On the canonical key formula

on the algebraic stack of principally polarized abelian varieties with theta characteristic

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1 Introduction

In this work we present a recent result on the "canonical key formula" (1985, [19], Appendix 1) for abelian schemes with theta characteristic, that shows that the determinant bundle is a torsion element in the Picard group of the moduli stack of abelian varieties. In 1990, it has been improved showing that the order of the determinant bundle is exactly 4 ([4], Chapter I, Theorem 5.1). In this thesis, we will construct an invertible sheaf called the 'theta multiplier bundle', that we will show to be isomorphic to the determinant bundle. The isomorphism will be expressed over the moduli stack of abelian varieties with theta characteristic.
There are many aspects that show the importance of this result: the most important lies in the theory of theta functions. The latter has been developed in algebraic context by Mumford in the sixties in the celebrated papers [16] and [17]. There are many aspects still unknown. This result, for example, gives the first expression of the functional equation of the theta function in a completely algebraic language. We will not study this interpretation, but we will focus on the moduli problem.
The first two chapters are meant to introduce the notions that are needed to the present the construction of the theta multiplier bundle, thus we briefly discuss the theory of invertible sheaves on stacks and of abelian schemes. The third chapter is devoted to the construction of it, highlighting the natural properties, that allow to think of it as an invertible sheaf on the stack of principally polarized abelian varieties with theta characteristic. In the fourth chapter we give the proof of the main theorem, it relies on a technique of Mumford as explained in the ground-breaking paper [15].
This work is fruit of the last semester of the author's studies in Leiden. During this period, he has learnt a lot, but it was not possible to investigate all the aspects of this theory neither to give a fair explanation of all the machinery used in this thesis. This could be the subject for further studies.

2 Moduli problems and stacks

In the final chapter we prove that two invertible sheaves are isomorphic on the algebraic stack of principally polarized abelian varieties with theta characteristic. We devote this chapter to present the moduli problem in the special case of elliptic curves.

We start with the definition of family of elliptic curves:

Definition 1. A family of elliptic curves is a proper smooth morphism of schemes $E \to S$ of relative dimension 1, with geometrically connected fibers all of genus 1, together with a section $e : S \to E$.

The moduli problem (in the case of elliptic curves) consists in finding the universal object parametrizing all families of elliptic curves up to isomorphism. Precisely, the latter is the representing object (when it exists) of the functor:

$$\begin{aligned} F : S\text{-Sch} &\to \text{Sets} \\ (T \to S) &\mapsto \{\text{families of elliptic curves over } T\}/\cong . \end{aligned}$$

Since there exists a non-trivial isotrivial family of elliptic curves (e.g. [21] Example 2.3), there does not exist any fine moduli space. The problem arises from the fact that the objects that we are trying to parametrize have non-trivial automorphisms. A possible solution is to enlarge the category of schemes and consider not only presheaves with values in sets, but with values in groupoids. Thus, we are lead to the formalism of categories fibered in groupoids and one may restrict the attention to those that respect a sheaf condition for a Grothendieck topology over the category S-Sch; the latter are called stacks. They are quite abstract and they do not have many geometric properties, but one can recover some geometry looking at those special stacks called algebraic (for the precise definition that will be used later look at [3] Definition 4.6 p.98). For example, in the case of families of elliptic curves Mumford gave an informal description of the algebraic stack $\mathcal{M}_{1,1}$, that solves the moduli problem for elliptic curves in the sense that: for every scheme S and every morphism $S \to \mathcal{M}_{1,1}$ corresponds a unique family of elliptic curves over S.

The algebraic stack $\mathcal{M}_{1,1}$ can be described as the category whose objects are families of elliptic curves $E \to S$ and morphisms from $E \to S$ to $C \to T$, with sections $e : S \to E$ and $e' : T \to C$, are diagrams:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ S \arrow[r, "g"] & T \end{tikzcd}$$

which are cartesian and commute with the two sections, i.e. $f \circ e = e' \circ g$.

We are interested in some geometric properties of $\mathcal{M}_{1,1}$, in particular we want to study the group of invertible sheaves defined on $\mathcal{M}_{1,1}$, following the ideas in [15]. Thus, we come to the following definition:

Definition 2. An invertible sheaf $\mathcal{L}$ on $\mathcal{M}_{1,1}$ consists in an association $(E \to S) \mapsto \mathcal{L}(E \to S)$, where $\mathcal{L}(E \to S)$ is an invertible sheaf on S, and for a morphism F:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & C \arrow[d] \\ S \arrow[r, "g"] & T \end{tikzcd}$$

an isomorphism $\mathcal{L}(F) : \mathcal{L}(E \to S) \xrightarrow{\simeq} g^*\mathcal{L}(C \to T)$. Moreover for any two morphisms, F after G:

$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d] & E_2 \arrow[r, "f_2"] \arrow[d] & E_3 \arrow[d] \\ S_1 \arrow[r, "g_1"] & S_2 \arrow[r, "g_2"] & S_3 \end{tikzcd}$$

the data must satisfy the relation:

$$\mathcal{L}(F \circ G) = g_1^*\mathcal{L}(F) \circ \mathcal{L}(G).$$

Or equivalently the diagram:

$$\begin{tikzcd} \mathcal{L}(E_1 \to S_1) \arrow[r, "\mathcal{L}(G)"] \arrow[d, "\mathcal{L}(FG)"] & g_1^*\mathcal{L}(E_2 \to S_2) \arrow[d, "g_1^*\mathcal{L}(F)"] \\ (g_2g_1)^*\mathcal{L}(E_1 \to S_1) \arrow[r, equal] & g_1^*g_2^*\mathcal{L}(E_1 \to S_1) \end{tikzcd}$$

must commute.

There is a natural way to define morphisms between these objects, thus we can define the Picard group of $\mathcal{M}_{1,1}$ as the group of invertible sheaves up to isomorphism. It can be proved that $\mathcal{M}_{1,1}$ is an algebraic stack in the sense of Deligne-Mumford and that it is a geometrical object, very similar to a scheme, that best represents the space parametrizing the elliptic curves. The study of the Picard group of $\mathcal{M}_{1,1}$ helps then to understand the moduli problem of elliptic curves, because $\mathcal{M}_{1,1}$ is a universal object, thus any relation proved on $\mathcal{M}_{1,1}$ automatically follows for every family of elliptic curves.

3 Abelian schemes

First of all, we need to define the objects of our study, the families of abelian varieties; these are called abelian schemes. Then we will present some properties that will be useful later.

Definition 3. Let S be a scheme. An abelian scheme $A \xrightarrow{f} S$ of relative dimension g is a smooth, proper group scheme over S, such that the fibers are geometrically connected of dimension g.

The first important thing is that it is the correct definition in the sense that we may think of an abelian schemes as a family of abelian varieties. Indeed, for every point $s \in S$, every fiber $A \times \operatorname{Spec}(k(s))$ is an abelian variety; many of the properties of abelian scheme can be deduced by those of the abelian varieties. It can be proved that they are commutative group schemes (this follows from the rigidity lemma [14] Proposition 6.1), but in general they are not projective, in contrast with abelian varieties [6] Theorem 2.25.

Remark 4. *It is important to note that, by definition, an abelian scheme is locally of finite presentation (since it is smooth) and, since it is proper, it is actually of finite presentation.*

Example 5. The first examples of abelian schemes are given by families of elliptic curves: in particular one can prove that if $E \to S$ is a proper smooth curve with geometrically connected fibers all of genus 1, together with a section $e : S \to E$, then the scheme E is naturally endowed with a structure of group scheme over S, thus it is an abelian scheme of relative dimension 1. One can deduce the group structure, as in the classical case of an elliptic curve over a field k, finding a bijection with the connected component of the origin of the (relative) Picard group of $E \to S$; for details see [12] II.

Recall that for any morphism of schemes $f : X \to Y$ we can define the relative sheaf of differential forms $\Omega_{X/Y}$, whose formation is compatible under base change and compatible with taking products (e.g. [11] II.8), i.e. if $f : X \to S$ is a morphism of schemes, then:

- for any $S' \to S$ morphism of schemes, the canonical map

$$p^*\Omega_{X/S} \to \Omega_{X_{S'}/S'}$$

  is an isomorphism, where $p : X_{S'} \to X$ denotes the map given by definition of fiber product.

- for any $Y \to S$ morphism of schemes, the canonical map:

$$p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S} \to \Omega_{X \times Y/S}$$

  is an isomorphism, where $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the usual projections.

In the case of an abelian scheme, we have the pleasant result:

Theorem 6. *Let $f : A \to S$ be an abelian scheme of relative dimension g, let $\Omega_{A/S}$ be the sheaf of relative differential forms, then:*

- *$\Omega_{A/S}$ is a locally free $\mathcal{O}_A$-module of rank g*
- *there is a canonical isomorphism $f^*e^*\Omega_{A/S} \cong \Omega_{A/S}$.*

Proof. The first claim follows from the smoothness of f; while the isomorphism is obtained extending the invariant differentials, for more details see ([1] p. 102). $\square$

Remark 7. *In this case we see that $e^*\Omega_{A/S}$ is a locally free sheaf of rank g on S; we will denote with $\omega_{A/S}$ its determinant; its formation is still compatible under base change and with taking products. This is called the Hodge bundle and it is an invertible sheaf on S.*

3.1 Duals

The invertible sheaves play a prominent role in the study of families of abelian varieties and their geometry, hence the importance of the dual abelian scheme. The dual abelian scheme is defined as the connected component of the origin of the Picard scheme; the Picard functor is not always representable (the right context of representability are algebraic spaces, as Artin proved), but in case of abelian schemes the situation is simpler ([4], Chapter I, Theorem 1.9). In this chapter we highlight some properties of the dual abelian scheme, that will be useful later. We may assume, from general theory, the following result:

Theorem 8. *Let $A \to S$ be an abelian scheme of relative dimension g, then the relative Picard functor (in the Zariski topology) is representable by a group scheme $\mathrm{Pic}_{A/S}$ and the fiberwise-connected component of the unit section is an abelian scheme over S of relative dimension g. It will be called the dual abelian scheme and denoted with $A^\vee$.*

We start now our discussion, starting from the definition of the relative Picard functor: first of all, fix an S-scheme $f : X \to S$, then define the relative Picard functor:

$$\begin{aligned} \underline{\mathrm{Pic}}_{X/S} : \ & Sch_S \to Ab \\ & T \mapsto \mathrm{Pic}(X_T)/f_T^* \,\mathrm{Pic}(T) \end{aligned}$$

where X_T denotes the fiber product $X \times_S T$ and $f_T : X_T \to T$ is the natural map. On the morphisms it is defined via pullbacks. This functor is not a priori a sheaf, so it is better to consider its sheafification in a site (Zariski, étale or fppf). For example, the sheaf associated to it in the fppf-site can be described as functor of points in this way: for every S-scheme T an element of $\underline{\mathrm{Pic}}_{(X/S)(fppf)}(T)$ is represented by an invertible sheaf L defined over $X_{T'}$ for a fppf covering $T' \to T$.
Notation:
In the following, we will denote $\underline{\mathrm{Pic}}_{X/S}$ the relative Picard functor and by $\mathrm{Pic}_{X/S}$ the representing object, when it exists. Moreover, if $f : X \to S$ is a morphism of schemes, then the map $X_T := X \times_S T \to T$ induced by base change with any S-scheme T will be denoted by f_T. If $\mathcal{L}$ is a line bundle on X, then $\mathcal{L}_T$ will mean the line bundle induced on X_T via pullback along the canonical map $X_T \to X$.

3.2 Dual abelian scheme

In the case of abelian schemes, the Picard functor has a much simpler description. Indeed, we can work with invertible sheaves with extra rigidifications and we can use the identity section, provided by the structure of group scheme.

Definition 9. Let $f : A \to S$ be an abelian scheme and denote with $e : S \to A$ the identity section. Let $\mathcal{L}$ be an invertible sheaf defined over A. A rigidification of $\mathcal{L}$ is the choice of an isomorphism $\alpha : e^*\mathcal{L} \to \mathcal{O}_S$ (if it exists). A morphism between rigidified invertible sheaves $(\mathcal{L}, \alpha), (\mathcal{N}, \beta)$ is a morphism $\gamma : \mathcal{L} \to \mathcal{N}$ that respects rigidifications, i.e. $\beta \circ e^*\gamma = \alpha$.
The couples $(\mathcal{L}, \alpha)$ will be called normalized sheaves along the zero section.

Lemma 10. *Let $f : A \to S$ be an abelian scheme, then $f_*\mathcal{O}_A \cong \mathcal{O}_S$ holds universally, i.e for any S-scheme T $f_{T,*}\mathcal{O}_{A_T} \cong \mathcal{O}_T$.*

Proof. The hypotheses hold after any base change, so we are left to prove the relation for the scheme $A \to S$. Since f is finitely presented, we are reduced to consider S to be locally noetherian (by standard reduction argument, using [23], [TAG00F0]). Now, the thesis follows from Proposition 7.8.6 and Corollary 7.8.7 in [9]. □

Lemma 11. *Let $f : A \to S$ an abelian scheme, then:*

- *For any S-scheme T, the group of isomorphism classes $(\mathcal{L}, \alpha)$ of normalized invertible sheaves on A_T is isomorphic to* $\operatorname{Pic}(A_T)/f_T^* \operatorname{Pic}(T)$.
- *Let T be an S-scheme and $(\mathcal{L}, \alpha)$ be a normalized invertible sheaf on A_T, then every automorphism of $(\mathcal{L}, \alpha)$ is trivial.*

Proof. Thanks to the section $e : S \to A$, there is a natural way to associate to an invertible sheaf $\mathcal{L}$ a normalized one isomorphic to it:

$$\mathcal{L} \mapsto \mathcal{L} \otimes f_T^* e_T^* \mathcal{L}^{-1}.$$

One then verifies that it defines the isomorphism we were looking for. For details see ([24] Lemma 9.2.9 p. 255).
For the second point look at [24] Lemma 9.2.10 p.255. □

Theorem 12. *Let $f : A \to S$ be an abelian scheme, then the étale sheaf* $\operatorname{Pic}_{A/S}(et)$ *is canonically isomorphic to the Zariski sheaf* $\operatorname{Pic}_{A/S}(zar)$.
Moreover, for every S-scheme T we have

$$\operatorname{Pic}_{A/S(\acute{e}t)}(T) = \operatorname{Pic}(A_T)/f_T^* \operatorname{Pic}(T).$$

When the Picard scheme exists, that is the case for the abelian schemes as mentioned before, then one can prove:

Proposition 13. *Let $f : A \to S$ be an abelian scheme, then:*

- *the formation of the Picard scheme is stable under base change, i.e.:*

$$\operatorname{Pic}_{A/S} \times_S T = \operatorname{Pic}_{A_T/T}.$$

- *For any S-scheme T and for any invertible sheaf $\mathcal{L}$ on A_T there exists a unique closed subscheme $N \subset T$ and an invertible sheaf $\mathcal{N}$ on A_T, that satisfies the following properties: $\mathcal{L}_N \cong f_N^* \mathcal{N}$ and, for any $t : T' \to T$ such that $\mathcal{L}_{T'} \cong f_T^* \mathcal{N}'$ for some invertible sheaf $\mathcal{N}'$ on T', then t factors through N and $\mathcal{N}' \cong t^* \mathcal{N}$. These properties actually determine N uniquely and $\mathcal{N}$ up to isomorphism.*

Proof. The first claim follows just by comparing formally the T-points of the two functors. For the second one, consider the invertible sheaf O_A, it induces a section for the Picard scheme $s : S \to \mathrm{Pic}_{A/S}$. Indeed this is, by Yoneda, the zero map for the structure of group on $\mathrm{Pic}_{A/S}$. Call I the schematic image of s, set N as the inverse image of I via t', where t' classifies $\mathcal{L}$. If t' classifies $\mathcal{L}$, then we see that $\mathcal{L}_{\mathcal{N}}$ is classified by t'_N, but this map factors though N, thus it induces on A_N the invertible sheaf $\mathcal{O}_{A_N}$. This means that $\mathcal{L}_{\mathcal{N}}$ corresponds to the same class of $\mathcal{O}_{A_N}$ in $\underline{\mathrm{Pic}}_{A/S}$, i.e. that is trivial. Therefore, thanks to Theorem 12 we deduce the existence of such a $\mathcal{N}$. □

Remark 14. *When the Picard scheme exists, there exists a universal invertible sheaf defined on $A \times \mathrm{Pic}_{A/S}$: namely it is the one that corresponds to the identity of the Picard scheme. Actually this defines just a class in* $\mathrm{Pic}(A \times_S \mathrm{Pic}_{A/S})/f^* \mathrm{Pic}(\mathrm{Pic}_{A/S})$. *We can recover a unique invertible sheaf working with normalized invertible sheaf. The invertible sheaf that we get is called Poincaré bundle.*

Thus for an abelian scheme the Picard scheme exists, thus one can define $\mathrm{Pic}^0_{A/S}$ as the subgroup scheme obtained as union of the connected components of the unit element of $\mathrm{Pic}^0_{A_s/k(s)}$ for every $s \in S$. It can be shown that taking the zero connected component is stable under field extension. Therefore, at the end, one can associate to an abelian scheme $A \to S$ the group scheme $\mathrm{Pic}^0_{A/S}$ whose formation is stable under base change, in the sense that $\mathrm{Pic}^0_{A/S} \times_S T \cong \mathrm{Pic}^0_{A_T/T}$ for any S-scheme T. This is called the dual abelian scheme and it is denoted $A^\vee$: it is an abelian scheme over S.

Remark 15. *Actually, one usually defines the torsion component of the identity for the Picard scheme, i.e.* $\mathrm{Pic}^\tau_{A/S} := \bigcup_n [n]^{-1} \mathrm{Pic}^0_{A/S}$, *and gets all the results above for* $\mathrm{Pic}^\tau_{A/S}$. *Nevertheless, in the case of an abelian scheme* $\mathrm{Pic}^\tau_{A/S}$ *and* $\mathrm{Pic}^0_{A/S}$ *coincide, indeed one can check on the geometric fibers and prove the result for abelian varieties ([6] Corollary 7.25 p. 107).*

3.3 Polarizations from invertible sheaves

For any invertible sheaf $\mathcal{L}$ on A, set $\Lambda(\mathcal{L}) := m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1}$ on $A \times_S A$, where $p_i : A \times_S A \to A$ denote the projections on the first ($i = 1$) and on the second ($i = 2$) factor. If we think of $A \times_S A$ as an A-scheme via the second projection p_2, then, thanks to the universal property of $\mathrm{Pic}_{A/S}$, $\Lambda(\mathcal{L})$ defines a unique morphism $\varphi_{\mathcal{L}} : A \to \mathrm{Pic}_{A/S}$ such that the class of the Poincaré bundle corresponds via pullback along $(1, \varphi_{\mathcal{L}})$ to the class $\Lambda(\mathcal{L})$. On T-points we can describe this map as:

$$\begin{aligned} \varphi_{\mathcal{L}}(T) : A(T) &\to \mathrm{Pic}_{A/S}(T) \\ x &\mapsto \left[t_x^*\mathcal{L}_T \otimes \mathcal{L}_T^{-1}\right] \end{aligned}$$

where t_x^* denotes the translation by x and $\mathcal{L}_T$ the invertible sheaf induced by pullback on A_T. In particular, if we consider the special S-point $e : S \to A$ given by the group structure on A, we have

$$\varphi_{\mathcal{L}_T}(T) : e_T \mapsto \left[t_{e_T}^* \mathcal{L}_T \otimes \mathcal{L}_T^{-1}\right] = [\mathcal{O}_T]$$

thus we conclude that it is a morphism of groups, as follows from the rigidity lemma ([14] Corollary 6.4 p.117). Moreover, since the fibers are connected (by definition of abelian scheme) we deduce that $\varphi_{\mathcal{L}}$ factors through the dual abelian scheme $A^\vee$.

Definition 16. We will denote with $\mathcal{K}(\mathcal{L})$ the kernel of $\varphi_{\mathcal{L}}$.

Remark 17. *The kernel $\mathcal{K}(\mathcal{L})$ has been described already in Proposition 13: it is indeed the maximal subscheme of A, such that the restriction $\mathcal{L}|_{A\times_S\mathcal{K}(\mathcal{L})}$ is trivial as element in $\mathrm{Pic}(A\times_S\mathcal{K}(\mathcal{L}))/p_2^*\mathcal{K}(\mathcal{L})$. Thus, it is clear that its formation is stable under base change, i.e.:*
let $A \to S$ be an abelian scheme and $\mathcal{L}$ be an invertible sheaf on A, then for any $t : T \to S$ one has $\mathcal{K}(\mathcal{L}_T) \cong \mathcal{K}(\mathcal{L}) \times_S T$, where t' is the natural map $A_T \to A$.

Moreover, from the construction it easily follows, that, for any two invertible sheaves $\mathcal{L}, \mathcal{N}$ defined on A:

$$\varphi_{\mathcal{L}\otimes\mathcal{N}} = \varphi_{\mathcal{L}} + \varphi_{\mathcal{N}}.$$

Theorem 18. *Let $f : A \to S$ be an abelian scheme and $\mathcal{L}$ be a relatively ample invertible sheaf on A, then:*

1 $R^i f_*(\mathcal{L}) = 0$, *if* $i > 0$.

2 $f_*\mathcal{L}$ *is a locally free sheaf on* S, *say* r *its rank.*

3 $\varphi_{\mathcal{L}} : A \to A^\vee$ *is finite and flat, its degree is* r^2.

4 The formation of $f_*\mathcal{L}$ *is compatible under base change.*

Proof. [14] Proposition 6.13 p.123, for the $4th$ point [19], Corollaire 3.5.3 p.144. □

4 Constructions

In this chapter we will construct two special invertible sheaves for some abelian schemes with an extra structure; here is the definition we need:

Definition 19. An abelian scheme with theta characteristic is a pair $(A \to S, \Theta)$, where $A \to S$ is an abelian scheme and Θ is a relatively ample, normalized along the zero section, symmetric, invertible sheaf of degree 1.

Proposition 20. *Given two abelian schemes with theta characteristic* $(f_A : A \to S, \Theta_A)$, $(f_B : B \to T, \Theta_B)$, *the pair* $((f_A \times f_B) : A \times B \to S \times T, \Theta_A \boxtimes \Theta_B)$ *is naturally an abelian scheme with theta characteristic.*

Proof. To see that it is an abelian scheme is sufficient to notice that $A \times B$ is isomorphic to $(A \times T) \times_{S \times T} (B \times S)$ as scheme over $S \times T$. From this, we see that the structure of group scheme is given by taking the products of the maps (e.g. the unit section is $e_{A \times B} := e_A \times e_B$, where e_A, e_B are the unit sections for A and B). Thus, it is clear that $\Theta_A \boxtimes \Theta_B$ is symmetric and normalized. Further, $\Theta_A \boxtimes \Theta_B$ is ample since it holds on geometric fibers; to see that it has degree 1, one applies the cohomology and base change theorem combined with the Künneth formula ([23], TAGOBED). □

4.1 Determinant bundle

Definition 21. For every abelian scheme $f : A \to S$ and invertible sheaf Θ that is symmetric, relatively ample, normalized along the zero section and of degree 1, set $\Delta(\Theta) := (f_*\Theta)^{\otimes 2} \otimes \omega_{A/S}$. It is called the determinant line bundle.

Remark 22. *Thanks to Theorem 18,* $\Delta(\Theta)$ *is an invertible sheaf over* S.

Theorem 23. *The invertible sheaf* $\Delta(\Theta)$ *lies in* $\mathrm{Pic}(S)\,[4]$, *the 4- torsion subgroup of the Picard group of* S, *i.e.* $\Delta(\Theta)^{\otimes 4} \cong \mathcal{O}_S$.

Proof. See [4] Chapter I, Theorem 5.1. □

Proposition 24. *Moreover the formation of the determinant bundle is*

- *compatible with taking products, i.e. for any two pairs* $(A \to S, \Theta_1), (B \to T, \Theta_2)$ *there is a canonical isomorphism of invertible sheaves* $\Delta(\Theta_1 \boxtimes \Theta_2) \cong \Delta(\Theta_1) \boxtimes \Delta(\Theta_2)$
- *compatible under base change, i.e. for two pairs* $(A \to S, \Theta_1)$, $(B \to T, \Theta_2)$ *and a diagram:*

$$\begin{tikzcd} A \overset{\psi}{\cong} B_S \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d] \\ S \arrow[r, "h"] & T \end{tikzcd}$$

where ψ *is an isomorphism of abelian schemes, together with an isomorphism of invertible sheaves* $\Theta_1 \cong \psi^* g^* \Theta_2$*; then there is a canonical isomorphism* $\Delta(\Theta_1) \cong h^* \Delta(\Theta_2)$.

Proof. The first claim comes from the Künneth formula, while the second comes from Theorem 18 point 4 and the good properties of the Hodge bundle (Theorem 6). □

4.2 Again on invertible sheaves

In order to introduce the theta multiplier bundle we need to study in more detail the information carried by an invertible sheaf $\mathcal{L}$ on an abelian scheme. In this section we work over the category of $Sch_{\mathbb{Z}[1/2]}$, this assumption is crucial to deal with the 2-torsion of abelian schemes.

Consider an abelian scheme $A \to S$ of relative dimension g with a relatively ample, invertible sheaf $\mathcal{L}$ of degree d. We have seen how to associate to this a morphism $\varphi_{\mathcal{L}} : A \to A^\vee$ and its kernel $\mathcal{K}(\mathcal{L})$, a finite flat group scheme of rank d^2. Define $\mathcal{G}(\mathcal{L})$ as the group functor that to every S-scheme T associates:

$$\mathcal{G}(\mathcal{L})(T) := \{(x, \psi) : x \in A(T), \psi : t_x^* \mathcal{L}_T \xrightarrow{\sim} \mathcal{L}_T\}.$$

Recalling the description of $\mathcal{K}(\mathcal{L})$(Definition 16), there is a natural morphism of group functors described on points as:

$$\mathcal{G}(\mathcal{L})(T) \to \mathcal{K}(\mathcal{L})(T)$$
$$(x, \psi) \mapsto (x)$$

The kernel of this can be clearly identified with $\mathbb{G}_m$, with its natural embedding in the automorphism group of the invertible sheaf $\mathcal{L}$. Thus, we have the following exact sequence:

$$0 \to \mathbb{G}_m \to \mathcal{G}(\mathcal{L}) \to \mathcal{K}(\mathcal{L}) \to 0$$

In general $\mathcal{G}(\mathcal{L})$ is not commutative and we can consider the commutator pairing $\mathcal{G}(\mathcal{L}) \times \mathcal{G}(\mathcal{L}) \to \mathcal{G}(\mathcal{L})$, defined on points as $(x, y) \to (xyx^{-1}y^{-1})$. Since $\mathbb{G}_m$ is central in $\mathcal{G}(\mathcal{L})$ and $\mathcal{K}(\mathcal{L})$ is commutative, this defines a pairing $e_{\mathcal{L}} : \mathcal{K}(\mathcal{L}) \times \mathcal{K}(\mathcal{L}) \to \mathbb{G}_m$. In particular, one can prove:

Theorem 25. *The form $e_{\mathcal{L}}$ is a symplectic pairing.*

Proof. For a proof see [16], Theorem 1, p.293. □

Further, consider $\mathcal{L}$ to be symmetric and normalized so that there is a unique isomorphism $[-1]^* \mathcal{L} \cong \mathcal{L}$ of normalized invertible sheaves. Then, for every point $x \in A[2]$, we have $-x = x$, thus we can consider the automorphism of $\mathcal{L}_x$ given by:

$$\mathcal{L}_x = \mathcal{L}_{-x} = \left([-1]^* \mathcal{L}\right)_x \cong \mathcal{L}_x$$

that can be identified with an element in $\mathbb{G}_m$. Actually, the latter lies in μ_2, indeed

$$[-1]^* \psi \circ \psi = id$$

Thus, for any normalized, symmetric invertible sheaf $\mathcal{L}$, this can be used to define a map $e_*^{\mathcal{L}} : A[2] \to \mu_2$. Here is the result that establishes a relation with the previous pairing:

Theorem 26. *The function $e_*^{\mathcal{L}}$ is quadratic for the symplectic pairing $e_{\mathcal{L}^2}$ restricted to $A[2] \times A[2]$, i.e. :*

$$e_*^{\mathcal{L}}(x + y) = e_*^{\mathcal{L}}(x) e_*^{\mathcal{L}}(y) e_{\mathcal{L}^2}(x, y)$$

for every x, y in $A[2]$.

Proof. For a proof see [16], Corollary 1, p.314. □

4.3 Theta multiplier bundle, part I

We need an important result proved in [2], where the author constructs a special group morphism, say λ, from the automorphism group of a symplectic 4-group with theta characteristic to the cyclic group with four elements (see Theorem 29, below, for the precise statement). In the following, we limit ourselves to introduce the notions we need to state the theorem, then we report some properties of λ (see Propositions 30, 31 below), that will be useful later. All the proofs can be found in [2], Section 2.

Definition 27.

- A 4-group with theta characteristic of rank $2g$ is a triple (V, ψ, q), where V is a free $\mathbb{Z}/4\mathbb{Z}$-module of rank $2g$, $\psi : V \times V \to \mathbb{Z}/4\mathbb{Z}$ is a non-degenerate alternating bilinear form and $q : \bar{V} := V/2V \to \mathbb{Z}/2\mathbb{Z}$ a quadratic form for $\bar{\psi} := 2\psi : \bar{V} \times \bar{V} \to \mathbb{Z}/2\mathbb{Z}$ induced by ψ, i.e.
$$q(v_1 + v_2) - q(v_1) - q(v_2) = \bar{\psi}(v_1, v_2)$$
for every $v_1, v_2 \in \bar{V}$.
- Given a 4-group with theta characteristic (V, ψ, q) of rank $2g$, we define $\Gamma := Aut(V, \psi, q)$ as the group of $\mathbb{Z}/4\mathbb{Z}$-linear automorphisms $\phi : V \to V$ such that ϕ preserves ψ and the induced $\bar{\phi} : \bar{V} \to \mathbb{Z}/2\mathbb{Z}$ preserves q.
- Given a 4-group with theta characteristic of rank $2g$ (V, ψ, q), with automorphism group Γ, we define an *anisotropic transvection* to be a linear map $t \in \Gamma$ of the form:
$$t(x) = x + \psi(v, x)v$$
where v is a vector such that $q(\bar{v}) \neq 0$.

Remark 28. *Any non-degenerate quadratic form in* $2g$ *variables over* $\mathbb{Z}/2\mathbb{Z}$ *is equivalent to* $\sum_{i=1}^{g} a_i b_i$ *or* $a_1^2 + a_1 b_1 + b_1^2 + \sum_{i=2}^{g} a_i b_i$ *(see [17] lemma 4.19 p.60). According to this, we will say that the theta characteristic is even (respectively odd) when the form* q *is equivalent to the first (respectively to the second) one.*

Given a 4-group with theta characteristic of rank $2g$ (V, ψ, q), one can study the structure of Γ in detail and prove that there is an exact sequence
$$0 \to K \to \Gamma \overset{\gamma \mapsto \bar{\gamma}}{\to} O(\bar{V}, q) \to 0$$
where K and $O(\bar{V}, q)$ are naturally endowed with two group morphisms (see [2]):

- the Dickson invariant $D_q : O(\bar{V}, q) \to \mathbb{Z}/2\mathbb{Z}$
- $\bar{q} : K \to \mathbb{Z}/2\mathbb{Z}$.

We now state the following:

Theorem 29. *Let* (V, ψ, q) *be a symplectic* 4*-group with theta characteristic, and let* $\Gamma = Aut(V, \psi, q)$ *be the automorphism group defined above. Then there is a unique group homomorphism*
$$\lambda : \Gamma \to \mathbb{Z}/4\mathbb{Z}$$
such that:

- $\lambda|_K = 2 \cdot \bar{q}$, *where* $\mathbb{Z}/2\mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z}/4\mathbb{Z}$ *is the canonical injection*
- $\lambda(\gamma) \equiv D_q(\bar{\gamma})$ *for every* $\gamma \in \Gamma$, *where* D_q *is the Dickson invariant*
- $\lambda(t) = 1$, *for any anisotropic transvection* t.

Proposition 30. • *In the case of the symplectic* 4*-group with odd theta characteristic given by* $(\mathbb{Z}/4\mathbb{Z})^2$ *together with the standard symplectic form:*

$$\psi(x_1, x_2, y_1, y_2) = x_1 y_2 - x_2 y_1$$

and quadratic form:

$$q(x, y) = x^2 + y^2 + xy$$

Γ *is isomorphic to* $SL_2(\mathbb{Z}/4\mathbb{Z})$ *that is generated by the matrices* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. *Further,* $\lambda(S) = \lambda(T) = 1$.

- *In the case of the symplectic* 4*-group with even theta characteristic given by* $(\mathbb{Z}/4\mathbb{Z})^2$ *together with the standard symplectic form:*

$$\psi(x_1, x_2, y_1, y_2) = x_1 y_2 - x_2 y_1$$

and quadratic form:

$$q(x, y) = xy$$

Γ *is generated by the matrices* $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right), T^2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. *Further,* $\lambda(T^2) = 0$ *and* $\lambda(S) = -1$.

Moreover, one can show that such λ behaves nicely with products, indeed:

Proposition 31. *Given two symplectic* 4*-groups with theta characteristic* $(V, \psi, q), (V', \psi', q')$ *the diagram:*

$$\begin{tikzcd} \Gamma(V) \times \Gamma(V') \arrow[r] \arrow[d, "\lambda(V)+\lambda(V')"] & \Gamma(V \oplus V') \arrow[d, "\lambda(V \oplus V')"] \\ \mathbb{Z}/4\mathbb{Z} \arrow[r, "="] & \mathbb{Z}/4\mathbb{Z} \end{tikzcd}$$

commutes.

4.4 Theta multiplier bundle, part II

Now we are ready to construct the theta multiplier bundle $\mathcal{M}(\Theta)$ associated to an abelian scheme with theta characteristic; its construction relies on the character λ of the previous theorem, indeed we just need to extend the character to the case of group schemes. We treat first a slightly more general case, then we see how the construction applies to abelian schemes.

We now work on the category of S-schemes where S is any scheme over $\mathbb{Z}[1/2, i]$, that is the ring of integers with a primitive fourth root of unity and 2 invertible. Over such a category the sheaf μ_4 and the constant sheaf $\mathbb{Z}/4\mathbb{Z}$ are isomorphic, though not canonically, thus for the rest of this thesis we fix an isomorphism $\Psi : \mu_4 \xrightarrow{\sim} \mathbb{Z}/4\mathbb{Z}$. This is the same as to choose in a functorial way a primitive fourth root of unity in every ring that we will encounter.

Definition 32. First of all, consider an arbitrary scheme S over $\mathbb{Z}[1/2, i]$. A symplectic 4-group scheme over S of rank $2g$ with theta characteristic is a triple (K, e_K, e_*^K), where:

- $K \to S$ a finite étale commutative group scheme with geometric fibers isomorphic to $(\mathbb{Z}/4\mathbb{Z})^{2g}$
- $e_K : K \times K \to \mu_4$ is a non-degenerate symplectic pairing (μ_4 is the S-group scheme of the fourth roots of unity). We can consider $e_K^2 : K \times K \to \mu_4$: this actually factorizes through $\mu_2 \hookrightarrow \mu_4$ and induces a bilinear form $e_{\bar{K}} : \bar{K} \times \bar{K} \to \mu_2$, where $\bar{K} := K/2K$
- $e_*^K : \bar{K} \to \mu_2$ is a quadratic form for $e_{\bar{K}}$, i.e. on points:

$$e_*^K(x_1 + x_2) \cdot e_*^K(x_1)^{-1} \cdot e_*^K(x_2)^{-1} = e_{\bar{K}}(x_1, x_2)$$

  for every S-scheme T and for every $x_1, x_2 \in K_T(T)$.

For such a triple we can consider the contravariant group functor F from the category of S-schemes to the category of groups, defined as:

$$\underline{Aut}(K, e_K, e_*^K) : (T \to S) \mapsto Aut_T(K_T, e_{K_T}, e_{K_T}^*)$$

that to every S-scheme T associates the group of automorphisms of K_T that respect the additional structure given by $e_{K_T}, e_{K_T}^*$ (in a similar fashion of Γ, the group defined in 27).

Remark 33. *With the notation of the definition above, for every point s of the base scheme S, we can define the parity of the theta characteristic of $K|_s$ as the parity of the quadratic function $e_*^K|_{\bar{K}_s} : \bar{K}_s \to \mu_2$. It is clear that the parity is locally constant as function on S, thus that it is constant on connected components of S.*

Theorem 34. *For a symplectic 4-group scheme (K, e_K, e_*^K) over S of rank $2g$ with theta characteristic, the group functor F described above is representable by a finite étale group scheme, denoted by $\underline{\Gamma}_K$.*
Further, there exists a group scheme morphism $\lambda : \underline{\Gamma}_K \to \mu_4$, that on every geometric fiber concides with the one described in Theorem 29.

Proof. First, to prove the representability of F, notice that it is clearly locally (étale) constant and it has finite stalks, thus using an étale descent argument we are done (for details see [13], Proposition 1.1 p.155).
Once we have the representability of F, we can recover the group scheme morphism by étale descent. Indeed, by definition of symplectic 4-group scheme with theta characteristic, we can find an étale cover over which K is isomorphic to $\left(\underline{(\mathbb{Z}/4\mathbb{Z})_S^{2g}}, e_4, e_*\right)$, i.e. the constant group scheme with standard symplectic structure, with even or odd theta characteristic depending on K's one.

Caution. Recall that the parity of theta characteristic is well defined only on every connected component of S, see Remark 33. Here we mean that $(\mathbb{Z}/4\mathbb{Z})_S^{2g}$ has a theta characteristic, whose parity varies on every connected component S_c of S, accordingly to the $K|_{S_C}$'s one.

From the sheafification of the character λ (given in Theorem 29), we obtain a group scheme morphism $\underline{\lambda} : \underline{\Gamma}_K \to \mathbb{Z}/4\mathbb{Z}$ on every open of the cover. Thanks to the uniqueness of the special group morphism (as constructed in Theorem 29), we see that this is a descent datum for the étale topology, thus by étale descent we obtain $\underline{\lambda} : \underline{\Gamma}_K \to \mathbb{Z}/4\mathbb{Z}$. By composing this with the fixed morphism Ψ (Section 4.4), we get the thesis. $\square$

We now proceed in constructing a $\underline{\Gamma}_K$-torsor.

Lemma 35. *Given a 4-group scheme (K, e_K, e_*^K) over S of rank $2g$ with theta characteristic the functor F on S-schemes defined as:*

$$F : (T \to S) \mapsto Isom_{T\text{-}GrpSch}\left(\left(K_T, e_{K_T}, e_*^{K_T}\right), \left(\underline{(\mathbb{Z}/4\mathbb{Z})}_T^{2g}, e_4, e_*\right)\right)$$

where $\underline{(\mathbb{Z}/4\mathbb{Z})}_S^{2g}$ denotes the constant group sheaf endowed with the standard symplectic form and even or odd theta characteristic depending on K's one (see Caution in the proof of the previous theorem), is representable by a $\underline{\Gamma}_K$-torsor over the base scheme S.

Proof. Since K is finite étale, we can choose an étale cover on which K is constant and isomorphic to the restriction of $\left(\underline{(\mathbb{Z}/4\mathbb{Z})}_S^{2g}, e_4, e_*\right)$, thus it is clear that F is a $\underline{\Gamma}_K$-torsor. We can choose an étale cover of S trivializing F; on this cover it is representable, indeed it is isomorphic to $\underline{\Gamma}_K$. Thus, thanks to $\underline{\Gamma}_K \cong \underline{\mathrm{Spec}}(f_*\mathcal{O}_{\underline{\Gamma}_K})$, we can shift the problem to quasi-coherent (sheaves of) commutative algebras: they form a stack ([24] thm 4.29 p.90) and this concludes the proof. $\square$

Remark 36. *Thanks to the group homomorphism $\lambda : \Gamma_K \to \mu_4$ of Theorem 34, the Γ_K-torsor constructed in Lemma 35 induces a μ_4-torsor (for details [7], Chapitre III Proposition 1.4.6).*

In the two last propositions we have seen how to associate to a triple $\left(K_T, e_{K_T}, e_*^{K_T}\right)$ a μ_4-torsor, we now apply the construction to abelian schemes with theta characteristic.

Corollary 37. *For every abelian scheme with theta characteristic $(A \to S, \Theta)$, the triple $(\mathcal{K}(\Theta^4), e_{\Theta^4}, e_*^{\Theta})$ is a symplectic 4-group scheme of rank $2g$ with theta characteristic.*
Thus, we can associate to it, as in the previous Remark, a μ_4-torsor.

Proof. By construction of $\mathcal{K}$ and φ_Θ, we know that:

$$\mathcal{K}(\Theta^4) = \mathrm{Ker}(\varphi_{\Theta^4}) = \mathrm{Ker}(4 \cdot \varphi_\Theta) \cong A\,[4]$$

that is a finite flat group scheme of rank 4^{2g} thanks to Theorem 18, but it is even étale since in our setting 2 is invertible. By the results on abelian varieties (e.g. [6] Prop. 5.9, p. 74) we know that $A\,[4]$ has geometric fibers isomorphic to $(\mathbb{Z}/4\mathbb{Z})^{2g}$ (note that the characteristic of the residue field is not 2). Then, by Theorems 26 and 25, the claim follows. $\square$

Remark 38. *By abuse of language, we will say that a pair $(A \to S, \Theta)$ has even (respectively odd) theta characteristic, when the characteristic of the associated triple $(\mathcal{K}(\Theta^4), e_{\Theta^4}, e_*^{\Theta})$ has constant parity and it is even (respectively odd).*

We would like to give the structure of invertible sheaf to the μ_4-torsor just defined, thus consider the sequence of sheaves:

$$0 \to \mu_4 \to \mathbb{G}_m \xrightarrow{\cdot 4} \mathbb{G}_m \to 0.$$

Since in our setting 2 is a unit, this is exact for the étale topology and it induces a long exact sequence in cohomology, which reads:

$$... \to H^1(S_{\acute{e}t}, \mu_4) \to H^1(S_{\acute{e}t}, \mathbb{G}_m) \xrightarrow{\cdot 4} H^1(S_{\acute{e}t}, \mathbb{G}_m) \to ...$$

Thus the μ_4-torsor induces an invertible sheaf and actually it lies in the 4-torsion subgroup $\mathrm{Pic}(S)\,[4]$. (see for example [13], p.125).

Definition 39. For every abelian scheme with theta characteristic $(A \to S, \Theta)$ we will denote with $\mathcal{M}(\Theta)$ and will call theta multiplier bundle the invertible sheaf constructed above.

Moreover, one can prove that the formation of $\mathcal{M}(\Theta)$ is compatible under base change and compatible with taking products, the precise statement is:

Proposition 40.

- *For two abelian schemes with theta characteristic $(A \to S, \Theta_1)$, $(B \to T, \Theta_2)$ and a diagram:*

$$\begin{tikzcd} A \overset{\psi}{\cong} B_S \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d] \\ S \arrow[r, "h"] & T \end{tikzcd}$$

  where ψ is an isomorphism of abelian schemes, together with an isomorphism of invertible sheaves $\Theta_1 \cong \psi^* g^* \Theta_2$; there is a canonical isomorphism $\mathcal{M}(\Theta_1) \cong h^* \mathcal{M}(\Theta_2)$.

- *For two abelian schemes with theta characteristic $(A \to S, \Theta_1)$, $(B \to T, \Theta_2)$ (even of different relative dimension), there is a canonical isomorphism*

$$\mathcal{M}(\Theta_1 \boxtimes \Theta_2) \cong \mathcal{M}(\Theta_1) \boxtimes \mathcal{M}(\Theta_2)$$

  as invertible sheaves on $S \times T$.

Proof. The first claim holds because of the naturality of the construction of $\mathcal{M}$. For the second claim: it easy to see that $(\mathcal{K}(\Theta_A^4) \oplus \mathcal{K}(\Theta_B^4), e_{\Theta_A^4} \oplus e_{\Theta_B^4}, e_*^{\Theta_A} \oplus e_*^{\Theta_B})$ is isomorphic to $(\mathcal{K}(\Theta_A \boxtimes \Theta_B{}^4), e_{(\Theta_A \boxtimes \Theta_B)^4}, e_*^{\Theta_A \boxtimes \Theta_B})$, as symplectic 4-group schemes with theta characteristic. Moreover, the formation of λ behaves nicely under taking products in the sense of Proposition 29 and, by construction, the induced group scheme morphism $\lambda : \underline{\Gamma}_K \to \mu_4$ respects products too. Thus, the claim follows. $\square$

5 Result

We devote this chapter to present the result. In the first section we state the theorem, in the second one we compute determinant bundles on elliptic curves and in the last section we provide the proof of the main theorem.

5.1 The stack of principally polarized abelian varieties with theta characteristic

In this section we first define $\mathcal{A}_g$, the stack of principally polarized abelian varieties with theta characteristic over the base category of schemes; then we prove that is an algebraic stack. For a definition of the latter we will follow [3].

Definition 41.
- The objects of $\mathcal{A}_g$ are pairs $(A \stackrel{f}{\to} S, \Theta)$ where A is an abelian scheme of relative dimension g and Θ is a symmetric, normalized, relatively ample invertible sheaf on A of degree 1.
- A morphism from $(A_1 \stackrel{f_1}{\to} S_1, \Theta_1)$ to $(A_2 \stackrel{f_2}{\to} S_2, \Theta_2)$ is the data of: a morphism of schemes $S_1 \stackrel{g}{\to} S_2$, an isomorphism of abelian schemes (i.e. it respects the structure of group schemes) $A_1 \cong A_2 \times_{S_1} S_2$ and an isomorphism of invertible sheaves between Θ_1 and the pullback of Θ_2 on A_1.

Clearly, $\mathcal{A}_g$ is endowed with the "forgetful" functor towards the category of schemes that to every pair $(A \stackrel{f}{\to} S, \Theta)$ associates the scheme S and to every morphism of pairs the morphism between the base schemes. Thus, from the definition we just see that $\mathcal{A}_g$ is a category fibered in groupoids, but not that is a stack.
The fibered category $\mathcal{A}_g$ classifies the families of principally polarized abelian varieties with theta characteristic, in the sense that every pair $(A \to S, \Theta)$ corresponds to a unique morphism $S \to \mathcal{A}_g$, where we have denoted with S the stack associated to the representable functor h_S.

Theorem 42. *The fibered category $\mathcal{A}_g$ is an algebraic stack, with finite and unramified diagonal.*

Proof. We follow for this the paper [3]. We need to prove three facts: descent condition, existence of products in the moduli topology (or equivalently, the representability of the diagonal morphism $\Delta : \mathcal{A}_g \to \mathcal{A}_g \times \mathcal{A}_g$) and the existence of an étale and surjective morphism from a scheme (or better, its associated stack) to $\mathcal{A}_g$. Some of the argumentations are very involved and we sketch just the ideas of the proofs:

- descent:
  Since we are classifying the pairs $(A \to S, \Theta)$ where Θ (among the other assumptions) is relatively ample, we can rely on descent theory of ample invertible sheaves (indeed, by definition of morphisms in $\mathcal{A}_g$ we have a functorial association of invertible sheaves) as shown in [24] Theorem 4.38 p.93.
- étale cover:
  the proof of this is involved, but it relies again on the theory of Hilbert

schemes. One indeed classifies abelian schemes with an extra rigidification. Indeed, if $(f : A \to S, \Theta)$ is an abelian scheme with theta characteristic, then $f_*\Theta^3$ defines a closed embedding $\Phi : A \to \mathbb{P}(f_*\Theta^3)$ [14] Proposition 6.13 p.123. Thus, one classifies the abelian scheme with a linear rigidification, i.e. with a fixed isomorphism $\psi : \mathbb{P}(f_*\Theta^3) \to \mathbb{P}^{m-1} \times S$, where $m = rk(f_*\Theta^3)$. The moduli space results as a closed scheme of the Hilbert scheme. Then, the forgetful functor towards $\mathcal{A}_g$ defines the étale cover we were looking for. (More details can be found in [14] p.129 and [19] p.197).

- diagonal:
  it suffices to show that for every couple of abelian schemes over the same base $A \to S, B \to S$ with theta characteristic, the contravariant functor:

  $$\underline{Isom}_S(A, B) : (T \to S) \mapsto \{\text{isomorphisms of group T-schemes between } A_T \text{ and } B_T\}$$

  is representable by an unramified and finite scheme over S. Since the schemes are projective (as shown in the previous point), thanks to the theory of Hilbert schemes we see that the functor is representable by a quasi-projective scheme over S ([24], Chapter 5, Theorem 5.23), in particular an open subscheme of $Hilb_{A\times_S B/S}$. Actually, the theorem works just for generic morphisms of schemes; but the proof can be adapted to our situation: the condition of respecting the group structure can be translated in a closed condition thus we get the representability.
  The unramified condition can be checked on geometric fibers, thus we are reduced to abelian varieties and this can be shown as in [6],Proposition 7.14 p.103: the proof relies on the fact that the global vector fields of an abelian variety are precisely the translation-invariant vector fields.
  Since the scheme is unramified at every point, then it is quasi-finite over S ([23], TAG02V5); thus we are left to show that is proper over S. Therefore, one can use the valuative criterion for properness and the statement is valid because an abelian scheme over a DVR is a Néron Model of its generic fiber ([1] Proposition 8 p. 15) and hence determined by the generic fiber up to unique isomorphism.

□

Remark 43. *In the previous chapter we have defined for every abelian scheme with theta characteristic $(A \to S, \Theta)$ two invertible sheaves $\Delta(\Theta)$ and $\mathcal{M}(\Theta)$. In particular, thanks to Propositions 24 and 40 they both behave well under base change, thus we see that the associations*

$$\Delta_g : (A \to S, \Theta) \mapsto \Delta(\Theta)$$
$$\mathcal{M}_g : (A \to S, \Theta) \mapsto \mathcal{M}(\Theta)$$

define two invertible sheaves on the stack $\mathcal{A}_g$ in the sense of [15].

Remark 44. *The statement can be sharpened: $\mathcal{M}_g$ and Δ_g lie in* $\mathrm{Pic}(\mathcal{A}_g)$ [4]. *For $\mathcal{M}$ it is clear because of its construction. We have already mentioned that for an abelian scheme with theta characteristic $(A \to S, \Theta)$, the determinant bundle lies in* $\mathrm{Pic}(S)$ [4], *but it can be shown that the equality in Theorem 23 can be realized by an isomorphism stable under base change (for a proof look at [22] Remark 1 after Theorem 0.2, p.222).*

We can finally state the main theorem: we will work over the category of schemes over $\mathbb{Z}[1/2, i]$, in particular we will study the localization of the stack $\mathcal{A}_g$ on the schemes over $\mathbb{Z}[1/2, i]$. This hypothesis is crucial for several reasons: in order to construct $\mathcal{M}$ as μ_4-torsor we needed to fix a group scheme isomorphism from $\mu_4 \xrightarrow{\cong} \mathbb{Z}/4\mathbb{Z}$ (cf. 4.4) and, in general, we need to avoid problems that arise from characteristic 2.

Theorem 45. *For any g, the two invertible sheaves $\mathcal{M}_g$ and Δ_g, defined on the stack $\mathcal{A}_g$ over $Sch_{\mathbb{Z}[1/2,i]}$, represent the same class in the Picard group of $\mathcal{A}_g$.*

Before starting the proof we need to make some observations: for every two pairs $(A \to S, \Theta_1)$ in $\mathcal{A}_{g_1}$ and $(B \to T, \Theta_2)$ in $\mathcal{A}_{g_2}$ we can consider the pair described as $(A \times B \to S \times T, \Theta_1 \boxtimes \Theta_2)$, this can be seen in a natural way as an abelian scheme with theta characteristic, of relative dimension $g_1 + g_2$ (cf. Proposition 20). It is clear that the association is functorial, thus we have just defined a functor:

$$m_{g_1,g_2} : \mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_{g_1+g_2}$$

Thanks to Propositions 40 and 24, the formation of Δ and $\mathcal{M}$ respects products, thus we can restate our considerations:

Proposition 46. *For every g_1, g_2, consider the morphism of stacks:*

$$\begin{gathered} m_{g_1,g_2} : \mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_{g_1+g_2} \\ ((A \to S, \Theta_A), (B \to T, \Theta_B)) \mapsto (A \times B \to S \times T, \Theta_A \boxtimes \Theta_B) \end{gathered}$$

and the group homomorphism induced by pullback:

$$m^*_{g_1,g_2} : \operatorname{Pic}(\mathcal{A}_{g_1+g_2}) \to \operatorname{Pic}(\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}).$$

Under this morphism we have:

$$\begin{aligned} m^*_{g_1,g_2}(\mathcal{M}_{g_1+g_2}) &\cong p_1^*(\mathcal{M}_{g_1}) \otimes p_2^*(\mathcal{M}_{g_2}) \\ m^*_{g_1,g_2}(\Delta_{g_1+g_2}) &\cong p_1^*(\Delta_{g_1}) \otimes p_2^*(\Delta_{g_2}) \end{aligned}$$

where $p_1 : \mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_{g_1}$ and $p_2 : \mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_{g_2}$ denote the projections.

Moreover, the stack $\mathcal{A}_g$ has two irreducible components $\mathcal{A}_g^+, \mathcal{A}_g^-$ ([20]) each of them classifying the pairs (A, Θ^+), (respectively (A, Θ^-)) of abelian schemes with even (resp. odd) theta characteristic. The morphism m_{g_1,g_2} restricts to these components respecting the parity of the characteristic, i.e.

$$\begin{aligned} m_{g_1,g_2}|_{\mathcal{A}^+_{g_1} \times \mathcal{A}^+_{g_2}} &: \mathcal{A}^+_{g_1} \times \mathcal{A}^+_{g_2} \to \mathcal{A}^+_{g_1+g_2} \\ m_{g_1,g_2}|_{\mathcal{A}^-_{g_1} \times \mathcal{A}^-_{g_2}} &: \mathcal{A}^-_{g_1} \times \mathcal{A}^-_{g_2} \to \mathcal{A}^+_{g_1+g_2} \\ m_{g_1,g_2}|_{\mathcal{A}^+_{g_1} \times \mathcal{A}^-_{g_2}} &: \mathcal{A}^+_{g_1} \times \mathcal{A}^-_{g_2} \to \mathcal{A}^-_{g_1+g_2} \end{aligned}$$

and consequently $m^*_{g_1,g_2}$ does, i.e.:

$$\begin{aligned} m^*_{g_1,g_2}|_{\operatorname{Pic}(\mathcal{A}^+_{g_1+g_2})} &: \operatorname{Pic}(\mathcal{A}^+_{g_1+g_2}) \to \operatorname{Pic}(\mathcal{A}^+_{g_1} \times \mathcal{A}^+_{g_2}) \\ m^*_{g_1,g_2}|_{\operatorname{Pic}(\mathcal{A}^-_{g_1+g_2})} &: \operatorname{Pic}(\mathcal{A}^+_{g_1+g_2}) \to \operatorname{Pic}(\mathcal{A}^-_{g_1} \times \mathcal{A}^-_{g_2}) \\ m^*_{g_1,g_2}|_{\operatorname{Pic}(\mathcal{A}^-_{g_1+g_2})} &: \operatorname{Pic}(\mathcal{A}^-_{g_1+g_2}) \to \operatorname{Pic}(\mathcal{A}^+_{g_1} \times \mathcal{A}^-_{g_2}). \end{aligned}$$

5.2 Determinant on elliptic curves

We devote this section to compute theta characteristics and determinant bundles for families of elliptic curves, i.e. abelian schemes of dimension 1. In particular, we have summed up the results in these two lemmata:

Lemma 47.

- *Let $E \xrightarrow{f} S$ be a family of elliptic curves, where S is a scheme over* $\mathrm{Spec}(\mathbb{Z}[1/2])$*. Let $P : S \to E$ a 2-torsion point such that on every geometric fiber P is not trivial, then $\Theta := \mathcal{O}_E(P)$ is an even theta characteristic.*
- *in this case $\Delta(\Theta) = \omega_{E/S}$.*

Lemma 48.

- *Let $E \xrightarrow{f} S$ a family of elliptic curves (S any scheme) and $e : S \to E$ the identity section; then $\Theta := \mathcal{O}_E(e) \otimes \Omega_{E/S}$ is an odd theta characteristic.*
- *in this case $\Delta(\Theta) = \omega_{E/S}^{\otimes 3}$.*

Remark 49. *It is worth to notice that the condition on the fibers for the point P in Lemma 47 is crucial. Indeed it can happen that on a point $s \in S$ with residue field with characteristic* 2*, P specifies to the trivial point (i.e. $P|_s = e|_s$). This can be avoided by working over* $\mathbb{Z}[1/2]$*. In this setting we can just demand that P is not trivial when restricted to every connected component of S.*

Proof. (Lemma 47)
Recall that in both cases we need to prove that Θ is a normalized, symmetric, relatively ample invertible sheaf of degree 1.

- it is symmetric, because $[-1]^* \mathcal{O}(P) \cong \mathcal{O}([-1]P) = \mathcal{O}(P)$, where the last equality holds since P is a 2-torsion point.
- it is clearly normalized (i.e. $e^*\mathcal{O}(P) \cong \mathcal{O}_S$), since P is not trivial.
- to see that Θ is ample, we can check on the geometric fibres: indeed, since $E \to S$ is of finite presentation, we can reduce to S locally noetherian ([10] Proposition 8.9.1); then, thanks to ([9], Théorème 4.7.1), if Θ is ample on a fiber, then it is ample on a Zariski neighbourhood; finally since the map is quasi-compact (it is proper by assumption) we can check ampleness on a Zariski cover ([8], Corollaire 4.4.5) and we are done. Thus we are left to show that $\mathcal{O}_E(P)$ is ample for E an elliptic curve over an algebraically closed field k and P a 2-torsion point. In this case, it is a basic fact of the theory of elliptic curves that $\mathcal{O}(3P)$ is very ample.
- To see that Θ has degree 1, we proceed as follows: consider the exact sequence, defining P:

$$0 \to \mathcal{O}_E(-P) \to \mathcal{O}_E \to P_*P^*\mathcal{O}_E \to 0$$

  tensor it with $\mathcal{O}_E(P)$ and use the projection formula for the last term:

$$0 \to \mathcal{O}_E \to \mathcal{O}_E(P) \to P_*P^*\mathcal{O}_E(P) \to 0.$$

Consider the long exact sequence in cohomology obtained by applying π_*:

$$0 \to f_*\mathcal{O}_E \to f_*\mathcal{O}_E(P) \to f_*(P_*P^*\mathcal{O}_E(P)) \to$$
$$R^1f_*\mathcal{O}_E \to R^1f_*\mathcal{O}_E(P) \to R^1f_*(P_*P^*\mathcal{O}_E \otimes \mathcal{O}_E(P)) \to ...$$

I claim that the middle term in the second row is zero: indeed, we can reduce to check on geometric fibers; but for every geometric point $\bar{x}$ the Riemann-Roch theorem implies that $H^1(E_{\bar{x}}, \Theta_{\bar{x}})$ is trivial. Thus we conclude that the connecting morphism is surjective and, since only line bundles are involved (indeed $f_*(P_*P^*\mathcal{O}_E(P)) = P^*\mathcal{O}_E(P)$, since $f \circ P = id_S$), it is an isomorphism. At the end we get $f_*\mathcal{O}_E(P) \cong f_*\mathcal{O}_E \cong \mathcal{O}_S$. Hence, Θ has degree 1.

- Finally, just by definition and the previous point, we have $\Delta(\Theta) = (f_*\Theta)^{\otimes 2} \otimes \omega_{E/S} \cong \mathcal{O}_S \otimes \omega_{E/S} \cong \omega_{E/S}$.
- the parity of the theta characteristic is discussed in Remark 50.

□

Proof. (Lemma 48)

- Θ is symmetric: indeed, obviously $[-1]^*\,\mathcal{O}(e) \cong \mathcal{O}([-1]\,e) = \mathcal{O}(e)$ and $[-1]^*\,\Omega_{E/S} \cong \Omega_{E/S}$ holds, because $\Omega_{E/S}$ is the sheaf of invariant differentials.
- $\mathcal{O}(e)$ is ample, indeed $\mathcal{O}(3e)$ is very ample and gives the well-known embedding; while $\Omega_{E/S}$ is globally generated (by the invariant differentials). Thus we conclude that Θ is ample.
- to see that Θ is normalized, consider again the exact sequence:

$$0 \to \mathcal{O}_E \to \mathcal{O}_E(e) \to e_*e^*\mathcal{O}_E(e) \to 0$$

  and the long exact sequence in cohomology:

$$0 \to f_*\mathcal{O}_E \to f_*\mathcal{O}_E(e) \to f_*(e_*e^*\mathcal{O}_E(e)) \to$$
$$R^1f_*\mathcal{O}_E \to R^1f_*\mathcal{O}_E(e) \to R^1f_*(e_*e^*\mathcal{O}_E \otimes \mathcal{O}_E(e)) \to ...$$

  reasoning as above, we get the triviality of the middle term in the second row and consequently $R^1f_*\mathcal{O}_E \cong f_*e_*e^*\mathcal{O}_E(e) = e^*\mathcal{O}_E(e)$. By Serre duality we get $R^1f_*\mathcal{O}_E \cong \omega_{E/S}^{-1}$ and finally, collecting all together we conclude $e^*\Theta \cong e^*\mathcal{O}_E(e) \otimes e^*\Omega_{E/S} = \omega_{E/S}^{-1} \otimes \omega_{E/S} \cong \mathcal{O}_S$, i.e. Θ is normalized.
- we are left to show that the degree is one. For this, consider:

$$f_*\Theta = f_*(\mathcal{O}(e) \otimes \Omega_{E/S}) \cong f_*\mathcal{O}(e) \otimes \omega_{E/S} \cong \omega_{E/S}$$

  where the middle equation holds thanks to the isomorphism $\Omega_{E/S} \cong f^*\omega_{E/S}$ and the projection formula.
- By definition of determinant line bundle and the previous point, we have:

$$\Delta(\Theta) = (f_*\Theta)^{\otimes 2} \otimes \omega_{E/S} \cong \omega_{E/S}^{\otimes 3}.$$

- the parity of the theta characteristic is discussed in Remark 50.

□

Remark 50. *Here we need a little digression: the study of theta characteristics on abelian varieties arises naturally in the context of curves. Indeed, there is a canonical way to associate to a curve $C \to S$ an abelian variety, called the Jacobian variety, (that is just the connected component of the identity of the Picard variety of the curve). There is even a natural way to associate a divisor, called theta divisor and denoted with Θ. The latter lies in $\mathrm{Pic}^{g-1}(C/S)$ the connected component classifying the $(g-1)$-degree divisors. Now, consider an invertible sheaf $\mathcal{L}$, whose square is isomorphic to the canonical line bundle. By an application of Riemann-Roch theorem we find that $\mathcal{L}$ has degree $g-1$; let us call ℓ the point in $\mathrm{Pic}^{g-1}(C/S)$ corresponding to it. This can be used to translate the theta divisor Θ to $\mathrm{Pic}^0(C/S)$: let us call this new divisor $\Theta_{\mathcal{L}} := t_\ell^*\Theta$. It is indeed a theta characteristic for the Jacobian variety: it is ample, symmetric, it has degree 1 and it can be canonically normalized (via $\Theta_{\mathcal{L}} \mapsto \Theta_{\mathcal{L}} \otimes \pi^* e^* \Theta_{\mathcal{L}}^{-1}$). Moreover, Mumford proved that in this case the parity of the theta characteristic coincides with the parity of $h^0(\mathcal{L})$, that can be 0 or 1.*

Using this other description, it is easy to study the classes of theta characteristics of an elliptic curve (indeed, it is isomorphic with its Jacobian) and it can be shown that on an elliptic curve over $\mathbb{Z}[1/2]$ there are only one odd theta characteristic and three even ones. (For a detailed discussion see [18] Appendix to 3, p.162).

From the remark, we deduce:

Corollary 51. *The forgetful functor:*

$$\mathcal{A}_1^- \to \mathcal{M}_{1,1}$$
$$(E \to S, \Theta) \mapsto (E \to S)$$

is an equivalence of categories over $Sch_{\mathbb{Z}[1/2]}$.

5.3 Proof of the theorem

We devote this section to the proof of Theorem 45 stated above. We have split the proof in a few steps:

Remark 52. *Recall that in our setting, we are working on the category $Sch_{\mathbb{Z}[1/2,i]}$ and we have chosen an isomorphism on $\Psi : \mu_4 \xrightarrow{\simeq} \mathbb{Z}/4\mathbb{Z}$ (cf. section 4.4). This corresponds to fix a primitive fourth root of unity in every ring, we will encounter and it will be simply denoted by i. In particular if you consider an algebraically closed field of characteristic not 2, Ψ defines an isomorphism between the group of fourth roots of unity and the cyclic group with four elements.*

- reduction
  consider the morphism of stacks described in Proposition 46:

$$m_{g_1,g_2} : \mathcal{A}_{g_1} \times \mathcal{A}_{g_2} \to \mathcal{A}_{g_1+g_2}$$
$$((A_1,\Theta_1),(A_2,\Theta_2)) \mapsto (A_1 \times A_2, \Theta_1 \boxtimes \Theta_2)$$

that induces the group morphism via composition:

$$m^*_{g_1,g_2} : \mathrm{Pic}(\mathcal{A}_{g_1+g_2}) \to \mathrm{Pic}(\mathcal{A}_{g_1} \times \mathcal{A}_{g_2}).$$

Thanks to Proposition 24 (respectively 40), it is clear that $m^*_{g_1,g_2}\Delta_{g_1+g_2} \cong \Delta_{g_1} \otimes \Delta_{g_2}$ (respectively $m^*_{g_1,g_2}\mathcal{M}_{g_1+g_2} \cong \mathcal{M}_{g_1} \otimes \mathcal{M}_{g_2}$). Moreover the morphism m_{g_1,g_2} restricts to the components $\mathcal{A}^{\pm}_g$ respecting the parity of the characteristics, i.e.:

$$\begin{aligned}
m_{g_1,g_2}|_{\mathcal{A}^+_{g_1}\times\mathcal{A}^+_{g_2}} &: \mathcal{A}^+_{g_1} \times \mathcal{A}^+_{g_2} \to \mathcal{A}^+_{g_1+g_2}\\
m_{g_1,g_2}|_{\mathcal{A}^-_{g_1}\times\mathcal{A}^-_{g_2}} &: \mathcal{A}^-_{g_1} \times \mathcal{A}^-_{g_2} \to \mathcal{A}^+_{g_1+g_2}\\
m_{g_1,g_2}|_{\mathcal{A}^+_{g_1}\times\mathcal{A}^-_{g_2}} &: \mathcal{A}^+_{g_1} \times \mathcal{A}^-_{g_2} \to \mathcal{A}^-_{g_1+g_2}
\end{aligned}$$

and consequently $m^*_{g_1,g_2}$ does, e.g.:

$$\begin{aligned}
m^*_{g_1,g_2}|_{\mathrm{Pic}(\mathcal{A}^+_{g_1+g_2})} &: \mathrm{Pic}(\mathcal{A}^+_{g_1+g_2}) \to \mathrm{Pic}(\mathcal{A}^+_{g_1} \times \mathcal{A}^+_{g_2})\\
m^*_{g_1,g_2}|_{\mathrm{Pic}(\mathcal{A}^-_{g_1+g_2})} &: \mathrm{Pic}(\mathcal{A}^+_{g_1+g_2}) \to \mathrm{Pic}(\mathcal{A}^-_{g_1} \times \mathcal{A}^-_{g_2})\\
m^*_{g_1,g_2}|_{\mathrm{Pic}(\mathcal{A}^-_{g_1+g_2})} &: \mathrm{Pic}(\mathcal{A}^-_{g_1+g_2}) \to \mathrm{Pic}(\mathcal{A}^+_{g_1} \times \mathcal{A}^-_{g_2}).
\end{aligned}$$

Now, suppose that we have proven the isomorphism over $\mathcal{A}^{\pm}_g$ for the pairs ($g = 1,$ odd characteristic), and ($g \geq 3$, even characteristic), then I claim the theorem holds for any irreducible component. For this, in order to show the theorem on $\mathcal{A}^-_g$ for $g \geq 2$, consider $m^* : \mathrm{Pic}(\mathcal{A}^+_{g+1}) \to \mathrm{Pic}(\mathcal{A}^-_g \times \mathcal{A}^-_1)$ and compare the image of Δ_{g+1} and $\mathcal{M}_{g+1}$ that by assumption we know to be isomorphic:

$$\Delta_g \otimes \Delta_1 \cong m^*\Delta_{g+1} \cong m^*\mathcal{M}^{-1}_{g+1} \cong \mathcal{M}^{-1}_g \otimes \mathcal{M}^{-1}_1$$

by assumption $\Delta_1 \cong \mathcal{M}^{-1}_1$, hence $\Delta_g \cong \mathcal{M}^{-1}_g$ over $\mathcal{A}^-_g$. (For the last deduction, notice that $p^*_{\mathcal{A}^-_g} : \mathrm{Pic}(\mathcal{A}^-_g) \to \mathrm{Pic}(\mathcal{A}^-_g \times \mathcal{A}^-_1)$ is injective). Similarly, one can deduce the theorem in all the other cases: in particular in order to prove the theorem, in the case $g = 2$ with odd theta characteristic (respectively, $g = 1$ even theta characteristic) one considers the morphism of stacks:

$$m^* : \mathrm{Pic}(\mathcal{A}^-_3) \to \mathrm{Pic}(\mathcal{A}^-_1 \times \mathcal{A}^+_2)$$

respectively:

$$m^* : \mathrm{Pic}(\mathcal{A}^-_2) \to \mathrm{Pic}(\mathcal{A}^-_1 \times \mathcal{A}^+_1).$$

In both cases, we know already the theorem over two of the stacks involved, thus we can deduce for the other. Thus we are left to show the theorem in the following cases:

- case: g=1, odd characteristic

Recall that we know $\mathcal{A}^-_1 \cong \mathcal{M}_{1,1}$ (Corollary 51) and that $\mathrm{Pic}(\mathcal{M}_{1,1}) \cong \mathbb{Z}/12\mathbb{Z}$ ([5]). To show the theorem in this case we are going to construct a surjective group morphism $\alpha : \mathrm{Pic}(\mathcal{M}_{1,1}) \to \mathbb{Z}/4\mathbb{Z}$, so that α restricts to an isomorphism on $\mathrm{Pic}(\mathcal{M}_{1,1})[4]$. Thus, since the two invertible sheaves Δ, $\mathcal{M}^{-1}$ lie in the 4-torsion subgroup, comparing their image under α would conclude the proof.

In order to construct α, consider the object $(E \to \mathrm{Spec}(k), \Theta)$ in $\mathcal{A}_1^-$, where k is an algebraically closed field with characteristic not 2, E is the elliptic curve defined by the equation $y^2 = x^3 - x$ and $\Theta := \mathcal{O}(e) \otimes \Omega^1_{E/k}$ is an odd theta characteristic for E (Lemma 48). The curve E has a special automorphism β of order 4 given by

$$\begin{aligned} \beta &: E \to E \\ (x, y) &\mapsto (-x, iy) \end{aligned}$$

where i is the chosen primitive fourth root of unity in k (cf. Remark 52). Plainly, this automorphism is defined for the pair (E, Θ), since it respects the characteristic.
Therefore we can define α as follows: to every invertible sheaf $\mathcal{L}$ defined on $\mathcal{A}_1^-$, $\mathcal{L}(\beta)$ is an automorphism for the invertible sheaf $\mathcal{L}((E, \Theta))$, it has order 4, thus it can be identified with an element in μ_4 and via the isomorphism Ψ (of Remark 52) with an element in $\mathbb{Z}/4\mathbb{Z}$. Set this as the image of $\mathcal{L}$ under α.
the image of Δ:
By Lemma 48 we know that $\Delta(E, \Theta) \cong \omega_{E/k}^{\otimes 3}$, that is a 1-dimensional k-vector space generated by the invariant differential $\left(\frac{dx}{y}\right)^{\otimes 3}$, thus we compute:

$$\Delta(\beta) : \left(\frac{dx}{y}\right)^{\otimes 3} \mapsto \left(\frac{d(-x)}{iy}\right)^{\otimes 3} = i^3 \left(\frac{dx}{y}\right)^{\otimes 3}$$

hence $\alpha(\Delta) = \Psi(i^3) = \Psi(-i) = -1$.
the image of $\mathcal{M}$:
To compute the image of $\mathcal{M}$ we need to recall that, by Definition 39, $\mathcal{M}$ is a μ_4-torsor obtained as a pushforward along λ from a $\underline{\Gamma}_S$-torsor. Now in this case $\mathcal{K}(\Theta^4) = E[4] \cong (\mathbb{Z}/4\mathbb{Z})^2$ and since the characteristic is odd we can choose a basis of $E[4]$ such that the quadratic form is given by $e_*^{\Theta}(a, b) = (-1)^{a+b+ab}$ (cf. Remark 28) and the action of β is given by the matrix $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, which respects the quadratic form. Therefore the automorphism β extends to the pair (E, Θ). Then, thanks to Proposition 30 and seeing $\mathcal{M}(\Theta)$ as a $\mathbb{Z}/4\mathbb{Z}$ torsor, β acts as $\lambda(S) = 1 \in \mathbb{Z}/4\mathbb{Z}$. Finally under the choice of isomorphism $\mathbb{Z}/4\mathbb{Z} \cong \mu_4$, we identify the action of β as $i \in k^*$. In conclusion, $\alpha(\mathcal{M}) = i$.
Plainly α is surjective and the thesis follows comparing the images of the two invertible sheaves.

- case: g=2n+1$\geq$ 3, even characteristic
  We reason as before: in particular we are going to define a surjective group homomorphism $\alpha : \mathrm{Pic}(\mathcal{A}_g^+) \to \mathbb{Z}/4\mathbb{Z}$, this is sufficient since $\mathrm{Pic}(\mathcal{A}_g^+)[4] \cong \mathbb{Z}/4\mathbb{Z}$ (as showed in [22] Theorem 5.6). Consider the pair (E, Θ) in $\mathcal{A}_1^+$, where E is the elliptic curve defined by $y^2 = x^3 - x$ over an algebraically closed field k and $\Theta = \mathcal{O}(P)$ for some non-trivial 2 torsion point. The pair

$(E^{\times g}, \Theta^{\boxtimes g})$ is an object of $\mathcal{A}_g^+$; the abelian scheme $E^{\times g}$ has an automorphism of order 4, namely $\beta^{\times g}$. Plainly, the automorphism respects the theta characteristic and it extends to the pair $(E^{\times g}, \Theta^{\boxtimes g})$. Therefore we can define α as follows: to every invertible sheaf $\mathcal{L}$ defined on $\mathcal{A}_g^+$, $\mathcal{L}(\beta^{\times g})$ is an automorphism for the invertible sheaf $\mathcal{L}((E^{\times g}, \Theta^{\boxtimes g}))$, it has order 4, thus it can be identified with an element in μ_4 and via Ψ as an element in $\mathbb{Z}/4\mathbb{Z}$. Set this as the image of $\mathcal{L}$ under α.
<u>the image of Δ under α:</u>
By Lemma 47, we know that $\Delta_1((E, \Theta^+)) \cong \omega_{E/S}$, moreover we know that the formation of Δ respects products (cf. Proposition 24), thus we can compute:

$$\Delta_g((E^{\times g}, \Theta^{\boxtimes g})) : \left(\frac{dx}{y}\right)^{\boxtimes g} \mapsto \left(\frac{d(-x)}{iy}\right)^{\boxtimes g} = i^g \cdot \left(\frac{dx}{y}\right)^{\boxtimes g}$$

hence $\alpha(\Delta_g) = \Psi(i^g) = [g] \in \mathbb{Z}/4\mathbb{Z}$.
<u>the image of $\mathcal{M}$ under α:</u>
analogously we can compute the image of $\mathcal{M}_g$: indeed λ is compatible with products (Proposition 40), thus

$$\alpha(\mathcal{M}_g) = \Psi(\lambda(S^g)) = \Psi(\lambda(S)^g) = \Psi((i^{-1})^g) = [-g] \in \mathbb{Z}/4\mathbb{Z}.$$

Notice that in this case $\lambda(S) = -i$, since the characteristic is even (cf. Proposition 30). Hence, α is surjective (i^g is still primitive, since g is odd) and comparing the image of the invertible sheaves under α we conclude $\Delta_g \cong \mathcal{M}_g^{-1}$ over $\mathcal{A}_g^+$.

- <u>case: g=2n$\geq$ 3, even characteristic</u>
  we can proceed analogously as the previous point, using the pair $(E^{\times g}, \Theta^{\boxtimes g})$ and the automorphism $\beta^{\times(g-1)} \times id$ to define the group morphism α. Following the same computations we conclude $\alpha(\Delta_g) = \alpha(\mathcal{M}_g^{-1}) = \Psi(i^{g-1}) = [g-1] \in \mathbb{Z}/4\mathbb{Z}$. Since g is odd α is still surjective (indeed i^{g-1} is still primitive) and we deduce $\Delta_g \cong \mathcal{M}_g^{-1}$ over $\mathcal{A}_g^+$.

This concludes the proof of the main theorem.

This result is an improvement of the canonical key formula (Theorem 23), as we will explain now: let $(f : A \to \mathcal{A}_g, \Theta)$ be the universal pair, we can use Theorem 23 to write:

$$(f_*\Theta^{-1})^{\otimes 8} \cong \omega^{\otimes 4}$$

where ω denotes the Hodge bundle for the universal family of abelian varieties with theta characteristic. According to theorem 23 we cannot say anything more, on the other hand thanks to Theorem 45 we can write:

$$(f_*\Theta^{-1})^{\otimes 2} \cong \mathcal{M}(\Theta) \otimes \omega. \tag{1}$$

Thus we can think of $f_*\Theta^{-1}$ as a square root of the Hodge bundle, tensored with $\mathcal{M}(\Theta)$.
This is very important in the theory of the theta functions, indeed the analytical theta function satisfies the classical functional formula (e.g. [18] Theorem 7.1 p.32) and thus it is a modular form of weight 1/2. The content of the formula (1) above is exactly the same, but in algebraic context, as formalized in the theory of Katz modular forms.

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