense that if $\varphi : X \longrightarrow Y$ is a G-invariant morphism and if there is an open cover $\{U_i\}_i$ of Y such that the restriction $\varphi| : \varphi^{-1}(U_i) \longrightarrow U_i$ is a categorical quotient for each i, then φ is a categorical quotient [13, Remark 3.24]. But a categorical quotient may fail to have the desired geometric properties of interest.

Example 5.3.4. Let the multiplicative group $G = \mathbb{G}_m$ act on the affine variety $\mathbb{A}^2$ by $t \cdot (x, y) = (tx, t^{-1}y)$. The orbits of this action are

- the conics $C_\alpha := \{(x, y) : xy = \alpha\}$, for $\alpha \in \mathbb{A}^1 - \{0\}$; corresponding to the points (x, y) such that $xy \neq 0$,
- the punctured x-axis, which is the orbit of the points of the form $(x, 0)$ with $x \neq 0$,
- the punctured y-axis, which is the orbit of the points of the form $(0, y)$ with $y \neq 0$, and
- the origin which is the orbit of the point $(0, 0)$.

The quotient of $\mathbb{A}^2$ by the action of $\mathbb{G}_m$ in this example is a categorical quotient, but it is not separated. Indeed, the punctured axes both are not closed, and they contain the origin in their orbit closures. Moreover, the dimension of the orbit at the origin is strictly smaller than the dimension of $\mathbb{G}_m$, which suggests that its stabilizer has positive dimension. So $\mathrm{k}^2/\mathrm{k}^*$ cannot be separated. To get a separated quotient, we need to combine the punctured axes and the origin, and we obtain a categorical quotient $\mathrm{k}^2/\mathrm{k}^* \cong \mathrm{k}$. The quotient morphism is given by $(x, y) \longmapsto xy$. However, even if we remove the origin, we cannot obtain a separated variety.

Example 5.3.5. Let k^* operate on k^n through the multiplication, that is $\lambda.(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$ (with $n \geq 2$). Then any G-invariant morphism $X \longrightarrow Y$ is necessarily constant because the origin is contained in the orbit closure of each orbit. Therefore, the categorical quotient is a single point, and is given by $\varphi : \mathrm{k}^n \longrightarrow \{\mathrm{pt}\}$, $(x_1, \ldots, x_n) \longmapsto \mathrm{pt}$. This quotient cannot be an orbit space because the preimage of $\{\mathrm{pt}\}$ is not a single orbit.

Example 5.3.6. Let k^* operate on $\mathrm{k}^n - \{(0, \ldots, 0)\}$ through the multiplication. It is obvious that the quotient is the projective space $\mathbb{P}^{n-1}$ which is both a categorical quotient and an orbit space.

Remark 5.3.7. Deleting only one point has an important impact on the quotient as shown in the examples above. As the ultimate goal is to construct categorical quotients with nice geometric properties, we will do this using the GIT by throwing away bad points.

Definition 5.3.8. Let G be an affine algebraic group acting on a scheme X over k via σ. A subset U of X is *invariant* under σ if $\sigma_g(U) \subseteq U$ for all $g \in G$, that is $g.U \subseteq U$. The set

$$\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) : g \cdot f = f \text{ for all } g \in G\} \tag{5.2}$$

is called the subalgebra of invariant functions. For any invariant subset U of X, the above action induces an action on U, and we denote by $\mathcal{O}(U)^G$ the corresponding subalgebra of invariant functions.

The categorical quotient exists under a rather weak assumption in the category of affine schemes. However, Example 5.3.5 above shows that categorical quotients may not at all look like orbit spaces. Now we introduce the notion of good quotient for an action of a group G on an affine scheme X.

5.3.9 Good and geometric quotient

Let G be an affine algebraic group operating on a scheme X over k.

Definition 5.3.10. A *good quotient* of X for the action of G on X is a scheme Y and a morphism $\varphi : X \longrightarrow Y$ satisfying:

(1) φ is G-invariant, i.e; $\varphi \circ \sigma = \varphi \circ p_2$ where $p_2 : G \times X \longrightarrow X$ is the projection on X, and σ the action of G on X.

(2) φ is surjective and affine (where affine means that the preimage of every affine open is affine).

(3) If U is an open subset of Y, the homomorphism $\varphi^* : \mathcal{O}_Y(U) \longrightarrow \mathcal{O}_X(\varphi^{-1}(U))$ maps the algebra $\mathcal{O}_Y(U)$ isomorphically onto the algebra of G-invariant functions $\mathcal{O}_X(\varphi^{-1}(U))^G$.

(4) For every closed subscheme Z of X which is G-invariant, its image $\varphi(Z)$ is closed in Y.

(5) If W and W' are two disjoint G-invariant closed subsets of X, then $\varphi(W)$ and $\varphi(W')$ are disjoint.

Remark 5.3.11. If we assume that φ is surjective, then the assertions (4) and (5) are equivalent to:

(4′) If W and W' are disjoint closed subsets, then the closures $\overline{\varphi(W)}$ and $\overline{\varphi(W')}$ are disjoint.

In general, good quotients may not always exist: take $X = \mathbb{P}^n$ with a nontrivial action of $\mathbb{G}_m$ given by

$$\lambda \cdot (x_0, \dots, x_n) = (\lambda^{i_0} x_0, \dots, \lambda^{i_n} x_n), \text{ with } \sum_{l=0}^{n} i_l = 0.$$

A generic orbit has dimension 1. A good quotient, if it existed, is then of dimension $< n$. On the other hand, the points e_i of coordinates $x_j = 0$ for $j \neq i$ are invariant under this action and their orbits are therefore closed and disjoint. Hence the quotient, if it exists, must have dimension > 0. But $\mathbb{P}^n$ admits no non-trivial affine morphism on an algebraic variety Y of dimension $< n$. (The fibers of such a morphism are affine and closed in $\mathbb{P}^n$, therefore have dimension 0).

The good quotients are desirable to the extent that we have:

Lemma 5.3.12. The fibers of a good quotient $\varphi : X \longrightarrow Y$ for an action of G are in bijection with the closed orbits of G.

Proof. If $G \cdot x$ and $G.y$ are distinct closed orbits, then they are disjoint. The fifth property shows that $\varphi(x)$ and $\varphi(y)$ are distinct points of Y. It remains to understand why every point of Y parameterizes a closed orbit. This follows from the fact that for every orbit $G \cdot x$, there is a closed orbit $G.x'$ contained in the Zariski-closure $\overline{G.x}$ of $G.x$. To see this, note that if $G \cdot x$ is not closed, $\overline{G \cdot x} \setminus G \cdot x$ is G-invariant and of dimension $< \dim G.x$. It suffices to take an orbit of minimal dimension contained in $\overline{G.x}$. □

Remark 5.3.13. This Lemma tells us that a good quotient parametrizes closed orbits.

Lemma 5.3.14. [5, Proposition 6.1] A good quotient is necessarily a categorical quotient.

Definition 5.3.15. A *geometric quotient* is a good quotient in the sense of Definition 5.3.9, such that for every $y \in Y$, the fiber $\varphi^{-1}(y)$ is a single orbit. (In particular the orbits are closed). In other words, a geometric quotient is a good quotient $\varphi : X \longrightarrow Y$ such that the points of Y correspond one-to-one to the orbits of the geometric points of X via φ. Thus, the underlying set of a good quotient is the orbit space. This notion of quotient is the best possible because then the quotient is both categorical by Lemma 5.3.14 and Y is the set of orbits by Lemma 5.3.12.

Proposition 5.3.16. Let an affine algebraic group G act on X. Then a good quotient $\varphi : X \longrightarrow Y$ enjoys the following properties:

(i) $\varphi(x_1) = \varphi(x_2)$ if and only if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$.

(ii) For each element $y \in Y$, the fiber $\varphi^{-1}(y)$ contains exactly one closed orbit. In particular, if the action is closed (that is, all the orbits are closed), then φ is a geometric quotient.

Proof. (i) Assume that $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$, and let us prove that $\varphi(x_1) = \varphi(x_2)$. Since φ is constant on each orbit $G \cdot x_i$, then it is constant on each orbit closure $\overline{G \cdot x_i}$, $i = 1, 2$. As x_i is both in $\overline{G \cdot x_i}$ and in $G \cdot x_i$, we deduce that $\varphi(G \cdot x_i) = \varphi(x_i) = \varphi(\overline{G \cdot x_i})$. Now consider an element $x \in \overline{G \cdot x_1} \cap \overline{G \cdot x_2}$. We have that $\varphi(x) = \varphi(\overline{G \cdot x_1}) = \varphi(G \cdot x_1) = \varphi(x_1)$ and $\varphi(x) = \varphi(\overline{G \cdot x_2}) = \varphi(G \cdot x_2) = \varphi(x_2)$, so that $\varphi(x_1) = \varphi(x_2)$ as desired.

Conversely, suppose that $\varphi(x_1) = \varphi(x_2)$. By contradiction, let us assume that $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} = \emptyset$. Then by the fifth assertion of the definition of a good quotient, it follows that $\varphi(\overline{G \cdot x_1}) \cap \varphi(\overline{G \cdot x_2}) = \emptyset$. But the hypothesis $\varphi(x_1) = \varphi(x_2)$ and the fact that φ is constant on orbits imply that $\varphi(x_1) = \varphi(x_2) \in \varphi(\overline{G \cdot x_1}) \cap \varphi(\overline{G \cdot x_2}) = \emptyset$. This is absurd, and therefore $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$.

(ii) Let $y \in Y$. We want to show that $\varphi^{-1}(y)$ contains a unique orbit. We proceed by contradiction assuming that it contains two distinct orbits, say W_1 and W_2. Then $\varphi(W_1) = \{y\} = \varphi(W_2)$, which contradicts the assertion (5) of the definition of a good quotient.

□

This proposition implies that the good quotient Y does not, in general, parametrize the G-orbits on X; instead it parametrizes the closed G-orbits on X. Also, it implies that, topologically, the quotient Y is obtained from X by modulo a new equivalence relation [28, Remark 2.3.7]:

$$x \sim x' \text{ if and only if } \overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset.$$

Corollary 5.3.17. [22, Corollary 3.5.1] If $\varphi : X \longrightarrow Y$ is a good (resp. geometric) quotient, then for every open subset U of Y, the restriction $\varphi| : \varphi^{-1}(U) \longrightarrow U$ is also a good (resp. geometric) quotient of G acting on $\varphi^{-1}(U)$.

5.4 Coarse moduli space and quotients

The aim of this section is to give the relation between the notion of coarse moduli spaces and quotient spaces. In fact, one of the oldest method to construct moduli spaces is via the geometric invariant theory (GIT) developed earlier by Mumford. GIT translates the problem of the construction of moduli spaces into the problem of the construction of suitable quotients as described in Theorem 5.4.8.

5.4.1 The local universal property

We have seen that, in many interesting cases, a fine moduli space fails to exist. This means that it is difficult or impossible in many cases to fine a universal family since there is an equivalence between producing a fine moduli space and a universal family. But it is not hard to produce a family with the local universal property, a notion that we define now.

Definition 5.4.2. [21] For any given moduli problem, a family X parameterized by S is said to have the *local universal property* if for any family X' parametrized by S' and any point $s \in S'$, there exists a neighborhood U of s such that $X'|_U$ is equivalent to the family induced from f^*X by some morphism $f : U \longrightarrow S$.

Note that in this definition the morphism f is not required to be unique. If there is a family parametrised by S with the local universal property as defined above, we say that the variety S has the local universal property.

The following is an example of a moduli problem which does not does admit a universal family but does admit a local universal family.

Example 5.4.3 (Moduli problem of endomorphisms or moduli problem of $n \times n$ matrices)**.** Let k be an algebraically closed field. In what follows, we will always consider our vector spaces defined over k. Let us consider the moduli problem of endomorphisms. For this, we set the following.

(i) $\mathcal{M}$ denotes the collection of all endomorphisms $T : V \longrightarrow V$. That is,

$$\mathcal{M} = \{(T, V) : \dim V = n, \text{ and } T \text{ is a homomorphism}\}.$$

(ii) We say that two endomorphisms (T, V) and (T', V') are equivalent if there exists an isomorphism $h : V \longrightarrow V'$ such that $T = hT'h^{-1}$.

(iii) We define a family of endomorphisms parametrized by S to be a vector bundle E of rank n along with a morphism of vector bundles $T : E \longrightarrow E$.

(iv) Two families of endomorphisms (E, T) and (E', T') are equivalent if there exists an isomorphism $h : E \longrightarrow E'$ of vector bundles such that $T = hT'h^{-1}$.

(v) For any variety S, let $\mathrm{F}(S)$ denotes the set of isomorphism classes of families parametrized by S. This defines a contravariant functor $\eta : \mathrm{Var} \longrightarrow \mathrm{Set}$.

So we have in this way defined a moduli problem. We have seen in Example 2.6.5 that this moduli problem does not have a coarse moduli space due to the jump phenomenon. Therefore a universal family for this moduli problem cannot exist. However, we can construct a family possessing the local universal property as follows.

Proposition 5.4.4. Let $S = M(n)$ be the set of $n \times n$ matrices and let $E = I_n = S \times \mathrm{k}^n$ be the trivial vector bundle along with the homomorphism

$$\begin{array}{rcl} T : E & \longrightarrow & E \\ (f, v) & \longmapsto & (f, fv). \end{array}$$

Then the family (E, T) possesses the local universal property.

Proof. Let us show that the family (E, T) has the local universal property. For this, consider any family (E', T') parametrized by S' and $s \in S'$. As vector bundles are locally trivial, then there exists an open neighborhood $U \subset S'$ of s such that the restriction $E'|_U$ is isomorphic to $U \times \mathrm{k}^n$. This means, there exists an isomorphism $h : U \times \mathrm{k}^n \longrightarrow E$ making the following diagram commutative.

$$\begin{array}{ccc} U \times \mathrm{k}^n & \xrightarrow{h} & E \\ & \searrow^{pr_1} & \downarrow{\scriptstyle p|_U} \\ & & U \end{array}$$

Here $pr_1 : U \times \mathrm{k}^n \longrightarrow U$ denotes the first projection and $p\ |_U : E' \longrightarrow U$ denotes the restriction of E' to U. Now consider the composition

$$\begin{array}{rcl} pr_1 \circ T \circ h : U \times \mathrm{k}^n & \longrightarrow & M(n) \\ (s, v) & \longmapsto & pr(T(h(s, z))). \end{array}$$

This gives rise to a morphism

$$\begin{array}{rcl} \phi : U & \longrightarrow & M(n) \\ s & \longmapsto & pr(T(h(s, 1_n))), \end{array}$$

where 1_n denotes the vector $(1, \ldots, 1) \in \mathbb{C}^n$. Moreover, it is clair that the family $(E'_{|_U}, T'_{|_U})$. □

We now turn our attention to the problem of constructing a coarse moduli given a family satisying the local universal property.

Proposition 5.4.5. Suppose that there is given a family X/S with the local universal property.

(1) Let $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ be any natural transformation. Let $\phi : S \longrightarrow M$ be the morphism corresponding to the family X. Then the morphism ϕ is constant on equivalence classes. That is, if the fibers X_s and $X_{s'}$ are equivalent, then $\phi(s) = \phi(s')$.

(2) Conversely, if $\phi : S \longrightarrow M$ is any morphism which is constant on equivalence classes, then there exists a natural transformation $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ such that ϕ is the morphism corresponding to the family X.

Proof. (1) Let $\eta : \mathrm{F} \longrightarrow \mathrm{h}_M$ be any natural transformation, and let $\phi : S \longrightarrow M$ be the morphism corresponding to the family X via the map $\eta(S) : \mathrm{F}(S) \longrightarrow \mathrm{h}_M(S)$. Let us show that ϕ is constant on equivalence classes. For this, let $s, s' \in S$ such that $X_s \sim X_{s'}$, and let us show that $\phi(s) = \phi(s')$. We think of any single point s as a morphism $s : \{\mathrm{pt}\} \longrightarrow M$. By the naturality of η, we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{\eta(S)} & \mathrm{h}_M(S) \\ {\scriptstyle s^*}\downarrow & & \downarrow{\scriptstyle s^*} \\ \mathcal{F}(\mathrm{pt}) & \xrightarrow[\eta(\mathrm{pt})]{} & \mathrm{h}_M(\mathrm{pt}) \end{array}$$

In other words, $s^* \circ \eta(S) = \eta(\mathrm{pt}) \circ s^*$. It follows that $s^* \circ \eta(S)(X) = \eta(\mathrm{pt}) \circ s^*(X)$. But $s^* \circ \eta(S)(X) = s^*(\phi) = \phi(s)$ and $\eta(\mathrm{pt}) \circ s^*(X) = \eta(\mathrm{pt})(X_s)$. We deduce that $\phi(s) = \eta(\mathrm{pt})(X_s)$.

Similarly, we have $\phi(s') = \eta(\mathrm{pt})(X_{s'})$. As $X_s \sim X_{s'}$, it follows that $\phi(s') = \eta(\mathrm{pt})(X_{s'}) = \eta(\mathrm{pt})(X'_s) = \phi(s)$, so that $\phi(s) = \phi(s')$ as desired.

(2) Conversely, let $\phi : S \longrightarrow M$ be a morphism which is constant on equivalence classes, and let us show that there exists a natural transformation $\eta : \mathcal{F} \longrightarrow \mathrm{h}_M$ such that $\eta(S)(X) = \phi$. So, for each $S' \in \mathcal{C}$, we want to define a map $\eta(S') : \mathcal{F}(S') \longrightarrow \mathrm{h}_M(S')$. For this, let $X' \in \mathcal{F}(S')$, that is X' is a family parametrized by S'. We want to find a morphism $\phi' : S' \longrightarrow M$ such that $\eta(S')(X') = \phi'$. As the family X satisfies the local universal property, for each $s \in S'$, there exists an open set U containing s and there exists a morphism $\phi_1 : U \longrightarrow S$ such that $\phi_1^* X \sim X'_{|U}$. By composing ϕ_1 with ϕ, we obtain a mophism $\phi \circ \phi_1 : U \longrightarrow M$. If U' is another open set containing s with a morphism $\phi'_1 : U' \longrightarrow S$, we compose again with ϕ to obtain another morphism $\phi \circ \phi'_1 : U \longrightarrow M$. These two morphisms should be identical on $U \cap U'$ since ϕ is constant on equivalence classes. Hence, patching these morphisms together, we obtain a morphism $\phi : S' \longrightarrow M$ as desired. The data of such morphism for each S defines a natural transformation $\eta : \mathcal{F} \longrightarrow \mathrm{h}_M$ as required.

□

5.4.6 Coarse moduli space

Now, assume that there exists a family X parametrized by S with the local universal property. Then the problem of constructing a coarse moduli space can be reduced to the following.

Lemma 5.4.7. Let X be a local universal family parametrized by S. Then the coarse moduli space is the variety M along with a morphism $\phi : S \longrightarrow M$ which is constant on equivalence classes and which satisfies

(1) if $\psi : S \longrightarrow N$ is any morphism which is constant on equivalence classes, then there exists a unique morphism $\gamma : M \longrightarrow N$ such that $\gamma \circ \phi = \psi$;

(2) each fiber of ϕ consists of only one equivalence class.

The proof of this lemma follows from the second part Theorem 5.4.8.

Theorem 5.4.8. Let $(\mathcal{M}, \sim_{\mathcal{M}})$ be a moduli problem and let $X \longrightarrow S$ be a family over a scheme S. Suppose that X has the local universal property. Let G be an algebraic group operating on S such that two k-points s and t belong to the same G-orbit if and only if the fibers X_s and X_t are equivalent. Then

(i) Any coarse moduli space is a categorical quotient for the action of G on S.

(ii) A categorical quotient for the action of G on S is a coarse moduli space if and only if it is an orbit space.

Proof. (i) Let M be a coarse moduli space for the moduli problem $\mathcal{M}$, and F the moduli functor associated with $\mathcal{M}$. We want to prove that M is a categorical quotient. Let us first show that there exists a bijective correspondence

$$\{\text{natural transformations } \psi : \mathrm{F} \longrightarrow \mathrm{h}_M\} \longleftrightarrow \{G\text{-invariant morphisms } f : S \longrightarrow M\}.$$

If $\psi : \mathrm{F} \longrightarrow \mathrm{h}_M$ is a natural transformation, then $\psi_S : \mathrm{F}(S) \longrightarrow \mathrm{h}_M(S) = \mathrm{Hom}_{\mathrm{Sch}}(S, M)$ sends the family $X \longrightarrow S$ over S to the morphism

$$\begin{aligned} \psi_{S,X} : S &\longrightarrow M \\ s &\longmapsto [X_s] \end{aligned}$$

which is obviously G-invariant thanks to the assumption on the action of G on S.

Recall that the family $\pi : X \longrightarrow S$ has the local universal property means that for any family $X' \longrightarrow S'$ over S' and any point $s \in S'$, there exists an open neighborhood U of s such that $X'|_U$ is equivalent to the family f^*X induced from X by some morphism $f : U \longrightarrow S$.

Conversely, let $f : S \longrightarrow M$ be a G-invariant morphism. We want to find a natural transformation $\psi : \mathrm{F} \longrightarrow \mathrm{h}_M$. Let S' be a scheme. If $X' \longrightarrow S'$ is a family over S', then for any $s \in S'$ there exists an open neighborhood U_s of s and a morphism $g_s : U_s \longrightarrow S$ such that $X'|_{U_s} \sim_{U_s} g_s^*X$. The open subsets $\{U_s\}_{s\in S'}$ cover S'. If $u \in U_s \cap U_t$, with $s, t \in S$, then

$$X_{g_s(u)} \sim (g_s^*X)_u \sim X'_u \sim (g_t^*X)_u \sim X_{g_t(u)}.$$

The assumption on the G-action on S implies that $g_s(u)$ and $g_t(u)$ lie in the same G-orbit. As $f : S \longrightarrow M$ is G-invariant, the composition $f \circ g_s : U_s \longrightarrow M$ can be glued to a morphism $\psi_{S',X'} : S \longrightarrow M$. Now, we define a natural transformation

$$\begin{aligned} \psi : \mathrm{F} &\longrightarrow \mathrm{h}_M \\ S' &\longmapsto \psi_{S'} : \mathrm{F}(S') \longrightarrow \mathrm{Hom}_{\mathrm{Sch}}(S, M) \\ &\qquad\quad X' \longmapsto \psi_{S',X'} \end{aligned}$$

which gives the inverse of the correspondence above as desired.

Let $(M, \psi : \mathrm{F} \longrightarrow \mathrm{h}_M)$ be a coarse moduli space. Let us prove that it is a categorical quotient. By the correspondence above, there exists a G-invariant morphism $\psi_{S,X} : S \longrightarrow M$ is a G-invariant. It is enough to show that $\psi_{S,X}$ is universal among such morphisms. It follows from the universality of $\psi : \mathrm{F} \longrightarrow \mathrm{h}_M$, since M is a coarse moduli space. This proves the first part of the theorem. Moreover, the G-invariant morphism $\psi_{S,X} : S \longrightarrow M$ is an orbit space if and only if ψ_{Speck} is a bijection. This proves the second part of the theorem. □

This theorem gives the relationship between coarse moduli spaces and orbit spaces. It tells us that the search for coarse moduli spaces reduces to the computation of orbit spaces. We have seen that quotients may not be orbit space. Therefore, it is natural to look for the conditions under which orbit spaces exist.

5.5 Reducible groups, finite generation and Nagata's theorem

In this section, we introduce the groups we are interested in and for which the construction of the geometric invariant theory quotient is feasible. We will define and relate the notions of reductive group, linearly reductive group, and geometrically reductive group. Our goal is to show the existence of a good quotient when an affine algebraic reductive group G acts linearly on an affine scheme X of finite type over k.

5.5.1 Reductive Groups

Most of the algebraic groups that we will deal with in thesis are reductive groups; namely, $\mathrm{SL}(n)$, $\mathrm{GL}(n)$, $\mathrm{PGL}(n)$ are reductive.

Definition 5.5.2. Let G be an affine algebraic group, V a vector space, and $\rho^* : V \longrightarrow \mathcal{O}(G) \otimes_{\mathrm{k}} V$ be a comodule associated with a linear representation of G on V. We say that

- $g \in G$ is *semisimple* if there is a faithful linear representation $\rho : G \hookrightarrow \mathrm{GL}_n$ such that $\rho(g)$ is diagonalizable.
- $g \in G$ is *unipotent* if there exists a faithful linear representation $\rho : G \hookrightarrow \mathrm{GL}_n$ such that $\rho(g)$ is unipotent (i.e; $\rho(g) - I_n$ is nilpotent).
- $v \in V$ is G-invariant if $\rho^*(v) = 1 \otimes v$. The set of all G-invariant vectors is a subspace of V, and is denoted by V^G.
- A subgroup H of an affine algebraic group G is *normal* if the conjugation action $H \times G \longrightarrow G$ given by $(h, g) \longmapsto ghg^{-1}$ factors through the inclusion $H \hookrightarrow G$.
- G is *reductive* if it is smooth and every smooth unipotent normal algebraic subgroup of G is trivial.
- G is *linearly reductive* if any finite-dimensional linear representation $\rho : G \longrightarrow \mathrm{GL}(V)$ is completely reducible. Equivalently, G is linearly reductive if for any finite-dimensional representation V of G, and any $v \in V^G$, there is a G-invariant linear function $f : V \to \mathrm{k}$ such that $f(v) \neq 0$.
- G is *geometrically reductive* if for any finite-dimensional linear representation $\rho : G \longrightarrow \mathrm{GL}(V)$ and every nonzero G-invariant point $v \in V^G$, there is a G-invariant nonconstant homogeneous polynomial $f : V \longrightarrow \mathrm{k}$ such that $f(v) \neq 0$.

Definition 5.5.3. An affine algebraic group G is *unipotent* if every nontrivial linear representation $\rho : G \longrightarrow \mathrm{GL}(V)$ has a nonzero G-invariant vector.

Proposition 5.5.4. Let G be an affine algebraic group, and $\mathbb{U}$ the subgroup of $\mathrm{GL}(V)$ consisting of upper triangular matrices with diagonal entries equal to 1. The following assertions are equivalent.

(1) G is unipotent.

(2) For any representation, $\rho : G \longrightarrow \mathrm{GL}(V)$ there exists a basis of V such that $\rho(G) \subset \mathbb{U}$.

(3) G is isomorphic to a subgroup of a standard unipotent group $\mathbb{U}_n \subset \mathrm{GL}_n$ consisting of upper triangular matrices with diagonal entries equal to 1.

Proof. $(1) \Leftrightarrow (2)$. Let us assume that G is unipotent. Consider any non-trivial representation of $\rho : G \longrightarrow \mathrm{GL}(V)$. Since G is unipotent, there exists a non-zero G-invariant vector $v \in V$, so that the subspace of G-invariant vectors V^G is non-zero. Let then $\{e_1, \ldots, e_m\}$ be a basis of V^G. By the incomplete basis theorem, we may complete this system into a basis of V by adding a basis $\{\overline{e}_{m+1}, \ldots, \overline{e}_n\}$ of V/V^G, such that the image of the induced representation contained in the upper triangular matrices with diagonal equal 1. Then, we choose a lift e_{m+i} for each $\overline{e}_{m+i}$, and we obtain a basis $\{e_1, \ldots, e_m, e_{m+1}, \ldots, e_n\}$ with the desired property.

Conversely, suppose that for every representation $\rho : G \longrightarrow \mathrm{GL}(V)$ there exists a basis of V, say $\{e_1, \ldots, e_n\}$, such that $\rho(G) \subset \mathbb{U}$. Then e_1 is fixed by ρ, showing that G is unipotent.

$(2) \Leftrightarrow (3)$. Assume that for any representation $\rho : G \longrightarrow \mathrm{GL}(V)$ there exists a basis of V such that $\rho(G) \subset \mathbb{U}$. But, since every affine algebraic group admits a faithful representation, say $\sigma : G \longrightarrow \mathrm{GL}(V)$, G embeds in $\sigma(G)$ which is contained in $\mathbb{U}_n$. The converse is also true by [18, XV Theorem 2.4]. □

Example 5.5.5. (1) Let G be a finite group scheme over k such that its order is prime to the characteristic of k. Then G is linearly reductive [19, Proposition 4.38].

Proof. Let $\rho : G \longrightarrow \mathrm{GL}(V)$ be a finite-dimensional linear representation of G on V and let $v \in V$ be invariant modulo a proper subrepresentation U. Define

$$w = \frac{1}{|G|} \sum_{g \in G} (g \cdot v).$$

It is clear that the vector w is G-invariant by construction and $w - v$ is contained in U. Then G is reductive. □

(2) Every algebraic torus $(\mathbb{G}_m)^N$ is linearly reductive [19, Proposition 4.41].

Proof. It suffices to show that $\mathbb{G}_m$ is linearly reductive. Let V be a finite dimensional representation and there is a vector v that is invariant modulo a proper subrepresentation U. We have weight decompositions $V = \bigoplus V_m$ and $U = \bigoplus U_m$. Now, to say that $v = \Sigma v_m$ is invariant modulo U means that $v_m \in U_m$ for all $m \neq 0$. Thus v_0 is torus invariant and an element of the coset $v + U$ as desired. □

(3) The group $G = \mathbb{G}_a$ is not linearly reductive [19, Example 4.42].

Proof. Consider the 2-dimensional representation given by $\mathbb{G}_a \to \mathrm{GL}_2$ which corresponds to $s \mapsto \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$. The map $\mathrm{k}[x, y] \to \mathrm{k}[x, y]/(y) = \mathrm{k}[x]$ is a surjection of G representations but the map on invariants is not surjective. □

(4) The multiplicative group $\mathbb{C}^\times$ is reductive.

(5) In characteristic 0, the linear special group SL_n is linearly reductive [19, Theorem 4.43]. In positive characteristic $p > 0$ a theorem of Haboush guarantees that every reductive group is geometrically reductive.

Lemma 5.5.6. [19, Proposition 4.37] Linear reductivity of G is equivalent to the following conditions:

(1) Given a finite dimensional representation $\rho : G \longrightarrow \mathrm{GL}(V)$ of G over V and a surjective G-invariant linear form $f : V \longrightarrow \mathrm{k}$, there exists an invariant vector $w \in V^G$ such that $f(w) \neq 0$.

(2) For each surjection of G-representation $V \longrightarrow W$ the induced map on G-invariants is surjective.

(3) For each surjection of finite dimensional representations as above, the induced map on invariants is surjective.

(4) For any finite-dimensional representation V, if $v \in V$ is G-invariant modulo a proper subrepresentation $U \subset V$, then the coset $v + U$ contains a nontrivial G-invariant vector.

Proof. Linear reductivity is equivalent to condition (1) by replacing V with its dual V^* and observing that the space of G-invariant forms is $\mathrm{Hom}_G(V^*, \mathrm{k}) = V^G$, where G oprerates trivially on k.

(2) implies (3) is trivial. (3) implies (4) is easy by looking at the quotient map $V \to V/U$.

For (4) implies (2): if $L : V \to W$ is a surjection of representations, suppose that $L(v) = w \in W^G$. The vector v is contained in a finite-dimensional subrepresentation $V_0 \subset V$. This vector is invariant modulo $U_0 = V_0 \cap \ker(L)$, so by (4), there is a G-invariant vector v_0 such that $v - v_0 \in U_0$. Then $L(v_0) = w$, so that the map on invariants is surjective.

(2) implies (1) is clear by taking $W = \mathrm{k}$ with trivial action.

(1) implies (3): If $w \in W^G$ then W decomposes as a representation of G as $W = \mathrm{k} \cdot w \oplus W'$. Then by (1), the composition $V^G \to V \to W \to \mathrm{k} \cdot w$ is surjective. □

Proposition 5.5.7. [19, Theorem 4.43, Corollary 4.44] The Special Linear group $\mathrm{SL}(n)$, the Linear Group $\mathrm{GL}(n)$ are linearly reductive.

Theorem 5.5.8 (Weyl, Nagata, Mumford, Haboush)**.** [4]

(1) Every linearly reductive group is geometrically reductive.

(2) In characteristic zero, every reductive group is linearly reductive.

(3) A smooth affine algebraic group is reductive if and only if it is geometrically reductive.

In particular, for smooth affine algebraic group schemes, we have

linearly reductive $\Longrightarrow$ geometrically reductive $\Longleftrightarrow$ reductive

and all three notions coincide in characteristic zero.

In positive characteristic p, the groups GL_n , SL_n and PGL_n are not linearly reductive for $n > 1$.

5.5.9 Nagata's Theorem

We recall again that our main objective is to construct coarse modui spaces, a notion which is related in some way to the notion of categorical quotient. Suppose that X is a G-space. Then a G-morphism $\phi : X \longrightarrow Y$, that is ϕ is constant on G-orbit, induces a morphism of k-algebras $\mathcal{O}(Y) \longrightarrow \mathcal{O}(X)$. This morphism factors through the algebra $\mathcal{O}(X)^G$ of G-invariant elements of $\mathcal{O}(X)$. Assume that the categorical quotient Y of X by the G-action is affine. Then $\mathcal{O}(Y) = \mathcal{O}(X)^G$. In order to have an affine quotient, we need to know whether the $\mathcal{O}(X)^G$ is finitely generated.

Definition 5.5.10. Let G be an algebraic group and A be a k-algebra. A rational action of G on A is a map

$$A \times G \longrightarrow A, \ (f, g) \longmapsto f^g$$

such that

(a) $f^{gg'} = (f^g)^{g'}$ and $f^e = f$ for all $f \in A$ and $g, g' \in G$;

(b) The map $f \longmapsto f^g$ is a k-algebra automorphism of A for all $g \in G$;

(c) every element of A is contained in a finite-dimensional invariant subspace on which G acts by a rational representation.

Question 5.5.11. Let G be an algebraic group acting on a finitely generated k-algebra R. Is R^G finitely generated?

This question is the content of the Hilbert's fourteenth problem posed in 1900 at the international Congress of Mathematicians in Paris. Many contibutions were made in the particular case. Hilbert himself proved the result for the action of $\mathrm{GL}(n)$. It was only in 1958 that Nagata constructed a counterexample (see [20, Chapter 3]). Moreover, Nagata proved that for rational action of geometrically reductive groups, the question above has an affirmative answer. This is the content Nagata's theorem.

Theorem 5.5.12 (Nagata's Theorem)**.** [20, Chapter 1][22, Theorem 3.4] Let G be a geometrically reductive group which acts rationally on a finitely generated k-algebra A. Then A^G is a finitely generated k-algebra.

As noted above, a reductive group is a geometrically reductive group, so that we can apply the theorem above with reductive groups.

Theorem 5.5.13 (Hilbert, Mumford)**.** [4][19, Theorem 4.53] Let G be a linearly reductive group acting rationally on a finitely generated k-algebra A. Then A^G is finitely generated.

We close this subsection with the following result.

Theorem 5.5.14 (Popov)**.** Let G be an affine algebraic group over k. Then G is reductive if and only if, for any rational G-action on a finitely generated k-algebra A, A^G is finitely generated.

The main idea behind the Geometric Invariant Theory is to go from local to global. That is, we have to cover a G-scheme X with invariant affine open subsets, for each affine open subset we take an affine quotient and then patch these resulting affine quotients together to form a desired quotient $X//G$. By definition, any scheme can be covered by affine schemes, that is, it is locally an affine scheme, so this suggests that we have to consider affine schemes first. In the next section, we will use the techniques we developed so far to the construction of affine quotients.

5.6 Construction of the affine geometric invariant theory quotients

Let X be an affine scheme of finite type over k and G a reductive group operating on X. The action of G induces an action on $\mathcal{O}(X)$ as described in formula (5.1). Also, $\mathcal{O}(X)$ is a finitely generated k-algebra. It follows from Nagata's theorem that $\mathcal{O}(X)^G$ is finitely generated too. Conversely, if A is a finitely generated k-algebra, then $\mathrm{Spec}\mathrm{k}$ is an affine of finite type over k. This gives a bijective correspondence between finitely generated k-algebras and affine schemes of finite type over k. The main goal of this section is to prove the existence of a good quotient for X.

Theorem 5.6.1 (First main theorem)**.** Let G be a reductive group acting on an affine scheme X. Then $\mathcal{O}(X)^G$ is finitely generated and the natural surjection $\varphi : X = \mathrm{Spec}(\mathcal{O}(X)) \longrightarrow X//G := \mathrm{Spec}\mathcal{O}(X)^G$ induced by the inclusion $\varphi^* : \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ is a good quotient. Furthermore, $X//G$ is an affine scheme.

Definition 5.6.2. The *affine GIT quotient* or *affine geometric invariant theory quotient* is the morphism $\varphi : X = \mathrm{Spec}(\mathcal{O}(X)) \longrightarrow X//G := \mathrm{Spec}\mathcal{O}(X)^G$ of affine schemes induced by the inclusion map $\varphi^* : \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$.

In order to prove that the reductive GIT quotient is a good quotient, we first prove the following lemma.

Lemma 5.6.3. [21, Lemma 3.3] Let G be a geometrically reductive group operating on X, and let W and W' be two disjoint G-invariant closed subsets of X. Then, there exists a G-invariant function $f \in \mathcal{O}(X)^G$ such that $f(W) = 0$ and $f(W') = 1$.

Proof. Since W and W' are disjoint and closed, we have:

$$(1) = I(\emptyset) = I(W \cap W') = I(W) + I(W').$$

Therefore we may write $1 = f_1 + f_2$, for some $f_1 \in I(W)$ and $f_2 \in I(W')$. Since $f_1 \in I(W)$, which means $f_1(x) = 0$ for all $x \in W$, it follows from the equality $1 = f_1 + f_2$ that $f_2(x) = 1$ for all $x \in W$. Hence $f_1(W) = 0$ and $f_1(W') = 1$. By Lemma 5.2.8 there exists a finite dimensional G-invariant subspace V of $\mathcal{O}(X)$ containing

f_1. Let V' be the space generated by $\{g \cdot f_1 \mid g \in G\}$. Since V is G-invariant and $f_1 \in V$, then $\{g \cdot f_1 \mid g \in G\} \subset V$, so that V' is a subspace of V. Hence V' is also finite dimensional. Moreover, it is clear that V' is G-invariant. Let n be the dimension of V' and let $\{h_1, \ldots, h_n\}$ be a basis of V'. Then we can build up a morphism

$$\begin{aligned} h : X &\longrightarrow \mathbb{A}^n \\ x &\longmapsto (h_1(x), \ldots, h_n(x)) . \end{aligned}$$

We can write the function h_i as a linear combination of translates of f_1, so that $h_i = \sum_{l=1}^{n_i} c_{il} g_{il} \cdot f_1$ for some constants c_{ij} and some elements g_{il} of G. By definition of the induced action of G on $\mathcal{O}(X)$, it follows that:

$$h_i(x) = \sum_{l=1}^{n_i} c_{il} f_1(g_{il}^{-1} \cdot x).$$

Since the subsets W and W' are G-invariant and $f_1(W) = 0$ (resp. $f_1(W') = 1$), it follows that $h(W) = 0$ and $h(W') = v \neq 0$.

For all $h_i \in \{h_1, \ldots, h_n\}$ and for all $g \in G$ the function $g \cdot h_i$ is again an element of V'. Since $\{h_1, \ldots, h_n\}$ is a basis of V' we can express $g \cdot h_i$ as a linear combination of the elements of that basis, that is

$$g \cdot h_i = \sum_{j=1}^{n} a_{ij}(g) h_j.$$

Now we can use the coefficients a_{ij} above to define a linear representation for the group G as follows:

$$\begin{aligned} \rho : G &\longrightarrow \mathrm{GL}_n, \\ g &\longmapsto a_{ij}(g). \end{aligned}$$

It follows that the function $h : X \longrightarrow \mathbb{A}^n$ is G-equivariant with respect to the action of G on X and the G-action on $\mathbb{A}^n$ via this representation ρ. Hence $v = h(W_2)$ is a nonzero G-invariant point. As G is geometrically reductive, by definition there exists a nonconstant homogeneous polynomial $P \in \mathrm{k}[x_1, \ldots, x_n]^G$ such that $P(v) \neq 0$ and $P(0) = 0$. Then $f = cP \circ h$ is the desired invariant function where $c = \dfrac{1}{P(v)}$. □

Theorem 5.6.1 is the main result of this section, so will take a particular attention to its proof.

Proof of Theorem 5.6.1. The reductivity of G together with the Nagata's Theorem imply that $\mathcal{O}(X)^G$ is finitely generated. Therefore the geometric invariant theory quotient $Y := X//G = \operatorname{Spec} \mathcal{O}(X)^G$ is an affine scheme of finite type over k. Let $\varphi : X \longrightarrow Y$ be the morphism of affine schemes associated to the inclusion $\varphi^{\#} : \mathcal{O}(X)^G \hookrightarrow \mathcal{O}(X)$ as in the definition of a good quotient. Then φ is G-invariant and affine. We now prove that φ is surjective. Let y be an element of $Y(k)$, and let us show that there is an element $x \in X(\mathrm{k})$ such that $y = \varphi(x)$. Let N_x be the maximal ideal in $\mathcal{O}(Y) = \mathcal{O}(X)^G$

of the point y. Now N_x is also finitely generated, so we can choose a set of generators $\{f_1, \dots, f_l\}$ of N_x. As G is reductive a result from [21, Lemma 3.4.2] shows that

$$\sum_{i=1}^{l} f_i \mathcal{O}(X) \neq \mathcal{O}(X).$$

Therefore, there exists a maximal ideal $N_x \subset \mathcal{O}(X)$ containing $\sum_{i=1}^{l} f_i \mathcal{O}(X)$, and corresponding to a closed point $x \in X(k)$. In particular, we have $f_i(x) = 0$ for all $i \in \{1, \dots, l\}$ and $\varphi(x) = y$. Hence, every closed point belongs to $\operatorname{Im}\varphi$ and since $\operatorname{Im}\varphi$ is a constructible subset by Chevaley's Theorem, it follows that φ is actually surjective. This proves axioms (1) and (2) of the definition of good quotient.

Let us show that φ satisfies the axiom (3) of a good quotient. Since the sets $D = Y_f$ form a basis open subsets for the topology of Y, it is enough to prove the statement for any open subset of the form $U = Y_f$, $f \in \mathcal{O}(Y) = \mathcal{O}(X)^G$. In this case, $\varphi^{-1}(U) = X_f$. So the statement is simply "taking invariants commutes with localization"

$$(\mathcal{O}(X)^G)_f = (\mathcal{O}(X)_f)^G,$$

and let us then prove this statement. In fact,

$$(\mathcal{O}(X)^G)_f = \left\{ \frac{u}{f^n} : u \in \mathcal{O}(X)^G, n \in \mathbb{N} \right\}.$$

As f is G-invariant, so is f^n, for all $n \in \mathbb{N}$. Therefore, for all G-invariant function $u \in \mathcal{O}(X)^G$, we have

$$\left(\frac{u}{f^n} \right)(g \cdot x) = \frac{u(g \cdot x)}{f^n(g \cdot x)} = \frac{u(x)}{f^n(x)},$$

so that $\frac{u}{f^n}$ belongs to $(\mathcal{O}(X)_f)^G$. This shows that $(\mathcal{O}(X)^G)_f \subset (\mathcal{O}(X)_f)^G$. Conversely,

$$(\mathcal{O}(X)_f)^G = \left\{ \frac{u}{f^n} : \frac{u(g \cdot x)}{f^n(g \cdot x)} = \frac{u(x)}{f^n(x)}, n \in \mathbb{N}, \forall g \in G, \forall x \in X \right\}.$$

But for each $\frac{u}{f^n} \in (\mathcal{O}(X)_f)^G$, using the fact that f is G-invariant, we have

$$\frac{u(x)}{f^n(x)} = \frac{u(g \cdot x)}{f^n(g \cdot x)} = \frac{u(g \cdot x)}{f^n(x)},$$

which implies that u is G-invariant. Hence $\frac{u}{f^n} \in (\mathcal{O}(X)^G)_f$, so that $(\mathcal{O}(X)_f)^G \subset (\mathcal{O}(X)^G)_f$. This proves the statement above.

Let us show that the axiom $(4')$ is satisfied for φ. Let W_1 and W_2 be two disjoint invariant closed subsets of X, and let us show their images via φ are disjoint. By

Lemma 5.6.3, there exists a G-invariant function $f \in \mathcal{O}(X)^G$ such that

$$f(W_1) = 0, \; f(W_2) = 1.$$

Since $\mathcal{O}(X)^G = \mathcal{O}(Y)$, we may view f as an element of $\mathcal{O}(Y)$, and then we have:

$$f(\varphi(W_1)) = 0, \; f(\varphi(W_2)) = 1.$$

Thus

$$\overline{\varphi(W_1)} \cap \overline{\varphi(W_2)} = \emptyset,$$

which completes the proof. ■

This allows us to define the Geometric Invariant Theory quotient of $\mathrm{Spec}(A)$ by the group G to be

$$\mathrm{Spec}(A)//G = \mathrm{Spec}(A^G).$$

Corollary 5.6.4. Let G be a reductive group acting on an affine scheme X and let $\varphi : X \longrightarrow Y := X//G$ be an affine geometric invariant theory quotient. Then $\varphi(x) = \varphi(y)$ if and only if $\overline{G \cdot x} \cap \overline{G \cdot y} \neq 0$.

Moreover, $\varphi^{-1}(y)$ for each $y \in Y$ contains a single closed orbit. In particular, if the G-action is closed, then φ is a geometric quotient.

Proof. By Theorem 5.6.1, φ is also a good quotient, so that the result follows from Corollary 5.3.16. □

5.7 Construction of geometric quotients on open subsets

Let G be a reductive group operating on an affine scheme X. It is natural to consider X/G as a candidate for our quotient. Unfortunately, this quotient X/G may fail to have the structure of a scheme because the action is not necessarily closed. But, the situation is not too bad; this is because of a result due to Rosenlicht which states that there is always a dense G-invariant open subset U of X for which the quotient U/G has a structure of a scheme. The main goal of this section is to define an open subset $X^s \subset X$ for which there is a geometric quotient.

5.7.1 Stability

Stable points are fundamental for the construction of moduli spaces. Many moduli spaces, such as the moduli space of hypersurfaces in $\mathbb{P}^n$, can be realized as the quotients of the space of stable points of some subset X^s of the projective space $\mathbb{P}^n$ by the action of some group G on X^s. More often, we can compactify these spaces by adding certain equivalence classes of semistable points. There is a bijective correspondence between orbits of stable and the points of quotients X^s/G. However, two different orbits of semistable points may correspond to the same point in the quotient if the intersection of their orbit closures is nonempty.

Definition 5.7.2. [13, Definition 4.35] Let G be a reductive group operating on an affine scheme X. An element $x \in X$ is said to be

(1) *stable* if its orbit $G \cdot x$ is a closed subset of X and $\dim G_x = 0$ (or G_x is finite), where G_x denotes the stabilizer of x.

(2) *unstable* if 0 is in the closure of its orbit.

(3) *semi-stable* if 0 is not in the closure of its orbit.

The set of all stable points of X is denoted X^s, and will be one of the objects of our interest here. The set of all semistable points of X is denoted by X^{ss}, and the set of untable points of X is denoted by X^u.

Proposition 5.7.3. [13, Proposition 4.36] Let G be a reductive group which operates on an affine scheme X, and let $\varphi : X \longrightarrow Y := X//G$ be the affine geometric quotient. Then X^s is an open G-invariant subset of X, and $Y^s := \varphi(X^s)$ is an open subset of Y and $X^s = \varphi^{-1}(Y^s)$. Furthermore, $\varphi : X^s \longrightarrow Y^s$ is a geometric quotient.

Proof. Let us show that X^s is open. It suffices to show that every point $x \in X^s(\mathrm{k})$ admits an open neighborhood in X^s. Let $X_+ := \{x \in X : \dim G_x > 0\} = \{x \in X : \dim G_x \geq 1\}$. This is a closed subset of X (by Proposition 5.2.11). As x is stable, by definition, the orbit $G.x$ is closed and $\dim G_x = 0$, so that G_x and X_+ are disjoint closed subsets of X. By Lemma 5.6.3, there exists a G-invariant function $f \in \mathcal{O}(X)^G$ such that $f(X_+) = 0$ and $f(G_x) = 1$. Then x belongs to the non-vanishing set of f, $X_f := \{a \in X : f(a) \neq 0\}$, which is open by definition of the Zariski topology. It is enough to show that $X_f \subset \mathrm{X}^s$. For every $a \in X_f$, we have that $f(a) \neq 0$, which implies that $a \notin X_+$ (because $f(X_+) = 0$). Thus $\dim G_a = 0$ for all $a \in G$. It remains only to show that the orbits of the elements of X_f are closed. By contradiction, let us assume that there exists $z \in X_f(\mathrm{k})$. So every $w \in G.z$ belongs to the orbit closure $\overline{G.z}$. Then w belongs to X_f too as f is G-invariant and so $\dim G_w = 0$. It follows from Proposition 5.2.10 that the boundary $\overline{G \cdot z} - G \cdot z$ of the orbit of z is a union of orbits of dimension strictly smaller. Therefore the orbit of w must be of dimension strictly smaller than that of z, that is, $\dim G \cdot w < \dim G.z$ so that $\dim G_w > 0$ which contradicts the fact that $\dim G_w = 0$. This proves that X_f is an open neighborhood of x so that X^s is open.

Let us prove that $Y^s = \varphi(X^s)$ is an open subset of Y. By similar arguments as in the previous case, it is sufficient to show that Y_f is open. This follows from the equality $\varphi(X_f) = Y_f$, which is open in Y. Also $X_f = \varphi^{-1}(Y_f)$, it follows that Y^s is open. To show that $X^s = \varphi^{-1}(Y^s)$, it suffices to use the equality $X_f = \varphi^{-1}$, so that $X^s = \varphi^{-1}(\varphi(X^s))$.

Let us prove that the map $\varphi : X^s \longrightarrow Y^s$ is a geometric quotient. To do this, it suffices to apply Corollary 5.3.17 with $U = Y^s$. But we have to check that $X^s = \varphi^{-1}(U)$. This was already been proved, so the restriction morphism $\varphi : X^s \longrightarrow Y^s$ is a geometric quotient. □

5.8 Geometric invariant theory quotients of projective schemes

Let G be a reductive group operating on a projective scheme X. We would like to construct a GIT quotient of X for this action. To apply the previous section, we need to consider affine open subsets of X which are G-invariant, and then construct our GIT quotient by patching together affine GIT quotients.

5.8.1 Construction of geometric invariant theory quotient for projective schemes

Definition 5.8.2. [13, Definition 5.1] Let G be an algebraic group acting on a projective scheme X. A *linear G-equivariant projective embedding* of X is a group homomorphism $G \longrightarrow \mathrm{GL}_{n+1}$ and a G-equivariant projective embedding $X \hookrightarrow \mathbb{P}^n$. For seek of simplicity, we will say that the action of G on $X \hookrightarrow \mathbb{P}^n$ is linear to mean that we have a linear G-equivariant projective embedding.

Definition 5.8.3. [13, Definition 5.2] Let $X \subset \mathbb{A}^{n+1}$ be an affine algebraic set. We say that X is a cone if it is not empty and for all $\lambda \in \mathrm{k}$,

$$(x_0, \dots, x_n) \in X \Longrightarrow (\lambda x_0, \dots, \lambda x_n) \in X.$$

Let $X \subset \mathbb{P}^n$ be a projective algebraic set. We define the affine cone over X to be the set (which is obviously a cone)

$$\tilde{X} := \{(x_0, \dots, x_n) \in \mathbb{A}^{n+1} | (x_0 : \dots : x_n) \in X\} \cup \{(0, \dots, 0)\} = \pi^{-1}(X) \cup \{(0, \dots, 0)\},$$

where $\pi : \mathbb{A}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$ is the standard projection, which, to each $(x_0, \dots, x_n)$ associates $(x_0 : \dots : x_n)$. If $x = (x_0 : \dots : x_n) \in X$, we denote by $\hat{x} = (x_0, \dots, x_n) \in \tilde{X}$ the point lying over x. For instance, the affine space $\mathbb{A}^{n+1}$ is the affine cone over the projective space $\mathbb{P}^n$.

Now we suppose that G is a reductive group which acts linearly (as above) on a projective scheme $X \subset \mathbb{P}^n$. Then the action of G on the projective space $\mathbb{P}^n$ lifts to an action of G on the affine cone $\mathbb{A}^{n+1}$. Let $\tilde{X} \subset \mathbb{A}^{n+1}$ be the affine cone over $X \subset \mathbb{P}^n$. As the G-action on X is G-equivariant, there is an induced G-action on $\tilde{X} \subset \mathbb{A}^{n+1}$. Precisely, we have

$$\mathcal{O}(\mathbb{A}^{n+1}) = \mathrm{k}[x_0, \dots, x_n] = \bigoplus_{r \geq 0} [x_0, \dots, x_n]_r.$$

and if $X \subset \mathbb{P}^n$ is the closed subscheme associated with a homogeneous ideal $I(X) \subset \mathrm{k}[x_0, \dots, x_n]$, then $\tilde{X} = \mathrm{Spec} R(X)$ where $R(X) = \mathrm{k}[x_0, \dots, x_n]/I(X)$.

The k-algebras $\mathcal{O}(\mathbb{A}^{n+1})$ and $R(X)$ are graded by the rings of homogeneous degree r polynomials $\mathrm{k}[x_0, \dots, x_n]_r$, $r \in \mathbb{N}$. Since the action of G on the affine space $\mathbb{A}^{n+1}$ is linear it preserves the graded components $\mathrm{k}[x_0, \dots, x_n]_r$. It follows that the invariant subalgebra

$$\mathcal{O}(\mathbb{A}^{n+1})^G = \bigoplus_{r \geq 0} \mathrm{k}[x_0, \dots, x_n]_r^G$$

is a graded algebra. In the same way, the invariant subalgebra $R(X)^G = \bigoplus_{r\geq 0} R(X)_r^G$ is a graded algebra. Since G is reductive, a use of Nagata's theorem shows that $R(X)$ is finitely generated. The inclusion of finitely generated graded k-algebras $R(X)^G \hookrightarrow R(X)$ determines a rational morphism of projective schemes $X \dashrightarrow \operatorname{Proj} R(X)^G$ whose indeterminacy locus is the closed subscheme of X defined by the homogeneous ideal $R(X)_+^G = \bigoplus_{r>0} R(X)_r^G$.

Definition 5.8.4. Let G be a reductive group operating linearly on a projective scheme $X \subset \mathbb{P}^n$. The null cone over X, is defined to be the closed subscheme of X defined the homogeneous ideal $R(X)_+^G$ in $R(X)$. The semistable set, denoted X^{ss}, is defined to be the complement in X of the null cone over X. That is, $X^{ss} = X - N$. The elements of X^{ss} are called the semistable points of X. The definition of semistable may be rephrased as follows. A point $s \in X$ is semistable if there exists a homogeneous function $f \in R(X)_r^G$ for some $r > 0$ which is G-invariant and satisfies $f(x) \neq 0$.

By construction, the semistable set X^{ss} of X is the open subset which is the domain of definition of the rational map

$$X \dashrightarrow \operatorname{Proj} R(X)^G.$$

The morphisms $X^{ss} \longrightarrow X//G := \operatorname{Proj} R(X)^G$ are called the GIT quotient of the action of G on X.

Theorem 5.8.5 (Second main Theorem)**.** [13, Theorem 5.3] Let G be a reductive group which acts linearly on the projective scheme $X \subset \mathbb{P}^n$. Then, the GIT quotient $\varphi : X^{ss} \longrightarrow X//G := \operatorname{Proj} R(X)^G$ is a good quotient of the G-action on the open subset X^{ss} of semistable points in X. Furthermore, $X//G$ is a projective scheme.

Corollary 5.8.6. [13, p. 36] Let G be a reductive group which acts linearly on a projective scheme $X \subset \mathbb{P}^n$ and $x_1, x_2 \in X^{ss}$ two stable points. Then, $\varphi(x_1) = \varphi(x_2)$ if and only if $\overline{G.x_1} \cap \overline{G.x_2} \cap X^{ss} \neq \emptyset$. Moreover, for each $y \in Y = X//G$, the preimage $\varphi^{-1}(y)$ contains a unique closed orbit.

As in the affine case, we would like to prove the existence of an open subset $X^s \subset X^{ss}$, consisting of stable points of X for which the quotient becomes a geometric quotient.

Definition 5.8.7. Let G be a reductive group operating on a projective closed scheme $X \subset \mathbb{P}^n$. We say that a point $x \in X$ is called stable if $\dim G_x = 0$ and there exists a G-invariant homogeneous polynomial $f \in R(X)_+^G$ such that $x \in X_f$ (i.e; $f(x) \neq 0$) and the action of G on X_f is closed. When it is not semistable, we say that it is unstable.

Let us denote the set of stable points of X by X^s, and $X^{us} = X - X^{ss}$ the set of unstable points of X. So it is natural to consider the set of stable points X^s as a candidate. The next step is to show that this is really an open set with the required property. The next two results show in fact that it is.

Lemma 5.8.8. [13, Lemma 5.5] The sets of stable points X^s and the set of semistable points X^{ss} are open in X.

Theorem 5.8.9 (Third Main Theorem)**.** [13, Theorem 5.6] Let G be a reductive group acting linearly on a closed subscheme $X \subset \mathbb{P}^n$, and let $\varphi : X \longrightarrow Y := X//G$ denote the GIT quotient. Then, there exists an open subscheme $Y^s \subset Y$ consisting of stable points of Y such that $\varphi^{-1}(Y^s) = X^s$ and that the restriction $\varphi : X^s \longrightarrow Y^s$ is a geometric quotient.

As we have seen above, the good properties we are looking for are related to the set of stable points. The next question that we will be concerned is that of determining stable points. In other words, how can we verify the stability of a point $x \in X$? The following Lemma comes in that direction.

Lemma 5.8.10. [13, Lemma 5.9] Let G be a reductive group which operates on a projective scheme $X \subset \mathbb{P}^n$. A k-point x in $X(\mathrm{k})$ is stable if and only if it is semistable and its orbit $G \cdot x$ is closed in the set of semistable points X^{ss} and $\dim G_x = 0$.

Chapter 6

The Hilbert-Mumford Numerical Criterion for Stability

In this chapter, we will look for a criterion that analyse the stability numerically. The main references for ths chapter are [13, 22].

6.1 A numerical criterion for stability

In Theorem 5.6.1, we saw that the Geometric Invariant Theory quotient of $\mathrm{Spec}(A)$ by the group G is $\mathrm{Spec}(A)//G = \mathrm{Spec}(A^G)$. Therefore, finding a GIT quotient reduces to the computation of the G-invariant set A^G. In general, computing the ring of invariants A^G is very delicate. So we have to find another way the analyse the stability without computing explicitely A^G. The Hilbert-Mumford criterion is a numerical method to compute stable and semistable points.

The next proposition characterizes stability not involving projective space.

Proposition 6.1.1. [13, Proposition 6.1] Let G be a reductive group acting on a projective scheme $X \subset \mathbb{P}^n$, and let $\tilde{X}$ denote the affine cone over X. Consider a k-point $x \in X(\mathrm{k})$ and let $\tilde{x} \in \tilde{X}$ be a point lying over x. Then

(i) x is semistable if and only if $0 \notin \overline{G.\tilde{x}}$.

(ii) x is stable if and only if $G.\tilde{x}$ is closed in $\tilde{X}$ and $\dim G_{\tilde{x}} = 0$.

It follows from (i) that in order to understand the semistability of a point x, it is important to understand the closure orbit of the lift of x. The idea to find points in the closure of the orbit is to find them via one parameter subgroups. Now we introduce 1-parameter subgroups of G as a tool to understand orbit closure.

Definition 6.1.2. Let G be a reductive group operating on a projective scheme $X \subset \mathbb{P}^n$. A 1-parameter subgroup of G, shortened to 1-PS of G, is a nontrivial homomorphism of groups $\lambda : \mathbb{G}_m \longrightarrow G$.

Let us fix a k-point $x \in X(\mathrm{k})$ and a 1-parameter subgroup $\lambda : \mathbb{G}_m \longrightarrow G$. Then for any $t \in \mathbb{G}_m$, $\lambda(t) \cdot x$ belongs to X. This gives a well-defined morphism $\lambda_x : \mathbb{G}_m \longrightarrow X$ given by:

$$\lambda_x(t) = \lambda(t) \cdot x.$$

So, if G operates on X, then the multiplicative group $\mathbb{G}_m$ acts on X via the 1-parameter subgroup λ:

$$\begin{aligned} \rho : \mathbb{G}_m \times X &\longrightarrow X \\ (t, x) &\longmapsto \lambda(t) \cdot x. \end{aligned}$$

We would like to express semistability and stability for a 1-PS of G in a numerical manner. The multiplicative group $\mathbb{G}_m$ acts linearly on the affine cone $\mathbb{A}^{n+1}$ via:

$$\begin{aligned} \sigma : \mathbb{G}_m \times \mathbb{A}^{n+1} &\longrightarrow \mathbb{A}^{n+1} \\ (t, (x_0, \dots, x_n)) &\longmapsto (tx_0, \dots, tx_n). \end{aligned}$$

In this case the 1-parameter subgroup associates to each $t \in \mathbb{G}_m$, the value t. As this action is linear, then it is diagonalizable. Therefore, there exists a basis $\{e_0, \dots, e_n\}$ of k^{n+1} such that for $0 \leq i \leq n$,

$$\lambda(t).e_i = t^{r_i} e_i, \tag{6.1}$$

for some integer $r_i \in \mathbb{Z}$. The integers r_i are called the λ-weights of the action on $\mathbb{A}^{n+1}$. For a k-point $x \in X(\mathrm{k})$, a non-zero lift $\tilde{x} \in \tilde{X}$ of x can be uniquely written with respect to this basis, as $\tilde{x} = \sum_{i=0}^{n} x_i e_i$. Then

$$\lambda(t) \cdot \tilde{x} = \sum_{i=0}^{n} t^{r_i} x_i e_i.$$

The λ-weights of x is defined to be the set $\lambda\text{-}wt(x) := \{r_i : x_i \neq 0\}$, which does not depend on the choice of a lift for x.

Definition 6.1.3. The Hilbert-Mumford weight of x at λ is defined to be the quantity

$$\mu(x, \lambda) := -\min\{r_i : x_i \neq 0\} := \max\{-r_i : x_i \neq 0\}.$$

Proposition 6.1.4. The Hilbert-Mumford weight satisfies the following properties:

1. $\mu(x, \lambda)$ is the unique integer μ such that $\lim_{t \mapsto 0} t^{\mu} \lambda(t) \cdot \tilde{x}$ exists and is non-zero.
2. $\mu(x, \lambda^n) = n\mu(x, \lambda)$ for positive n.
3. $\mu(g \cdot x, g\lambda g^{-1}) = \mu(x, \lambda)$ for all $g \in G$.
4. $\mu(x, \lambda) = \mu(y, \lambda)$ where $y = \lim_{t \mapsto 0} \lambda(t) \cdot x$.

Lemma 6.1.5. Let λ be a 1-parameter subgroup of G and let $x \in X(\mathrm{k})$. Let $\{e_0, \dots, e_n\}$ be a basis of k^{n+1} in which the $\lambda(\mathbb{G}_m)$-action is diagonal. Consider a non-zero lift $\tilde{x} = \sum_{i=0}^{n} x_i e_i$ of x. Then

(1) $\mu(x, \lambda) < 0$ if and only if $\tilde{x} = \sum_{r_i > 0}^{n} x_i e_i$, if and only if $\lim_{t \mapsto 0} \lambda(t) \cdot \tilde{x} = 0$.

(2) $\mu(x,\lambda) = 0$ if and only if $\tilde{x} = \sum_{r_i \geq 0}^{n} x_i e_i$, and there exists some $r_i = 0$ such that $x_i \neq 0$ if and only if $\lim_{t \mapsto 0} \lambda(t) \cdot \tilde{x}$ exists and is non-zero.

(3) $\mu(x,\lambda) > 0$ if and only if $\tilde{x} = \sum_{r_i}^{n} x_i e_i$, and there exists some $r_i = 0$ such that $x_i \neq 0$ if and only if $\lim_{t \mapsto 0} \lambda(t) \cdot \tilde{x}$ does not exist.

Proof. (1) By definition, $\mu(x,\lambda) < 0$ if and only if $\max\{-r_i : x_i \neq 0\} < 0$. Therefore, $-r_i < 0$ for all i. This is the same as $r_i > 0$ for all i. The last statement of (1) follows immediately from the defintion.

(2) By definition, $\mu(x,\lambda) = 0$ if and only if $\max\{-r_i : x_i \neq 0\} = 0$; which means that $r_i \geq 0$ for all i and there is an in index j such that $r_j = 0$. For such index j, we have $x_j \neq 0$ by definition. The other case is proved similarly.

□

We can study $\lim_{t \mapsto 0} \lambda(t).\tilde{x}$ using λ^{-1} by setting:

$$\lim_{t \mapsto 0} \lambda^{-1}(t) \cdot \tilde{x} = \lim_{t \mapsto \infty} \lambda^{-1}(t) \cdot \tilde{x}.$$

In that case, with the same hypotheses of the previous, we have:

Lemma 6.1.6. (1) $\mu(x,\lambda^{-1}) > 0$ if and only if $\tilde{x} = \sum_{r_i < 0}^{n} x_i e_i$, if and only if $\lim_{t \mapsto \infty} \lambda(t) \cdot \tilde{x} = 0$.

(2) $\mu(x,\lambda^{-1}) = 0$ if and only if $\tilde{x} = \sum_{r_i \leq 0}^{n} x_i e_i$, and there exists some $r_i = 0$ such that $x_i \neq 0$ if and only if $\lim_{t \mapsto \infty} \lambda(t) \cdot \tilde{x}$ exists and is non-zero.

(3) $\mu(x,\lambda^{-1}) < 0$ if and only if $\tilde{x} = \sum_{r_i}^{n} x_i e_i$, and there exists some $r_i = 0$ such that $x_i \neq 0$ if and only if $\lim_{t \mapsto \infty} \lambda(t) \cdot \tilde{x}$ does not exist.

Now we have a first result that characterizes the stability and the semistability of a k-point $x \in X(\mathrm{k})$.

Lemma 6.1.7. [22, Proposition 4.9] Suppose that G is a reductive group operating linearly on a projective scheme $X \subset \mathbb{P}^n$, and $x \in X(\mathrm{k})$. Then

(1) x is semistable for the 1-PS action $\lambda(\mathbb{G}_m)$ of G if and only if $\mu(x,\lambda) \geq 0$ and $\mu(x,\lambda^{-1}) \geq 0$.

(2) x is stable for the 1-PS action $\lambda(\mathbb{G}_m)$ of G if and only if $\mu(x,\lambda) > 0$ and $\mu(x,\lambda^{-1}) > 0$.

Proof. (1) Let $x \in X(\mathrm{k})$ and let $\tilde{x}$ be a lift of x. By Proposition 6.1.1, x is semistable for the 1-PS action $\lambda(\mathbb{G}_m)$ of G if and only if $0 \notin \overline{\lambda(\mathbb{G}_m) \cdot \tilde{x}}$. As any point in boundary point $\overline{\lambda(\mathbb{G}_m) \cdot \tilde{x}} - \lambda(\mathbb{G}_m) \cdot \tilde{x}$ is either

$$\lim_{t \mapsto 0} \lambda(\mathbb{G}_m) \cdot \tilde{x} \text{ or } \lim_{t \mapsto \infty} \lambda(t) \cdot \tilde{x} = \lim_{t \mapsto 0} \lambda^{-1}(t) \cdot \tilde{x},$$

by Lemma 6.1.5, it follows that x is semistable if and only if

$$\mu(x, \lambda) \geq 0 \text{ and } \mu(x, \lambda^{-1}) \geq 0.$$

(2) Let $x \in X(\mathrm{k})$ and let $\tilde{x}$ be a lift of x. By Proposition 6.1.1, x is stable for the 1-PS action $\lambda(\mathbb{G}_m)$ of G if and only if $\dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0$ and $\lambda(\mathbb{G}_m)_{\tilde{x}}$ is closed. The orbit is closed if and only if $\overline{\lambda(\mathbb{G}_m) \cdot \tilde{x}} = \lambda(\mathbb{G}_m) \cdot \tilde{x}$, that is, if and only if the boundary is empty. This is equivalent to the condition

$$\lim_{t \mapsto 0} \lambda(\mathbb{G}_m) \cdot \tilde{x} \text{ and } \lim_{t \mapsto \infty} \lambda(t).\tilde{x} = \lim_{t \mapsto 0} \lambda^{-1}(t) \cdot \tilde{x},$$

do not exist. It follows from third property of Lemma 6.1.5 that the latter condition is equivalent to say

$$\mu(x, \lambda) > 0 \text{ and } \mu(x, \lambda^{-1}) > 0.$$

If this happens, then $\lambda(\mathbb{G}_m)$ cannot fix $\tilde{x}$; because otherwise the both limits above would both exist. Hence, we must have $\dim \lambda(\mathbb{G}_m)_{\tilde{x}} = 0$. This completes the proof of the lemma.

□

This leads us the Hilbert-Mumford numerical criterion.

Theorem 6.1.8 (Hilbert-Mumford Criterion)**.** [21, Theorem 4.9] Let a reductive group G act linearly on a projective scheme $X \subset \mathbb{P}^n$. Then, if $x \in X(\mathrm{k})$:

$$x \in X^{ss} \text{ if and only if } \lambda(x, \lambda) \geq 0 \text{ for all 1-parameter subgroup } \lambda \text{ of } G.$$
$$x \in X^{s} \text{ if and only if } \lambda(x, \lambda) > 0 \text{ for all 1-parameter subgroup } \lambda \text{ of } G.$$

The Hilbert-Mumford Criterion is equivalent to the following result:

Theorem 6.1.9 (Fundamental Theorem in GIT)**.** Let a reductive group G acts on an affine space $\mathbb{A}^{n+1}$. If $x \in \mathbb{A}^{n+1}$ is a closed point and $y \in \overline{G \cdot x}$, then there exists a 1-PS λ of G such that $\lim_{t \mapsto 0} \lambda(t) \cdot x = y$.

6.2 Summary of the construction of GIT quotients

6.2.1 Summary of the Strategy for construction of affine quotient

To begin with, let us recall the correspondence between the category affine of schemes over k and the category of finitely generated nilpotent-free algebras. First, to any affine scheme X there corresponds a k-algebra $\mathcal{O}(X) = \{\text{regular functions } X \longrightarrow \mathrm{k}\}$. The following define a bijective correspondence:

If G is a group which operates on an affine scheme X, then it operates on $\mathcal{O}(X)$ and the construction of an affine quotient follows the following procedure:

$$X \text{ affine scheme} \rightsquigarrow \mathcal{O}(X) \rightsquigarrow \mathcal{O}(X)^G \rightsquigarrow \boxed{\operatorname{Spec} \mathcal{O}(X)^G}.$$

In this case, $\operatorname{Spec} \mathcal{O}(X)^G$ is the quotient we were looking for. This ultimately leads to the study of the invariant algebra $\mathcal{O}(X)^G$. In order to use the correspondence above, we have to verify that $\mathcal{O}(X)^G$ is a finitely generated nilpotent-free algebra. The nilpotence-free property follows from the nilpotence-freeness of the algebra $\mathcal{O}(X) \supset \mathcal{O}(X)^G$. However, the finite generation is a much harder question and is known as Hilbert's 14-th problem. Hilbert proved in the case where the group is the Linear Group GL_n that $\mathcal{O}(X)^G$ is finitely generated. The complete answer to this question was given by Nagata by constructing a counterexample in positive characteristic, and proving that $\mathcal{O}(X)^G$ is finitely generated when the group is reductive.

Therefore, to get a quotient with suitable geometric properties, it is natural to consider the action of a reductive group. Fortunately, many groups of our interest such as $\mathrm{GL}_n, \mathrm{SL}_n, \mathrm{PGL}_n$ are reductive.

6.2.2 Summary for the strategy for the construction of projective GIT quotients

The first step is to consider a reductive group acting linearly on a projective scheme $X \subset \mathbb{P}^n$. Then separate the scheme X into three sets: the set X^s of stable k-points, the set X^{ss} of semistable k-points and the set $X^{us} = X - X^{ss}$ of unstable points. The stable points are the well behaved. The semistable points are still well behaved and have good geometric properties. However, the unstable points are not well behaved and need to be discarded in order to get a quotient with good properties. Then cover the open set X^{ss} by affine open subsets U, and construct the affine quotient $\operatorname{Spec}\mathcal{O}(U)^G$. The final construction follows by glueing the affine quotient $\operatorname{Spec}\mathcal{O}(U)^G$.

On the other hand, there is a similar correspondence between the category of projective schemes over k and the category of finitely generated nilpotent-free graded k-algebras. The construction of projective quotient follows that of the affine case passing through the associated the affine cone. The previous construction is equivalent to the following:

$$\text{projective scheme } X \subset \mathbb{P}^n \rightsquigarrow \text{Affine cone } \hat{X} \subset \mathrm{k}^{n+1} \rightsquigarrow \mathcal{O}(\hat{X}) \rightsquigarrow \mathcal{O}(\hat{X})^G \rightsquigarrow \boxed{\operatorname{Proj} \mathcal{O}(\hat{X})^G}$$

The difficulty still remains when computing the algebra of invariant $\mathcal{O}(\hat{X})^G$. A direct way to analyse the stability is via the 1-parameter subgroup. We can then use the Mumford-Hilbert's theorem to determine the stable and semistable loci.

Chapter 7

The Moduli Space of Hypersurfaces in $\mathbb{P}^n$

Let us apply the theory we have developed so far to the classification of hypersufaces of a fixed degree d in a projective space $\mathbb{P}^n_{\mathrm{k}}$ up to the action of the automorphism group PGL_{n+1} of $\mathbb{P}^n_{\mathrm{k}}$. We assume that k is field of characteristic coprime to d. We will simply write $\mathbb{P}^n$ rather than $\mathbb{P}^n_{\mathrm{k}}$. The main references for this chapter are [21], [5], [19] and [13].

7.1 Definition and moduli problem of hypersurfaces in $\mathbb{P}^n$

Definition 7.1.1. Let F be a homogeneous polynomial of degree d, that is, $F \in \mathrm{k}[x_0, \ldots, x_n]_d - \{0\}$. The locus

$$V(F) = \{(b_0 : \cdots : b_n) \mid F(b_0, \ldots, b_n) = 0\} \subset \mathbb{P}^n$$

is called the projective degree d hypersurface defined by F.

Definition 7.1.2. Let $F \in \mathrm{k}[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree d, and $p \in \mathbb{P}^n$. We say that p is a singular point of the projective hypersurface $V(F)$ defined by F if F and its first partial derivatives vanish at the lift $\tilde{p} \in \mathbb{A}^{n+1} - \{0\}$ over p.

A smooth or nonsingular hypersurface is a hypersurface without singular points. If F is irreducible, we say that the associated hypersurface $V(F)$ is irreducible. If this happens, then $V(F)$ is a closed subvariety of the projective space $\mathbb{P}^n$, and has codimension 1. If however F decomposes into the product of irreducible forms $F = F_1^{d_1} \ldots F_1^{d_l}$, with F_i distinct irreducible forms, $i = 1, \ldots, l$, we say that X is reducible and the associated hypersurfaces $V(F_i)$ are its irreducible components. In this case $V(F)$ decomposes as:

$$V(F) = \bigcup_{i=1}^{l} d_i V(F_i).$$

There is a bijective correspondence between hypersurfaces of degree d and the set of non-zero homogeneous polynomial of degree d, $\mathrm{k}[x_0, \ldots, x_n]_d - \{0\}$. Every element $F \in \mathrm{k}[x_0, \ldots, x_n]_d - \{0\}$ can be written as

$$F = \sum a(\mu_0, \ldots, \mu_n) x_0^{\mu_0} \ldots x_n^{\mu_n},$$

where $\mu_0, \ldots, \mu_n$ are nonnegative integers less or equal than d and such that $\mu_0 + \mu_1 + \cdots + \mu_n = d$. Each non-zero homogeneous polynomial F is determined by the data of the exponents $\mu_0, \ldots, \mu_n$, with $0 \leq \mu_0, \ldots, \mu_n \leq d$, and $\mu_0 + \mu_1 + \cdots + \mu_n = d$. The combinatorial analysis shows that there are

$$N := \binom{n+d}{n}$$

choices for these exponents, so that the dimension of $\mathrm{k}[x_0, \ldots, x_n]_d - \{0\}$ is given by the number N above. Now, for any nonzero scalar λ and for any degree d homogeneous polynomial F, the equality

$$V(\lambda F) = \{(b_0 : \cdots : b_n) \mid \lambda F(b_0, \ldots, b_n) = 0\} = \{(b_0 : \cdots : b_n) \mid F(b_0, \ldots, b_n) = 0\} = V(F)$$

shows that the projectivation of the space $\mathrm{k}[x_0, \ldots, x_n]_d - \{0\}$, denoted

$$V_{d,n} = \mathbb{P}(\mathrm{k}[x_0, \ldots, x_n]_d)$$

is a smaller dimensional space that parametrizes degree d projective hypersurfaces in $\mathbb{P}^n$. Recall that for a vector space V, the projectivation of V, denoted $\mathbb{P}(V)$ is the set of all lines in V. For $v \in V$, we denote by $[v]$ the corresponding line generated by v. In other words, $\mathbb{P}(V) = \mathrm{G}(1, V)$, the Grassmannian of degree 1 in V.

7.1.3 Families of degree d hypersurfaces

By definition of a family, over each fiber X_s over a point s, we have a degree d hypersurface which is parametrized by that point. As in the case of Grassmannians, we will define our families parametrized by a variety as vector bundles.

Remark 7.1.4. First, let us consider a surjective morphism $f : X \longrightarrow Y$. For simplicity, suppose that the varieties X and Y are irreducible. Usually, fibers $f^{-1}(y)$, $y \in Y$ of f can vary highly discontinuously. A technical requirement on f that prevents such pathological behaviour is flatness. In particular, if we require flatness for f, then all components of all fibers $f^{-1}(y)$ have the same dimension. Flatness plays a very important part in the process of the formulation of the notion of families in the study of a moduli problem. A stronger concept that we will require here is that of local triviality [21].

Definition 7.1.5. [21, page 17] A *family of hypersurfaces in $\mathbb{P}^n$ of degree d over S* is a locally trivial line bundle $\pi : L \longrightarrow S$ over S and a tuple of sections $a_{i_0 \ldots i_n} : S \longrightarrow L$, with $i_j \geq 0$, $\sum_{j=0}^{n} i_j = d$, and such that for each $s \in S(\mathrm{k})$, the corresponding homogeneous polynomial

$$F(L, a, s) := \sum_{i_0, \ldots, i_n} a_{i_0 \ldots i_n}(s) x_0^{i_0} \ldots x_n^{i_n}$$

is nonzero.

Notation 7.1.6. We will simply denote a family of hypersurfaces above by the pair (L, a). The hypersurface over $s \in S(\mathrm{k})$ is denoted $(L, a)_s = V(F(L, a, s))$, and this is clearly the hypersurface associated with the homogeneous polynomial $F(L, a, s)$.

Definition 7.1.7. Let (L, a) and (L', a') be two families of hypersurfaces degree d, in $\mathbb{P}^n$, over the same base variety S. We say that (L, a) and (L', a') are equivalent over S if there exists an isomorphism of line bundles $\varphi : L \longrightarrow L'$ and $g \in \mathrm{GL}_{n+1}$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{a} & L \\ & \searrow^{g \cdot a'} & \downarrow \varphi \\ & & L' \end{array}$$

commutes.

Lemma 7.1.8. Let (L, a) be a family of hypersurfaces of degree d over a base variety, and λ be any nonzero scalar. Then (L, a) and $(L, \lambda a)$ are equivalent.

Proof. It suffices to take $\varphi = \lambda \cdot \mathrm{id}_L$ and g the identity map in GL_{n+1}. □

7.1.9 Moduli problem of hypersurfaces

Let us consider the moduli of hypersurfaces of degree d in the projective space $\mathbb{P}^n$. We formulate a moduli problem for hypersurfaces as follows.

(1) $\mathcal{M} = \{$all hypersurfaces of degree d in the projective space $\mathbb{P}^n\}$.

(2) Two hypersurfaces H_1 and H_2 are equivalent if there exists $g \in \mathrm{GL}(n + 1)$ such that $H_2 = g(H_1)$.

(3) A hypersurface of degree d is completely determined by the zeroes of a homogeneous polynomial $f = \sum\limits_{\substack{0 \le i_0, \ldots, i_n \le d \\ i_0 + \cdots + i_n = d}} a_{i_0 \ldots i_n} x_0^{i_0} \cdots x_n^{i_n}$. A family of hypersurfaces of degree d parametrized by a variety S is a pair (L, a), where L is a line bundle over S and $a = \{a_{i_0 \ldots i_n} \mid 0 \le i_0, \ldots, i_n \le d,\ i_0 + \cdots + i_n = d\}$ is a set of sections.

(4) The parameter category $\mathcal{C}$ can be considered as the category Var of all varieties.

(5) Two families (L, a) and (L', a') of hypersurfaces parametrized by S are isomorphic if there exists an isomorphism of vector bundles $h : L \longrightarrow L'$ such that $h(a) = a'$.

(6) Two families (L, a) and (L', a') of hypersurfaces parametrized by S are equivalent if there exists $g \in \mathrm{GL}(n + 1)$ such that the families (L, a) and (L', ga') are isomorphic.

(7) For each variety $S \in \mathcal{C}$, let $\mathrm{F}(S)$ denote the set of families parametrized S. Suppose that $h : S' \longrightarrow S$ is a morphism of varieties, and that (L, a) is a family of hypersurfaces of degree d. Then the pullback family along the morphism h is the fiber product $p : S' \times_S L \longrightarrow S'$ together with the section $a' : S' \longrightarrow S' \times_S L$

defined by $a'(s') = (s', a(h(s')) = (\mathrm{id}_{S'} \times (a \circ h))(s')$ for all $s' \in S'$. That is, $a' = \mathrm{id}_{S'} \times (a \circ h)$.

This defines a functor

$$\begin{array}{rcl} \mathrm{F} : \mathrm{Var}^{\mathrm{opp}} & \longrightarrow & \mathrm{Set} \\ S & \longmapsto & \mathrm{Fam}(S)/\sim_S \\ (h : S' \to S) & \longmapsto & [h^*] : \mathrm{Fam}(S)/\sim_S \longrightarrow \mathrm{Fam}(S')/\sim_S \\ & & (L, a)/\sim_S \longmapsto [(S' \times_S L, \mathrm{id}_{S'} \times (a \circ h))]/_{\sim_{S'}} \end{array}$$

This defines a moduli problem, called the moduli problem of hypersurfaces of degree d in $\mathbb{P}^n$. The linear group $\mathrm{GL}(n+1)$ acts on k^{n+1} by left multiplication. This action induces an action on $\mathbb{P}^{N-1}$, where $N = \binom{n+d}{n}$. One of the major problems in classical invariant theory was to find the quotient $\mathbb{P}^{N-1}//\mathrm{GL}(n+1)$.

Proposition 7.1.10. The moduli functor associated with the moduli problem of hypersurfaces is not representable.

Proof. Since a jump phenomenon prevents the existence of fine moduli spaces (see Proposition 2.6.2), it suffices to construct a family of hypersurfaces with the jump phenomenon. For this, let $f \in \mathrm{k}[x_0, \ldots, x_n]_d - \{0\}$ and $X = V(f)$ be the corresponding hypersurface. Let $0 \leq i < n$ be fixed. Consider $H := V\left(f(x_0, \ldots, x_i, tx_{i+1}, \ldots, tx_n)\right) \subset \mathbb{P}^n \times \mathbb{A}^1_t$ along with the projection $\pi : H \longrightarrow \mathbb{A}^1_t$.

Now we want to show that all the fibers H_t, for $t \neq 0$ are isomorphic. For this we will show that each H_t is isomorphic to X, for $t \neq 0$. Let $t \neq 0$. The map

$$\begin{array}{rcl} \phi : X & \longrightarrow & H_t \\ (x_0, \ldots, x_i, x_{i+1}, \ldots, x_n) & \longmapsto & (x_0, \ldots, x_i, t^{-1}x_{i+1}, \ldots, t^{-1}x_n) \end{array}$$

is clearly well-defined and bijective. Hence each fiber H_t, for $t \neq 0$, is isomorphic to X, so that all the fibers H_t, for $t \neq 0$, are isomorphic. Moreover, for $t = 0$, we have:

$$\begin{array}{rcl} X_0 & = & X \cap \{(x_0, \ldots, x_i, x_{i+1}, \ldots, x_n) \mid x_{i+1} = \cdots = x_n = 0\} \\ & = & V\left(f(x_0, \ldots, x_i, 0, \ldots, 0)\right) \subset \mathbb{P}^n \end{array}$$

which cannot be isomorphic to X. Therefore we have a jump phenomenon as defined in Definition 2.6.1. We conclude from Proposition 2.6.2 that the moduli functor of hypersurfaces is not representable. □

Proposition 7.1.11. The only closed point of the moduli functor of hypersurfaces is $\left[V\left(x_0^d\right)\right]$.

Proof. Let $X = V(F) \subset \mathbb{P}^n$ be a hypersurface of degree d. By a change of coordinates, we may assume that the coefficient appearing in x_0^d is nonzero. For $i = 0$, we consider again the family $H := V(x_0, tx_1, \ldots, tx_n) \subset \mathbb{P}^n \times \mathbb{A}^1_t$ along with the projection $\pi : H \longrightarrow \mathbb{A}^1_t$. Taking the fiber over 0, we get as above

$$\begin{array}{rcl} X_0 & = & X \cap \{(x_0, x_1, \ldots, \ldots, x_n) \mid x_1 = \cdots = x_n = 0\} \\ & = & V\left(f(x_0, 0, \ldots, 0)\right) \subset \mathbb{P}^n = V\left(x_0^d\right) \end{array}$$

Hence $[X]$ cannot be a closed point of the moduli functor of hypersurfaces, unless X is isomorphic to X_0. Furthermore, if $X \longrightarrow S$ is a flat family of hypersurfaces whose generic fiber is a d-fold plane, then every fiber is a d-fold plane. This shows that $\left[V\left(x_0^d\right)\right]$ is a closed point. □

Proposition 7.1.12. $V_{d,n}$ parametrizes a tautological family of degree d hypersurfaces in $\mathbb{P}^n$ with the local universal property.

This shows that the coarse moduli space for hypersurfaces is given by categorical quotient for the SL_{n+1}-action on $V_{d,n}$ as described in Theorem 5.4.8.

7.2 Projective GIT for hypersurfaces

We would like to construct a projective GIT quotient for hypersurfaces in $\mathbb{P}^n$. This is possible because the action of the reductive group $\mathrm{SL}(n+1)$ on $V_{d,n}$ is linear, and Theorem 5.8.5 tells us that the GIT quotient $\varphi : V_{d,n}^{ss} \longrightarrow V_{d,n}//\mathrm{SL}_{n+1} := \mathrm{Proj}\, R(V_{d,n})^{\mathrm{SL}(n+1)}$ on the open subset $V_{d,n}^{ss}$ is a good quotient. Moreover, it is a projective variety. By Theorem 5.8.9, there exists a subvariety $Y^s \subset Y := V_{d,n}//\mathrm{SL}_{n+1}$, such that $\varphi : V_{d,n}^s \longrightarrow Y^s$ is a geometric quotient. This leads to the study of the set of semistable points $V_{d,n}^{ss}$ and the set of stable points $V_{d,n}^s$.

7.2.1 Action of the automorphism group PGL_{n+1} on the projective space $\mathbb{P}^n$

The General Linear group GL_{n+1} acts on the projective space $\mathbb{P}^n$ via the left multiplication :

$$A(b_0 : \cdots : b_n) = \begin{pmatrix} x_{00} & x_{01} & x_{02} & \dots & x_{0n} \\ x_{10} & x_{11} & x_{12} & \dots & x_{1n} \\ \multicolumn{5}{c}{\dotfill} \\ x_{n0} & x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} x_{00}b_0 + x_{01}b_1 + \cdots + x_{0n}b_n \\ x_{10}b_0 + x_{11}b_1 + \dots x_{1n}b_n \\ \vdots \\ x_{n0}b_0 + x_{n1}b_1 + \cdots + x_{nn}b_n \end{pmatrix}.$$

This action induces a linear action on homogeneous polynomials, and is given by:

$$(A \cdot F)(b_0 : \cdots : b_n) = F(A^{-1}(b_0 : \cdots : b_n))$$

so that the hypersurfaces transform consistently: $A(V(F)) = V(AF)$. We obtain an action

$$\begin{array}{rcl} \mathrm{PGL}_{n+1} \times V_{d,n} & \longrightarrow & V_{d,n} \\ ([A],[F]) & \longmapsto & [A \cdot F]. \end{array}$$

We hope to prove that the moduli space for hypersurfaces of degree d in the projective space $\mathbb{P}^n$ is given by an orbit space of this action. If this happens, Theorem 5.4.8 tells us that this orbit space is the coarse moduli space we are looking for. To be able to apply Theorem 5.4.8, we need to determine a family with the local universal property. It will be enough to prove that $V_{d,n}$ parameterizes a family with the required property.

We note that the action of PGL_{+1} on $V_{d,n}$ is not linear. Fortunately, GL_{n+1} and SL_{n+1} act linearly on $V_{d,n}$. From the fact that there is a surjection $\mathrm{SL}_{n+1} \longrightarrow \mathrm{PGL}_{n+1}$ with finite kernel, the orbits of the action of SL_{n+1} on $V_{d,n}$ should be the same as the orbits of the action of PGL_{n+1} on $V_{d,n}$. This allows us to consider the action of SL_{n+1} rather than the action of PGL_{n+1}.

7.2.2 Stability of smooth hypersurfaces

Theorem 7.2.3 (Matsumura-Monsky-Mumford)**.** [4, Proposition 4.2.][5, Theorem 10.1] For $d > 1$, every smooth hypersurface of degree d, in $\mathbb{P}^n$, is semistable for the SL_{n+1}-action on $V_{d,n}$. Moreover, if $d > 2$, then all smooth hypersurfaces of degree d, in $\mathbb{P}^n$, are stable.

Example 7.2.4. For the special case $d = 1$, one checks that the invariant ring is [22, Example 4.1]:

$$\mathrm{k}[x_0, \dots, x_n]_1^{\mathrm{SL}_{n+1}} = \mathrm{k}.$$

So there are no semistable hypersurfaces of degree 1. Then $V_{1,n}^{ss} = V_{1,n}^{s} = \emptyset$.

Example 7.2.5 (Quadric hypersurfaces)**.** Suppose $d = 2$. The degree 2 hypersurfaces are called *quadric hypersurfaces* in $\mathbb{P}^n$. The space $V_{2,n}$ of quadric hypersurfaces can be identified with $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(\mathrm{k}))$ as follows. Any quadric hypersurface defined by the homogeneous polynomial

$$F(x_0, \dots, x_n) = \sum_{j=0}^{n} \sum_{i=0}^{n} a_{ij} x_i x_j \in \mathrm{k}[x_0, \dots, x_n]_2,$$

determines a matrix

$$B_F = (b_{ij}),$$

by setting $b_{ij} = b_{ji} = a_{ij}$ and $b_{ii} = 2a_{ii}$. Conversely, any symmetric $(n + 1) \times (n + 1)$ matrix $B = (b_{ij})$ determines a homogeneous polynomial of degree 2

$$F_B(x_0, \dots, x_n) = \sum_{j=0}^{n} \sum_{i=0}^{n} a_{ij} x_i x_j \in \mathrm{k}[x_0, \dots, x_n]_2,$$

by setting $a_{ij} = b_{ij}$ and $a_{ii} = b_{ii}/2$ as the characteristic of the ground field k is coprime to 2.

With this correspondence, the discriminant Δ on $V_{2,n}$ is identified with the determinant on the space $\mathbb{P}(\mathrm{Sym}_{(n+1)\times(n+1)}(\mathrm{k}))$. In this case $V(F)$ is smooth if and only if B_F is invertible. It follows from the theorem above that $V_{2,n}^{ss} \cong \mathrm{GL}_{n+1} \cap \mathrm{Sym}_{(n+1)\times(n+1)}(\mathrm{k})$.

Let us prove that there are no stable points in $V_{2,n}$. If F is the degree 2 homogeneous polynomial corresponding to a matrix B of rank $r + 1$, we know that by a linear change of coordinate system, it is projectively equivalent to the quadratic form [25, Lemma 1.3.2.1]

$$x_0^2 + \cdots + x_r^2.$$

Therefore, all smooth quadratic forms are projectively equivalent to the quadratic form

$$x_0^2 + \cdots + x_n^2.$$

But the stabilizer of this quadric is the Special Orthogonal group SO_{n+1}, which has positive dimension. Thus the hypersurface $V(x_0^2 + \cdots + x_n^2)$ is not stable, and so are all smooth hypersurfaces of degree 2. Moreover, the GIT quotient for hypersurfaces of degree 2 is just $\mathrm{Spec}k$, that is, it is a single point.

7.2.6 The numerical Hilbert-Mumford criterion for hypersurfaces

We use Hilbert-Mumford's criterion to find the set of stable points and that of semistable points for the SL_{n+1}-action on $V_{n,d}$.

Proposition 7.2.7. [19, Proposition 7.5] For every 1-PS $\lambda : \mathbb{G}_m \longrightarrow \mathrm{SL}_{n+1}$, there exist integers $r_0 \geq r_1 \geq \cdots \geq r_n$, $r_0 + \cdots + r_n = 0$, for which λ is conjugate in SL_{n+1} to the diagonal 1-PS:

$$\begin{array}{rcl} \lambda : \mathbb{G}_m & \longrightarrow & \mathrm{SL}_{n+1} \\ t & \longmapsto & \lambda(t) = \mathrm{diag}(t^{r_0}, t^{r_1}, \ldots, t^{r_n}). \end{array}$$

Let $\mathcal{I}_{d,n+1} = \{I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1} : |I| = i_0 + \cdots + i_n = d\}$ denote the set of multi-indices of order d, and let $x_I := x_0^{i_0} x_1^{i_1} \ldots x_n^{i_n}$ denote a monomial of degree d. Let $\tilde{V}_{d,n}$ denote the affine cone over $V_{d,n}$. Then λ acts diagonally with respect to the basis $\{x_I : I \in \mathcal{I}\}$. Applying the inverse of $\lambda(t)$, we get:

$$\lambda^{-1}(t).x_I = (t^{-r_0}x_0)^{i_0}(t^{-r_1}x_1)^{i_1} \ldots (t^{-r_n}x_n)^{i_n} = t^{-(r_0 i_0 + \ldots r_n i_n)} x_I,$$

so that

$$\mu(x_I, \lambda) = -\sum_{j=0}^{n} r_j i_j.$$

Let $V(F)$ be a hypersurface determined by a degree d homogeneous polynomial $F = \sum_{I \in \mathcal{I}} a_I x_I$, and let $[F] \in V_{d,n}$ be the corresponding line generated by F. We have:

$$\lambda(t) \cdot V(F) = V(\lambda(t) \cdot F) = V(F(\lambda(t)^{-1}(x_0, \ldots, x_n)).$$

But,

$$F(\lambda(t)^{-1}(x_0, \ldots, x_n) = F(t^{-r_0}x_0, \ldots, t^{-r_n}x_n) = \sum_{j=0}^{n} t^{-(r_0 i_0 + \cdots + r_n i_n)} x_0^{i_0} \ldots x_n^{i_n}.$$

One then deduces the Hilbert weight:

$$\mu([F], \lambda) \;\; = \;\; -\min\{-\sum_{j=0}^{n} r_j i_j : I \in \mathcal{I}\} = \max\{\sum_{j=0}^{n} r_j i_j : I \in \mathcal{I}\}.$$

This is actually a good formula, but it is difficult in general to describe explicitely the semistable and stable points for arbitrary (d, n). However, it is possible describe precisely the semisatble and stable points for small value of n. For concreteness, let us consider the case of binary forms of degree d.

7.3 Degree d binary forms in $\mathbb{P}^1$

Let SL_2 acts on $V_{d,1} = \mathbb{P}(\mathrm{k}[x, y]_d) \cong \mathbb{P}^d$ as above. We want to describe the semistable and stable loci by the use of Hilbert-Mumford criterion, and determine the GIT quotient. Let

$$F(x, y) = \sum_{i=0}^{d} a_i x^{d-i} y^i \in \mathrm{k}[x, y]_2.$$

Then F has exactly d zeroes in $\mathbb{P}^1$ counted with their multipliciies. The group GL_2 acts on $\mathrm{k}[x, y]$ via linear substitution:

$$\begin{array}{ccc} \mathrm{GL}_2 \times V_{d,1} & \longrightarrow & V_{d,1} \\ \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, F(x, y)\right) & \longmapsto & F(\alpha x + \beta y, \gamma x + \delta y) \end{array}$$

The case of $V_{1,1}$ has been treated in Example 7.2.4, so we may assume that $d \geq 2$. By Proposition 7.2.7, we know that every 1-parameter subgroup of SL_2 is conjugate to the 1-parameter subgroup given by:

$$\lambda(t) = \begin{pmatrix} t^r & 0 \\ 0 & t^{-r} \end{pmatrix},$$

for some positive integer r. Now we compute the Hilbert λ-weight:

$$\lambda(t)^{-1} \cdot F(x, y) = F(t^{-r}x, t^r y) = \sum_{i=0}^{d} a_i (t^r x)^{d-i} (t^{-r} y)^i = \sum_{i=0}^{d} t^{r(d-2i)} a_i x^{d-i} y^i.$$

Therefore,

$$\mu([F], \lambda) = -\min\{2i - d : a_i \neq 0\} = \max\{d - 2i : a_i \neq 0\}.$$

Let i_0 denote the smallest integer for which $a_i \neq 0$. Then

$$\mu([F], \lambda) = d - 2i_0.$$

Hence, the semistable and stable hypersurfaces of degree 2 are characterised by the following lemma.

Lemma 7.3.1. $\mu([F], \lambda) \geq 0$ if and only if $i_0 \leq \frac{d}{2}$, and $\mu([F], \lambda) < 0$ if and only if $i_0 > \frac{d}{2}$.

By Hilbert-Mumford numerical criterion, $[F] \in V_{d,1}$, is semistable if and only if it satisfies $\mu([F], \lambda') \geq 0$ for all 1-PS λ' of SL_2. But, for any such λ', there exists $g \in \mathrm{SL}_2$

such that $\lambda' = g^{-1}\lambda g$. Then

$$\mu([F], \lambda') = \mu(g \cdot [F], \lambda).$$

Let $p_1, \ldots, p_d \in \mathbb{P}^1$ be roots of F counted with their multiplicities. Then $g \cdot p_1, \ldots, g \cdot p_d \in \mathbb{P}^1$ are the roots of $g.F$. By the transitivity of the SL_2-action on $\mathbb{P}^1$, we deduce the following result that determines stable and semistable hypersurfaces of d in $\mathbb{P}^1$.

Proposition 7.3.2. [19, Proposition 7.9] Let $F \in \mathrm{k}[x, y]_d$ lie over $[F] \in V_{d,1}$. Then

(1) $[F]$ is semistable if and only if all roots of F in $\mathbb{P}^1$ have multiplicities less than or equal to $\frac{d}{2}$.

(2) $[F]$ is stable if and only if all roots of F in $\mathbb{P}^1$ have multiplicities strictly less than $\frac{d}{2}$.

In particular, if d is odd, then each semistable point is stable, $V_{d,1}^{ss} = V_{d,1}^{s}$, and the GIT quotient is a projective variety which is the geometric quotient of the stable degree d hypersurfaces in $\mathbb{P}^1$.

7.3.3 Application to small values of d

For $d = 2$, it follows from the first property of Proposition 7.3.2 that $V(F)$ is semistable hypersurface if and only if all the roots of F have multiplicities less or equal than 1. This means that the semistable loci are those corresponding to binary forms F with exactly two distinct roots in $\mathbb{P}^1$. The second property of Proposition 7.3.2 says that stable hypersurfaces $V(F)$ are those for which the roots of the defining polynomial F have multiplicity strictly less that 1. Hence, for this case, there are no stable hypersurfaces. By the transitivity of the SL_2-action on $V_{2,1}^{ss}$, there is only one orbit so that the GIT quotient is Speck.

For $d = 3$, which is odd, Proposition 7.3.2 asserts that the semistable and the stable hypersurfaces are the same. Moreover, an hypersurface $V(F)$ is semistable if and only if all the roots of F have multiplicity less or equal than 1. This means that F has exactly three distinct roots. By Theorem **??**, we know that any three distinct points (p_1, p_2, p_3) can be transformed into any other three distinct points (q_1, q_2, q_3) by some unique Möbius transformation, so that we have only a single orbit. This proves that the GIT quotient is again $\mathbb{P}^0 = \mathrm{Speck}$ which is a projective variety.

For $d = 4$, it follows from Proposition 7.3.2 that a semistable hypersurfaces are those which the roots of the corresponding binary form F have multiplicity less or equal that 2. Hence semistable hypersurfaces corresponds to binary forms with at most 2 repeated roots. Similarly, stable hypersurfaces correspond to binary forms F with exactly four distinct roots. Let $(p_1, \ldots, p_4)$ be four distinct ordered points. Using the cross-ratio formula, there exists a unique Möbius transformation which sends $(p_1, \ldots, p_4)$ to $(0, 1, \infty, \lambda)$, where $\lambda \in \mathbb{A}^1 - \{0, 1\}$ is the cross-ratio of these points as defined in Definition **??**. For the point p_1, we have three possibilities: it can be send to 0, or to 1, or to ∞. For each choice for the image of p_1, we choose the image of p_2 among the remaining two points, which gives two possibilities. For each choice of the image of p_1 and p_2, there is only one possibility for the choice for the image of p_3. This gives six possibilities for the value of the cross-ratio depending on a particular choice of ordering. This is justified by the following result.

Lemma 7.3.4. [11, Lemma 4.5] Let the symmetric group Σ_3 act on $\mathbb{A}-\{0,1\}$ as follows: given $\lambda \in \mathbb{A}-\{0,1\}$, permute the numbers $0, 1, \lambda$ according to $\alpha \in \Sigma_3$, then apply a linear transformation of x to send the first two back to $0, 1$ and let $\alpha(\lambda)$ be the image of the third. Then the orbit of λ consists of

$$\lambda, 1-\lambda, \frac{1}{\lambda}, \frac{\lambda-1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda}{1-\lambda}.$$

The morphism

$$\begin{array}{rcl} f : V^s_{4,1} & \longrightarrow & \mathbb{A}^1 \\ \lambda & \longmapsto & \left(\dfrac{(2\lambda-1)(\lambda-2)(\lambda+1)}{\lambda(\lambda-1)}\right)^3 \end{array}$$

satisfies

$$f(\lambda) = f(1-\lambda) = f\left(\frac{1}{\lambda}\right) = f\left(\frac{\lambda-1}{\lambda}\right) = f\left(\frac{1}{1-\lambda}\right) = f\left(\frac{\lambda}{1-\lambda}\right).$$

The last condition means that f is symmetric in the six values of the cross-ratio. Hence f is invariant under the SL_2-action. Moreover, f is surjective and six-to-one except for $\lambda = 0$ and $\lambda = 27$, where it is two-to-one and three-one-one respectively [26, II, Proposition 1.7]. These correspond to a unique stable orbit.

We know that the stable degree d hypersurfaces in $\mathbb{P}^1$ are those for which all the roots of the defining polynomial F has multiplicities strictly less than 2, that means F has only simple roots. This case is well understood. We may turn our attention to the case of strictly semistable hypersurfaces. This is completely characterised by hypersurfaces whose corresponding binary quartics F has atmost two repeated roots. As F has degree 4, we may distinguish two cases: F has one double root, or two distinct double roots. The SL_2-orbit of a double root contains only double roots and the orbit of simple root will contain only the simple root. So, we have two distinct orbits for the SL_2-action. The orbit consisting of one double root is not closed. In fact, its closure contains the orbits of points with two double roots. For instance, by considering a family of Möbius transformations h_t sending (p, q, q, r) to $(0, 1, 1, t)$, then as $t \longrightarrow 0$, the point $(0, 1, 1, 0)$ lies in this orbit closure. Hence the GIT quotient $V_{4,1}//\mathrm{SL}(2)$ is $\mathbb{P}^1$, and the good quotient is $\varphi : V^{ss}_{4,1} \longrightarrow \mathbb{P}^1$.

7.4 Plane cubics

The degree 3 hypersurfaces in $\mathbb{P}^2$ are also called plane cubic curves. The moduli problem for smooth plane cubic curves over the complex numbers has been studied in Chapter 4 using the representability of the moduli functor. In this section section, we study again this moduli problem in the GIT setting. Namely, we will describe the set of the stable, semistable points, the GIT quotient, and the geometric quotient. Homogeneous degree 3 polynomial $F \in \mathrm{k}[x, y, z]_3$ can be written as

$$F(x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x^{3-i-j} y^i z^j.$$

Let us consider the moduli problem of cubic plane curves that consists of classifying these curves up to projective equivalence. In other words, we would like to describe the quotient for the SL_3-action on $V_{3,2}$. We assume that k is a field of characteristic different from 2 or 3. Let us first recall the Bézout's theorem which will be useful at least once.

Theorem 7.4.1 (Bézout's Theorem)**.** Suppose two projective curves C and C' in $\mathbb{P}^2(\mathrm{k})$ of degree m, n have no common component. Then, they have precisely nm points of intersection counting multiplicities.

Proposition 7.4.2. [13, Proposition 7.16]

(1) Any non-singular projective plane curve $C \subset \mathbb{P}^2$ is irreducible

(2) Any irreducible projective plane curve $C \subset \mathbb{P}^2$ has at most finitely many singular points.

Proof. (1) Let C be a non-singular projective plane curve defined by the equation $P(x, y, z) = 0$. By contradiction, let us assume that there exist two polynomials $F, G \in \mathrm{k}[x, y, z]$ fo degree $\deg F \geq 1$, and $\deg G \geq 1$ such that $P = FG$. By Bezout's theorem, F and G have at least a common zero. Let $p = [a, b, c]$ be a common zero of F and G. Then $P(p) = \frac{\partial P}{\partial x}(p) = \frac{\partial P}{\partial y}(p) = \frac{\partial P}{\partial z}(p) = 0$. This contradicts the non-singularity of C.

(2) Let C be an irreducible projective plane curve defined by the equation $P(x, y, z) = 0$, where P is an irreducible polynomial. By changing coordinates, we may assume that $[1, 0, 0] \notin C$ so that the partial derivative $Q = \frac{\partial P}{\partial x}$ has degree $n - 1$. P and Q have no common factor, otherwise it will contradict the irreducibility of P. By Bezout's theorem, there are at most $n(n - 1)$ common factors of $P$and Q. So there are at most $n(n - 1)$ singular points in the curve C. This completes the proof.

□

An application of Bézout's Theorem allows us to classify the irreducible plane conics as follows.

Lemma 7.4.3. [13, Lemma 7.17] Any irreducible plane conic $C \subset \mathbb{P}^2$ is projectively equivalent to the conic defined by $x^2 + yz = 0$. Thus, it is nonsingular and isomorphic to $\mathbb{P}^1$.

Proof. Let C be an irreducible projective plane conic defined by the equation $P(x, y, z) = 0$. By Proposition 7.4.2, there are only finitely many singular points on the curve C. Now we choose a cooordinate system in such a way that $[0, 1, 0] \in C$ is smooth and the tangent line is defined by the equation $z = 0$. Hence, the curve C is defined by an equation of the form

$$P(x, y, z) = ayz + bx^2 + cxz + dz^2.$$

Now substitue $[x', y', z'] = [\sqrt{b}x, ay + cx + dz, -z]$ and obtain $P(x', y', z') = x'^2 + y'z'$ as desired.

For the second half of the lemma, we assume that C is defined by $x^2 + yz = 0$. We define an isomorphism $\varphi : C \longrightarrow \mathbb{P}^1$ such that

$$\varphi([x, y, z]) = \begin{cases} [x, y] & \text{if } y \neq 0 \\ [-z, x] & \text{if } z \neq 0, \end{cases}$$

and the inverse being the map $\mathbb{P}^1 \longrightarrow C$ to each $[x, y] \longmapsto [xy, y^2, x^2]$. □

We can use this result to classify reducible plane cubics up to projective equivalence. In fact, if C is a reducible plane cubic defined by a homogeneous degree 3 polynomial F, it follows that there are only two cases: either C is a union of a line and an irreducible conic or a union of three lines. In the first case, that is when C is a union of a line and an irreducible conic, then the set of the intersection of the line and the irreducible conic either contains two points or one point. If the set of intersection is reduced to a single point, then the line is tangent to the conic. By Lemma 7.4.3 we have that C is projectively equivalent to $y^2 + xz = 0$.

Lemma 7.4.4. [13, p. 56] Let C be a reducible plane cubic in $\mathbb{P}^2$. Suppose that C is a union of a conic and a line. Then C is projectively equivalent to one of the following: $(xz + y^2)y = 0$ or $(xz + y^2)z = 0$.

Lemma 7.4.5. [13, p. 56] Let C be a reducible plane cubic in $\mathbb{P}^2$. Suppose that C is a union of three lines. Then C is projectively equivalent to one of the following: $y^3 = 0$, or $y^2(y + z) = 0$, or $yz(y + z) = 0$, or $xyz = 0$.

7.4.6 Stability of cubics

Let us turn our attention to the analysis of the stability and the semistability of the plane cubics. As $d = 3$ is odd, we know that $V^{ss}_{3,2} = V^{s}_{3,2}$. As usual, we want to use the Hilbert-Mumford criterion to decribe the stable cubics. Let λ' be a 1-PS of SL_3. Then, by Proposition 7.2.7, there exists $g \in \mathrm{SL}_3$ such that $\lambda' = g^{-1}\lambda g$, where

$$\lambda(t) = \begin{pmatrix} t^{r_0} & 0 & 0 \\ 0 & t^{r_1} & 0 \\ 0 & 0 & t^{r_2} \end{pmatrix},$$

with $r_0 + r_1 + r_2 = 0$ and $r_0 \geq r_1 \geq r_2 = 0$. Let $V(F)$ be the hypersurface of degree 3 defined by

$$F(x, y, z) = \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} x^{3-i-j} y^i z^j.$$

By the action of the 1-PS λ, we have by definition:

$$\begin{aligned} \lambda(t) \cdot F(x, y, z) = F(t^{-r_0}x, t^{-r_1}y, t^{-r_2}z) &= \sum_{i=0}^{3} \sum_{j=0}^{3-i} a_{ij} (t^{-r_0}x)^{3-i-j} (t^{-r_1}y)^i (t^{-r_2}z)^j \\ &= \sum_{i=0}^{3} \sum_{j=0}^{3-i} t^{-(r_0(3-i-j)+r_1 i + r_2 j)} a_{ij} x^{3-i-j} y^i z^j. \end{aligned}$$

Hence,

$$\begin{aligned}\mu(F,\lambda) &= -\min\{-(3-i-j)r_0 - ir_1 - jr_2 : a_{ij} \neq 0\} \\ &= \max\{(3-i-j)r_0 + ir_1 + jr_2 : a_{ij} \neq 0\}.\end{aligned}$$

Lemma 7.4.7. [13, Lemma 7.25] Let $C \subset \mathbb{P}^2$ be a plane cubic. Then C is semistable if and only if it has no triple point and no double point with a unique tangent. Moreover, C is stable if and only if it is smooth.

Strictly semistable orbits are the following: nodal irreducible cubics, cubics which are a union of a conic and a non-tangential line, and cubics which are the union of three non-concurrent lines [13].

Unstable orbits are: cuspidal cubic curves, cubic curves that are union of three lines with a common intersction, cubic curves that are the union of a tangent line and a conic, cubic curves that are the union of a double line with distinct line and cubic curves given by triple lines [13].

It follows from the lemma above that the geometric quotient of smooth cubic curves in $\mathbb{P}^2$ parameterizes the isomorphism classes of smooth cubic curves. As in Chapter 5.4.8, the j-line $\mathbb{A}^1_j$ is the geometric quotient.

Conclusion

In this thesis, we were concerned with the construction of moduli spaces, which are important geometric objects in Algebraic Geometry. We first define rigorously the notion of moduli problems. From a moduli problem, one is able to define a functor called the moduli functor associated with this moduli problem. Then the study of a moduli problem reduces to the representability of the moduli functor. When the moduli functor is representable, we call its representing object a fine moduli space. A fine moduli space does not always exist, but when it exists it is unique up to unique isomorphism. Even when the moduli functor is not representable, we can still get a satisfactory parametrization, either using the notion of algebraic stacks, either rigifying the problem, or looking for a coarse moduli space which a notion weaker than that of fine moduli space, but still gets the job done. We then study the Grassmannians as an example of fine moduli space. They generalize the notion projective spaces. However, the moduli problem of nonsingular projective curves of fixed genus g is not representable, but admits a coarse moduli space. The obstruction to the representability is due to the presence of curves with nontrivial automorphism group. In general, objects having nontrivial automorphism group cannot admit a fine moduli space.

Constructing a moduli space is not an easy thask. However, there is a satisfactory theory, called Geometric Invariant Theory, developed by Mumford [4] that deals with the construction of moduli spaces as quotients. So, the study of moduli problems can be reduced to the computation of an orbit space as proved in Theorem 5.4.8. GIT provides an elegant way to compute quotients as illustrated in Theorem 5.6.1 for the affine case, and Theorem 5.8.5. The Hilbert-Mumford criterion is the main tool for the analysis of the stability, and we can use it to determine the objects we have to include in the class of our objects in order to get a suitable moduli space.

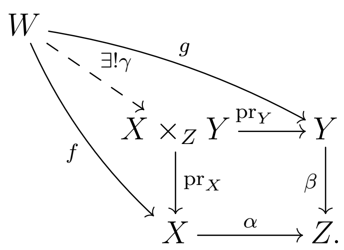
Appendix A

Basic Notions

The notion of fibered product is capital in the formulation of a moduli problem, this is because a moduli problem consists of the data of a class of objects of certain type along with an equivalence, and a notion of family of objects over a base scheme which is defined using the fibered product. The main goal here is to define the notions of fibered product and representable functor wich were used many times in Chapter 2 and Chapter 3.

A.1 Fibered product

Let us recall briefly the definition of pullback, also known as fibered product or cartesian product or a square. Suppose we are given two morphisms $\alpha : X \longrightarrow Z$ and $\beta : Y \longrightarrow Z$ (in any category). Then a fibered product of α and β, if it exists, is an object denoted $X \times_Z Y$, along with morphisms $\mathrm{pr}_X : X \times_Z Y \longrightarrow X$ and $\mathrm{pr}_Y : X \times_Z Y \longrightarrow Y$ (called the projections on X and Y respectively), where the two compositions $\alpha \circ \mathrm{pr}_X, \beta \circ \mathrm{pr}_Y : X \times_Z Y \longrightarrow Z$ agree, such that given any object W with maps to X and Y (whose compositions to Z agree), these maps factor through some unique $W \longrightarrow X \times_Z Y$:



In other words: a fibered product of two morphisms $\alpha : X \longrightarrow Y$ and $\beta : Y \longrightarrow Z$ is the triplet $(X \times_Z Y \longrightarrow Y,\ \ \mathrm{pr}_X : X \times_Z Y \longrightarrow X,\ \ \mathrm{pr}_Y : X \times_Z Y \longrightarrow Y)$ satisfying

$$\alpha \circ \mathrm{pr}_X = \beta \circ \mathrm{pr}_Y,$$

and that satisfies the universal property: for any object $W \in \mathcal{C}$, and for any morphisms $f : W \longrightarrow X$ and $g : W \longrightarrow Y$ satisfying

$$\alpha \circ f = \beta \circ g,$$

there exists a unique morphism $\gamma : W \longrightarrow X \times_Z Y$ such that

$$f = \mathrm{pr}_X \circ \gamma \qquad \text{and } g = \mathrm{pr}_Y \circ \gamma.$$

Note that the definition of the fibered product depends on α and β, even though they are omitted from the notation $X \times_Z Y$.

As a fibered product is defined by the universal property, one easily shows that, when a fibered product exists, it is unique to unique isomorphism.

Example A.1.1. In the category Set,

$$X \times_Z Y = \{(x, y) \in X \times Y : \alpha(x) = \beta(y)\}.$$

Definition A.1.2. We say that the category $\mathcal{C}$ has fibered products if the fibered product exists for any $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X, Z)$ and $\beta \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.

The following result shows that the category of S-schemes Sch/S has fibered products.

Theorem A.1.3. [11, Theorem 3.3, p.87] For any two schemes $X \longrightarrow S$ and $Y \longrightarrow S$ over a scheme S, the fibered product $(X \times_S Y, pr_X, pr_Y)$ exists, and is unique up to unique isomorphism.

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{pr_Y} & Y \\ {\scriptstyle pr_X}\downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

Definition A.1.4. Let $\mathcal{C}$ be any category, and $X, Y \in \mathcal{C}$. We say that a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is representable if for all $Z \in \mathcal{C}$ and all morphisms $g : Z \longrightarrow Y$ in $\mathcal{C}$ the fibered product $X \times_Y Z$ exists in $\mathcal{C}$. That is the diagram below commutes for all $Z \in \mathcal{C}$.

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{pr_Z} & Z \\ {\scriptstyle pr_X}\downarrow & & \downarrow{\scriptstyle \forall g} \\ X & \xrightarrow[f]{} & Y \end{array}$$

Lemma A.1.5. [14] Let $\mathcal{C}$ be a category. Let $f : X \longrightarrow Y$, and $g : Y \longrightarrow Z$ be representable. Then $g \circ f : X \longrightarrow Z$ is representable.

Definition A.1.6. [10] Let $\mathrm{F} : \mathcal{C}^{opp} \longrightarrow \mathrm{Set}$ be a functor. We say that F is representable if there exists $X \in \mathcal{C}$ such that F is isomorphic to the functor of points h_X; i.e., if there exists an isomorphism $\xi : \mathrm{h}_X \longrightarrow \mathrm{F}$ of functors.

If $\xi : \mathrm{h}_X \longrightarrow \mathrm{F}$ and $\xi : \mathrm{h}_{X'} \longrightarrow \mathrm{F}$ are two such isomorphisms, then by Yoneda's lemma, there exists an isomorphism $u : X \longrightarrow X'$. This proves that the representing pair (X, ξ), when it exists is unique up to unique isomorphism.

A.2 Fibered product of functors

Let $F, G, H \in \mathrm{Fct}(\mathcal{C}^{opp}, \mathrm{Set})$, and let $F \longrightarrow H$ and $G \longrightarrow H$ be morphisms of functors. Let $T \in \mathcal{C}$. Then $F(T), H(T), G(T)$ are objects in Set, and so the fibred product

$F(T) \times_{H(T)} G(T)$ makes sense in the category Set. This suggests that by setting

$$(F \times_H G)(T) := F(T) \times_{H(T)} G(T), \tag{A.1}$$

we obtain a well-defined notion of fibred product in the category $\mathrm{Fct}(\mathcal{C}^{opp}, \mathrm{Set})$. In fact for all $T \in \mathcal{T}$,

$$\begin{array}{ccc} F(T) \times_{H(T)} G(T) & \xrightarrow{pr_{G(T)}} & G(T) \\ {\scriptstyle pr_{F(T)}}\downarrow & & \downarrow \\ F(T) & \longrightarrow & H(T) \end{array}$$

the diagram above commutes, showing that $(F \times_H G)(T)$ is functorial with respect to T. Therefore, there is a well-defined functor $F \times_H G \in \mathrm{Fct}(\mathcal{C}^{opp}, \mathrm{Set})$, along with projection morphisms of functors $pr_F : F \times_H G \longrightarrow F$ and $pr_G : F \times_H G \longrightarrow G$. The tuple $(F \times_H G \longrightarrow F, pr_F, pr_G)$ is called the fibered product of the functors F and G over the functor H.

If F, G, and H are representable, say $\mathrm{F} \simeq \mathrm{h}_X, \mathrm{G} \simeq \mathrm{h}_Y$ and $\mathrm{H} \simeq \mathrm{h}_S$ the fiber product $\mathrm{F} \times_{\mathrm{H}} \mathrm{G}$ in $\mathrm{Fct}(\mathcal{C}^{opp}, \mathrm{Set})$ is representable by an object Z if and only if $X \times_S Y$ exists in $\mathcal{C}$ and in this case $Z = X \times_S Y$ [10].

A.3 Representable morphisms of functors

Let $\mathrm{F}, \mathrm{G} : (\mathrm{Sch/k})^{opp} \longrightarrow \mathrm{Set}$ be two functors, and let $f : \mathrm{F} \longrightarrow \mathrm{G}$ be a morphism of functors. Let **P** be a property of morphisms of schemes. The assertion f possesses the property **P** can make sense.

Definition A.3.1. Let $f : \mathrm{F} \longrightarrow \mathrm{G}$ be a morphism of functors in $\mathrm{Fct}((\mathrm{Sch/k})^{opp}, \mathrm{Set})$. We say that $f : F \longrightarrow \mathrm{G}$ is representable if for every $X \in \mathrm{Sch/k}$ and every $g : \mathrm{h}_X \longrightarrow \mathrm{G}$ in $\mathrm{Fct}((\mathrm{Sch/k})^{opp}, \mathrm{Set})$ the fibered product functor $\mathrm{F} \times_{\mathrm{G}} \mathrm{h}_X$ is represented by some $Z \in \mathrm{Sch/k}$.

Consider a scheme Z representing f, that is, there is an isomorphism $\zeta : \mathrm{h}_Z \longrightarrow \mathrm{F} \times_{\mathrm{G}} \mathrm{h}_X$. By composing with the projection $\mathrm{F} \times_{\mathrm{G}} \mathrm{h}_X \longrightarrow \mathrm{h}_X$, it follows from Yoneda's lemma, there is unique morphism $Z \longrightarrow X$ independent with respect to the choice of (Z, ζ) up to unique isomorphism.

We recall that a presheaf $mathrmF : \mathrm{Sch} \longrightarrow \mathrm{Set}$ is said to be a *sheaf* in the *Zariski topology* if for every scheme S and Zariski cover $\{S_i\}$ of S, the natural map

$$\{f \in \mathrm{F}(S)\} \quad \longrightarrow \quad \{(f_i \in \mathrm{F}(S_i)) : f_{i_{|S_i \cap S_j}} = f_{j|S_i \cap S_j} \qquad \text{for all } i, j\}$$

is a bijection. A presheaf is called a *separated presheaf* if these natural maps are injective.

Proposition A.3.2. The functor of points of a scheme is a sheaf in the Zariski topology.

In particular, a necessary condition for a presheaf to be representable it is that it must be a sheaf in the Zariski topology.

Proposition A.3.3. Let be a moduli problem $\mathcal{M}$ and F its associated moduli functor. If F is representable, then F is a Zariski-sheaf.

Proof. Since $\mathcal{M}$ has a fine moduli space, for any scheme S we have $\mathrm{F}(S) = \mathrm{Hom}(S, M)$. Furthermore, morphisms of schemes are determined locally, and can be glued if they are given locally and are compatible on overlaps. $\square$

Bibliography

[1] T.E.V. Balaji. *An Introduction to Families, Deformations and Moduli*. Universitätsdrucke Göttingen. Universitätsverlag Göttingen, 2010. ISBN: 9783941875326. URL: https://books.google.it/books?id=Ypp2kZiNXXEC.

[2] Montserrat Teixidor I Bigas. "Vector Bundles on Curves". In: (2017), p. 36. URL: http://emerald.tufts.edu/~mteixido/files/vectbund.pdf.

[3] A. Borel. *Linear Algebraic Groups*. Graduate Texts in Mathematics. Springer New York, 2012. ISBN: 9781461209416. URL: https://books.google.it/books?id=MoLTBwAAQBAJ.

[4] John Fogarty David Mumford and Frances Kirwan. *Geometric Invariant Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer-Verlag Berlin Heidelberg, 1994. ISBN: 9783540569633. URL: http://www.springer.com/us/book/9783540569633.

[5] I. Dolgachev. *Lectures on Invariant Theory*. Lecture notes. Cambridge University Press, 2003. ISBN: 9780521525480. URL: https://books.google.it/books?id=B9S_0iPvfrUC.

[6] D. Eisenbud and J. Harris. *The Geometry of Schemes*. Graduate Texts in Mathematics. Springer, 2000. ISBN: 9780387986371. URL: https://books.google.co.za/books?id=BpphspzsasEC.

[7] William Fulton. "On the irreducibility of the moduli space of curves - Appendix to the paper of Harris and Mumford". In: *Inventiones Mathematicae* 67.1 (Feb. 1982), pp. 87–88. ISSN: 0020-9910. DOI: 10.1007/BF01393372.

[8] Andreas Gathmann. *Algebraic Geometry*. Class Notes. TU Kaiserslautern, 2014. URL: www.mathematik.uni-kl.de/agag/mitglieder/professoren/gathmann/notes/alggeom/.

[9] E. Girondo and G. González-Diez. *Introduction to Compact Riemann Surfaces and Dessins D'Enfants*. London Mathematical Society Student Texts. Cambridge University Press, 2012. ISBN: 9780521519632. URL: http://books.google.cm/books?id=2SoLO_jknL4C.

[10] Ulrich Görtz and Trosten Wedhorn. *Algebraic Geometry 1, Schemes With Examples and Exercises*. Springer Fachmedien Wiesbaden GmbH. Vieweg +Teubner Verlag, 2010. ISBN: 9783834806765.

[11] Hartshorne, R. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1977. ISBN: 9780387902449. URL: http://books.google.cm/books?id=3rtX9t-nnvwC.

[12] R. Hartshorne. *Deformation Theory*. Graduate Texts in Mathematics. Springer New York, 2009. ISBN: 9781441915955. URL: https://books.google.it/books?id=bwhEX01J1XkC.

[13] Victoria Hoskins. *Moduli Problem and Geometric Invariant Theory*. Lecture notes. Freie Universität, Berlin, 2015/2016.

[14] Johan De Jong. *Stacks Project*. Copyright (C) 2005 – 2016. Version 936b790, compiled on Apr 01, 2016.

[15] J. Kock and I. Vainsencher. *An Invitation to Quantum Cohomology: Kontsevich's Formula for Rational Plane Curves*. Progress in Mathematics. Birkhäuser Boston, 2007. ISBN: 9780817644956. URL: https://books.google.cm/books?id=NCNxbJ5_r9UC.

[16] V. Lakshmibai and J. Brown. *The Grassmannian Variety: Geometric and Representation-Theoretic Aspects*. Developments in Mathematics. Springer New York, 2015. ISBN: 9781493930821. URL: https://books.google.it/books?id=8AShCgAAQBAJ.

[17] Janina Letz. *Flag Varieties*. Lecture notes. October 26, 2015. URL: http://www.math.utah.edu/~goller/F15_SAGS/Janina_FlagVarieties.pdf.

[18] J.S. Milne. *Algebraic Groups*. e-book: www.jmilne.org/math/CourseNotes/iAG.pdf, Version 2.00, December 20, 2015.

[19] S. Mukai and W.M. Oxbury. *An Introduction to Invariants and Moduli*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2003. ISBN: 9780521809061. URL: https://books.google.co.za/books?id=oUcMZbiM7eAC.

[20] Masayoshi Nagata. *Lectures on the fourteenth problem of Hilbert*. Ed. by M. P. Murthy. Lectures on mathematics and physics. Mathematics. Bombay: Tata Institute of Fundamental Research, 1965. URL: http://ezfind.technion.ac.il/vufind/Record/002102071.

[21] P.E. Newstead. *Introduction to Moduli Problems and Orbit Spaces*. Lectures on Mathematics and Physics. Published for the TIFR (Tata Institute of Fundamental Research), 2012. ISBN: 9788184871623. URL: https://books.google.it/books?id=W7WYuAAACAAJ.

[22] P.E. Newstead. *Vector Bundles On Agebraic Curves*. Lecture notes. Luckecin, Poland, 9-14 September, 2002.

[23] Brian Osserman. *Limit linear series*. https://www.math.ucdavis.edu/ osserman/classes/280-W13/llsbook.pdf. URL: https://www.math.ucdavis.edu/~osserman/classes/280-W13/llsbook.pdf.

[24] Deligne Pierre and Mumford David. "The irreducibility of the space of curves of given genus". eng. In: *Publications Mathématiques de l'IHÉS* 36 (1969), pp. 75–109. URL: http://eudml.org/doc/103899.

[25] Alexander H. W. Schmitt. "Geometric Invariant Theory Relative to a Base Curve". In: *Algebraic Cycles, Sheaves, Shtukas, and Moduli: Impanga Lecture Notes*. Ed. by Piotr Pragacz. Basel: Birkhäuser Basel, 2008, pp. 131–183. ISBN: 978-3-7643-8537-8. DOI: 10.1007/978-3-7643-8537-8_8. URL: http://dx.doi.org/10.1007/978-3-7643-8537-8_8.

[26] J.H. Silverman. *The Arithmetic of Elliptic Curves*. Graduate Texts in Mathematics. Springer, 2009. ISBN: 9780387094946. URL: http://books.google.cm/books?id=Z90CA_EUCCkC.

[27] Ravi Vakil. *The Rising Sea, Foundations of Algebraic Geometry*. Lecture notes. December 29, 2015 draft. URL: https://math216.wordpress.com.

[28] Yihu. *Lectures On Quotients And Moduli Spaces*. Lecture notes. University of Arizona, 2011. URL: http://phyawd.msc.tsinghua.edu.cn/upload/news_2012529155417.pdf.