QUANTIFIER ELIMINATION FOR ALGEBRAICALLY CLOSED NON-ARCHIMEDEAN VALUED FIELDS AND AN APPLICATION TO TROPICAL GEOMETRY.

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Contents

1. Introduction.

In this paper, we prove quantifier elimination for the theory of Algebraically Closed Valued Fields in the 2-sorted language $\mathcal{L}_\Gamma$. We will present the necessary model theoretic tools for this purpose. We will omit some technical details that are necessary for a proper construction, but that are well documented in various sources. In general, we follow the notations and approach used in [TZ12] and [PD11], where the interested reader also can find more details about model theory in general.

2. Languages and structures.

Definition 2.1. *Let S be a set. A **language of sort** S is a set consisting of the following distinct symbols:*

(1) *Logical symbols:* $\neg$ *(not)* $\wedge$ *(and)* $\forall$ *(for all)* $\doteq$ *(equals).*

(2) *Variables:* *For any $s \in S$, a collection of symbols x_{si} with $i \in \mathbb{N}$.*

(3) *Relation symbols:* *For any $n \in \mathbb{N}$ and any $\bar{s} = (s_1, \ldots, s_n) \in S^n$, a collection of symbols $R_{\bar{s}i}$ with $i \in I_{\bar{s}}$, where $I_{\bar{s}}$ is an arbitrary, possibly empty, index set.*

(4) *Function symbols:* *For any $n \in \mathbb{N}$ and any $\bar{s} = (s_1, \ldots, s_n, s) \in S^{n+1}$, a collection of symbols $f_{\bar{s}j}$ with $j \in J_{\bar{s}}$, where $J_{\bar{s}}$ is an arbitrary, possibly empty, index set.*

(5) *Constant symbols:* *For any $s \in S$ a collection of symbols c_{sk}, with $k \in K_s$, where K_s is an arbitrary, possibly empty, index set.*

(6) *Punctuation:* ,) (

*In this situation, we call S a **set of sorts** and we say that the variables, the relation symbols, the function symbols and the constant symbols are the **sorted symbols** of the language $\mathcal{L}$. The tuple $(s_1, \ldots, s_n) \in S^n$ associated to any sorted symbol $\mathfrak{s}$, as described above, is the called the **sort** of $\mathfrak{s}$.*

There are slight variations on how to define a language, and we use the notation in [PD11]. The reason for using $\doteq$ instead of $=$ is to emphasise the difference between a formal expression and assigning using equality in the naive way. For example, if we want to say that φ is the string $x_{s1} \doteq x_{s2}$, we write $\varphi = x_{s1} \doteq x_{s2}$.

What distinguishes two languages of the same sort are their relation symbols, function symbols and constant symbols. Hence, we will denote a language of sort S by

$$\mathcal{L} = \{R_{\bar{s}i}, f_{\bar{s}j}, c_{sk} \mid s \in S, \bar{s} \in S^n, n \in \mathbb{N}\}.$$

For simplicity, we will often write just $\mathcal{L} = \{R_{\bar{s}i}, f_{\bar{s}j}, c_{sk}\}$. We will only consider languages where the cardinality of S is finite. If $|S| = n$, we say that $\mathcal{L}$ is an n-**sorted** language.

It is often convenient to use different letter for variables of different sorts. For example, for a 2-sorted language of sort $S = \{s, t\}$ we will often write $(x_i)_{i \in \mathbb{N}}$ instead of $(x_{si})_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ or $(\xi)_{i \in \mathbb{N}}$ instead of $(x_{ti})_{i \in \mathbb{N}}$. By convention, we will write $x_0 \not\doteq x_1$ for $\neg(x_0 \doteq x_1)$.

The reader might have noticed that we excluded the commonly used symbols $\vee$, $\rightarrow$, $\leftrightarrow$ and $\forall$ from the logical symbols. This is only for convenience when making inductive proofs, and we will soon see that these symbol can also be introduced as an abbreviation.

Definition 2.2. *Fix a language* $\mathcal{L} = \{R_{\bar{s}i}, f_{\bar{s}j}, c_{sk}\}$. *An* $\mathcal{L}$-***structure*** $\mathcal{M}$ *is given by*

(1) *A set of non-empty sets* $\{s(\mathcal{M}) \mid s \in S)$.

(2) *For each relation symbol* $R \in \mathcal{L}$ *of sort* $(s_1, \dots, s_n)$*, a subset*
$$R^{\mathcal{M}} \subset s_1(\mathcal{M}) \times \cdots \times s_n(\mathcal{M}),$$
called the ***interpretation*** *of* R *in* $\mathcal{M}$.

(3) *For each function symbol of sort* $(s_1, \dots, s_n, s)$*, a function*
$$f^{\mathcal{M}} : s_1(\mathcal{M}) \times \cdots \times s_n(\mathcal{M}) \to s(\mathcal{M}).$$
called the ***interpretation*** *of* f *in* $\mathcal{M}$.

(4) *For each* $c \in \mathcal{C}$ *of sort* S*, an element* $c^{\mathcal{M}} \in S(\mathcal{M})$.

The disjoint union
$$\coprod_{s \in S} s(\mathcal{M})$$
is called the ***universe*** *of* $\mathcal{M}$.

We will use the convention to write structures of a language with curly letters, and their corresponding universes with normal letters. For example, if $\mathcal{M}$ and $\mathcal{N}$ are structures of some language of sort S, we write
$$M = \coprod_{s \in S} s(\mathcal{M}) \quad \text{and} \quad N = \coprod_{s \in S} s(\mathcal{M}).$$
We will denote an $\mathcal{L}$-structure by $\mathcal{M} = (M, f_i^{\mathcal{M}}, R_j^{\mathcal{M}}, c_k^{\mathcal{M}} \mid i \in I,\, j \in J,\, k \in K)$.

Example 2.3. Let $\mathcal{L}_G = \{\cdot, 0\}$ be a language of sort $\{G\}$, where $\cdot$ is a binary function symbol and 1 is a constant symbol. Then any group $\mathcal{H}$ gives a natural $\mathcal{L}_G$-structure $\mathcal{H} = (H, \cdot^{\mathcal{H}}, 1^{\mathcal{H}})$, where $\cdot^{\mathcal{H}}$ is the group operation on H and $1^{\mathcal{H}}$ is the identity element of H.

Example 2.4. Let $\mathcal{L}_R = \{+, -, \cdot, 0, 1\}$ be a language of sort $\{R\}$, where $+$, $-$ and $\cdot$ are binary function symbols, and 0 and 1 are a constant symbol. Then any ring S has a natural $\mathcal{L}$-structure.

Example 2.5. Consider the set of sorts $\{R, G\}$ and let $\mathcal{L}_M = \mathcal{L}_R \sqcup \mathcal{L}_G \sqcup \{\Phi\}$, with $\mathcal{L}_R$ and $\mathcal{L}_G$ being as above, and Φ being a function symbol of sort (R, G, G). Then $\mathcal{L}_M$ is a 2-sorted language of sort $\{G, R\}$. For any ring S and any S-module A, we then have that A gives an $\mathcal{L}_M$ structure by interpreting Φ as the action of S on A.

Example 2.6. As we will see in Section 8, there is a natural way to consider Algebraically Closed Valued Fields as structures of a 2-sorted language.

We will now use the logical symbols and the symbols of a language $\mathcal{L}$ to express properties of a given $\mathcal{L}$-structure. Properties are expressed by finite strings of symbols, which we will define inductively.

Definition 2.7. *Fix a language $\mathcal{L}$ of sort S. The* ***terms*** *of $\mathcal{L}$, or $\mathcal{L}$-**terms**, and their sorts are defined inductively by:*

(1) *For each $s \in S$, the variables x_{si} with $i \in \mathbb{N}$ are terms of sort s.*

(2) *For each $s \in S$, the constant symbols c_{sk} are terms of sort s.*

(3) *For each $\bar{s} = (s_1, \ldots, s_n, s) \in S^{n+1}$, each function symbol $f_{\bar{s}j}$ with $j \in J$ and all terms $t_1, \ldots, t_n$ with t_i having sort s_i, we have that $f_{\bar{s}j}(t_1, \ldots, t_n)$ is a term of sort s.*

Given a term t, we will write $t(x_1, \ldots, x_m)$ to indicate that the variables occurring in t are among $x_1, \ldots, x_m$. In this case, we also write t as $t(\bar{x})$.

Definition 2.8. *Fix a language $\mathcal{L}$. The* ***formulas*** *of $\mathcal{L}$, or $\mathcal{L}$-**formulas**, are defined inductively by:*

(1) *If t_1 and t_2 are $\mathcal{L}$-terms, then $t_1 \doteq t_2$ is an $\mathcal{L}$-formula*

(2) *If $t_1, \ldots, t_n$ are terms of sorts $s_1, \ldots, s_n$ respectively and R is a relation symbol of sort $(s_1, \ldots, s_n)$, then $R(t_1, \ldots, t_n)$ is an $\mathcal{L}$-formula.*

(3) *If φ is an $\mathcal{L}$-formula, then $\neg\varphi$ is an $\mathcal{L}$-formula.*

(4) *If φ_1 and φ_2 are $\mathcal{L}$-formulas, then $(\varphi_1 \wedge \varphi_2)$ is an $\mathcal{L}$-formula.*

(5) *If φ is an $\mathcal{L}$-formula and if x is a variable, then $\exists x\varphi$ is an $\mathcal{L}$-formula.*

Formulas of the form $t_1 \doteq t_2$ or $R(t_1, \ldots t_n)$ are called ***atomic****.*

We will use the abbreviations

(1) $(\varphi_1 \vee \varphi_2)$ for $\neg(\neg\varphi_1 \wedge \varphi_2)$

(2) $(\varphi_1 \to \varphi_2)$ for $\neg(\varphi_1 \wedge \neg\varphi_2)$

(3) $(\varphi_1 \leftrightarrow \varphi_2)$ for $(\varphi_1 \to \varphi_2) \wedge (\varphi_2 \to \varphi_1)$

(4) $\forall x\varphi$ for $\neg\exists x\neg\varphi$.

Definition 2.9. *Let x be a variable. We define that x is* ***free*** *in a formula φ inductively as follows:*

(1) *x is free in $t_1 \doteq t_2$ if x occurs in t_1 or t_2.*

(2) *x is free in $R(t_1, \dots, t_n)$ if x occurs in one of the t_i.*

(3) *x is free in $\neg\varphi$ if x is free in φ*

(4) *x is free in $(\varphi_1 \wedge \varphi_2)$ if x is free in φ_1 or if x is free in φ_2.*

(5) *x is free in $\exists y \varphi$ if $x \neq y$ and x is free in φ.*

*Variables in a formula that are not free are called **bound**. We write $\varphi(x_1, \dots, x_n)$ or $\varphi(\bar{x})$ to indicate that the only free variables of φ are among $x_1, \dots, x_n$. A formula without free variables is called a **sentence**.*

Remark 2.10. Note that the expressions $t(x_1, \dots, x_n)$ in Definition 2.7 $\varphi(x_1, \dots, x_n)$ in Definition 2.9 is not uniquely determined by t or φ. For example, let $t_1 = x_1$, $t_2 = x_2$ and $\varphi = t_1 \doteq t_2$. Then we can write $t_1(x_1)$, $t_2(x_2)$ and $\varphi(x_1, x_2)$ or $t_1(x_1, x_2)$, $t_2(x_2, x_3)$ and $\varphi(x_1, x_2, x_3)$ or even $t_1(x_1, x_2, \dots, x_n)$, $t_2(x_1, x_2, \dots, x_n)$ and $\varphi(x_1, x_2, \dots, x_n)$.

Example 2.11. Let $\mathcal{L}_G = \{\cdot, 1\}$. Then

$$\varphi = \forall x \forall y (x \cdot y = y \cdot x)$$

is a formula.

Example 2.12. Let $\mathcal{L}_R = \{+, -, \cdot, 0, 1\}$. Then

$$\varphi_n = \underbrace{1 + \dots + 1}_{n \text{ times}} \doteq 0$$

is a formula.

Example 2.13. Let $\mathcal{L}_R = \{+, -, \cdot, 0, 1\}$. Then

$$\bigwedge_{n \in \mathbb{N}_{>0}} \underbrace{1 + \dots + 1}_{n \text{ times}} \dot{\neq} 0$$

is not a formula, since it is an infinite string.

3. Satisfaction.

When we do mathematics, we often use intuitively what it means for a formula to be satisfied by some elements in a structure. For example, consider the $\mathcal{L}_G$-formula $\varphi(x_1) = \exists x_2 (x_2 \cdot x_2 \doteq x_1)$, with $\mathcal{L}_G$ being the language of groups defined in Example 2.3. This formula expresses the statement that an element is a square, which we know is true for any element in the $\mathcal{L}_G$-structure $(\mathbb{R}, \cdot)$, but not only for certain elements in the $\mathcal{L}_G$-structure $(\mathbb{Q}, \cdot)$. In this section, we will make this notion precise.

We fix a language $\mathcal{L}$ and an $\mathcal{L}$-structure $\mathcal{M}$. If $t(x_1, \ldots, x_n)$ is a term of sort s with each x_i having sort s_i, then by composition of functions, we can define a function

$$t^{\mathcal{M}} : \prod_{i=1}^{n} s_i(M) \to s(M).$$

By Remark 2.10, the function $t^{\mathcal{M}}$ is not canonical. This is however not a problem, as the choice of such a function does not change the definition below.

Definition 3.1. *Let $\mathcal{M}$ be an $\mathcal{L}$ structure. For $\mathcal{L}$-formulas φ of a given sort $(s_1, \ldots, s_n)$ and $\bar{a} = (a_1, \ldots, a_n) \in \prod_{i=1}^{n} s_i(\mathcal{M})$, we define the relation*

$$\mathcal{M} \models \varphi[\bar{a}]$$

inductively as follows:

(1) *If $t_1(x_1, \ldots, x_n)$ and $t_2(x_1, \ldots, x_n)$ are terms and if $t_1^{\mathcal{M}}[\bar{a}] = t_2^{\mathcal{M}}[\bar{a}]$, then $\mathcal{M} \models (t_1 \doteq t_2)[\bar{a}]$*

(2) *If R is a relation symbol of sort $s_1, \ldots, s_n$, if $t_1, \ldots, t_n$ are terms of sorts $s_1, \ldots, s_n$ respectively and if $R^{\mathcal{M}}(t_1^{\mathcal{M}}[\bar{a}], \ldots, t_m^{\mathcal{M}}[\bar{a}])$, then $\mathcal{M} \models R(t_1, \ldots, t_m)[\bar{a}]$*

(3) *If it is not the case that $\mathcal{M} \models \varphi[\bar{a}]$, then $\mathcal{M} \models \neg\varphi[\bar{a}]$. We also write $\mathcal{M} \not\models \varphi[\bar{a}]$.*

(4) *If $\mathcal{M} \models \varphi_1[\bar{a}]$ and $\mathcal{M} \models \varphi_2[\bar{a}]$, then $\mathcal{M} \models (\varphi_1 \wedge \varphi_2)[\bar{a}]$.*

(5) *If there exists an element $b \in M$ such that $\mathcal{M} \models \varphi\left[\bar{a}\frac{b}{x}\right]$, where $\bar{a}\frac{b}{x} = (a_0, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n)$ if $x = x_i$, then $\mathcal{M} \models \exists x\varphi$.*

Example 3.2. Again, let $\mathcal{L}_G = \{\cdot, 1\}$, where $\cdot$ is a binary function symbol and 1 is a constant symbol. Let $\varphi(x)$ be the formula $\forall y(y \cdot x \doteq y)$. If H is a group, then we have that $H \models \varphi[1_H]$, hence $H \models \exists x\varphi$.

Example 3.3. As in Example 2.11, consider the language $\mathcal{L}_G$-formula

$$\varphi = \forall x\forall y(x \cdot y = y \cdot x).$$

Let H be a group and consider H as an $\mathcal{L}_G$-structure. It can be verified, using Definition **??** and the fact that $\forall$ is in fact an abbreviation as described after Definition 2.8 that $H \models \varphi$ if and only if H is commutative.

Example 3.4. Let $\mathcal{L}_R$ and φ_n be as in Example **??**. Let A be a ring and consider A as an $\mathcal{L}_R$-structure. It follows from Definition **??** that $A \models \varphi_n$ if and only if A has characteristic a divisor of n. Therefore, a ring A has characteristic zero if and only if $A \models \neg\varphi_n$ for all $n \in \mathbb{N}_{>0}$. With this in mind, it might be tempting to say that a ring A has characteristic zero if and only if $A \models \bigwedge_{n\in\mathbb{N}_{>0}} \neg\varphi_n$. However, this is not well defined since $\bigwedge_{n\in\mathbb{N}_{>0}} \neg\varphi_n$ is not a formula, as noted in Example **??**. One might ask if it is possible to find another sentence ψ such that $A \models \psi$ if and only if A has characteristic 0. In section 5, we will see that this cannot be done when A is an algebraically closed fields.

Definition 3.5. *Let $\mathcal{L}$ be a language of sorts S and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. A map $g : M \to N$ is called an $\mathcal{L}$-**homomorphism** if for all $(s_1, \dots, s_n, s) \in \mathcal{S}$ and all $(a_1, \dots, a_n) \in \prod_{i=1}^n s_i(\mathcal{M})$,*

$$g(c^{\mathcal{M}}) = c^{\mathcal{N}}$$
$$g(f^{\mathcal{M}}(a_1, \dots, a_n)) = f^{\mathcal{N}}(g(a_1), \dots, g(a_n))$$
$$R^{\mathcal{M}}(a_1, \dots, a_n) \Rightarrow R^{\mathcal{N}}(g(a_1), \dots, g(a_n))$$

where c is any constant symbol, f is any function symbol of sort $(s_1, \dots, s_n, s)$ and R is any relation symbol of sort $(s_1, \dots, s_n)$. When $g : M \to N$ is an $\mathcal{L}$ homomorphism, we write

$$g : \mathcal{M} \to \mathcal{N}.$$

If in addition g is injective and

$$R^{\mathcal{M}}(a_1, \dots, a_n) \Leftrightarrow R^{\mathcal{N}}(g(a_1), \dots, g(a_n))$$

*we say that g is an $\mathcal{L}$-**embedding**. If g is a surjective embedding, we say that g is an $\mathcal{L}$-isomorphism, and we write*

$$h : \mathcal{M} \tilde{\to} \mathcal{N}.$$

Definition 3.6. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. If $M \subset N$ and the natural inclusion is an embedding of $\mathcal{L}$-structures, we say that $\mathcal{M}$ is a **substructure** of $\mathcal{N}$. In this case, we call $\mathcal{N}$ an **extension** of $\mathcal{M}$.*

Definition 3.7. *Let $\mathcal{L}$ and let $\mathcal{M}$ and $\mathcal{N}$ be $\mathcal{L}$-structures. We say that $\mathcal{M}$ is an **elementary substructure** of $\mathcal{N}$, and that $\mathcal{N}$ is an **elementary extension** of $\mathcal{M}$, if for any formula $\varphi(x_1, \dots, x_n)$ and any $(a_1, \dots, a_n) \in M^n$, we have*

$$\mathcal{M} \models \varphi[\bar{a}] \Leftrightarrow \mathcal{N} \models \varphi[\bar{a}].$$

Example 3.8. Consider the language $\mathcal{L} = \{+, <\}$, where $+$ is a binary function and $<$ is a binary relation. We view $\mathbb{Z}$ and $\mathbb{Q}$ as $\mathcal{L}$-structures, with $+$ interpreted as the usual addition and $<$ interpreted as the usual ordering. Then the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an $\mathcal{L}$-embedding, so $(\mathbb{Q}, +, <)$ is an extension of $(\mathbb{Z}, +, <)$. However, it is not an elementary extension. To see this, let $\varphi(x, y)$ be the formula

$$\exists z(x < z \wedge z < y).$$

Then we have that $\mathbb{Z} \not\models \varphi[0, 1]$, but $\mathbb{Q} \models \varphi[0, 1]$.

Example 3.9. It is difficult to prove that an $\mathcal{L}$-embedding is an elementary embedding, since a priori we have to consider all possible formulas $\mathcal{L}$-formulas. We will see in Section 5 a general situation where elementary extensions naturally arise. This will allow us to show that $\mathbb{C}$ is an elementary extension of the algebraic closure of $\mathbb{Q}$, viewed as $\mathcal{L}_R$-structures.

4. Theories.

Definition 4.1. *Let $\mathcal{L}$ be a language. An $\mathcal{L}$-**theory** is a set of sentences of the language $\mathcal{L}$. A **model of a theory** T is an $\mathcal{L}$-structure $\mathcal{M}$ which satisfies all sentences of T. We write $\mathcal{M} \models T$ if $\mathcal{M}$ is a model of T. A theory is said to be **consistent** if it has a model. A theory which is not consistent is called **inconsistent**.*

The class of all models of a theory T is denoted $\mathrm{Mod}(T)$. If $\mathcal{K}$ is a class of $\mathcal{L}$-structures, then $\mathrm{Th}(\mathcal{K})$ denotes the set of all sentences true in all elements of $\mathcal{K}$. We denote $\mathrm{Th}(\{\mathcal{M}\})$ by $\mathrm{Th}(\mathcal{M})$. If φ is a sentence that holds in all models of a theory T, then we write $T \models \varphi$.

Definition 4.2. *Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are said to be **elementary equivalent**, denoted $\mathcal{M} \equiv \mathcal{N}$ if $\mathrm{Th}(\mathcal{M}) = \mathrm{Th}(\mathcal{N})$, i.e. if they satisfy the same sentences. A theory T is **complete** if given a sentence φ, either $T \models \varphi$ or $T \models \neg\varphi$.*

If $\mathcal{M}$ is an $\mathcal{L}$-structure and φ is a sentence, then $\mathcal{M} \models \varphi$ or $\mathcal{M} \not\models \varphi$. The latter being the same as $\mathcal{M} \models \neg\varphi$ by definition, so we have that $\mathrm{Th}(\mathcal{M})$ is complete.

Example 4.3. Let $\mathcal{L}_G = \{\cdot, 1\}$ and let $T_{\text{group}} = \{\varphi_1, \varphi_2, \varphi_3\}$, where

$$\begin{aligned}
\varphi_1 &= \forall x_1 \forall x_2 \forall x_3 ((x_1 \cdot x_2) \cdot x_3 \doteq x_1 \cdot (x_2 \cdot x_3)) \\
\varphi_2 &= \forall x ((1 \cdot x \doteq x) \wedge (x \cdot 1 \doteq x)) \\
\varphi_3 &= \forall x_1 \exists x_2 ((x_1 \cdot x_2 \doteq 1) \wedge (x_2 \cdot x_1 \doteq 1)).
\end{aligned}$$

Then the models for T_{group} coincide with the category of groups.

Example 4.4. Let $\mathcal{L}_R = \{+, -, \cdot, 0, 1\}$ and let T_{field} be the theory consisting of the field axioms:

(1) $\forall x_1 \forall x_2 \forall x_3 ((x_1 + x_2) + x_3 \doteq x_1 + (x_2 + x_3))$

(2) $\forall x_1 \forall x_2 (x_1 + x_2 \doteq x_2 + x_1)$

(3) $\forall x (x + 0 \doteq x)$

(4) $\forall x (x - x \doteq 0)$

(5) $\forall x_1 \forall x_2 \forall x_3 ((x_1 \cdot x_2) \cdot x_3 \doteq x_1 \cdot (x_2 \cdot x_3))$

(6) $\forall x_1 \forall x_2 (x_1 \cdot x_2 \doteq x_2 \cdot x_1)$

(7) $\forall x (x \cdot 1 \doteq x)$

(8) $\forall x_1 \exists x_2 ((x_1 \not\doteq 0) \to (x_1 \cdot x_2 \doteq 1))$

Then any field K is a model for T_{field}. Furthermore, if

$$T_{\text{closed}} = \left\{ \forall x_0 \cdots \forall x_n \exists y \left(\sum_{i=0}^{n} x_i \cdot y^i = 0 \right) \mid n \in \mathbb{N}_{>0} \right\},$$

then any algebraically closed field is a model for $T_{\text{field}} \cup T_{\text{closed}}$, and we call this theory ACF.

The following theorem is a very important standard result in model theory. We will not prove it here, but it can be found for example as Theorem 1.5.6 in [PD11].

Theorem 4.5 (Compactness Theorem)**.** *Let T be an $\mathcal{L}$-theory. Then T is consistent if and only if it is finitely consistent, i.e. if every finite subset of T is consistent.*

The compactness theorem is very useful to show the existence of models. We will use it in combination with the following lemma, which we also just state for reference. See [PD11, p.27] for a proof.

Lemma 4.6. *Let T be an $\mathcal{L}$-theory and let φ be an $\mathcal{L}$-sentence. Then $T \models \varphi$ if and only if $T \cup \{\neg\varphi\}$ is inconsistent.*

5. Quantifier elimination.

Definition 5.1. *Let $\mathcal{L}$ be a language containing a constant symbol. An $\mathcal{L}$-theory T has **quantifier elimination**, or **eliminates quantifiers**, if for any formula $\varphi(\bar{x})$ there exists a quantifier free formula $\psi(\bar{x})$ which is equivalent to $\varphi(\bar{x})$ modulo T, i.e. if*

$$T \models \forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x})).$$

Note that the latter notation makes sense even if $\varphi(\bar{x})$ is not a sentence, since all free variables in φ will be bounded in the sentence $\forall \bar{x}(\varphi(\bar{x}) \leftrightarrow \psi(\bar{x}))$.

Example 5.2. Consider the language $\mathcal{L}_R$ of rings and the theory of algebraically closed fields, ACF from Example 4.4. The formula $\varphi(x) = \exists y(y^2 \doteq x)$ is true for all elements in an algebraically closed fields. Hence, we get

$$\text{ACF} \models \forall x(\varphi(x) \leftrightarrow x \doteq x)$$

since $x \doteq x$ is also true for all elements in an algebraically closed field.

The example above shows that a particular formula is equivalent to a quantifier free formula modulo ACF. The following well known result, which can be found for example in [Mar02, p.85], shows that this is true for any formula in ACF:

Theorem 5.3. *The theory of algebraically closed fields,* ACF *from Example 4.4 eliminates quantifiers.*

Example 5.4. The theory of fields, T_{field} from Example 4.4 does not eliminate quantifiers. We justify this by the following counterexamples:

(1) Let $\varphi = \exists x(x^2 \doteq -1)$. If T_{field} would eliminate quantifier, there would exist a formula ψ without quantifiers such that
$$T_{\text{field}} \models \varphi \leftrightarrow \psi.$$
In particular, we have that $\mathbb{Q} \not\models \psi$, since $\mathbb{Q} \not\models \varphi$. Since ψ is quantifier free, it only relates to the characteristic of the field as described in the proof of 5.7. Hence, we also have that $\mathbb{C} \not\models \psi$ and so $\mathbb{C} \not\models \varphi$. But this is a contradiction, since $i \in \mathbb{C}$ satisfies $x^2 \doteq -1$. This shows that T_{field} does not eliminate quantifiers.

(2) Let $\varphi(x) = \exists y(x \doteq y^2)$ and consider the definable set
$$S = \{q \in \mathbb{Q} \mid \mathbb{Q} \models \varphi[q]\},$$
i.e. the set of all squares in $\mathbb{Q}$. We claim that S is not definable by a quantifier free formula. To see this, suppose for contradiction that S is defined by a quantifier free formula $\psi(x)$. Note that ψ then can be written as
$$\psi(x) = \bigvee_{j=1}^{k} \bigwedge_{i=1}^{\ell} P_{i,j}(x) \simeq_{i,j} 0$$
where $\simeq_{i,j} \in \{\doteq, \not\doteq\}$ and $P_{i,j} \in \mathbb{Z}[X]$. Since S is infinite, there must be a j_0 such that
$$\bigwedge_{i=1}^{\ell} P_{i,j_0}(x) \simeq_{i,j_0} 0$$
is satisfied by infinitely many elements in $\mathbb{Q}$. Since a polynomial has only finitely many roots, all $\simeq_{i,j_0}$ are equal to $\not\doteq$. Let
$$V = \{\omega \in \mathbb{C} \mid P_{i,j_0}(\omega) = 0 \text{ for all } i \in \{1, \dots \ell\}\}.$$
Since V is finite, we can define $m = \max_{\omega \in V}(|\omega|)$. Let N be an integer strictly larger than m. Then $\mathbb{Q} \models \psi[-N]$, since $-N \notin V$. But $-N \notin S$, so S is not defined by a quantifier free formula.

(3) The same argument as in (1) shows that the set of square in $\mathbb{R}$ is not defined by a quantifier free formula.

Example 5.5. Consider the language $\mathcal{L}_R$ of rings and the theory of algebraically closed fields. We now return to the question whether the property of having characteristic 0 is expressible by a sentence. Suppose that there is such a sentence, i.e. a sentence φ_0 such that $K \models \varphi_0$ if and only if K has characteristic 0, where K is an algebraically closed field. By quantifier elimination, φ_0 is equivalent to a sentence ψ without quantifiers. Let ℓ be the total number of symbols in ψ, and let m be the maximal number that can be obtained using ℓ instances of 1, addition and multiplication. Since a quantifier free formula in $\mathcal{L}_R$ can only be finite disjunctions and conjuctions of equalities involving 1 and 0, ψ can only contain disjunctions and conjunctions of equalities and inequalities involving $0, 1, \dots, m$. Let p be a prime number greater than m. Then any equality and inequality involving $0, 1, \dots, m$ is true in and algebraically closed field of characteristic 0 if and only if it is true in an algebraically closed field of characteristic p. This contradicts our choice of φ_0, so there is no such φ_0.

Example 5.6. We can use the previous example to show that there is no formula for the property of a group being torsion free in the language $\mathcal{L}_G$. Indeed, suppose that there is such a formula. Then we can use this formula in the language $\mathcal{L}_R$, to describe that the group $(K, +)$ is torsion free, for a field K. But this means exactly that K has characteristic zero, so there is no such formula.

The following two theorems are direct applications of quantifier elimination for algebraically closed fields.

Theorem 5.7. *Let $P_1, \dots, P_m \in \mathbb{Q}[X_1, \dots, X_n]$. Then the following are equivalent:*

(1) *There exists $(\alpha_1, \dots, \alpha_n) \in (\mathbb{Q}^{\mathrm{alg}})^n$ such that $P_i(\alpha_1, \dots, \alpha_n) = 0$ for all $i \in \{1, \dots, m\}$.*

(2) *There exists $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ such that $P_i(\alpha_1, \dots \alpha_n) = 0$ for all $i \in \{1, \dots, m\}$.*

Proof. Let D be the least common multiple of the denominators of all the coefficients of all the polynomials P_i. Then $DP_i \in \mathbb{Z}[X_1, \dots, X_n]$ and $DP_i(\alpha_1, \dots, \alpha_n) = 0$ if and only if $P_i(\alpha_1, \dots, \alpha_n) = 0$. We can now write each coefficient a of DP_i as $\pm(\underbrace{1 + \cdots + 1}_{|a| \text{ times}})$. Hence

$$\exists \bar{x} \left(\bigwedge_{i=1}^{m} DP_i(\bar{x}) = 0 \right)$$

is a formula in the language $\mathcal{L}_R$. Therefore, it is equivalent to a quantifier free formula φ modulo ACF. Since any such formula can only express equalities involving 0 and 1, and both $\mathbb{Q}^{\mathrm{alg}}$ and $\mathbb{C}$ have characteristic zero, we get that if φ is true in one of the fields, it is also true in the other. □

Theorem 5.8. *Let K and L be algebraically closed fields such that K is a subfield of L. Let $P_1, \dots, P_m \in K[X_1, \dots, X_n]$. Then the following are equivalent:*

(1) *There exists $(\alpha_1, \dots, \alpha_n) \in K^n$ such that $P_i(\alpha_1, \dots, \alpha_n) = 0$ for all $i \in \{1, \dots, m\}$.*

(2) *There exists $(\alpha_1, \dots, \alpha_n) \in L^n$ such that $P_i(\alpha_1, \dots \alpha_n) = 0$ for all $i \in \{1, \dots, m\}$.*

This theorem cannot be proved directly in the same way as Theorem 5.7, as we cannot write polynomials in K multiplied by an integer as formulas.

Proof. Let $a_1, \dots, a_N$ be all the coefficients of the polynomials $P_1, \dots, P_m$. Then we can write $P_i(X_1, \dots, X_n) = Q_i(X_1, \dots, X_n, a_1, \dots, a_N)$ for some polynomial $Q_i \in K[X_1, \dots X_n, Y_1, \dots, Y_N]$. Consider the formula

$$\exists x_1, \dots, x_n \bigwedge_{i=1}^{m} Q_i(x_1, \dots, x_n, y_1, \dots, y_N) = 0.$$

This formula has free variables among $y_1, \dots, y_N$. Hence, it is equivalent to a quantifier free formula $\psi(y_1, \dots, y_N)$. That all the P_i:s have a common root $\bar{x} \in L^n$ is therefore the same as $\psi[a_1, \dots, a_N]$ is satisfied by L. But since $(a_1, \dots, a_N) \in K^N$, this means that $\psi[a_1, \dots, a_N]$ is also satisfied by K, by the same argument as above, since K and L have the same characteristic. □

As seen in Theorem 5.7 and Theorem 5.8, rather strong results sometimes follows almost immediately from quantifier elimination of a theory. In the light of this, it is not unreasonable to suspect that quantifier elimination can sometimes be difficult to prove. Indeed, the main point with this whole paper is to prove quantifier elimination for the theory of Algebraically Closed Valued Fields. For this, we need to introduce some more model theoretic machinery.

Let $\mathcal{L}$ be a language of sort S and let $\{c_{s\alpha} \mid s \in S, \alpha \in I\}$ for some index set I be a set of constant symbols not appearing in $\mathcal{L}$. We then obtain an expanded language $\mathcal{L}' = \mathcal{L} \cup \{c_\alpha \mid \alpha \in I\}$ of sort S. For any sorted symbol $\mathfrak{s}$ in $\mathcal{L}'$ we denote by $S(\mathfrak{s})$ the sort of $\mathfrak{s}$. Similarly, we denote the sort of an $\mathcal{L}$-term t by $S(t)$. Let $s_\alpha = S(c_\alpha)$. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\prod_{\alpha \in I} s_\alpha(\mathcal{M})$ is non-empty, then $\mathcal{M}$ can be viewed as an $\mathcal{L}'$-structure by the interpretation $c_\alpha^{\mathcal{M}} = a_\alpha \in s_\alpha(\mathcal{M})$.

Definition 5.9. *With the notation above, if $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subset \coprod_{s \in S} s(\mathcal{M})$ we will denote by $\mathcal{L}(A)$ the language obtained by for each $a \in A$ adjoining to $\mathcal{L}$ a constant symbol c_a with $S(c_a) = S(a)$. In this case, $\mathcal{M}$ gives an $\mathcal{L}'$ structure as described, since $a \in s_a(\mathcal{M})$ for each $a \in A$.*

If T is a theory in the language $\mathcal{L}$, we can view T as a theory T' in the language $\mathcal{L}'$, since every $\mathcal{L}$-sentence is also an $\mathcal{L}'$-sentence. If $\mathcal{M}$ is a model of T which can be viewed as a $\mathcal{L}'$-structure in the sense described above, then $\mathcal{M}$ induces a model $\mathcal{M}'$ of T'. Similarly, any model $\mathcal{M}'$ of T' can be viewed as a model $\mathcal{M}$ of T.

Lemma 5.10. *Let T be a theory in a language $\mathcal{L}$ of sorts S and let $\varphi(x_1, \dots, x_m)$ be an $\mathcal{L}$-formula. Let $c_1, \dots, c_m$ be constant symbols not appearing in $\mathcal{L}$ and define $\mathcal{L}' = \mathcal{L} \cup \{c_1, \dots, c_m\}$ with c_i and x_i having the same sort. Denote by T' the theory T viewed as an $\mathcal{L}'$-theory. Then $T \models \forall x_1 \cdots \forall x_m \varphi(x_1, \dots, x_m)$ if and only if $T' \models \varphi[c_1, \dots, c_m]$.*

Proof. Suppose that $T \models \forall x_1 \cdots \forall x_m \varphi(x_1, \dots, x_m)$. Let $\mathcal{M}'$ be any model of T' and let $a_i = c_i^{\mathcal{M}'}$. Let $\mathcal{M}$ be the corresponding model of T. Then $\mathcal{M} \models \forall x_1 \cdots \forall x_m \varphi(x_1, \dots, x_m)$. In particular, $\mathcal{M} \models \varphi[a_1, \dots, a_m]$ and so

$$\mathcal{M}' \models \varphi[c_1^{\mathcal{M}'}, \dots, c_m^{\mathcal{M}'}].$$

Conversely, assume that $T' \models \varphi[c_1, \dots, c_m]$. Let $\mathcal{M}$ be a model of T. Let $s_i = S(x_i)$. Since $s(\mathcal{M})$ is non-empty for each $s \in S$, by 2.2, the set $\prod_{i=1}^m s_i(\mathcal{M})$ is non-empty. Let $(a_1, \dots, a_m)$ be any element in $\prod_{i=1}^m s_i(\mathcal{M})$ and let $\mathcal{M}'$ be the $\mathcal{L}'$-structure obtained by interpreting $c_i^{\mathcal{M}'} = a_i$. As noted before, $\mathcal{M}'$ is a model of T', so by assumption, $\mathcal{M}' \models \varphi[c_1^{\mathcal{M}'}, \dots, c_m^{\mathcal{M}'}]$. Hence, $\mathcal{M} \models \varphi[a_1, \dots, a_m]$ and the lemma is proved. □

Definition 5.11. *Let $\mathcal{L}$ be a language of sort S and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\bar{x} = (x_1, \dots, x_n)$ be a tuple of variables with x_i having sort $s_i \in S$. Let $\Sigma(\bar{x})$ be a set of $\mathcal{L}$-formulas with free variables among the x_i. We say that $\Sigma(\bar{x})$ is **realised** in $\mathcal{M}$ if there exists an element $\bar{a} \in \prod_{i=1}^n s_i(\mathcal{M})$ such that for each $\psi(\bar{x}) \in \Sigma(\bar{x})$, we have*

$$\mathcal{M} \models \psi[\bar{a}].$$

In this situation, we write

$$\mathcal{M} \models \Sigma[\bar{a}].$$

Definition 5.12. *Let $\mathcal{L}$, $\mathcal{M}$ and $\Sigma(\bar{x})$ be as in Definition 5.11. We say that $\Sigma(\bar{x})$ is **finitely satisfiable** in $\mathcal{M}$ if every finite subset of $\Sigma(\bar{x})$ is realised in $\mathcal{M}$.*

Definition 5.13. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. For an infinite cardinal κ, we say that $\mathcal{M}$ is κ-saturated if whenever A is a subset of M such that $|A| < \kappa$ and $\Sigma(\bar{x})$ is a set of $\mathcal{L}(A)$-formulas (with $\mathcal{L}(A)$ being the language defined in Definition 5.9) which is finitely satisfiable in $\mathcal{M}$, then $\Sigma(\bar{x})$ is realised in $\mathcal{M}$.*

We will use, but not prove, the following result which can be found for example as Theorem 2.5.2 in [PD11].

Theorem 5.14. *Let $\mathcal{L}$ be a language and denote by $\kappa_{\mathcal{L}}$ the cardinality of $\mathcal{L}$. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. Then, for any cardinal $\kappa > \kappa_{\mathcal{L}}$, there exists an elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that $\mathcal{N}$ is κ-saturated.*

Remark 5.15. We will only consider countable languages, and so the $\mathcal{L}$-structures we will consider will always have κ-saturated elementary extensions for any $\kappa > \aleph_0$.

Remark 5.16. Let T be an $\mathcal{L}$-theory, let $\mathcal{M}$ be a model of T and let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$. By definition we have that $\mathcal{N}$ satisfy exactly the same sentences as $\mathcal{M}$. In particular, $\mathcal{N}$ is also a model of T. Using Theorem 5.14, we get that for any model $\mathcal{M}$ of a theory T, there exists an elementary extension $\mathcal{N}$ such that $\mathcal{N}$ is an $\aleph_1$-saturated model of T.

Lemma 5.17. *Let T be a theory in a language $\mathcal{L}$ of sorts S containing a constant symbol c. Let $\varphi(\bar{x})$ be a formula of sort $\bar{s} = (s_i)_{1 \leq i \leq m} \in S^m$ which is not equivalent to a quantifier free formula modulo T. Then, there exists two models $\mathcal{M}$ and $\mathcal{N}$ of T and $\bar{a} \in \prod_{i=1}^m s_i(\mathcal{M})$, $\bar{b} \in \prod_{i=1}^m s_i(\mathcal{N})$ such that $\bar{a}$ and $\bar{b}$ satisfy the same quantifier free formulas while $\mathcal{M} \models \neg\varphi(\bar{a})$ and $\mathcal{N} \models \varphi(\bar{b})$.*

We follow the proof by Anand Pillay in [Pil, p.22].

Proof. Let $\Sigma(\bar{x})$ be the set of quantifier free formulas $\psi(\bar{x})$ such that

$$T \models \forall x \left(\varphi(\bar{x}) \to \psi(\bar{x})\right).$$

Note that if $\psi_1(\bar{x}), \dots, \psi_n(\bar{x}) \in \Sigma(\bar{x})$, then $\bigwedge_{i=0}^{n} \psi_i(\bar{x}) \in \Sigma(\bar{x})$. Also note that $\varphi(\bar{x})$ is not false, since $c \neq c$ is quantifier free $\mathcal{L}$-formula. So if $\psi(\bar{x}) \in \Sigma(\bar{x})$ then $\neg\psi(\bar{x}) \notin \Sigma(\bar{x})$. Define the language $\mathcal{L}' = \mathcal{L} \cup \{c_1, \dots, c_m\}$ where $c_1, \dots, c_m$ are new constant symbols of sorts $s_1, \dots, s_m$ respectively and let $\Sigma[\bar{c}]$ be the set of $\mathcal{L}'$-sentences obtained by replacing x_i for c_i. Denote the theory T viewed as an $\mathcal{L}'$-theory by T'.

Claim 1. The theory $T' \cup \Sigma[\bar{c}] \cup \{\neg\varphi[\bar{c}]\}$ is consistent.

Proof of Claim 1. If $T' \cup \Sigma[\bar{c}] \cup \{\neg\varphi[\bar{c}]\}$ inconsistent, then $T' \cup \Sigma[\bar{c}] \models \varphi[\bar{c}]$ by Lemma 4.6. Hence, by the compactness theorem, there exists $\psi_1, \dots, \psi_n \in \Sigma(\bar{x})$ such that $T' \cup \{\psi_1[\bar{c}], \dots, \psi_n[\bar{c}]\} \models \varphi[\bar{c}]$ and so $T' \cup \{\psi_1[\bar{c}] \wedge \cdots \wedge \psi_n[\bar{c}]\} \models \varphi[\bar{c}]$. Writing $\psi_1 \wedge \cdots \wedge \psi_n$ as ψ, we get that $T' \models \psi[\bar{c}] \to \varphi[\bar{c}]$, so $T \models \forall(\bar{x}(\psi(\bar{x}) \to \varphi(\bar{x}))$ by Lemma 5.10. But $T \models \forall\bar{x}(\varphi \to \psi)$ by definition of $\Sigma(\bar{x})$, so $T \models \forall\bar{x}(\psi \leftrightarrow \varphi)$. This contradicts φ not being equivalent to a quantifier free formula, hence the claim.

Let $\mathcal{M}'$ be a model of $T' \cup \Sigma(\bar{c}) \cup \{\neg\varphi(\bar{c})\}$ and denote by $\mathcal{M}$ be the corresponding model of T. Let $\bar{a} = \bar{c}^{\mathcal{M}'} \in \prod_{i=1}^{m} s_i(\mathcal{M})$. Then $\mathcal{M} \models \neg\varphi(\bar{a})$ and $\mathcal{M} \models \psi(\bar{a})$ for all $\psi \in \Sigma$. Let $\Delta(\bar{x})$ be the set of quantifier free formulas $\psi(\bar{x})$ of sort $\bar{s}$ such that $\mathcal{M} \models \psi[\bar{a}]$ and let $\Delta[\bar{c}]$ be the set of corresponding $\mathcal{L}'$-sentences. By definition, $\Sigma(\bar{x}) \subset \Delta(\bar{x})$. If $\psi \in \Delta$, then $\neg\psi \notin \Delta$ since, by definition, $\mathcal{M} \models \neg\psi[\bar{a}]$ if and only if $\mathcal{M} \not\models \psi[\bar{a}]$. Note also that if $\psi(\bar{x})$ is a quantifier free formula of sort $\bar{s}$ and $\psi \notin \Delta$, then $\mathcal{M} \not\models \psi[\bar{a}]$ and so $\mathcal{M} \models \neg\psi[\bar{a}]$ by definition. Hence $\neg\psi \in \Delta$. This implies that Δ is maximal in the sense that for any quantifier free formula of sort $\bar{s}$, either ψ or $\neg\psi$ is in Δ.

Claim 2. The theory $T' \cup \Delta[\bar{c}] \cup \{\varphi[\bar{c}]\}$ is consistent.

Proof of Claim 2. We prove this by contradiction. Suppose that $T' \cup \Delta[\bar{c}] \cup \{\varphi[\bar{c}]\}$ is not consistent. Then by Lemma 4.6

$$T' \cup \Delta[\bar{c}] \models \neg\varphi[\bar{c}].$$

By the compactness theorem, there are $\psi_1, \dots, \psi_n \in \Delta(\bar{x})$ such that

$$T' \cup \{\psi_1[\bar{c}], \dots, \psi_n[\bar{c}]\} \models \neg\varphi[\bar{c}].$$

Hence, letting $\psi = \bigwedge_{i=1}^{n} \psi_i$ we get

$$T \models \forall\bar{x}(\psi \to \neg\varphi).$$

So by Boolean calculus we have

$$T \models \forall\bar{x}(\varphi \to \neg\psi).$$

By definition of $\Sigma(\bar{x})$, we get that $\neg\psi \in \Sigma(\bar{x})$. Since $\Sigma(\bar{x}) \subset \Delta(\bar{x})$ we then have $\neg\psi \in \Delta(\bar{x})$. By the same argument as for $\Sigma(\bar{x})$ we have that $\Delta(\bar{x})$ is closed under finite conjunction, so $\psi \in \Delta$. This is a contradiction, as we have shown that not both ψ and $\neg\psi$ can be in Δ, which proves the claim.

By Claim 2, there is a model $\mathcal{N}'$ of $T' \cup \Delta[\bar{c}] \cup \varphi[\bar{c}]$. Denote the corresponding model of T by $\mathcal{N}$ and let $\bar{b} = \bar{c}^{\mathcal{N}'} \in \prod_{i=1}^{n} s_i(\mathcal{N})$. Then $\mathcal{N} \models \Delta[\bar{b}]$ and $\mathcal{N} \models \varphi[\bar{b}]$. Since Δ is maximal, the quantifier free formulas satisfied by $\bar{b}$ in $\mathcal{N}$ is exactly the formulas in Δ. Hence, $\bar{a}$ and $\bar{b}$ satisfy the same quantifier free formulas, namely the formulas in $\Delta(\bar{x})$ while $\mathcal{M} \models \neg\varphi(\bar{a})$ and $\mathcal{N} \models \varphi(\bar{b})$. □

Remark 5.18. Let $\mathcal{L}$ be a language of sorts S and let $\mathcal{M}$ be an $\mathcal{L}$-structure. If $(\mathcal{N}_\alpha)_{\alpha\in I}$ is a family of substructures of $\mathcal{M}$ having at least one common element, then we get a substructure $\mathcal{N}$ of $\mathcal{M}$, defined as follows:

(1) For each $s \in S$, define $s(\mathcal{N}) := \bigcap_{\alpha\in I} s(\mathcal{N}_\alpha)$.

(2) For each relation symbol $R \in \mathcal{L}$, define $R^{\mathcal{N}} = \bigcap_{\alpha\in I} R^{\mathcal{N}_\alpha}$.

(3) For each function symbol $f \in \mathcal{L}$ of sorts $(s_1, \ldots, s_n, s)$, define
$$f^{\mathcal{N}} = s_1(\mathcal{N}) \times \cdots \times s_n(\mathcal{N}) \to s(\mathcal{N})$$
as the restriction of $f^{\mathcal{M}}$ to $s_1(\mathcal{N}) \times \cdots \times s_n(\mathcal{N})$.

(4) For each constant symbol $c \in \mathcal{L}$, define $c^{\mathcal{N}} := c^{\mathcal{M}}$.

We denote this substructure $\mathcal{N}$ by $\bigcap_{\alpha\in I} \mathcal{N}_\alpha$.

Definition 5.19. *Let $\mathcal{L}$ be a language and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $A \subset M$ and let $(\mathcal{N}_\alpha)_{\alpha\in I}$ be the family of substructure of $\mathcal{M}$ such that $A \subset N_\alpha$. We denote by $\langle A\rangle_{\mathcal{L}}$ the substructure of $\mathcal{M}$ generated by A, i.e. the substructure of $\mathcal{M}$*
$$\bigcap_{\alpha\in I} \mathcal{N}_\alpha,$$
as defined in Remark **??**.

The substructure $\langle A\rangle_{\mathcal{L}}$ can be constructed inductively in the following way:

Proposition 5.20. *Let $\mathcal{L}$ be a language of sorts S and let $\mathcal{M}$ and A be as in Definition 5.19. For any subset $B \subset M$, we denote $s(B) = B \cap s(\mathcal{M})$. For any sorted symbol $\mathfrak{s} \in \mathcal{L}$, denote by $S(\mathfrak{s})$ the sort of $\mathfrak{s}$. Let $\mathcal{C}$ be the set of constant symbols in $\mathcal{L}$ and let $\mathcal{F}$ be the set of function symbols in $\mathcal{L}$. Let*
$$A_0 = A \cup \{c^{\mathcal{M}} \mid c \in \mathcal{C}\}.$$
For any function symbol f, denote by $D(f)$ the domain of $f^{\mathcal{M}}$ and let
$$A_{i+1} = A_i \cup \{f^{\mathcal{M}}(a_1, \ldots, a_n) \mid f \in \mathcal{F},\ (a_1, \ldots, a_n) \in D(f) \cap A_i^n\}.$$
Then $\langle A\rangle_{\mathcal{L}} = \bigcup_{i\in\mathbb{N}} A_i$

Proof. See Proposition 1.2 in **??**. □

Example 5.21. Let $\mathcal{L}_R$ be the language of rings, as in Example 2.4. Let K be a field and let A be a subset of K. Then $\langle A\rangle_{\mathcal{L}_R}$ is the subring of K generated by the elements in A. For example, if $K = \mathbb{Q}$, and $A = \{0, \sqrt{2}\}$, then $\langle A\rangle_{\mathcal{L}_R} = \mathbb{Z} \oplus \mathbb{Z}\sqrt{2}$.

Lemma 5.22. *Let T be a theory in a language $\mathcal{L}$ of sorts S containing a constant symbol. Let $\mathcal{M}$ and $\mathcal{N}$ be two $\aleph_1$-saturated models of T with universes M and N respectively. Then the following are equivalent:*

(1) *The theory T eliminates quantifiers.*

(2) *Every existence formula with only one quantifier is equivalent to a quantifier free formula.*

(3) *If $A_0 \subset M$ and $B_0 \subset N$ are at most countable subsets and if there exists an isomorphism $f : \langle A_0 \rangle_{\mathcal{L}} \xrightarrow{\sim} \langle B_0 \rangle_{\mathcal{L}}$ of $\mathcal{L}$-structures sending A_0 to B_0 then for every element $a \in M$, there exists an isomorphism of $\mathcal{L}$-structures $g : \mathcal{M}' \xrightarrow{\sim} \mathcal{N}'$ extending f with $a \in M'$, $\mathcal{M}' \subset \mathcal{M}$ and $\mathcal{N}' \subset \mathcal{N}$.*

(4) *If $A_0 \subset M$ and $B_0 \subset N$ are finite subsets and if there exists an isomorphism $f : \langle A_0 \rangle_{\mathcal{L}} \xrightarrow{\sim} \langle B_0 \rangle_{\mathcal{L}}$ of $\mathcal{L}$-structures sending A_0 to B_0 then for every element $a \in M$, there exists an isomorphism of $\mathcal{L}$-structures $g : \mathcal{M}' \xrightarrow{\sim} \mathcal{N}'$ extending f with $a \in M'$, $\mathcal{M}' \subset \mathcal{M}$ and $\mathcal{N}' \subset \mathcal{N}$.*

Proof. (1) $\Leftrightarrow$ (2):
The implication (1) $\Rightarrow$ (2) is immediate. The other direction is proved by induction on the formulas. Intuitively, the idea is to remove one quantifier at the time. For the details, we refer to Lemma 3.2.4 in [TZ12].

(1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4):
Let A_0, B_0, f and a be as in (3). Denote $\mathcal{A} = \langle A_0 \rangle_{\mathcal{L}}$ and $\mathcal{B} = \langle B_0 \rangle_{\mathcal{L}}$. If A_0 and B_0 are countable, define $\mathcal{L}' = \mathcal{L} \cup \{c_0, c_1, \ldots\}$ with c_i being new constant symbols. Fix $\varphi_{A_0} : \mathbb{N} \xrightarrow{\sim} A_0$. If A_0 and B_0 are finite of cardinality n, define instead $\mathcal{L}' = \mathcal{L} \cup \{c_0, \ldots, c_n\}$ and $\varphi_{A_0} : \{1, \ldots, n\} \xrightarrow{\sim} A_0$. Let $\mathcal{M}'$ be the $\mathcal{L}'$ structure having the same universe as $\mathcal{M}$, with the interpretation $c_i^{\mathcal{M}'} = \varphi_{A_0}(i)$, as described before Definition 5.9. Let $\varphi_{B_0} = f \circ \varphi_{A_0}$ and let $\mathcal{N}'$ be the $\mathcal{L}'$-structure having the same universe as $\mathcal{N}$, with the interpretation $c_i^{\mathcal{N}'} = \varphi_{B_0}(i)$. Let Σ_{A_0} be the set of quantifier free $\mathcal{L}'$-sentences ψ such that $\mathcal{M}' \models \psi$ and let Σ_{B_0} be the set of quantifier free $\mathcal{L}'$-sentences ψ such that $\mathcal{N}' \models \psi$. Denote by $\mathcal{C}$ the set of new constant symbols added to $\mathcal{L}$ to obtain $\mathcal{L}'$.

Claim 1. In the situation above, we have that $\Sigma_{A_0} = \Sigma_{B_0}$.

Proof of Claim 1. Let $\psi(x_1, \ldots, x_k)$ be an $\mathcal{L}$-formula such that $\psi[c_{i_1}, \ldots, c_{i_k}] \in \Sigma_{A_0}$, where $c_{i_1}, \ldots, c_{i_k} \in \mathcal{C}$. By definition, $\mathcal{M} \models \psi[\bar{a}]$, where $\bar{a} = (\varphi_{A_0}(i_j)_{1 \leq j \leq k})$. Since $\mathcal{A}$ is a substructure of $\mathcal{M}$, we have that $\mathcal{A} \models \psi[\bar{a}]$. Since f is an isomorphism of $\mathcal{L}$-structures, this implies that $\mathcal{B} \models \psi[f(\bar{a})]$ and consequently that $\mathcal{N} \models \psi[f(\bar{a})]$. By definition, we then have that $\mathcal{N}' \models \psi$, so $\psi \in \Sigma_{B_0}$. The same argument gives that $\Sigma_{B_0} \subset \Sigma_{A_0}$, by noting that $f^{-1} \circ \varphi_{B_0} = \varphi_{A_0}$ and that f^{-1} is an isomorphism. This proves the claim.

Let Σ_a be the set of all $\mathcal{L}'$-formulas satisfied by a in $M' = M$. By definition, Σ_a is closed by finite conjunction. Also, Σ_a is maximal in the sense that if φ is an $\mathcal{L}'$-formula of the same sort as a, then either $\mathcal{M}' \models \varphi(a)$ or $\mathcal{M}' \models \neg\varphi(a)$, so either $\varphi \in \Sigma_a$ or $\neg\varphi \in \Sigma_a$. Furthermore, we can write all elements in Σ_a as $\varphi[c_0, \ldots, c_n, x]$, with φ being an $\mathcal{L}$-formula. We will use the notation $\varphi_{\bar{c}}(x)$. Since $\mathcal{M}' \models \exists x \varphi_{\bar{c}}(x)$ we get from quantifier elimination that $\mathcal{M}' \models \phi_{\bar{c}}$ for some quantifier free $\mathcal{L}'$-sentence $\phi_{\bar{c}}$ with constant symbols among $c_0, \ldots, c_n$. Hence $\phi_{\bar{c}}$ is in Σ_{A_0} and Σ_{B_0}, which gives that $\mathcal{N}' \models \phi_{\bar{c}}$ and so $\mathcal{N}' \models \exists x \varphi_{\bar{c}}(x)$. By definition of the interpretation of $c_0, \ldots, c_n$ in $\mathcal{N}'$, we thus get that $\mathcal{N} \models \exists x \varphi[f(\bar{a}), x]$. By $\aleph_1$-saturation of $\mathcal{N}$ and since Σ_a is closed under conjunction, we get that Σ_a is realised in $\mathcal{N}'$, i.e. there exists an element b in N satisfying all formulas in Σ_a. Let Σ_b be the set of all $\mathcal{L}'$-formulas satisfied by b in $N' = N$. By maximality of Σ_a, we get that $\Sigma_a = \Sigma_b$. Now consider the language $\widetilde{\mathcal{L}} = \mathcal{L}' \cup \{c\}$, where c is a new constant symbol of the same sort as the elements a and b. If we regard $\mathcal{M}'$ and $\mathcal{N}'$ as $\widetilde{\mathcal{L}}$-models $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ by interpreting $c^{\widetilde{\mathcal{M}}} = a$ and $c^{\tilde{\mathcal{N}}} = b$, we get that $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ satisfy the same quantifier free $\widetilde{\mathcal{L}}$-formulas.

Now, note that any element in A can be written as $t[\bar{a}, a]$ with t being a term and $\bar{a} = (a_1, \ldots, a_n) \in A_0^n$. Similarly, any element in B can be written as $t[\bar{b}, b]$ with $\bar{b} = (b_1, \ldots, b_n) \in B_0^n$. We define the map

$$\begin{aligned} h : A &\longrightarrow B \\ t[\bar{a}, a] &\longmapsto t[g(\bar{a}), b]. \end{aligned}$$

To see that it is well-defined and injective, note that $t_1[\bar{a}, a] = t_2[\bar{a}, a]$ if and only if $\widetilde{\mathcal{M}}$ satisfy the quantifier free $\widetilde{L}$-formula $t_1[\bar{c}, c] \doteq t_2[\bar{c}, c]$. This is equivalent to the fact that $\widetilde{\mathcal{N}}$ satisfy the same formula. Hence, $t_1[\bar{a}, a] = t_2[\bar{a}, a]$ if and only if $t_1[g(\bar{a}), b] = t_2[g(\bar{a}), b] \in N$, which shows that h is well-defined and injective. It is surjective by construction, since g maps A_0 onto B_0. To see that h is an isomorphism of $\mathcal{L}$-structures we need to verify that

$$R^{\mathcal{A}}(t_1[\bar{a}, a], \ldots, t_n[\bar{a}, a]) \Leftrightarrow R^{\mathcal{B}}(t_1[g(\bar{a}), b], \ldots, t_n[g(\bar{a}), b])$$

for any relation symbols R of $\mathcal{L}$ But since $R(t_1[\bar{c}, c], \ldots, t_n[\bar{c}, c])$ is a quantifier free $\widetilde{\mathcal{L}}$-formula, this follows from the fact that $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ satisfy the same quantifier free $\widetilde{\mathcal{L}}$-formulas.

(3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2):
Suppose there exists a formula $\exists x \varphi(\bar{y}, x)$ of sorts $(S_1, \ldots, S_n, S)$ which is not equivalent modulo T to a formula without quantifiers. Then, by Lemma 5.17, there are models $\mathcal{M}$, $\mathcal{N}$ of T and elements

$$\bar{a} = (a_1, \ldots, a_n) \in \prod_{i=1}^{n} S_i(\mathcal{M}) \qquad \text{and} \qquad \bar{b} = (b_1, \ldots, b_n) \in \prod_{i=1}^{n} S_i(\mathcal{N})$$

with $\bar{a}$ and $\bar{b}$ satisfying the same quantifier free formulas but $\mathcal{M} \models \exists x \varphi(\bar{a}, x)$ while $\mathcal{N} \models \forall x \neg\varphi(\bar{b}, x)$.

As every model has an $\aleph_1$-saturated elementary extension, we may assume that $\mathcal{M}$ and $\mathcal{N}$ are $\aleph_1$-saturated. Now consider the substructure $\mathcal{A} = \langle \bar{a} \rangle_{\mathcal{L}} \subset \mathcal{M}$ and $\mathcal{B} = \langle \bar{b} \rangle_{\mathcal{L}} \subset \mathcal{N}$. Every element in A is on the form $t[\bar{a}]$, with t being an $\mathcal{L}$-term. We define the homomorphism

$$g : \mathcal{A} \to \mathcal{B}$$

inductively on the elements $t^{\mathcal{M}}[\bar{a}] \in A$ by

(1) $g(a_i) = b_i$ for $i \in \{1, \ldots, n\}$.

(2) If t is a constant symbol, then $g(t^{\mathcal{M}}) = t^{\mathcal{N}}$.

(3) If $t^{\mathcal{M}}[\bar{a}] = f^{\mathcal{M}}(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in A$, then
$$g(t^{\mathcal{M}}[\bar{a}]) = f^{\mathcal{M}}(g(\alpha_1), \ldots, g(\alpha_n)).$$

That g is well defined and in fact an isomorphism follows by construction and from the fact that $\bar{a}$ and $\bar{b}$ satisfy the same quantifier free formulas. This is verified in the same way as for the morphism h in the previous part of the proof.

Let $a \in M$ be such that $\mathcal{M} \models \varphi(\bar{a}, a)$. But, then there is no isomorphism h extending g, having a in its domain and its image contained in N, as $g(\langle \bar{a}, a\rangle_{\mathcal{L}}) \models \varphi(\bar{b}, h(a))$ but $\mathcal{N} \not\models \varphi(\bar{b}, g(a))$. This contradicts (3) and (4), and so $\exists\varphi(\bar{y}, x)$ is equivalent to a quantifier free formula modulo T. □

6. Valued Fields

Let $(\Gamma, <)$ be an ordered abelian group. Consider the set $\Gamma \cup \{\infty\}$. We extend $<$ to this set by letting $\gamma < \infty$ for all $\gamma \in \Gamma$. Furthermore, we extend the group operation of Γ to $\Gamma \cup \{\infty\}$ by letting $\infty + \infty = \gamma + \infty = \infty + \gamma = \infty$ for all $\gamma \in \Gamma$.

We say that a field K is a **non-Archimedean valued field** with **value group** Γ if there exists a surjective mapping
$$\nu : K \longrightarrow \Gamma \cup \{\infty\}$$
satisfying the following properties:

(1) $\nu(a) = \infty$ if and only if $a = 0$.

(2) $\nu(ab) = \nu(a) + \nu(b)$.

(3) $\nu(a + b) \geq \min(\nu(a), \nu(b))$.

We call the map ν a **valuation** on K. We recall the following important properties, that can be found for example in [EP05].

(1) $\nu(1) = 0$.

(2) $\nu(a^{-1}) = -\nu(a)$ for all $a \in K^*$.

(3) $\nu(a) < \nu(b) \Rightarrow \nu(a + b) = \nu(a)$.

(4) The set $\mathcal{O}_\nu := \{a \in K \mid \nu(a) \geq 0\}$ is a **valuation ring** of K, i.e. a subring of K such that $a \in \mathcal{O}_\nu$ or $a^{-1} \in \mathcal{O}_\nu$, for all $a \in K^*$.

(5) The units of $\mathcal{O}_v$ is given by $\{a \in K \mid \nu(a) = 0\}$ and $M_\nu := \{a \in K \mid \nu > 0\}$ is the only maximal ideal of $\mathcal{O}_\nu$. The field $\mathcal{O}_\nu / M\nu$ is called the **residue field** of ν.

(6) For any valuation ring $\mathcal{O}$ of K, there exists a valuation ν on K such that $\mathcal{O}_\nu = \mathcal{O}$.

The following result, found for example as Theorem 3.2.15 in [EP05] will be an important tool when we look at the model theory of Algebraically Closed Valued Fields in the next section.

Theorem 6.1. *Suppose that K is a valued field with valuation ring $\mathcal{O}_K$ and that L/K is a normal extension of K. Let $\mathcal{O}_L$ and $\mathcal{O}'_L$ be valuation rings in L, such that $\mathcal{O}_L \cap K = \mathcal{O}'_L \cap K = \mathcal{O}_K$. Then, there exists an automorphism $\sigma \in \mathrm{Aut}(L/K)$ with $\sigma(\mathcal{O}_L) = \mathcal{O}'_L$.*

We will particularly look at algebraically closed valued fields. In that case we have that the value group Γ is a divisible group. That is, for any $\alpha \in \Gamma$ and any $n \in \mathbb{N}$, there exists an element $\beta \in \Gamma$ such that $\underbrace{\beta + \cdots + \beta}_{n \text{ times}} = n\beta = \alpha$. We write $\beta = \alpha/n$.
To see this, let $a \in K$ be such that $\nu(a) = \alpha$. Since K is algebraically closed, there exists an element $b \in K$ such that $b^n = a$. By properties of the valuation, we get that $n\nu(b) = \nu(b^n) = \nu(a) = \alpha$ and $\nu(b) = \alpha/n$. Furthermore, for any two element $\alpha, \beta \in \Gamma$ with $\alpha < \beta$, the element $\frac{\alpha+\beta}{2} \in \Gamma$. Since $\alpha < \frac{\alpha+\beta}{2} < \beta$, we have that $<$ is a dense order on Γ.

Also, if K is algebraically closed it follows that the residue field k is algebraically closed. Indeed, let

$$\bar{P}(X) = X^n + \sum_{i=0}^{n-1} \bar{a}_i X^i \in k[X],$$

and consider a lift

$$P(X) = X^n + \sum_{i=0}^{n-1} a_i X^i \in \mathcal{O}_\nu[X].$$

Let $b \in K$ be a root of P. Since $b^n = -\sum_{i=0}^{n-1} a_i b^i$ and $\nu(a_i) \geq 0$, we have by the strong triangle inequality that

$$n\nu(b) \geq \min_{0 \leq i \leq n-1} \{\nu(a_i) + i\nu(b)\} \geq \min_{1 \leq i \leq n-1} \{i\nu(b)\}.$$

If $n\nu(b) = \min_{1 \leq i \leq n-1}\{i\nu(b)\}$, then $n\nu(b) = i\nu(b)$ for some $i < n$, so $\nu(b) = 0$. If the inequality is strict, then $\frac{n}{n-1}\nu(b) > \nu(b)$, so $\nu(b) > 0$. So in any case, we have that $b \in \mathcal{O}_\nu$, and so $\bar{b} \in k$ is a root of $\bar{P}$.

Let K be an algebraically closed valued field with value group Γ. With $<$ being the order on Γ, we equip the group $\Gamma \oplus \mathbb{Q}$ with an order $\lessdot$, defined by

$$(\gamma_1, q_1) \lessdot (\gamma_2, q_2) :\Leftrightarrow \begin{cases} \gamma_1 < \gamma_2 & \text{or} \\ \gamma_1 = \gamma_2 & \text{and } q_1 < q_2. \end{cases}$$

It is immediate from the definition that $(\Gamma \oplus \mathbb{Q}, <)$ is a totally ordered divisible abelian group. If we regard Γ as a subgroup of $\Gamma \oplus \mathbb{Q}$ under the embedding $\gamma \mapsto (\gamma, 0)$, we get that the order defined on $\Gamma \oplus \mathbb{Q}$ extends the order on Γ. Furthermore, we have that $(0,0) < (0,q) < (\gamma, 0)$ for all $\gamma \in \Gamma$. We will denote the element

$(0,1)$ by ϖ and write $\Gamma \oplus \mathbb{Q}\varpi$ for the totally ordered group divisible abelian group $(\Gamma \oplus \mathbb{Q}, <)$. Similarly, we will denote by

$$\Gamma \bigoplus_{i=1}^{n} \mathbb{Q}\varpi_i$$

the totally ordered abelian group obtained by repeating the above construction n times. That is, $\Gamma \bigoplus_{i=1}^{n} \mathbb{Q}\varpi_i := (\Gamma \bigoplus_{i=1}^{n-1} \mathbb{Q}\varpi_i) \oplus \mathbb{Q}$ with the order $\lessdot$ defined above.

Proposition 6.2. *Let G be a torsion-free abelian group. Let*

$$\mathbb{Q}G = \{(g,n) \mid g \in G,\ n \in \mathbb{N}_{>0}\}/\sim$$

where $\sim$ is the equivalence relation defined by

$$(g,n) \sim (h,n) :\Leftrightarrow mg = nh.$$

Then the following hold:

(1) *The set $\mathbb{Q}G$ together with the operation $[(g,n)] + [(h,n)] = [(mg+nh, mn)]$ is a divisible abelian group, and $i : G \hookrightarrow \mathbb{Q}G;\ g \mapsto [(g,1)]$ is an embedding.*

(2) *If H is a divisible group and $j : G \hookrightarrow H$ is an embedding, then there exists an embedding $h : \mathbb{Q}G \hookrightarrow H$ such that $j = h \circ i$.*

Proof. The first part verified just as when one construct the rational numbers. For the second part, one can verify that $h : \mathbb{Q}G \hookrightarrow H;\ [(g,n)] \mapsto j(g)/n$ is an embedding satisfying $j = h \circ i$. For a complete proof, see Lemma 3.1.8 in [Mar02]. □

Lemma 6.3. *Let K be a valued field with valuation ν_K and value group Γ. Let ν'_K be an extension of ν_K to an algebraic closure K^{alg} of K. Then $\nu_K(K^{\text{alg}}) \cong \mathbb{Q}\Gamma$, with $\mathbb{Q}\Gamma$ being as in Proposition 6.2*

Proof. Let Γ' be the value group of K^{alg}. Since Γ' is divisible and Γ is a subgroup of Γ', we have that $\mathbb{Q}\Gamma$ embeds in Γ'. For the remainder of the proof, we identify $\mathbb{Q}\Gamma$ with its image under this embedding in Γ'. It rests to show that $\Gamma' \subset \mathbb{Q}\Gamma$. For contradiction, suppose that this does not hold. Then, there exists an element $\beta \in \Gamma'$ such that $\beta \notin \mathbb{Q}\Gamma$. Let $b \in K^{\text{alg}}$ be such that $\nu'_K(b) = \beta$. Then $P(b) = 0$ for some $P(X) = \sum_{i=1}^{n} a_i X^i \in K[X]$. We get that $b = -\sum_{i=1}^{n} a_i b^i$. Since $\nu_L(a_i b^i)\nu_L(a_i)+i\beta$ for any $i \in \{1,\dots,n\}$ and since $\beta \notin \mathbb{Q}\Gamma$, we have that $\nu_L(a_i b^i) \neq \nu_L(a_j b^j)$ for any distinct i and j in $\{1,\dots,n\}$. By the strong triangle inequality, we get that $\beta = \min_{1\leq i\leq n}(\nu_L(a_i b^i)) = \nu_L(a_{i_0}) + i_0\beta$ for some $i_0 \in \{1,\dots,n\}$. This implies that $\beta = \nu_L(a_{i_0})/(1-i_0)$. But since $\mathbb{Q}\Gamma$ is divisible, we get that $\beta \in \Gamma \oplus \mathbb{Q}\varpi$. This contradicts our choice of β, so $\Gamma' \subset \mathbb{Q}\Gamma$, and we are done. □

Lemma 6.4. *Let K be an algebraically closed valued field with value group Γ and let $\Gamma \oplus \mathbb{Q}\varpi$ be the totally ordered abelian group defined above. Then, there exists an algebraically closed valued field extension L/K such that L has value group $\Gamma \oplus \mathbb{Q}\varpi$.*

This lemma can be proved by constructing such a valuation explicitly. However, since this text focuses on the applications of Model Theory to Valued Fields, we will give a Model Thoeretic proof instead.

Proof. Let (K, Γ) be a model of ACVF and denote by κ be the cardinality of K. Let $(\widetilde{K}, \widetilde{\Gamma})$ be a model of ACVF such that $(\widetilde{K}, \widetilde{\Gamma})$ is a κ-saturated elementary extension of (K, Γ). Let $A = \Gamma_{>0}$ and let $\Sigma(\gamma)$ be the set of $\mathcal{L}(A)$-formulas defined as

$$\Sigma(\gamma) = \{0 < \gamma \wedge \gamma < \beta \mid \beta \in A\}.$$

Since Γ is divisible, we have that $\Sigma(\gamma)$ is finitely satisfiable in $(\widetilde{K}, \widetilde{\Gamma})$. Since A has cardinality less than κ and since $(\widetilde{K}, \widetilde{\Gamma})$ be κ-saturated, there is an element $\varpi \in \widetilde{\Gamma}$ such that $0 < \varpi < \alpha$, for all $\alpha \in \Gamma_{>0}$. Let $a \in \widetilde{K}$ be an element such that $\nu_{\widetilde{K}}(a) = \varpi$. Denote by ν'_K the restriction of $\nu_{\widetilde{K}}$ to $K(a)$. This is a valuation which extends ν_K. Denote by Γ' value group of $K(a)$. We have that Γ' is contained in $\Gamma \oplus \mathbb{Q}\varpi$ and it contains ϖ. Hence, it follows that $\mathbb{Q}\Gamma' \cong \Gamma \oplus \mathbb{Q}\varpi$. Denote by L the algebraic closure of $K(a)$ and extend ν'_K to a valuation ν_L on L. Denote by Γ_L the value group of (L, ν_L). By Lemma 6.3, we get that $\Gamma_L \cong \Gamma \oplus \mathbb{Q}\varpi$, which proves the lemma. $\square$

7. Definable sets.

Definition 7.1. *Let $\mathcal{L}$ be a language of sort S and let $\mathcal{M}$ be an $\mathcal{L}$-structure. Let $\varphi(\bar{x}, \bar{y})$ be a formula, with $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_m)$. We allow the situation $n = 0$, in this case φ has no free variables among the x_i. Suppose that the sort of each variable x_i is s_i and that the sort of each variable y_i is s'_i. Let $\bar{a} \in \prod_{i=1}^n s_i(M)$. Then the set*

$$B = \left\{\bar{b} \in M^m \mid \mathcal{M} \models \varphi[\bar{a}, \bar{b}]\right\}$$

*is called a **definable set**. We also say that B is **defined** over $\bar{a}$ by $\varphi[\bar{a}, \bar{y}]$, or that it is $\bar{a}$-**definable**. In this situation, $\bar{a}$ is called a **parameter** of $\varphi[\bar{a}, \bar{y}]$. If φ has no free variables among the x_i, we say that B is **defined without parameters**.*

Example 7.2. Consider $\mathbb{Q}$ as a model of the $\mathcal{L}_R$-theory T_{field}, as described in Example 4.4. Then the empty set is a definable set in $\mathbb{Q}$, defined for example by the formula $\varphi(x) = (x^2 \doteq 1 + 1)$. That is, we have that

$$\emptyset = \{q \in \mathbb{Q} \mid \mathbb{Q} \models \varphi[q]\}.$$

If we let $\psi(x)$ be the formula $x^2 \doteq -1$, we also have that

$$\emptyset = \{q \in \mathbb{Q} \mid \mathbb{Q} \models \psi[q]\}.$$

If we now consider $\mathbb{R}$ as a model of T_{field}, we have that

$$\emptyset = \{r \in \mathbb{R} \mid \mathbb{R} \models \psi[r]\} \neq \{r \in \mathbb{R} \mid \mathbb{R} \models \psi[r]\}.$$

This shows that two formulas that define the same set in a model of some theory T, does not in general define the same set in an extension of that model. The following result however will show that in a theory which admits quantifier elimination, two formulas define the same set in a model if and only if they define the same set in every extension of that model.

Lemma 7.3. *Let $\mathcal{L}$ be a language of sorts S and let T be an $\mathcal{L}$-theory which admits quantifier elimination. Let $\mathcal{M}$ be a model of T and let $\varphi(\bar{x},\bar{y})$ be an $\mathcal{L}$-formula with $\bar{x}=(x_1,\dots,x_m)$ and $\bar{y}=(y_1,\dots,y_n)$. Denote by s_i the sort of x_i and denote by s'_i the sort of y_i. For $\bar{a}=(a_1,\dots,a_m)\in\prod_{i=1}^m s_i(\mathcal{M})$, we denote by $\varphi_{\bar{a}}$ the set defined over $\bar{a}$ by φ, i.e.*

$$\varphi_{\bar{a}}(\mathcal{M}) := \left\{ \bar{b} \in \prod_{i=1}^{n} s'_i(\mathcal{M}) \mid \mathcal{M} \models \varphi[\bar{a},\bar{b}] \right\}.$$

Then, the following are equivalent:

(1) $\varphi_{\bar{a}}(\mathcal{M})=\emptyset$

(2) *For every model $\mathcal{M}'$ of T which is an extension of $\mathcal{M}$, we have that*
$$\varphi_{\bar{a}}(\mathcal{M}')=\emptyset.$$

Proof. Suppose that $\varphi_{\bar{a}}(\mathcal{M})=\emptyset$ and let $\mathcal{M}'$ be an extension of $\mathcal{M}$ which is a model of T. Define the formula $\psi(\bar{y}) := \exists\bar{x}\varphi(\bar{x},\bar{y})$. By quantifier elimination, we have that ψ is equivalent to a quantifier formula χ modulo T. Hence, $\psi[\bar{a}]$ is an $\mathcal{L}(\bar{a})$-sentence which is equivalent to a quantifier free $\mathcal{L}(\bar{a})$-sentence modulo T', where T' is the theory T regarded as an $\mathcal{L}(\bar{a})$-theory. Since $\mathcal{M}'$ is an extension of $\mathcal{M}$, we have that any $\mathcal{L}(\bar{a})$-sentence is true in $\mathcal{M}'$ if and only if it is true in $\mathcal{M}$. In particular, $\mathcal{M}\models\psi[\bar{a}]$ if and only if $\mathcal{M}'\models\psi[\bar{a}]$. Since $\mathcal{M}\models\psi[\bar{a}]$ if and only if $\varphi_{\bar{a}}(\mathcal{M})\neq\emptyset$ and $\mathcal{M}'\models\psi[\bar{a}]$ if and only if $\varphi_{\bar{a}}(\mathcal{M}')\neq\emptyset$, it follows that $\varphi_{\bar{a}}(\mathcal{M}')=\infty$. Hence, (1) $\Rightarrow$ (2). The other implication is immediate since $\mathcal{M}$ is an extensions of itself. □

Corollary 7.3.1. *Let $\mathcal{L}$ be a language of sorts S and let T be an $\mathcal{L}$-theory which admits quantifier elimination. Let $\mathcal{M}$ be a model of T and let $\bar{x}$, $\bar{y}$ and $\bar{a}$ be as in Lemma 7.3. Let $\varphi(\bar{x},\bar{y})$ be an $\mathcal{L}$-formula and let $\psi(\bar{x},\bar{y})$ be a quantifier free $\mathcal{L}$-formula such that $\varphi_{\bar{a}}(\mathcal{M})=\psi_{\bar{a}}(\mathcal{M})$. Let $\mathcal{M}'$ be an extension of $\mathcal{M}$. Then*
$$\varphi_{\bar{a}}(\mathcal{M}')=\psi_{\bar{a}}(\mathcal{M}').$$

Proof. If $\varphi_{\bar{a}}(\mathcal{M})\subset\psi_{\bar{a}}(\mathcal{M})$, then $\varphi_{\bar{a}}(\mathcal{M})\setminus\psi_{\bar{a}}(\mathcal{M})=\emptyset$. Hence, $(\varphi\wedge\neg\psi)_{\bar{a}}(\mathcal{M})=\emptyset$, and by Lemma 7.3, $(\varphi_{\bar{a}}\wedge\neg\psi)_{\bar{a}}(\mathcal{M}')=\emptyset$ and $\varphi_{\bar{a}}(\mathcal{M}')\subset\psi_{\bar{a}}(\mathcal{M}')$. The same argument with $(\psi\wedge\neg\varphi)_{\bar{a}}$ instead of $(\varphi\wedge\neg\psi)_{\bar{a}}$ shows the inclusion $\psi_{\bar{a}}(\mathcal{M}')\subset\varphi_{\bar{a}}(\mathcal{M}')$. □

8. Quantifier Elimination of Algebraically Closed Valued Fields

Consider the 2-sorted language
$$\mathcal{L}_\Gamma = \{+_R, -_R, \cdot, 1, 0_R\} \cup \{0_\Gamma, +_\Gamma, -_\Gamma, <, \infty\} \cup \{\nu\}$$
of sorts R and Γ, where $\{+_R,-_R,\cdot,1,0_R\}$ is the language of rings, $\{0_\Gamma,+,-_R,\Gamma,<\}$ is the language or ordered abelian monoids, the symbol ∞ is a constant symbol of sort Γ and ν is a function symbols of sort (R,Γ). We use $t_1\le t_2$ as an abbreviation for $t_1<t_2\vee t_1\doteq t_2$. When it is clear from context, we will omit the subscripts indicating the sort of the symbols $+_R$, $+_\Gamma$, 0_R and 0_Γ.

Any valued field K is an L_Γ-structure. Furthermore, we can define the theory ACVF of Algebraically Closed Valued Fields by the following sentences together with the theory of algebraically closed fields, where we use the notation x, y for variables of sort R and α, β, γ for variables of sort Γ:

(1) $\forall\alpha\forall\beta(\alpha + \beta \doteq \beta + \alpha)$

(2) $\forall\alpha(\infty + \alpha = \infty)$

(3) $\forall\alpha(\alpha \not\doteq \infty \to \exists\beta(\alpha + \beta \doteq 0))$

(4) $\forall\alpha\forall\beta(\alpha \leq \beta \vee \beta \leq \alpha)$

(5) $\forall\alpha\forall\beta((\alpha \leq \beta \wedge \beta \leq \alpha) \to \alpha \doteq \beta)$

(6) $\forall\alpha\forall\beta\forall\gamma((\alpha \leq \beta \wedge \beta \leq \gamma) \to \alpha \leq \gamma)$

(7) $\forall x(\nu(x) \doteq \infty \leftrightarrow x \doteq 0)$

(8) $\forall x\forall y(\nu(xy) \doteq \nu(x) + \nu(y))$

(9) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

It is also possible to formulate the theory of ACVF in a one-sorted language $\mathcal{L}_{\text{div}}$ which consists of the language of ring and a binary relation symbol div. In this language we interpret $a \operatorname{div} b \Leftrightarrow$ as $\nu(a) \leq \nu(b)$. It is well documented, see for example [PD11], that the theory of ACVF eliminates quantifiers in this one-sorted language. However, this language is not sufficient for the application we want to consider further on EXPLAIN HOW FURTHER ON, and we therefore prove that quantifier elimination holds also for the 2-sorted language described above. The proof of this is an adaptation of the proof for the one-sorted case found in [PD11].

Theorem 8.1. *The theory* ACVF *admits quantifier elimination in the language* $\mathcal{L}_\Gamma$.

Proof. We will show that ACVF admits quantifier elimination by proving the equivalent statement (4) in Lemma 5.22. To do this, let (K, Γ_K) and (L, Γ_L) be $\aleph_1$-saturated models of ACVF in $\mathcal{L}_\Gamma$. Such models exists due to Theorem 5.14. Let $A_0 \subset K$, $B_0 \subset L$, $\Gamma_{A_0} \subset \Gamma_K$ and $\Gamma_{B_0} \subset \Gamma_L$ all be finite subsets. Denote by (A, Γ_A) and (B, Γ_B) the $\mathcal{L}_\Gamma$-structures generated by (A_0, Γ_{A_0}) and (B_0, Γ_{B_0}) respectively and suppose that $g : A \sqcup \Gamma_A \to B \sqcup \Gamma_B$ is an $\mathcal{L}_\Gamma$-isomorphism such that $g(A_0) = B_0$ and $g(\Gamma_{A_0}) = \Gamma_{B_0}$. Now let $a \in K$. To show that (4) in Lemma 5.22 holds, we first need to show that g extends to an $\mathcal{L}_\Gamma$-isomorphism $h : A' \sqcup \Gamma_{A'} \to B' \sqcup \Gamma_{B'}$ where $a \in A' \subset K$, $\Gamma_A \subset \Gamma_{A'}$, $\Gamma_B \subset \Gamma_{B'}$ and $B' \subset L$.

First, we assume that A and B are fields. This can be done since the extension of g to the quotient fields of A and B respects the ordering on Γ_A. Note however that (A, Γ_A) and (B, Γ_B) are not necessarily valued fields, in the sense that $\nu_K : A \to \Gamma_A$ and $\nu_L : B \to \Gamma_B$ are not necessarily surjective. As a first step, we will show that the existence of g induces an isomorphism of valued fields containing (A, Γ_A) and (B, Γ_B) as substructures.

If we denote by h the restriction of g to the $\mathcal{L}_\Gamma$-structure $(A, \nu_K(A))$ we get an isomorphism of valued fields

$$h\ : A \sqcup \nu_K(A) \longrightarrow B \sqcup \nu_L(B).$$

Let $a \in K$ and suppose that a is algebraic over A. Let A^{alg} and B^{alg} be the algebraic closures of A and B respectively and extend h to an isomorphism between A^{alg} and B^{alg}. Then $h(\mathcal{O}_{A^{\mathrm{alg}}})$ is a valuation ring of B^{alg} such that $h(\mathcal{O}_{A^{\mathrm{alg}}}) \cap B = \mathcal{O}_B$. Hence, $h(\mathcal{O}_{A^{\mathrm{alg}}})$ defines a valuation ν'_L on B^{alg} that extends ν_L on B. By Theorem 6.1, there exists $\sigma \in \mathrm{Aut}(B^{\mathrm{alg}}/B)$ such that $\nu'_L = \nu_L \circ \sigma$. Hence, for $c \in A^{\mathrm{alg}}$ we have

$$\begin{aligned}\nu_K(c) \geq 0 \Leftrightarrow c \in \mathcal{O}_{A^{\mathrm{alg}}} \Leftrightarrow h(c) \in \mathcal{O}_{\nu'_L} \geq 0 \\ \Leftrightarrow \nu'_L(h(c)) \geq 0 \Leftrightarrow \nu_L \circ \sigma \circ h(c) \geq 0.\end{aligned}$$

This shows that the isomorphism $\sigma \circ h$ respects the ordering on $\nu_K(A^{\mathrm{alg}})$. Therefore, we can assume that A and B are algebraically closed. Let $S = \{\alpha_0, \ldots, \alpha_n\} \subset \Gamma_{A_0}$ be the set of elements such that $\alpha_i \notin \nu_K(A)$ and let $\beta_i = g(\alpha_i)$. If $\beta_i \in \nu_L(B)$ then

$$g(\alpha_i) = \beta_i = \nu_L(g(c)) = g(\nu_K(c))$$

for some $c \in A$, where the last equality follows from the fact that g is an isomorphism of $\mathcal{L}_\Gamma$-structures. This contradicts $\alpha_i \notin \nu_K(A)$, so $\beta_i \notin \nu_L(B)$. Let $a_i \in K$ be such that $\nu_K(a_i) = \alpha_i$. Then a_i is transcendental over A, since A is assumed to be algebraically closed. Similarly, there are elements $b_i \in L$ such that $\nu_L(b_i) = \beta_i$ which is transcendental over B. We thus get that h extends to an isomorphism of $\mathcal{L}_\Gamma$-structures

$$h: \ A(a_n) \sqcup \nu_K(A(a_n)) \longrightarrow B(b_n) \sqcup \nu_L(B(b_n)).$$

Again taking the algebraic closures of $A(a_n)$ and $B(b_n)$, we can extend h to an isomorphism between these algebraically closed valued fields, with value groups containing α_n and β_n respectively. Thus, we can assume that $S = \{\alpha_0, \ldots, \alpha_{n-1}\}$. By induction, we can then assume that $S = \emptyset$, i.e. that $\nu_K|_A$ and $\nu_L|_B$ are surjective. This means that we only have to consider the case where $g : A \sqcup \Gamma_A \to B \sqcup \Gamma_B$ is an isomorphism of algebraically closed valued fields. In this situation, we will regard three different cases.

(1) The value group $\Gamma_{A(a)} := \nu_K(A(a))$ is not equal to Γ_A.

(2) The residue field $k_{A(a)}$ is not equal to k_A.

(3) All other cases, i.e. $\Gamma_{A(a)} = \Gamma_A$ and $k_{A(a)} = k_A$.

Case 1. In this case, there exists element $c \in A(a)$ such that $\gamma := \nu_K(c) \notin \Gamma_A$. Since A and B are algebraically closed, the value groups Γ_A and Γ_B are divisible. Furthermore, we have that

$$\nu_K(A(c)^\times) = \Gamma_A \oplus \langle\gamma\rangle.$$

Let $S_- = \{\alpha \in \Gamma_A \mid \gamma > \alpha\}$ and let $S_+ = \{\alpha \in \Gamma_A \mid \gamma < \alpha\}$. Consider now the set of $\mathcal{L}_\Gamma(\Gamma_A)$-formulas

$$\Sigma(\xi) = \{\xi > \alpha \mid \alpha \in S_-\} \cup \{\xi < \alpha \mid \alpha \in S_+\}$$

and its image under g of $\mathcal{L}_\Gamma(\Gamma_B)$-formulas

$$\Sigma^g(\xi) = \{\xi > g(\alpha) \mid \alpha \in S_-\} \cup \{\xi < g(\alpha) \mid \alpha \in S_+\}.$$

Recall that we have obtained B by taking the algebraic closure of the quotient field of a finitely generated ring. Thus, B is countable, and so is Γ_B since it is the image of B under ν_L. The set $\Sigma^g(x)$ is finitely satisfiable in (L, Γ_L), since Γ_L is dense linearly ordered without endpoints. Thus, by $\aleph_1$-saturation, there exists an element $\delta \in \Gamma_L$ satisfying all formulas in $\Sigma^g(\xi)$ since Γ_B is countable. Note that $\delta \notin \Gamma_B$.

We now extend the isomorphism g on the value groups to $g : \Gamma_A \oplus \langle\gamma\rangle \longrightarrow \Gamma_B \oplus \langle\delta\rangle$ by setting $g(\gamma) = \delta$. To see that this is indeed an isomorphism of ordered groups, let $\alpha + m\gamma, \alpha' + m'\gamma \in \Gamma_A \oplus \langle\gamma\rangle$ and suppose that $\alpha + m\gamma < \alpha' + m'\gamma$. If $m = m'$, then $\alpha < \alpha'$ and $g(\alpha) + m\delta < g(\alpha') + m'\delta$. If $m \neq m'$ we get

$$\alpha + m\gamma < \alpha' + m'\gamma \iff \frac{\alpha - \alpha'}{m - m'} < \gamma.$$

Since Γ_A is divisible, $\frac{\alpha-\alpha'}{m-m'} \in \Gamma_A$. By definition of $\Sigma^g(x)$ we then get that

$$g\left(\frac{\alpha - \alpha'}{m - m'}\right) = \frac{g(\alpha) - g(\alpha')}{m - m'} < \delta,$$

which is equivalent to

$$g(\alpha) + m\delta < g(\alpha') + m'\delta.$$

Hence, g preserves the ordering. Using this, we also get that g is injective since any non-zero element is mapped to a non-zero element.

Let $a_0, \dots, a_n \in A$. Suppose that $\nu_K(a_i c^i) = \nu_K(a_j c^j)$ for some $i \neq j$. Then

$$\nu_K(a_i) - \nu_K(a_j) = (j - i)\gamma.$$

This contradicts that $\gamma \notin \Gamma_A$, since Γ_A is divisible. Therefore we have

$$\nu_K\left(\sum_{i=1}^{n} a_i c^i\right) = \min_{1\leq i\leq n}\{\nu_K(a_i) + i\gamma\}.$$

By the same reasoning as above, we get that

$$\nu_L\left(\sum_{i=0}^{n} b_i d^i\right) = \min_{1\leq i\leq n}\left\{\nu_L(b_i) + i\delta\right\}.$$

Let $a_0, \dots, a_n \in A$ and

$$\nu_K\left(\sum_{i=0}^{i} a_i c^i\right) = a_j + j\gamma.$$

This is then equivalent to $\nu_K(a_i) + i\gamma > \nu_K(a_j) + j\gamma$ for all i in $\{0, \dots, n\} \setminus \{j\}$, which is equivalent to

$$\frac{\nu_K(a_i) - \nu_K(a_j)}{j - i} > \gamma.$$

Since Γ_A is divisible, we have that $(\nu_K(a_i) - \nu_K(a_j))/(j - i) \in \Gamma_A$. By definition of $\Sigma(\xi)$ and $\Sigma^g(\xi)$, we thus get

$$\frac{\nu_L(g(a_i)) - \nu_L(g(a_j))}{j - i} > \delta \iff \nu_L(g(a_i)) + i\delta > \nu_L(g(a_j)) + j\delta$$

for all such i. This gives

$$\nu_L\left(\sum_{i=0}^{i} g(a_i) d^i\right) = \nu_L(g(a_j)) + j\delta$$

and so the extension of g which sends c to d is an isomorphism of valued rings $h : A[c] \to B[d]$. By extending again to the quotient fields, we get an isomorphism of valued fields $h : A(c) \to B(d)$. Using the same argument as above, we can extend g to an isomorphism between $A(c)^{\mathrm{alg}}$ and $B(c)^{\mathrm{alg}}$. Since $c \in A(a)$, we have that

$$c = \frac{P(a)}{Q(a)}$$

for $P(a) \in A[a]$ and $Q(a) \in A[a] \setminus \{0\}$. So $c \cdot Q(a) - P(a) = 0$ and a is a root of the polynomial $c \cdot Q(X) - P(X) \in A(c)$. Hence, $a \in A(c)^{\mathrm{alg}}$ and we have proved the existence of h in Case 1.

Case 2. Suppose there exists an element $c \in \mathcal{O}_{A(a)}$ such that $\mathrm{res}(c) \notin k_A$. Since k_A is algebraically closed, this means that $\mathrm{res}(c)$ is transcendental over k_A. Consider the set of $\mathcal{L}_\Gamma(\mathcal{O}_L)$-formulas

$$\Sigma(x) = \left\{\nu(x) \doteq 0\right\} \cup \left\{\nu(x-b) \doteq 0 \mid b \in \mathcal{O}_B\right\}.$$

By definition, an element $d \in L$ satisfying $\Sigma(x)$ is a unit in $\mathcal{O}_B$ such that $\mathrm{res}(d) \neq \mathrm{res}(b)$ for all $b \in \mathcal{O}_L$. This set is finitely satisfiable in L, since k_L is infinite (SHOW THIS). So by $\aleph_1$-saturation it is realisable in (L, Γ_L). Since k_B is algebraically closed, $\mathrm{res}(d)$ is transcendental over k_B.

Let $a_0, \ldots, a_n \in A$. We will show that $\nu_K\left(\sum_{i=1}^n a_i c^i\right) = \min_i\{\nu_K(a_i)\}$. Suppose that all $\nu_K(a_i)$ are equal. Then $\nu_K(a_i/a_1) = 0$ and $a_i/a_1 \in \mathcal{O}_K$. We get that

$$\mathrm{res}\left(\sum_{i=1}^n \frac{a_i}{a_1} c^i\right) = \sum_{i=1}^n \mathrm{res}\left(\frac{a_i}{a_1}\right) \mathrm{res}(c)^i.$$

This element is not zero, since $\mathrm{res}(c)$ is transcendental over k_A. Hence

$$\sum_{i=1}^n \frac{a_i}{a_1} c^i \notin M_K$$

and

$$\nu_K\left(\sum_{i=1}^n a_i c^i\right) = \nu_K\left(a_1 \sum_{i=1}^n \frac{a_i}{a_1} c^i\right) = \nu_K(a_1) + \nu_K\left(\sum_{i=1}^n \frac{a_i}{a_1} c^i\right) = \nu_K(a_1).$$

If not all $\nu_K(a_i)$ are equal, we write

$$\sum_{i=1}^n a_i c^i = \sum_{i \in I_{\alpha_1}} a_i c^i + \cdots + \sum_{i \in I_{\alpha_k}} a_i c^i$$

with $I_\alpha = \{i \mid \nu_K(a_i) = \alpha\}$. Then, as we have just shown,

$$\nu_K\left(\sum_{i \in I_\alpha} a_i c^i\right) = \alpha$$

and so, due to the ultrametric inequality,

$$\nu_K\left(\sum_{i \in I_{\alpha_1}} a_i c^i + \cdots + \sum_{i \in I_{\alpha_k}} a_i c^i\right) = \min_{1 \leq j \leq k}\{\alpha_j\} = \min_{1 \leq i \leq n}\{\nu_K(a_i)\}.$$

By the same reasoning,

$$\nu_L\left(\sum_{i=1}^n b_i d^i\right) = \min_i\left\{\nu_L(b_i)\right\}$$

for $b_0, \dots, b_n \in B$. This shows that the ring isomorphism $h : A[c] \to B[d]$ that extends g and sends c to d is an isomorphism of valued rings, since g preserves the order on A. The isomorphism h extends uniquely to an isomorphism of valued fields $A(c) \to B(d)$. Since a is algebraic over $A(c)$, as shown in Case 1, the result of the theorem follows by extending g to the algebraic closure of $A(c)$ in the same way as above.

Case 3. Suppose that $k_{A(a)} = k_A$ and $\Gamma_{A(a)} = \Gamma_A$. Let $I = \{\nu_K(a-c) \mid c \in A\}$. If $\nu_K(a-c) \in I$ and $\nu_K(d) < \nu_K(a-c)$, then $\nu_K(a-c+d) = \nu_K(d)$ and so $\nu_K(d) \in I$. Let $e \in A$ such that $\nu_K(e) = \nu_K(a-c)$. Then $\nu_K((a-c)/e) = 0$. Since $k_{A(a)} = k_A$, there exists an element $d \in \mathcal{O}_A$ such that $\operatorname{res}((a-c)/e) = \operatorname{res}(d)$. Hence $(a-c)/e - d \in M_{A(a)}$ and $\nu_K((a-c)/e - d) > 0$. This gives

$$\nu_K(a-c-de) = \nu_K((a-c)/e - d) + \nu_K(a-c) > \nu_K(a-c),$$

and so I has no maximal element.

Consider now the set of $\mathcal{L}_\Gamma(B)$-formulas

$$\Sigma(x) = \{\nu(x - g(c)) = \nu(f(d)) \mid c, d \in A,\ \nu_K(a-c) = \nu_K(d)\}.$$

This set is finitely satisfiable in L. Indeed, let $(c_1, d_1), \dots, (c_n, d_n) \in A^2$ be such that $\nu_K(a-c) = \nu_K(d)$ and let $e \in A$ such that $\nu_K(e-a) > \nu_K(a-c_i)$ for all $i \in \{1, \dots, n\}$. Then

$$\nu_K(e-c_i) = \nu_K(e-a+a-c_i) = \nu_K(a-c_i).$$

Since g is an $\mathcal{L}_\Gamma$-isomorphism, we have that $g(e)$ satisfies $\nu_L(x - g(c_i)) = \nu_L(g(d_i))$ for all $i \in \{1, \dots, n\}$. By $\aleph_1$-saturation of (L, Γ_L), there is therefore an element $b \in L$ satisfying $\Sigma(x)$. Note that this element is not in B, since otherwise we would have that b satisfies the formula

$$\nu(x - g(g^{-1}(b)) = \nu(g(d))$$

with $\nu_K(a - g^{-1}(b)) = \nu_K(d)$. But then $\nu_L(g(d)) = \infty$ and so $\nu_K(d) = \infty$, which contradicts that I has no maximal element. Hence, b is transcendental over B, since B is algebraically closed. We get that $g|_A$ extends to an isomorphism $h : A(a) \to B(b)$ sending a to b.

It now rests to show that h is an isomorphism of valued fields. Let $P(T) \in A[T]$ be a polynomial. Since A is algebraically closed, we can write

$$P(T) = c \prod_{i=1}^{n} (T - a_i)$$

with $c, a_i \in A$. Consequently,

$$\nu_K\left(P(a)\right) = \nu_K(c) + \sum_{i=1}^{n} \nu_K(a - a_i).$$

We get that

$$\begin{aligned} \nu_L(h(P(a)) = \nu_L\left(h(c)\prod_{i=1}^{n} h(a-a_i)\right) &= \nu_L(g(c)) + \prod_{i=1}^{n} \nu_L(b - g(a_i)) \\ &= g(\nu_K(c)) + \sum_{i=1}^{n} g(\nu_K(a-a_i)), \end{aligned}$$

where the last equality follows from the fact definition of $\Sigma(x)$. This shows that h is an isomorphism of valued fields, since $\Gamma_{A(a)} = \Gamma_A$, which concludes Case 3.

We have now finished to construct an isomorphism h extending g, with the element a in its domain and its image contained in $(L, \varGamma_L)$. Note however that showing this is not enough to show the condition (4) in Lemma 5.22 since the condition for the lemma is that a is an arbitrary element from the universe of $(K, \varGamma_K)$. Since we are working in a 2-sorted language, we will also have to consider the case where the element is taken from $\varGamma_K$. However, if $\alpha \in \varGamma_A$, then there exists an element $a \in K$ such that $\nu_K(a) = \alpha$. Thus, by extending $g : A \sqcup \varGamma_A \to B \sqcup \varGamma_B$ to an isomorphism $h : A' \sqcup \varGamma_{A'} \to B' \sqcup \varGamma_{B'}$ with $a \in A'$ as we have done above, we get that $\alpha \in \varGamma_{A'}$, and so h satisfies condition (4) in Lemma 5.22. This concludes the proof. $\square$

9. Definable sets in the value group

Lemma 9.1. *Let $(K, \varGamma_K)$ be a model of* ACVF. *Let $a_1, \ldots, a_m \in K^*$ and let $D \subset \varGamma_K^n$ be a non-empty set defined in $(K, \varGamma_K)$ by a formula $\phi(\bar{x}, \bar{\xi})$ over $a_1, \ldots, a_m \in K$, with $\bar{x} = (x_1, \ldots, x_m)$ having sort R and $\bar{x} = (\xi_1, \ldots, \xi_n)$ having sort $\varGamma$ in $\mathcal{L}_\varGamma$. Then there exists a quantifier free formula of the form*

$$\psi(\bar{x}, \bar{\xi}) = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{\ell} \varphi_{i,j}(\bar{x}, \bar{\xi}) \bowtie_{i,j} 0$$

for some $k, \ell \in \mathbb{N}$, where $\varphi_{i,j}[\bar{a}, \bar{\xi}] = \sum_{s=1}^{n} m_{s,i,j} y_s + C_{i,j}(\bar{a})$ with $m_{s,i,j} \in \mathbb{Z}$, $C_{i,j}(\bar{a}) \in \varGamma$ and $\bowtie_{i,j} \in \{<, \leq\}$ such that

$$D = \{\bar{\gamma} \in \varGamma_K \mid (K, \varGamma_K) \models \psi[\bar{a}, \bar{\xi}]\}.$$

Proof. First, note that a quantifier free formula $\psi(\bar{x}, \bar{\xi})$ in $\mathcal{L}_\varGamma$ can be written as

$$\psi(\bar{x}, \bar{\xi}) = \bigvee_{i=1}^{k} \bigwedge_{j=1}^{\ell} \phi_{i,j}^R(\bar{x}) \wedge \phi_{i,j}^\varGamma(\bar{x}, \bar{\xi})$$

where the variables $\bar{x}$ have sort R and the variables $\bar{\xi}$ have sort $\varGamma$ and where $\phi_{i,j}^R$ and $\phi_{i,j}^\varGamma$ are atomic formulas, or negation of atomic formulas, such that all terms in $\phi_{i,j}^K$ have sort K and all terms in $\phi_{i,j}^\varGamma$ have sort $\varGamma$. That $\phi_{i,j}^\varGamma$ can have free variables among the $\bar{x}$ is because the function symbol ν has sort $\varGamma$ but takes terms of sort K as arguments. So by quantifier elimination in ACVF, we have that

$$D = \bigcup_{i=1}^{k} \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \varGamma_K^n \mid (K, \varGamma_K) \models \phi_{i,j}^R(\bar{a}) \wedge \phi_{i,j}^\varGamma(\bar{a}, \bar{y})\},$$

for some $\phi_{i,j}^R$ and $\phi_{i,j}^\varGamma$ as described. Without loss of generality, we assume that

$$D_i := \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \varGamma_K^n \mid (K, \varGamma_K) \models \phi_{i,j}^R(\bar{a}) \wedge \phi_{i,j}^\varGamma(\bar{a}, \bar{y})\}$$

is non-empty for each i, since we can simply omit every i for which the corresponding set D_i is empty.

By considering the possible atomic formulas in the language $\mathcal{L}_\varGamma$, we have that the formulas $\phi_{i,j}^R$ can be written as

$$P_{i,j}(\bar{x}) \simeq_{i,j} 0$$

where $P_{i,j} \in \mathbb{Z}[\bar{x}]$ and $\simeq_{i,j} \in \{\doteq, \neq\}$.

Now consider

$$\psi[\bar{a},\bar{\xi}] = \bigvee_{i=1}^{k}\bigwedge_{j=1}^{\ell}\phi^R_{i,j}[\bar{a}]\wedge\phi^{\Gamma}_{i,j}[\bar{a},\bar{\xi}].$$

If $(K,\Gamma_K)\not\models\phi^R_{i,j}[\bar{a}]$, then $(K,\Gamma_K)\not\models\phi^R_{i,j}[\bar{a}]\wedge\phi^{\Gamma}_{i,j}[\bar{a},\bar{\gamma}]$ for any $\bar{\gamma}\in\Gamma^n_K$. That is,

$$\{\bar{\gamma}\in\Gamma^n_K \mid (K,\Gamma_K)\models\phi^R_{i,j}[\bar{a}]\wedge\phi^{\Gamma}_{i,j}[\bar{a},\bar{\gamma}]\} = \emptyset.$$

In particular, this implies that $D_i=\emptyset$, which contradicts the assumption above. So we get that $(K,\Gamma_K)\models\phi^R_{i,j}[\bar{a}]$. By definition, $(K,\Gamma_K)\models\phi^R_{i,j}[\bar{a}]\wedge\phi^{\Gamma}_{i,j}[\bar{a},\bar{\xi}]$ if and only if $(K,\Gamma_K)\models\phi^R_{i,j}[\bar{a}]$ and $(K,\Gamma_K)\models\phi^{\Gamma}_{i,j}[\bar{a},\bar{\gamma}]$ so we have

$$D=\bigcup_{i=1}^{k}\bigcap_{j=1}^{\ell}\{\bar{\gamma}\in\Gamma^n_K \mid (K,\Gamma_K)\models\phi^{\Gamma}_{i,j}[\bar{a},\bar{\gamma}]\}.$$

Furthermore, we have that $\phi^{\Gamma}_{i,j}$ can be written as

$$\sum_{s=1}^{n} m_{s,i,j}\xi_s+\nu(t_{i,j}(\bar{x}))\bowtie_{i,j} 0 \quad\text{or}\quad \sum_{s=1}^{n} m_{s,i,j}\xi_s+\nu(t_{i,j}(\bar{x}))\bowtie_{i,j}\infty$$

where $m_{s,i,j}\in\mathbb{Z}$, $\bowtie_{i,j}\in\{<,\leq\}$ and $t_{i,j}$ is a term of sort K, i.e. a polynomial in $\mathbb{Z}[\bar{x}]$. Suppose that

$$\phi^{\Gamma}_{i,j}=\sum_{s=1}^{n} m_{s,i,j}\xi_s+\nu(t_{i,j}(\bar{x}))+\bowtie_{i,j}\infty$$

and that $\nu_K(t_{i,j}[\bar{a}])=\infty$. Then for any $\bar{\gamma}$, we have that $(K,\Gamma_K)\models\phi^{\Gamma}_{i,j}[\bar{a},\bar{\gamma}]$ if and only if $\bowtie_{i,j}$ is equal to $\leq$. Since we can assume that $\{\bar{\gamma}\in\Gamma^n_K \mid (K,\Gamma_K)\models\phi^{\Gamma}_{i,j}[\bar{a},\bar{\gamma}]\}$ is non-empty, we get that $\bowtie_{i,j}$ is equal to $\leq$, and therefore $\phi^{\Gamma}_{i,j}[\bar{a},\bar{y}]$ is satisfied by all $\bar{\gamma}\in\Gamma$. Hence, $\phi^{\Gamma}_{i,j}$ is equivalent in (K,Γ_K) to the formula $0\leq 0$, which is on the form we want. If

$$\phi^{\Gamma}_{i,j}=\sum_{s=1}^{n} m_{s,i,j}y_s+\nu(t_{i,j}(\bar{x}))+\bowtie_{i,j}\infty$$

but $\nu_K(t_{i,j}[\bar{a}])<\infty$, then $\phi^{\Gamma}_{i,j}$ is still satisfied by all $\bar{\gamma}\in\Gamma^n_K$, and $\phi^{\Gamma}_{i,j}$ is equivalent in (K,Γ_K) to $0\leq 0$.

Hence, in any case we get that $\phi^{\Gamma}_{i,j}$ is equivalent in (K,Γ_K) to a formula of the form

$$\sum_{s=1}^{n} m_{s,i,j}y_s+\nu(t_{i,j}(\bar{x}))\bowtie_{i,j} 0$$

which is what we wanted to prove. □

Definition 9.2. *Let (K,Γ_K) be a totally ordered divisible abelian group. A **pavé** of dimension d of Γ^n_K is a definable set of the form*

$$A\left(\prod_{i=1}^{d}(r_i,R_i)\times\{\gamma_{d+1},\dots,\gamma_n\}\right)$$

where $r_i,R_i,\gamma_j\in\Gamma$ and $r_i<R_i$ and $A\in\mathrm{GL}_n(\mathbb{Q})$.

Definition 9.3. *Let (K, Γ_K) be a model of* ACVF *and let $D \subset \Gamma_K^n$ be a non-empty definable subset. We define the* ***dimension*** *of D to be the largest number d such that D contains a pavé of dimension d.*

Remark 9.4. Let $D \subset \Gamma_K^n$ be a definable set of dimension d, defined over $\bar{a} \in K^m$ by a formula $\varphi(\bar{x}, \bar{\xi})$, with $\bar{x} = (x_1, \ldots, x_m)$ having sort R and $\bar{\xi} = (\xi_1, \ldots, \xi_n)$ having sort Γ. Let $A \in \mathrm{GL}_n(\mathbb{Q})$ and define the set

$$AD = \{A\bar{\gamma} \mid \bar{\gamma} \in D\}.$$

Let $A = (q_{i,j})_{1 \le i,j \le n}$ with $q_{i,j} \in \mathbb{Q}$ and let m be the least common multiple of all the the $q_{i,j}$. Let $m_{i,j} := mq_{i,j} \in \mathbb{Z}$. Then, $A\bar{\gamma} \in AD$ if and only if $m\bar{\gamma} \in (mA)D$. Hence, AD is a definable set, defined over $\bar{a}$ by the formula

$$\psi(\bar{x}, \bar{\zeta}) = \exists \xi_1 \cdots \exists \xi_n \left(\varphi(\bar{x}, \bar{\xi}) \wedge \bigwedge_{i=1}^{n} \left(m\zeta_i \doteq \sum_{j=1}^{n} m_{i,j}\xi_j \right) \right),$$

with $\bar{\zeta} = (\zeta_1, \ldots, \zeta_n)$ having sort Γ. Since the dimension of a pavé is not changed by multiplying it with a matrix in $\mathrm{GL}_n(\mathbb{Q})$, we have that the dimension of D is equal to the dimension of AD.

Lemma 9.5. *Let (K, Γ_K) be a model of* ACVF *and let $D \subset \Gamma_K^n$ be a definable set and let $P \subset D$ be a pavé of dimension d in D, given by*

$$P = A\left(\prod_{i=1}^{d} (r_i, R_i) \times \{\gamma_{d+1}, \ldots, \gamma_n\} \right)$$

for some $A \in \mathrm{GL}_n(\mathbb{Q})$. Let (L, Γ_L) be a valued algebraically closed extension of (K, Γ_K). Then the pavé

$$P(\Gamma_L) = A\left(\prod_{i=1}^{d} (r_i, R_i) \times \{\gamma_{d+1}, \ldots, \gamma_n\} \right) \subset \Gamma_L^n$$

is contained in $D(\Gamma_L)$.

Proof. From the description of AD as a definable set in Remark 9.4, we can see that $(AD)(\Gamma_L) = AD(\Gamma_L)$. So, $P(\Gamma_L)$ is a pavé in $D(\Gamma_L)$ if and only if $A^{-1}P(\Gamma_L)$ is a pavé in $A^{-1}D(\Gamma_L)$, and we can assume that

$$P = \prod_{i=1}^{d} (r_i, R_i) \times \{\gamma_{d+1}, \ldots, \gamma_n\}$$

Let $\bar{\rho} = (\rho_1, \ldots, \rho_d)$, $\bar{\mathrm{P}} = (\mathrm{P}_1, \ldots, \mathrm{P}_d)$, $\bar{\xi} = (\xi_{d+1}, \ldots, \xi_n)$ and $\bar{\zeta} = (\zeta_1, \ldots, \zeta_n)$ be variables of sort Γ. Define the formula

$$\psi(\bar{\rho}, \bar{\mathrm{P}}, \bar{\xi}, \bar{\zeta}) = \bigwedge_{i=1}^{d} (\rho_i < \zeta_i \wedge \zeta_i < \mathrm{P}_i) \bigwedge_{i=d+1}^{n} (\zeta_i \doteq \xi_i)$$

Then we have that P is a definable set given as

$$P = \{\bar{\alpha} \in \Gamma_K^n \mid (K, \Gamma_K) \models \psi[\bar{r}, \bar{R}, \bar{\gamma}, \bar{\alpha}]\},$$

with $\bar{r} = (r_1, \dots, r_d)$, $\bar{R} = (R_1, \dots, R_d)$ and $\bar{\gamma} = (\gamma_{d+1}, \dots, \gamma_n)$. Similarly, we get that

$$P(\Gamma_L) = \{\bar{\alpha} \in \Gamma_L^n \mid (L, \Gamma_L) \models \psi[\bar{r}, \bar{R}, \bar{\gamma}, \bar{\alpha}]\}.$$

Now, let $\varphi(\bar{x}, \bar{\zeta})$ be a formula defining D over some parameters $\bar{a}$. Since P is in D, we have that

$$P = \{\bar{\alpha} \in \Gamma_K^n \mid (K, \Gamma_K) \models \psi[\bar{r}, \bar{R}, \bar{\gamma}, \bar{\alpha}] \wedge \varphi[\bar{a}, \bar{\alpha}]\}.$$

Hence, by Corollary 7.3.1, we have that $P(\Gamma_L)$ is also defined by $\psi \wedge \varphi$, i.e. $P(\Gamma_L)$ is contained in $D(\Gamma_L)$. □

Lemma 9.6. *Let (K, Γ_K) be a model of* ACVF *and let $D \subset \Gamma_K^n$ be a definable set of dimension d and let (L, Γ_L) be a valued algebraic extension of (K, Γ_K). Then $D(\Gamma_L)$ has dimension d.*

Proof. If $P \subset D$ is a pavé of dimension d in D, then $P(\Gamma_L)$ is a pavé of dimension d in $D(\Gamma_L)$, by Lemma 9.5. So $D(\Gamma_L)$ has dimension at least d. Suppose that

$$P = A\left(\prod_{i=1}^{c}(r_i, R_i) \times \{\gamma_{c+1}, \dots, \gamma_n\}\right) \subset D(\Gamma_L)$$

is a pavé of dimension $c > d$. We will show that this implies that there is a pavé of dimension c in D. By Remark 9.4 So, we can assume that A is the identity matrix.

Let D be defined by the formula $\varphi(\bar{x}, \bar{\zeta})$ over $\bar{a} = (a_1, \dots, a_m) \in (K \sqcup \Gamma_K)^m$, with $\bar{\zeta} = (\zeta_1, \dots, \zeta_n)$, $\bar{x} = (x_1, \dots, x_m)$ and x_i having the same sort as a_i. Let $\bar{\rho} = (\rho_1, \dots, \rho_c)$, $\bar{\mathrm{P}} = (\mathrm{P}_1, \dots, \mathrm{P}_c)$ and $\bar{\xi} = (\xi_{c+1}, \dots, \xi_n)$ be variables of sort Γ. We note that $D(\Gamma_L)$ contains a pavé of the form

$$\prod_{i=1}^{c}(r_i, R_i) \times \{\gamma_{c+1}, \dots, \gamma_n\} \subset D(\Gamma_L)$$

if and only if the formula $\psi(\bar{x})$ defined as

$$\exists\bar{\rho}\,\exists\bar{\mathrm{P}}\,\exists\bar{\xi}\,\forall\bar{\zeta}\left(\left(\bigwedge_{i=1}^{c}\rho_i < \mathrm{P}_i\right) \wedge \left(\left(\bigwedge_{i=1}^{c}\rho_i < \xi_i \wedge \xi_i < \mathrm{P}_i \bigwedge_{i=c+1}^{n}\zeta_i \doteq \xi_i\right) \to \varphi(\bar{x}, \bar{\zeta})\right)\right)$$

is satisfied over $\bar{a}$ in (L, Γ_L), i.e. if and only if $(L, \Gamma_L) \models \psi[\bar{a}]$. But since (L, Γ_L) is an elementary extension of (K, Γ_K), we have that $(K, \Gamma_K) \models \psi[\bar{a}]$, so D contains a pavé of the form

$$\prod_{i=1}^{c}(r'_i, R'_i) \times \{\gamma'_{c+1}, \dots, \gamma'_n\},$$

with $r'_i, R'_i, \gamma'_j \in \Gamma_K$ for $i \in \{1, \dots, c\}$ and $j \in \{c+1, \dots, n\}$. So D contains a pavé of dimension c, which is what we wanted to prove. □

10. Some Algebraic Geometry

Definition 10.1. *Let K be an algebraically closed field. An **affine algebraic set** of K^n is the zero set of a finite number of polynomials in $K[X_1, \ldots, X_n]$.*

If K is an algebraically closed field and X is an affine algebraic set of K^n, then the set

$$\mathcal{A}_X := \{P \in K[X_1, \ldots, X_n] \mid P(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in X\}$$

is an ideal of $K[X_1, \ldots, X_n]$.

Definition 10.2. *Let K be an algebraically closed field and let X be an affine algebraic set. We define the **coordinate ring** $K[X]$ to be the ring*

$$K[X_1, \ldots, X_n]/\mathcal{A}_X.$$

*If $K[X]$ is an integral domain, i.e. if $\mathcal{A}_X$ is a prime ideal, we say that X is an **affine variety**. In this case, the fraction field of $K[X]$, denoted $K(X)$, is called the **function field** of X.*

Definition 10.3. *Let K be an algebraically closed field and let B be an integral domain which is a finitely generated K algebra and let L be the quotient field of B. We define the **dimension** of B to be the transcendence degree of L over K, i.e. the largest cardinality of an algebraically independent subset of $K(X)$ over K. If X is an affine variety, we define the **dimension** of X to be the dimension of the coordinate ring $K[X]$.*

Definition 10.4. *Let K be an algebraically closed field. We define the **algebraic torus** of dimension n over K to be $(K^*)^n$. An **algebraic subset** of $(K^*)^n$ is a set $X \subset (K^*)^n$ such that X is the zero set in $(K^*)^n$ of some polynomials $P_1, \ldots, P_m \in K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$.*

Remark 10.5. If K is an algebraically closed field and X is an algebraic subset of $K(^*)^n$ defined by the polynomials

$$P_i(X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}) \in K[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$$

for $i \in \{1, \ldots, m\}$, we can consider X as a subset of K^{2n}, satisfying the polynomials $P_i(X_1, Y_1, \ldots, X_n, Y_n) \in K[X_1, Y_1, \ldots, X_n, Y_n]$ for $i \in \{1, \ldots, m\}$ and $X_jY_j - 1$ for $j \in \{1, \ldots, n\}$. Hence, X can be considered as an affine algebraic set. We will say that X is a **subvariety** of $(K^*)^n$ if X is an affine variety and we refer to the dimension of X as its dimensions as an affine variety.

Lemma 10.6. *Let X be a subvariety of $(K^*)^n$ and let L/K be an algebraically closed extension. Let $a_1, \ldots, a_n \in X(L)$ and define $K' := K(a_1, \ldots, a_n)$. Then $\operatorname{trdeg}(K'/K) \leq \dim(X)$.*

Proof. Let $\bar{a} := (a_1, \dots, a_n)$ and let $\mathcal{A}_{\bar{a}} \subset K[X_1, \dots, X_n]$ be the ideal of all polynomials vanishing at $\bar{a}$. Then $\mathcal{A}_X \subset \mathcal{A}_{\bar{a}}$, so we have a natural projection

$$\pi : \; K[X] = K[X_1, \dots, X_n]/\mathcal{A}_X \longrightarrow K[X_1, \dots, X_n]/\mathcal{A}_{\bar{a}}.$$

Since K' is isomorphic to the quotient field of $K[X_1, \dots, X_n]/\mathcal{A}_{\bar{a}}$, we have that $\operatorname{trdeg}(K'/K) = \dim(K[X_1, \dots, X_n]/\mathcal{A}_{\bar{a}})$.

Suppose that $\operatorname{trdeg}(K'/K) > \dim(X)$. By Theorem 1.8A in [Har77]), we have that the dimension of a ring B which is a finitely generated K-algebra is given by the maximum number $n \in \mathbb{N}$ such that there exists a chain

$$\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$$

of distinct prime ideals of B. But since π is surjective, any chain of distinct prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n \subset K[X_1, \dots, X_n]/\mathcal{A}_{\bar{a}}$$

gives a chain of distinct prime ideals

$$\pi^{-1}(\mathfrak{p}_0) \subset \cdots \subset \pi^{-1}(\mathfrak{p}_n) \subset K[X].$$

Hence, $\dim(K[X_1, \dots, X_n]/\mathcal{A}_{\bar{a}}) \leq \dim(X)$, and we are done. □

Definition 10.7. *Let K be an algebraically closed field. An **isogeny** of the algebraic torus is a map*

$$\begin{aligned} \Phi_M : (K^*)^n &\longrightarrow (K^*)^n \\ (a_1, \dots, a_n) &\longmapsto (\bar{a}^{\bar{m}_1}, \dots, \bar{a}^{\bar{m}_n}) \end{aligned}$$

where $\bar{m}_i = (m_{1,i}, \dots, m_{n,i}) \in \mathbb{Z}^n$, $\bar{a} = (a_1, \dots, a_n)$, $\bar{a}^{\bar{m}_i} = a_1^{m_{1,i}} \cdots a_n^{m_{n,i}}$ and $M = \begin{pmatrix} \bar{m}_1 & \cdots & \bar{m}_n \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Q})$.

Remark 10.8. If K is an algebraically closed valued field with valuation ν and value group Γ, and Φ_M is an isogeny on $(K^*)^n$, then Φ_M induces a map on Γ^n, which we will also denote by Φ_M, as follows

$$\begin{aligned} \Phi_M : \Gamma^n &\longrightarrow \Gamma^n \\ \bar{\gamma} &\longmapsto (\bar{m}_1\bar{\gamma}, \dots, \bar{m}_n\bar{\gamma}) \end{aligned}$$

where the $\bar{m}_i \in \mathbb{Z}^n$ are as in Definition 10.7 and $\bar{m}_i\bar{\gamma} = m_{1,i}\gamma_1 + \cdots + m_{n,i}\gamma_n$. We extend ν to $(K^*)^n$ by setting $\nu(a_1, \dots, a_n) = (\nu(a_1, \dots, \nu(a_n))$ and note that $\nu \circ \Phi_M = \Phi_M \circ \nu$ by construction. By Remark 9.4, we have that an isogeny preserve the dimension of a definable set $D \subset \Gamma^n$.

Lemma 10.9. *Let X be a subvariety of $(K^*)^n$. And let Φ_M be an isogeny of $(K^*)^n$. Then $\Phi_M(X)$ is a subvariety of $(K^*)n$ and $\dim(X) = \dim(\Phi_M(X))$.*

Proof. We won't prove this in detail, but the idea is to use that an isogeny is a finite morphism, so in particular a closed map [Har77]. This implies that the image of a subvariety under an isogeny will be a subvariety. The dimension of $\Phi_M(X)$ follows from the fact that the restriction of Φ_M to X is also a finite morphism, and a finite surjective morphism preserves dimension. □

Definition 10.10. *Let K be an algebraically closed field and let X be a subvariety of $(K^*)^n$, defined by the polynomials $P_1, \dots, P_m \in K[X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}]$. If L is an algebraically closed extension of K, we define*

$$X(L) := \left\{ \bar{b} \in (L^*)^n \mid \bigwedge_{i=1}^{m} P_i(\bar{b}) = 0 \right\}.$$

Remark 10.11. (1) Consider the language $\mathcal{L}_R$ and the theory ACF as in Example 4.4. Let $X \subset (K^*)^n$ be a subvariety defined by the polynomials

$$P_1, \dots, P_k \in K[X_1, \dots, X_n, X_1^{-1}, \dots, X_n^{-1}].$$

We identify X with the affine algebraic set in $(K^*)^{2n}$ defined by the polynomials

$$\widetilde{P}_i := P_i(X_1, \dots, X_n, Y_1, \dots, Y_n) \in K[X_1, \dots, X_n, Y_1, \dots, Y_n]$$

with $i \in \{1, \dots, m\}$ and the polynomials $X_j Y_j - 1$ with $j \in \{1, \dots, n\}$. Let $a_1, \dots, a_m \in K$ be the coefficients of all the polynomials $P_1, \dots, P_k \in K[\bar{X}, \bar{Y}]$ and let $Q_i \in K[T_1, \dots, T_m, X_1, \dots, X_n, Y_1, \dots, Y_n]$ be such that

$$\widetilde{P}_i = Q_i(a_1, \dots, a_m, X_1, \dots, X_n, Y_1, \dots, Y_n).$$

Let $\psi(\bar{x}, \bar{y}, \bar{z})$ be the $\mathcal{L}_R$-formula

$$\bigwedge_{i=1}^{k} Q_i(\bar{x}, \bar{y}, \bar{z}) \doteq 0 \bigwedge_{i=1}^{n} y_i \cdot z_i \doteq 1$$

with $\bar{x} = x_1, \dots, x_m$, $\bar{y} = y_1, \dots, y_n$ and $\bar{z} = z_1, \dots, z_n$. Then

$$X = \{(\bar{b}, \bar{c}) \in (K^*)^{2n} \mid K \models \psi[\bar{a}, \bar{b}, \bar{c}]$$

so X is a definable set, defined over $\bar{a}$. By Corollary 7.3.1, this shows that $X(L)$ in Definition **??** is well defined, since the theory ACF admits quantifier elimination.

(2) If K is an algebraically closed valued field, with valuation ν_K and with value group Γ_K, then ν_K naturally gives a map from $(K^*)^n$ to Γ^n, by sending $(a_1, \dots, a_n)$ to $(\nu_K(a_1), \dots, \nu_K(a_n))$. We will denote this map by ν_K as well. Let X be a subvariety of $(K^*)^n$, viewed as a subset of $(K^*)^{2n}$ defined over $\bar{a}$ by $\psi(\bar{x}, \bar{y}, \bar{z})$, as above. Let $\varphi(\bar{z}, \bar{w})$ be the formula

$$\bigwedge_{i=1}^{n} z_i \doteq \nu(w_i)$$

and let

$$\phi(\bar{x}, \bar{w}) = \exists \bar{y} \exists \bar{z} (\psi(\bar{x}, \bar{y}, \bar{z}) \wedge \varphi(\bar{y}, \bar{w})).$$

We then get

$$\nu_K(X) = \{\bar{\gamma} \in \Gamma_K^n \mid (K, \Gamma_K) \models \phi[\bar{a}, \bar{\gamma}]\},$$

so $\nu_K(X)$ is a definable set. Furthermore, if (L, Γ_L) is a valued algebraically closed extension of (K, Γ_K), we have that

$$\nu_L(X(L)) = \{\bar{\gamma} \in \Gamma_L^n \mid (L, \Gamma_L) \models \phi[\bar{a}, \bar{\gamma}]\}$$

so $\nu_L(X(L)) = \nu_L(X)((L, \Gamma_L))$. This shows that any formula defining $\nu_K(X)$ also defines $\nu_L(X(L))$, by Corollary 7.3.1.

Definition 10.12. *Let (K, Γ_K) be a model of* ACVF *and let $D \subset \Gamma^n$ be a definable set in (K, Γ_K), defined over $\bar{a}$ by an $\mathcal{L}_\Gamma$-formula $\varphi(\bar{x}, \bar{y})$. If L is an algebraically closed valued extension of K with value group Γ_L, we define*

$$D(\Gamma_L) := \left\{ \bar{\gamma} \in \Gamma_L^n \mid (L, \Gamma_K) \models \varphi[\bar{a}, \bar{y}] \right\}.$$

Remark 10.13. In the situation of Definition 10.12, let $\psi(\bar{x}, \bar{y})$ be a quantifier free formula such that φ and ψ both define the set D over $\bar{a}$. By Corollary 7.3.1, we then have that $D(\Gamma_L)$ is also defined by ψ over $\bar{a}$.

11. The Bieri-Groves Theorem

Definition 11.1. *Let Γ be a totally ordered divisible abelian group. A Γ-**polyhedron** is a subset $P \subset \Gamma^n$ for some $n \in \mathbb{N}$ such that*

$$P = \bigcap_{i=1}^{m} \left\{ \bar{\gamma} \in \Gamma^n \mid \varphi_i(\bar{\gamma}) \leq c_i \right\}$$

where $c_i \in \Gamma$ and $\varphi_1, \ldots, \varphi_m$ are functions on the form

$$\varphi_i : \Gamma^n \longrightarrow \Gamma$$

$$(\gamma_1, \ldots, \gamma_n) \longmapsto \sum_{j=1}^{n} a_{i,j} \gamma_j$$

*with $a_{i,j} \in \mathbb{Z}$. An $\mathbb{R}$-polyhedron is called a **real polyhedron**.*

Remark 11.2. For a totally ordered set, in particular a totally ordered divisible abelian group Γ, the **order topology** is the topology generated by the open intervals

$$(\alpha, \beta) = \{\gamma \in \Gamma \mid \alpha < \gamma < \beta\}$$

and the open rays

$$(\alpha, \infty) = \{\gamma \in \Gamma \mid \alpha < \gamma\} \quad \text{and} \quad (-\infty, \alpha) = \{\gamma \in \Gamma \mid \gamma < \alpha\}$$

for $\alpha, \beta \in \Gamma$. This topology coincides with the usual topology on $\mathbb{R}^n$.

The order topology gives a topology on Γ^n by using the product. It follows from the definition that a Γ-polyhedron is closed under this topology.

Lemma 11.3. *Let Γ be a non-zero totally ordered divisible abelian subgroup of $\mathbb{R}$. Then Γ is dense in $\mathbb{R}$ with respect to the order, i.e. for any $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we have that $(\alpha, \beta) \cap \Gamma \neq \emptyset$.*

The proof is completely analogous to the standard proof that $\mathbb{Q}$ is dense in $\mathbb{R}$, using the Archimedean property of $\mathbb{R}$, i.e. that for any $\xi \in \mathbb{R}$, there exists a natural number N such that $N > \xi$.

Proof. Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$ and let $\gamma \in \Gamma_{>0}$. Let $N \in \mathbb{N}$ be such that $N > \frac{\gamma}{\beta - \alpha}$. Define the set

$$A = \left\{ \frac{m\gamma}{N} \mid m \in \mathbb{N} \right\} \subset \Gamma.$$

We claim that $A \cap (a, b) \neq \emptyset$. Assume that $A \cap (a, b) = \emptyset$ and let $m_1 \in \mathbb{N}$ be the greatest positive integer such that $\frac{m_1 \gamma}{N} < \alpha$. Such an integer exists due to the well ordering principle. Since $\frac{m_1+1}{N}\gamma \in A$, we then get that $\frac{m_1+1}{N}\gamma > \beta$. But this implies

$$\beta - \alpha < \frac{m_1 + 1}{N}\gamma - \frac{m_1}{N}\gamma = \frac{\gamma}{N} < \beta - \alpha.$$

This is a contradiction, so $A \cap (\alpha, \beta) \neq \emptyset$, which proves the lemma. □

Theorem 11.4. *Let K be an algebraically closed valued field with a valuation ν_K and value group Γ_K. Let X be an algebraic variety over the algebraic torus and let d be the dimension of X. Then $\nu_K(X)$ is a finite union of Γ_K-polyhedra and $\dim(\nu_K(X)) \leq d$.*

This theorem appears in a more general setting as Theorem 1.2 A) in [Duc12], and we follow the proof given there.

Proof. Since $\nu_K(X)$ is a definable set in Γ_K, we have by 9.1 that

$$\nu_K(X) = \bigcup_{i=1}^{k} \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_{i,j}(\bar{\gamma}) \bowtie_{i,j} 0\}$$

where $\bowtie_{i,j} \in \{<, \leq\}$. We assume that $\bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_{i,j}(\bar{\gamma}) \bowtie_{i,j} 0\}$ is non-empty for each i. Since $\varphi_{i,j}(\bar{\gamma}) \bowtie_{i,j} 0$ implies that $\varphi_{i,j}(\bar{\gamma}) \leq 0$, we only need to show that

$$\bigcup_{i=1}^{k} \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_{i,j}(\bar{\gamma}) \leq 0\} \subset \bigcup_{i=1}^{k} \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_{i,j}(\bar{\gamma}) \bowtie_{i,j} 0\}.$$

To do this, it is enough to show that

$$\bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_{i,j}(\bar{\gamma}) \leq 0\} \subset \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_{i,j}(\bar{\gamma}) \bowtie_{i,j} 0\}$$

for each $i \in \{1, \ldots, k\}$.

For the remaining part, we fix $i \in \{1, \ldots, k\}$ and write φ_j and $\bowtie_j$ instead of $\varphi_{i,j}$ and $\bowtie_{i,j}$. Let $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$ be an element in Γ^n satisfying $\varphi_j(\bar{\alpha}) \leq 0$ for all j. We will show that $\varphi_j(\bar{\alpha}) \bowtie_j 0$ for all j, which proves the first part of the theorem. Let

$$P := \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_j(\bar{\gamma}) \bowtie_j 0\} \subset \nu_K(X).$$

By assumption, it is non-empty. We define the function

$$\begin{aligned} d : P &\longrightarrow \Gamma_K \\ (\gamma_1, \ldots, \gamma_n) &\longmapsto \max_{1 \leq i \leq n} (\alpha_i - \gamma_i, \gamma_i - \alpha_i). \end{aligned}$$

Note that $d(P)$ is a definable set in Γ_K, since $\max_i(\alpha_i - \gamma_i, \gamma_i - \alpha_i)$ is the element satisfying the formula with parameters $\bar{\alpha}$ and $\bar{\gamma}$:

$$\psi[x, \bar{\alpha}, \bar{\gamma}] = \bigwedge_{i=1}^{n} (\alpha_i - \gamma_i \leq x) \bigwedge_{i=1}^{n} (\gamma_i - \alpha_i \leq x).$$

More precisely, we have that

$$d(P) = \left\{ \gamma \in \Gamma_K \mid (K, \Gamma_K) \models \forall \bar{\gamma} \left(\left(\bigwedge_{j=1}^{\ell} \varphi_j(\bar{\gamma}) \bowtie_j 0 \right) \rightarrow \psi[\gamma, \bar{\alpha}, \bar{\gamma}] \right) \right\}$$

By Lemma 9.1, we have that $d(P)$ consists of the elements $\gamma \in \Gamma_K$ satisfying the disjunction and conjunction of equalities of the form $\gamma + \delta \bowtie 0$ with $\delta \in \Gamma_K$ and $n \in \mathbb{Z}$. That is, $d(P)$ is the finite union of intervals (which may be open or closed). Since $d(P) \subset \Gamma_K^+ := \{\gamma \in \Gamma_K \mid \gamma \geq 0\}$, we must have that each of these intervals have a lower endpoint in Γ_K greater than or equal to 0. Let $\beta \in \Gamma_K$ be the minimum of these lower endpoints. Since $d(P)$ is the union of finitely many intervals, β is well defined. Since β is the lower endpoint of an interval contained in $d((P)$ and since the order $<$ is dense on Γ_K, we have that for any $\beta' \in \Gamma_K$ such that $\beta < \beta'$, there is an element $\gamma \in d(P)$ such that $\beta < \gamma < \beta'$. I.e. β is the greatest lower bound of $d(P)$. We will show that $\beta = 0$.

For contradiction, suppose that $\beta > 0$.

Claim 1. There exists an element $\bar{\delta} = (\delta_1, \ldots, \delta_n) \in \Gamma_K^n$ such that the following hold:

(1) For all i we have the inequality $-\frac{3}{2}\beta \leq \delta_i \leq \frac{3}{2}\beta$.

(2) There exists an i_0 such that $\beta \leq \delta_{i_0}$ or $\delta_{i_0} \leq -\beta$.

(3) The element $(\alpha_1 + \delta_1, \ldots, \alpha_n + \delta_n)$ is in P.

Proof of Claim: Let $\bar{\gamma} \in P$. Then for all j we have

$$\varphi_j\left(\frac{\bar{\alpha} + \bar{\gamma}}{2}\right) = \frac{1}{2}\left(\varphi_j(\bar{\alpha}) + \varphi_j(\bar{\gamma})\right) \bowtie_j 0.$$

Hence $\frac{\bar{\alpha}+\bar{\gamma}}{2} \in P$, which shows that we can find an element in P arbitrarily close to $\bar{\alpha}$. So assume that $\bar{\gamma} = (\gamma_1, \ldots, \gamma_n) \in P$ is such that

$$\frac{-3}{2}\beta \leq \gamma_i - \alpha_i \leq \frac{3}{2}\beta$$

for all $i \in \{1, \ldots, n\}$. Now define $\delta_i := \gamma_i - \alpha_i$. Then all δ_i satisfy condition (1) in the claim. Note that $d(\bar{\gamma}) = \max_i(\delta_i, -\delta_i)$. Since $\beta \leq d(\bar{\gamma})$, there exists an i_0 such that $\beta \leq \delta_{i_0}$ or $\beta \leq -\delta_{i_0}$, which is equivalent to (2). Assertion (3) follows directly from the fact that $\alpha_i + \delta_i = \gamma_i$ and $\bar{\gamma} \in P$ by assumption. This finishes the proof of the claim.

Let $\bar{\delta}$ be as in Claim 1. Then

$$\varphi_j\left(\bar{\alpha} + \frac{1}{2}\bar{\delta}\right) = \varphi_j\left(\frac{\bar{\alpha} + (\bar{\alpha} + \bar{\delta})}{2}\right) = \frac{1}{2}(\underbrace{\varphi_j(\bar{\alpha})}_{\leq 0} + \underbrace{\varphi_j(\bar{\alpha} + \bar{\delta})}_{\bowtie_j 0}) \bowtie_j 0$$

for all j. So $\bar{\alpha} + \frac{1}{2}\bar{\delta} \in P$. But then

$$d\left(\bar{\alpha} + \frac{1}{2}\bar{\delta}\right) = \max_i \left(\frac{1}{2}\delta_i, -\frac{1}{2}\delta_i\right)$$

contradicts assertion (1) in Claim 1 and the minimality of β since

$$\frac{1}{2}\delta_i \leq \frac{3}{4}\beta < \beta \quad \text{and} \quad -\frac{1}{2}\delta_i \leq \frac{3}{4}\beta < \beta.$$

Hence, the assumption $\beta > 0$ is false, and we have that $\beta = 0$.

Now, let L be a valued algebraically closed extension of K with value group $\Gamma_L := \Gamma_K \oplus \mathbb{Q}\varpi$, with ϖ being an infinitesimal element strictly greater than 0, as in 6.4. Let

$$P' := P(\Gamma_L) = \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_L^n \mid \varphi_j(\bar{\gamma}) \bowtie_j 0\}$$

and define the function d as above but with P' as domain:

$$d : P' \longrightarrow \Gamma_L$$
$$(\gamma_1, \ldots, \gamma_n) \longmapsto \max_{1 \leq i \leq n} (\alpha_i - \gamma_i, \gamma_i - \alpha_i).$$

We can think of the function d as measuring the distance between an element $\bar{\gamma} \in P'$ and $\bar{\alpha}$. With this intuitive idea in mind, we will show that there is an element $\bar{\gamma} \in P'$ such that $\bar{\gamma}$ which is in a sense infinitely close to $\bar{\alpha}$.

By the same argument as above, $d(P')$ is a definable set of Γ' with greatest lower bound 0. So, there exists an element $\delta \in \mathbb{Q}\varpi$ such that $\delta \in d(P')$. Indeed, if $q\varpi \notin d(P')$ for any $q \in \mathbb{Q}$, then ϖ is a lower bound for $d(P')$ strictly greater than 0. Let $\bar{\gamma} = (\gamma_1, \ldots, \gamma_n) \in d^{-1}(\delta)$. Then $\max_i(\alpha_i - \gamma_i, \gamma_i - \alpha_i) = \delta$, and so $\bar{\gamma} - \bar{\alpha} = \bar{\delta} = (\delta_1, \ldots, \delta_n) \in (\mathbb{Q}\varpi)^n$. So $\bar{\alpha} + \bar{\delta} \in P'$. Since P' is a subset of $\nu_L(X(L))$, there exists an element $\bar{a} \in X(L)$ such that $\nu_L(\bar{a}) = \bar{\alpha} + \bar{\delta}$.

Now, identify $\Gamma'/\mathbb{Q}\varpi$ with Γ_K by factoring out all the infinitesimal elements. Denote by π the natural projection $\Gamma' \to \Gamma_K$. We define the map

$$\nu'_L : L \longrightarrow \Gamma$$
$$a \longmapsto \pi \circ \nu_L(a).$$

This is a valuation on L, since π is a homomorphism. We denote the valued field (L, ν'_L) by L'. Then $X(L') = X(L)$, since $X(L)$ only depends on the underlying field. Thus, $\bar{a} \in X(L')$. As noted in Remark 10.11, we have that $X(L')$ is defined by the same quantifier free formula as X and that $\nu'_L(X(L'))$ is defined by the same quantifier free formula as $\nu_K(X)$. This gives

$$\nu'_L(\bar{a}) \in \nu'_L(X(L')) = \bigcup_{i=1}^{k} \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma^n \mid \varphi_{i,j}(\bar{\gamma}) \bowtie_{i,j} 0\}.$$

But since $\nu_L(\bar{a}) = \bar{\alpha} + \bar{\delta}$, we have that $\nu'_L(\bar{a}) = \bar{\alpha}$. This shows that $\bar{\alpha} \in \nu_K(X)$, which was what we wanted to prove.

We will now show that $\dim(\nu_K(X)) \leq d$. Let $\mathcal{P}$ be a pavé of Γ_K^n contained in $\nu_K(X)$. Denote by δ the dimension of $\mathcal{P}$. We will show that $\delta \leq d$. Since an isogeny preserves dimension of both $\mathcal{P}$ and X, we can assume that

$$\mathcal{P} = \prod_{i=1}^{\delta} (r_i, R_i) \times \{\gamma_{\delta+1}, \ldots, \gamma_n\}$$

with $r_i, R_i, \gamma_j \in \Gamma_K$ and $r_i < R_i$. Let Γ_L be the totally ordered divisible abelian group

$$\Gamma_K \oplus \mathbb{Q}\varpi_1 \oplus \cdots \oplus \mathbb{Q}\varpi_\delta$$

as defined in Lemma 6.4 and let L be a valued algebraically closed extension of K having Γ' as value group. Denote by $\mathcal{P}(\Gamma_L)$ the pavé of Γ_L^n given by

$$\prod_{i=1}^{\delta}(r_i, R_i) \times \{\gamma_{\delta+1}, \ldots, \gamma_n\} \subset \Gamma_L^n.$$

Then $\mathcal{P}(\Gamma_L =\subset \nu_L(X(L))$ by Lemma 9.5 and $(r_1 + \varpi_1, \ldots, r_\delta + \varpi_\delta, \gamma_{\delta+1}, \ldots, \gamma_n) \in \mathcal{P}(\Gamma_L)$, so

$$(r_1 + \varpi_1, \ldots, r_\delta + \varpi_\delta, \gamma_{\delta+1}, \ldots, \gamma_n) \in \nu_L(X(L)).$$

Hence, there is a point $\bar{a} = (a_1, \ldots, a_n) \in X(L)$ such that

$$\nu_L(\bar{a}) = (r_1 + \varpi_1, \ldots, r_\delta + \varpi_\delta, \gamma_{\delta+1}, \ldots, \gamma_n).$$

By restricting ν_L, we get a valuation ν'_K on $K' = K(a_1, \ldots, a_\delta)$. Let Γ'_K be the corresponding value group. Then $r_i + \varpi_i \in \Gamma'_K$ for all $i \in \{1, \ldots, \delta\}$. We now claim that the $a_1, \ldots, a_\delta$ are algebraically independent over K. For contradiction, suppose that they are algebraically dependent over K. Then, there exists a polynomial $P \in K[X_1, \ldots, X_\delta]$ such that $P(a_1, \ldots, a_\delta) = 0$, and so $\nu_L(P(a_1, \ldots, a_\delta)) = \infty$. Using multi-index notation, we write

$$P(X_1, \ldots, X_\delta) = \sum_{|I|=1}^{m} b_I X^I$$

with $b_I \in K^*$ and $m \in \mathbb{N}$. By the strong triangle inequality, we get that

$$\nu_L(b_I a^I) = \nu_L(b_J a^J)$$

for some $I \neq J$, since $\nu_L(P(a_1, \ldots, a_\delta))$ would otherwise be equal to

$$\min_I\{\nu_L(b_I a^I)\} \neq \infty.$$

We get that $\nu_L(a^I) = \nu_L(a^J) + \nu_L(b_I b_J)$. Since $\nu_L(a^I)$ and $\nu_L(a^J)$ are in $\bigoplus i = 1^\delta \mathbb{Q}\varpi$, we get by the ordering on $\Gamma_K \oplus \bigoplus_{i=1}^{\delta} \mathbb{Q}\varpi$ that $\nu_L(b_I b_J) = 0$ and $\nu_L(a^I) = \nu_L(a^J)$. But since the $\nu_L(a_i) = r_i + \varpi_i$ are $\mathbb{Q}$-linearly independent, again due to the ordering on $\Gamma'_K \oplus \bigoplus_{i=1}^{\delta} \mathbb{Q}\varpi$, this contradicts that $I \neq J$. Hence, $a_1, \ldots, a_\delta$ are algebraically independent over K, which shows that the transcendence degree of K' over K is at least δ. Furthermore, from Lemma 10.6 we have that the transcendence degree of K' over K is bounded above by d. So $\delta \leq d$, which implies that $\dim(\nu_K(X)) \leq d$. □

From Theorem 11.4, we can now deduce the main result of this section, originally Theorem A in [BG].

Corollary 11.4.1 (Bieri-Groves Theorem)**.** *Let K be an algebraically closed valued field with a real valuation ν_K and value group $\Gamma_K \subset \mathbb{R}$. Let X be an algebraic variety over the algebraic torus $(K^*)^n$. Then the topological closure of $\nu_K(X)$ in $\mathbb{R}^n$ is a finite union of real polyhedra.*

Proof. From Theorem 11.4, we have that $\nu_K(X)$ is a finite union of Γ_K-polyhedra. Write

$$\nu_K(X) = P := \bigcup_{i=1}^{k} \bigcap_{j=1}^{\ell} \{\bar{\gamma} \in \Gamma_K^n \mid \varphi_{i,j}(\bar{\gamma}) \leq 0\}.$$

We claim that the closure of $\nu_K(X)$ in $\mathbb{R}^n$ is equal to

$$P_{\mathbb{R}} := \bigcup_{i=1}^{k}\bigcap_{j=1}^{\ell}\{\bar{\gamma} \in \mathbb{R}^n \mid \varphi_{i,j}(\bar{\gamma}) \leq 0\}.$$

Since $P \subset P_{\mathbb{R}}$ and $P_{\mathbb{R}}$ is closed, the inclusion $\bar{P} \subset P_{\mathbb{R}}$ is immediate. For the other inclusion, we will show that any point of $P_{\mathbb{R}}$ is a limit point of P.

Let $\bar{\alpha} \in P_{\mathbb{R}}$ and let $B_\epsilon(\bar{\alpha})$ be an open ball of radius $\epsilon > 0$ around $\bar{\alpha}$. We will show that $B_\epsilon(\bar{\alpha}) \cap P \neq \emptyset$. Let $\mathcal{F}_<$ be the set consisting of the $\varphi_{i,j}$ defining P such that $\varphi_{i,j}(\bar{\alpha}) < 0$ and let $\mathcal{F}_=$ be the set of $\varphi_{i,j}$ defining P such that $\varphi_{i,j}(\bar{\alpha}) = 0$. If $\mathcal{F}_= = \emptyset$, then $\varphi_{i,j}(\bar{\gamma}) < 0$ for all i, j. For $\bar{\delta} = (\delta_1, \ldots, \delta_n) \in \Gamma_K^n$, we have that

$$\varphi_{i,j}(\bar{\alpha} + \bar{\delta}) = \varphi_{i,j}(\bar{\alpha}) + \sum_{s=1}^{n} m_{s,i,j}\delta_s$$

for some $m_{i,j} \in \mathbb{Z}$. Since Γ_K is dense in $\mathbb{R}$ we can pick an arbitrarily small element $\bar{\delta} \in \mathbb{R}^n$ such shat $\bar{\alpha} + \bar{\delta} \in \Gamma^n$. In particular, we can choose $\bar{\delta}$ so that $\bar{\alpha} + \bar{\delta} \in \Gamma^n$, $\|\bar{\delta}\| < \epsilon$ and

$$\left|\sum_{s_1}^{n} m_{s,i,j}\delta_s\right| < |\varphi_{i,j}(\bar{\alpha})|$$

for all i, j. So we get that

$$\varphi_{i,j}(\bar{\alpha} + \bar{\delta}) = \varphi_{i,j}(\bar{\alpha}) + \sum_{s=1}^{n} m_{s,i,j}\delta_s < 0.$$

This shows that $\bar{\alpha} + \bar{\delta} \in P_\Gamma$ and so $\bar{\alpha} + \bar{\delta} \in B_\epsilon \cap P$, which is what we wanted to show.

Suppose that $\mathcal{F}_=$ is not empty. The set

$$P'_{\mathbb{R}} := \{\bar{\gamma} \in \mathbb{R}^n \mid \varphi_{i,j}(\bar{\gamma}) = 0 \text{ for all } \varphi_{i,j} \in \mathcal{F}_=\}$$

is an affine subspace of $\mathbb{R}^n$ containing

$$P' := \{\bar{\gamma} \in \Gamma^n \mid \varphi_{i,j}(\bar{\gamma}) = 0 \text{ for all } \varphi_{i,j} \in \mathcal{F}_=\}.$$

If $P'_{\mathbb{R}} = \{\bar{\alpha}\}$ for some $\bar{\alpha} \in \mathbb{R}^n$, then $\bar{\alpha}$ must be an element in Γ_K^n and so $\bar{\alpha}$ is already in P. One can see this by noting that in this situation, Gaussian elimination on the system of linear equations $\varphi_{i,j}(\bar{\gamma}) = 0$ will yield a unique solution on the form $(q_1\alpha_1, \ldots, q_n\alpha_n) = \bar{\alpha}$ with $q_i \in \mathbb{Q}$ and $\alpha_i \in \Gamma$. Each $q_i\alpha_i$ is in Γ since Γ_K is divisible, hence $\bar{\alpha} \in \Gamma_K^n$. So assume that P' is an affine subspace of dimension at least 1. Up to a rigid transformation, $P'_{\mathbb{R}}$ can be identified with $\mathbb{R}^m$ for some $m < n$, and P' is identified with Γ^m under this transformation. Just as above, we can pick an arbitrarily small $\bar{\delta} \in \mathbb{R}^m$ such that $\bar{\alpha} + \bar{\delta} \in \Gamma_K^m$, identifying $\bar{\alpha}$ with its image in $\mathbb{R}^m$. Since rigid transformations preserve distance, we have that there are elements $\bar{\gamma} \in P'_\Gamma$ arbitrarily close to $\bar{\alpha} \in P$. Hence, we can pick an element $\bar{\delta} \in P'_{\mathbb{R}}$ such that $\bar{\alpha} + \bar{\delta} \in P'_\Gamma$, $\bar{\alpha} + \bar{\delta} \in B_\epsilon(\bar{\alpha})$ and

$$\varphi_{i,j}(\bar{\alpha} + \bar{\delta}) < 0$$

for all $\varphi_{i,j} \in \mathcal{F}_<$. By construction, $\bar{\alpha} + \bar{\delta} \in P$ so we get that $P \cap B_\epsilon(\bar{\alpha}) \neq \emptyset$, and we are done proving the corollary. $\square$

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