UNIVERSITÀ DEGLI STUDI DI MILANO

Universität Regensburg

ALGANT Master's Thesis

Kan Complexes as a Univalent Model of Type Theory

Gabriele Lobbia

Supervisor: Prof. Denis-Charles Cisinski

Academic Year 2017-2018

Do not worry about your
difficulties in Mathematics.
I can assure you mine
are still greater.

Albert Einstein

A mathematician is a device for
turning coffee into theorems.
Therefore a comathematician is a device for
turning cotheorems into ffee.

Acknowledgements

First of all I would like to express my full gratitude to my supervisor Prof. Denis-Charles Cisisnki, for accepting me as his students, for beeing always available for everything I needed, for helping me in the search for my future academic occupation and for teaching me so many aspects of mathematics. Every meeting has been enlightening math-wise and enjoyable on a personal level.

I also have to thanks all the ALGaNT Consortium for the opportunity they gave me. In these two years I was able to meet fantastic people that helped me growing as a mathematician and as a person, and the credit is firstly theirs. The involment in this program has been crucial in the search for a PhD, and I am really grateful for this too.

Speaking of people I have met, a special thanks goes to every friend that helped making this experience unique and amazing.
Thanks to all my ”mathematics” friends that shared with me this incredible passion of mine and adviced me whenever I was struggling.
Thanks to all my friends from Milan. They have always been there for me and I am sure they will always be, even though I might be away from home for sometime.
Thanks to my ”Erasmus” friends, that beared me during all the writing of this thesis and helped me live the experience abroad in all his different parts.

Last but not least, grazie alla mia famiglia. Ai miei genitori, ai miei fratelli, ai miei zii, ai miei nonni, a mia cugina e alla mia zia Lella. Senza nessuno di loro sarei diventato la persona che sono, anche se in effetti potrebbe essere una colpa e non un merito. Ma sopratutto, senza il loro incondizionato supporto e aiuto non sarei mai riuscito a raggiungere nessun obbiettivo, quale Laurea o altro.

Contents

Introduction

The aim of this Thesis is to illustrate an example of a model of Martin-Löf Dependent Type Theory where Voedvosky's Univalence Axiom holds. The main results are explained in [KL16].

The first chapter will handle the construction of a model category in the enviroment of simplicial sets (*sSet*). In order to do that we will investigate the definitions of Kan complexes and Kan fibrations.

Secondly we will give a rough overview of type theory in general. In particular, we will start giving an idea of what Simple and Dependent Type Theory mean. Then, finally, we will se a natural model of Homotopy Type Theory in category theory, thanks to [Awo16].

The last chapter will deal with the Univalence Axiom itself and the theorems that explain why Kan complexes can be seen as a model of type theory where it holds.

1 Kan Complexes

1.1 Preliminary Definitions

We denote Δ the category with objects the sets $[n] = \{0, ..., n\}$ for any $n \in \mathbb{N}$, and morphism the non decreasing functions between them.

Definition 1.1.1. *We call* ***simplicial set*** *(or* ***complex****) a controvariant functor of the kind* $X : \Delta^{op} \to Set$. *We write* X_n *for the set* $X([n])$ *and we call its elements the* ***n-simplices of*** X. *Usually the image of a function* $f : [q] \to [p]$ *through the complex* X *is denoted by* f^*.

From now on we will write *sSet* for the category whose objects are the simplicial sets and morphisms the natural trasformations between them. A **subcomplex** will be just a suboject in this category (i.e. a subfunctor).

Definition 1.1.2. *We write* Δ^n *for the complex defined by the contravariant functor represented by* $[n]$*, more precisely*

$$\begin{tikzcd} {[q]} \arrow[r, mapsto] \arrow[d, "f"'] & \Delta([q], [n]) \\ {[p]} \arrow[r, mapsto] & \Delta([p], [n]) \arrow[u, "- \circ f"'] \end{tikzcd}$$

We call it the ***standard*** n***-simplex***.

Definition 1.1.3. *Let* X *be a simplicial set, and* $x \in X_m$ *an* m*-simplex of* X. *We say that* x *is* ***degenerate*** *if there exists an epimorphism* $s : [m] \twoheadrightarrow [n]$ *with* $n < m$ *and a* n*-simplex* $y \in X_n$ *such that* $x = X(s)(y)$. [1]

Proposition 1.1.4. *Let* X *be a simplicial set. Any morphism* $a : \Delta^n \to X$ *has a unique factorization*

$$\begin{tikzcd} \Delta^n \arrow[rr, "a"] \arrow[dr, two heads] & & X \\ & \Delta^m \arrow[ur, "b"'] & \end{tikzcd}$$

where the first map is an epimorphism and b *is not degenerate, i.e. the correspondent element in* X_m *(by the Yoneda Lemma[2]) is not degenerate.* [1]

[1][GZ67] Chapter II, §3.1.
[2]Lemma A.1.1

Definition 1.1.5. *Let X be a complex. We define $Sk^n X$ the* ***n-skeleton of*** *X to be the subcomplex of X defined as follows* [3]

$$(Sk^n X)_m := \{x \in X_m \mid x \text{ is degenerated from a } q\text{-simplex with } q \leq n\}$$

Remark 1.1.6. The epimorphisms $p : [m] \twoheadrightarrow [n]$ in Δ are simply the non decreasing surjections. Moreover any epimorphism has a section, i.e. there is a morphism $s : [n] \to [m]$ such that $p \circ s = Id_{[n]}$.
Similarly the monomorphisms $s : [n] \rightarrowtail [m]$ are the non decresing injections and they have a retraction $p : [m] \to [n]$ in Δ, i.e. $p \circ s = Id_{[n]}$. [4]

Definition 1.1.7. *We denote with $\dot{\Delta}^n \equiv \partial\Delta^n := Sk^{n-1}\Delta^n$ the* ***boundary of the standard n-simplex****.*

We can see that any complex X is the union of its skeletons. Moreover there is a filtration

$$\emptyset =: \mathrm{SK}^{-1}X \subseteq \mathrm{SK}^0 X \subseteq ... \subseteq \mathrm{SK}^n X \subseteq ... \subseteq X$$

But simplicial sets have even a more important property. In fact we can recover $\mathrm{SK}^n X$ from $\mathrm{SK}^{n-1}X$ thanks to a pushout diagram, exactly analogously to the construction of CW-complexes in Topology. We have the more general result:

Theorem 1.1.8. *Let $X \hookrightarrow Y$ be an inclusion of simplicial sets. Let Σ_n be the set of non degenerate elements of Y_n which do not belong in X_n. Let us denote $Y^i := Sk^i Y \cup X$. Then the following diagram is a pushout:*

$$\begin{tikzcd}
\coprod_{s\in\Sigma_n} \partial\Delta^n \arrow[r] \arrow[d, hook] & Y^{n-1} \arrow[d, hook] \\
\coprod_{s\in\Sigma_n} \Delta^n \arrow[r] & Y^n
\end{tikzcd}$$

where the bottom and top morphism are defined componentwise as the morphism $\Delta^n \to Y$ correspondet to $s \in Y_n$. Moreover we have that $Y \cong \varinjlim Y^i$.[5]

Example 1.1.9. *Let us define the simplicial set S^1 as the pushout of the following diagram*

$$\begin{tikzcd}
\partial\Delta^1 \arrow[r, hook] \arrow[d] & \Delta^1 \arrow[d, "\pi"] \\
\Delta^0 \arrow[r] & S^1
\end{tikzcd}$$

[3][GZ67] Chapter II, §3.5.
[4][GZ67] Chapter II, §2.3.
[5]Generalization of [GZ67] Chapter II, Proposition 3.8.

We can see that π is both not injective and not degenerate. In fact $\#S^1_0 = 1$ and π corresponds to a different element of S^1_1 than the map $\Delta^1 \to \Delta^0 \to S^1$.

Remark 1.1.10. (/Alternative definition)
We can see that actually

$$\partial\Delta^n = \bigcup_{q<n} \mathrm{Im}(\Delta^q \to \Delta^n)$$

In fact, we can prove the more general statement:

$$\mathrm{Sk}^n X = \bigcup_{q\leq n} \mathrm{Im}(\Delta^q \to X)$$

That is true because by Yoneda any map $\Delta^q \to X$ correspond to a unique element of X_q. The condition regarding s to be an epimorphism is satisfied because we take the images of these maps.

We recall now an important property of morphisms in Δ, which will be useful to make the definition above easier. First of all we have to give the definition of the i**-th coface map** $\delta^n_i : [n-1] \to [n]$

$$\delta^n_i(x) = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if not} \end{cases}$$

Another important class of morphisms is formed by the so called i**-th codegeneracy maps** $\sigma^n_i : [n+1] \to [n]$

$$\sigma^n_i(x) = \begin{cases} x & \text{if } x \leq i \\ x-1 & \text{if not} \end{cases}$$

From now on, for any simplicial set X, we will denote with $d^i_n := (\delta^n_i)^* : X_n \to X_{n-1}$ and $s^i_n := (\sigma^n_i)^* : X_n \to X_{n+1}$ the maps induced on the simplices. By straight forward calculation we can verify that these two classes of maps verify the following properties, which are called **simplicial relations**[6]:

1. $\delta^{n+1}_j \delta^n_i = \delta^{n+1}_i \delta^n_{j-1}$ for any $i < j$

2. $\sigma^n_j \sigma^{n+1}_i = \sigma^n_i \sigma^{n+1}_{j+1}$ for any $i \leq j$

3. $\sigma^{n-1}_j \delta^n_i = \begin{cases} \delta^{n-1}_i \sigma^{n-2}_{j-1} & \text{if } i < j \\ Id_{[n-1]} & \text{if } i = j \text{ or } i = j+1 \\ \delta^{n-1}_{i-1} \sigma^{n-2}_j & \text{if } i > j+1 \end{cases}$

[6][GZ67] Chapter II, §2.1.

Dually we have that:

1. $d^i_n d^j_{n+1} = d^{j-1}_n d^i_{n+1}$ for any $i < j$
2. $s^i_{n+1} s^j_n = s^{j+1}_{n+1} s^i_n$ for any $i \leq j$
3. $d^i_n s^j_{n-1} = \begin{cases} s^{j-1}_{n-2} d^i_{n-1} & \text{if } i < j \\ Id_{X_{[n-1]}} & \text{if } i = j \text{ or } i = j+1 \\ s^j_{n-2} d^{i-1}_{n-1} & \text{if } i > j+1 \end{cases}$

Lemma 1.1.11. *Any morphism $f : [q] \to [p]$ in Δ has a unique representation*

$$f = \delta^p_{i_k} \cdots \delta^{q-h+1}_{i_1} \sigma^{q-t}_{j_1} \cdots \sigma^{q-1}_{j_h} \tag{1}$$

with h, $k \geq 0$, $q + k - h = p$ and $p \geq i_k > ... > i_1 \geq 0$, $0 \leq j_1 < ... < j_h < q$. We will refer to it as the ***canonical decomposition of*** *f in Δ.*[7]

Proof A non decreasing function f is determined in a unique way by its image in $[p]$ and by the $j \in [q-1]$ such that $f(j) = f(j+1)$. Let us choose $i_1 < ... < i_k$ as the elements of the image of f in $[p]$, and $j_1 < ... < j_h$ as the elements satisfying $f(j) = f(j+1)$. By definition of coface and codegeneracy maps we have that (1) holds.
To see the uniqueness of this factorization is helpful to consider the equivalence of categories between Δ and the category fo finite non-empty totally oredered set (*Toset*). First of all we notice that any morphism in *Toset* has a unique (up to isomorphism) epi-mono factorization (passing through the image). Moreover we can prove, by induction, that any epimorphism/monomorphism can be factorize in a unique way as a composition of "minimal" epimorphism/monomorphism, exactly as required.

□

Thanks to this Lemma we can reformulate the definition of the boundary of the standard n-simplex as it follows:

Definition 1.1.12. *We define the* ***i-th face of the standard n-simplex*** *as the following subcomplex of Δ^n:* [8]

$$\partial_i \Delta^n := Im(\bar{\delta}^n_i : \Delta^{n-1} \to \Delta^n)$$

[7] [GZ67] Chapter II, §2.2.
[8] [JT99] §1.3.

where $\bar{\delta}_i^n$ is the unique morphism between Δ^{n-1} and Δ^n corresponding to $\delta_i^n \in \Delta_{n-1}^n$ through the Yoneda isomophism[9]*. Then we redefine the* ***boundary of the standard*** n***-simplex*** *as*

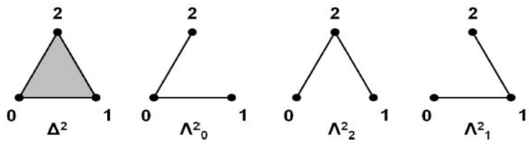
$$\partial\Delta^n := \bigcup_{0 \leq i \leq n} \partial_i \Delta^n$$

Finally we can give the definition that we will need to define later on the Kan fibration (and so the Kan complexes).

Definition 1.1.13. *We the* k***-th horn of the standard*** n***-simplex*** *as the following subcomplex of* Δ^n*:*

$$\Lambda_k^n := \bigcup_{i \neq k} \partial_i \Delta^n$$

To visualize better the concept of horns we can look at their geometric realization. For instance if we consider the standard 2-simplex we would get the following pictures: [10]



Definition 1.1.14. *Let X and Z be two simplicial sets. We define the* ***the product of*** Y ***and*** X *as the simplicial set defined in the following way:*

- *For any $n \in \mathbb{N}$ we set $(Z \times X)_n := Z_n \times X_n$;*
- *For any arrow $f : [m] \to [n]$ in Δ we set*

$$(Z \times X)(f) := (Z(f), X(f)) : Z_n \times X_n \longrightarrow Z_m \times X_m$$

Clearly this definition gives rise to a product in the category *sSet*. Moreover the construction is functorial in both components. Therefore, for any simplicial set X, we have that the cartesian product is an internal functor, i.e.

$$- \times X : sSet \longrightarrow sSet$$

[9]Lemma A.1.1.

[10]When we write that the standard simplices or the horns correpsond to a figure we are just using these to visualize them better. Actually it is possible to find a functor $| \ | : sSet \to Top$ called *geometric realization*, and we can see that the images of the simmplicial sets mentioned above are exactly the figures drawn. Some references are [JT99], [Rie11] and [GJ99].

Nonetheless, as any presheaves category, *sSet* is cartesian closed, i.e. there exists a right adjoint $(-)^X$ to $- \times X$. In categories of this kind the right adjoint of the cartisian product is usually called **internal hom**. More explicitly, for any simplicial sets Y and Z, we would have

$$sSet(Z \times X, Y) \cong sSet(Z, Y^X)$$

But by Yoneda we know that, for any simplicial set W, $W_n \cong sSet(\Delta^n, W)$. Obviously even Y^X has to respect this condition. Therefore we can find a more explicit description of it:

$$Y_n^X \cong sSet(\Delta^n, Y^X) \cong sSet(\Delta^n \times X, Y)$$

where the second isomophism is given by the adjuction.

Definition 1.1.15. *Let X and Y be two simplicial set. We define the simplicial set Y^X as follows:*

- *For any $n \in \mathbb{N}$ we set $Y_n^X := sSet(\Delta^n \times X, Y)$;*
- *For any arrow $f : [m] \to [n]$ in Δ we set*

$$Y^X(f) := - \circ \bar{f} \times Id_X : sSet(\Delta^n \times X, Y) \to sSet(\Delta^m \times X, Y)$$

where $\bar{f}$ is the element corresponding to $f \in \Delta_m^n$ through Yoneda.

We can see that this definition really gives us the right adjoint of $- \times X$ using the following maps:

$$sSet(Z \times X, Y) \rightleftarrows sSet(Z, Y^X)$$

$$\alpha \mapsto \alpha^\flat$$

$$\beta^\sharp \leftarrow\!\shortmid \beta$$

We define $\alpha_n^\flat$ as the maps sending any $z \in Z_n$ to $\alpha_n^\flat(z) : \Delta^n \times X \to Y$ defined as $(f, x) \mapsto \alpha_m(f^*(z), x)$ as the m-component. On the other hand the n-component of $\beta^\sharp$ is defined by $(z, x) \mapsto \beta_n(z)(Id_{[n]}, x)$. It can be proven that these constructions are inverse of each other and therefore we have an isomorphism. Clearly everything is natural in all the entries. [11]

Notation: We write $ev^X : (- \times X) \circ (-)^X \stackrel{.}{\to} Id_{sSet}$ for the counit of the adjunction, since when we are working with sets this morphism is actually the "evaluation" morphism.

[11] [GZ67] Chapter II, §2.5.3 and [Rie11] Example 4.8.

1.2 Nerve of a Category

Let us denote with *Cat* the category of small categories, where the morphisms are functors. We can consider any object $[n]$ of *sSet* as a (small) category where the objects are all $j \in [n]$ and morphisms are defined as follows

$$I[n](j,i) = \begin{cases} \{\emptyset\} & \text{if } j \leq i \\ \emptyset & \text{if not} \end{cases}$$

In order to avoid any possible confusion we will denote this category with $I[n]$. In a natural way any morphism $f : [m] \to [n]$ gives rise to a functor between the two posets seen as small categories, since it respects the order. Therefore we find a functor

$$I : \Delta \longrightarrow \mathit{Cat} \quad [n] \longmapsto I[n]$$

There is a natural functor $\mathcal{N} : \mathit{Cat} \to \mathit{sSet}$ which sends any small category $\mathbb{C}$ to the simplicial set

$$\mathcal{N}\mathbb{C} := \mathit{Cat}(-, \mathbb{C})$$

More precisely the n-simplices of $\mathcal{N}\mathcal{C}$ would be $\mathit{Cat}(I[n], \mathbb{C})$. For any morphism $f : [m] \to [n]$ in Δ we have that $f^* : \mathcal{N}\mathcal{C}_m \to \mathcal{N}\mathcal{C}_n$ is defined as the precomposition $G \mapsto G \circ If$ for any $G \in \mathcal{N}\mathcal{C}_m$.[12]

Definition 1.2.1. *For any small category $\mathbb{C}$ we call the simplicial set $\mathcal{N}\mathbb{C}$ described above **the nerve of the category** $\mathbb{C}$.*

Naturally a functor between small categories $F : \mathbb{C} \to \mathbb{D}$ gives rise to a morphism of simplicial sets between the nerves of these categories, whose $[n]$-component sends $G \mapsto F \circ G$ for any $G \in \mathcal{N}\mathcal{C}_n$. Therefore we get a functor

$$\mathcal{N} : \mathit{Cat} \longrightarrow \mathit{sSet}$$

Remark 1.2.2. More explicitly we can see that the n-simplices of the nerve of a small category $\mathbb{C}$: [13]

$\mathcal{N}\mathbb{C}_0 = \mathit{Cat}(I[0], \mathbb{C}) \cong \mathrm{Ob}(\mathbb{C})$
$\mathcal{N}\mathbb{C}_1 = \mathit{Cat}(I[1], \mathbb{C}) \cong \mathrm{Mor}(\mathbb{C})$
$\mathcal{N}\mathbb{C}_2 = \mathit{Cat}(I[2], \mathbb{C}) \cong \{\text{pairs of composable arrows } \cdot \to \cdot \to \cdot \text{ in } \mathbb{C}\}$
...
$\mathcal{N}\mathbb{C}_n = \mathit{Cat}(I[n], \mathbb{C}) \cong \{\text{strings of } n \text{ composable arrows } \cdot \to \cdots \to \cdot \text{ in } \mathbb{C}\}$

[12] [Cis18] §1.4.
[13] [Rie11] Example 3.2.

With this description the action of d_i^n, for $0 < i < n$, is to "remove" the i-th object and replace the morphisms there with the composition. For $i = 0$, n it leaves out the first or the last morphism. Instead s_i^n acts adding an identity morphism at the i-th spot.

On the other hand we can find a funtor $\tau : sSet \to Cat$. Let us consider a simplicial set X. We define $\mathrm{Ob}(\tau X)$ as the set X_0. We recover the morphisms in τX freely from the elements of X_1 subject to relations given by X_2 (i.e. we consider the free graph generated by X_1 with some conditions determined by X_2). More precisely, the unique map $\sigma_0^0 : [1] \to [0]$ induces a the morphism $s_0^0 : X_0 \to X_1$ which chooses the identities maps inside X_1, i.e. for any object $x \in X_0$ we have $Id_x := s_0^0(x)$. Then given any element $f \in X_1$ we define its domain as $d_1^0(f)$ and its codomain as $d_1^1(f)$ ($d_1^0, d_1^1 : X_1 \to X_0$). Lastly we say that $h = g \circ f$ if there exists $p \in X_2$ such that $d_2^2(p) = f$, $d_2^1(p) = g$ and $d_2^0(p) = h$:

$$\begin{tikzcd} & 1 \arrow[dr, "g"] & \\ 0 \arrow[ur, "f"] \arrow[rr, "h"'] & & 2 \end{tikzcd}$$

We have already that the composition is associative, since it is in the free graph. To prove that any Id_x truly works as the identity, we use s_1^0 and s_1^1 (respectively for the right and left identity axioms). In fact, let us consider $f \in X_1$, we need to prove $f \circ Id_x = f$. Setting $p := s_1^0(f)$ and using the dual simplicial relations we have that the property stated above give us the result wanted. To prove the left identity axiom we can use the same argument with s_1^1. All of this makes τX a (small) category.[14]

Remark 1.2.3. In the construction of τX we used just the 0-,1- and 2-simplices of X and the morphism between them. Therefore, for any simplicial set X, we have the following equivalence of categories:

$$\tau X \cong \tau \mathrm{Sk}^2 X$$

Proposition 1.2.4. *The functors above form an adjoint pair $\tau \dashv \mathcal{N}$.*

Proof Let X be a simplicial set and $\mathbb{C}$ a small category. Let us consider a morphism of simplicial sets $\varphi : X \to \mathcal{N}\mathbb{C}$, we want to find a $\widetilde{\varphi} : \tau X \to \mathbb{C}$. We can define it as follows:

- On objects we can define $\widetilde{\varphi}$ as $\varphi_0 : X_0 \to \mathcal{N}\mathbb{C}_0$, since $\mathrm{Ob}(\tau X) = X_0$ by definition and by remark 1.2.2 we have a bijection $\mathcal{N}\mathbb{C}_0 \cong \mathrm{Ob}(\mathbb{C})$.

[14] [Rie11] Example 4.6.

- Any arrow $x \to y$ in τX is represented by a finite sequence of element $f_i \in X_1$ for $i = 1, ..., n$ (the arrow is the composition of them), such that $d_1^0(f_1) = x$ and $d_1^1(f_n) = y$ (and $d_1^1(f_i) = d_1^0(f_{i+1})$ for any $i \geq 1$). Now we can define its image through $\widetilde{\varphi}$ as the composition of $\varphi_1(f_i) \in \mathcal{N}\mathbb{C}_1$, again thanks to remark 1.2.2.

Now we have to prove that the definition above gives actually a functor, i.e. that it sends identity maps to identity maps and it respects the composition law.
Since φ is a natural transformation, we have that, for any $x \in X_0$, $\varphi_1(Id_x) = \varphi_1(s_0^0(x)) = Id_{\varphi_0(x)}$. Moreover let us consider $f_0 = f_1 \circ f_2$ in τX, i.e. exist $p \in X_2$ such that $d_2^i(p) = f_i$ for any $i \in [2]$. Let us consider the following commutative diagrams

$$\begin{tikzcd} X_2 \arrow[r, "d_2^i"] \arrow[d, "\varphi_2"'] & X_1 \arrow[d, "\varphi_1"] \\ \mathcal{N}\mathbb{C}_2 \arrow[r, "-\circ\delta_i^2"'] & \mathcal{N}\mathbb{C}_1 \end{tikzcd}$$

They tell us that the diagram related to $\varphi_2(p)$ is

$$\begin{tikzcd} \varphi_0(s_0^0(f_2)) \arrow[r, "\varphi_1(f)"] \arrow[rr, bend right, "\varphi_1(h)"'] & \varphi_0(s_0^0(f_1)) \arrow[r, "\varphi_1(g)"] & \varphi_0(s_0^0(f_0)) \end{tikzcd}$$

and therefore the description above gives us a functor. Moreover this construction is functorial, and so it induces a natural morphism

$$\widetilde{(-)} : sSet(X, \mathcal{N}\mathbb{C}) \longrightarrow Cat(\tau X, \mathbb{C})$$

On the other hand let us consider now a functor $\psi : \tau X \to \mathbb{C}$. We want to find a morphism of simplicial sets $\bar{\psi} : X \to \mathcal{N}\mathbb{C}$, i.e. a family of morphisms of sets $\bar{\psi}_n : X_n \to \mathcal{N}\mathbb{C}_n$ for any $n \in \mathbb{N}$ such that for any $\alpha : [m] \to [n]$ in Δ the following diagram commute

$$\begin{tikzcd} X_m \arrow[r, "\alpha^*"] \arrow[d, "\bar{\psi}_m"'] & X_n \arrow[d, "\bar{\psi}_n"] \\ \mathcal{N}\mathbb{C}_m \arrow[r, "-\circ\alpha"'] & \mathcal{N}\mathbb{C}_n \end{tikzcd}$$

By the description of the nerve in remark 1.2.2, it sufficies to give the description of $\bar{\psi}_n$ for $n = 0, 1$. In fact for any $n > 1$ the image of $\bar{\psi}_n$ has to be a string of n (composable) morphisms. Therefore, using d_i^n's, we can reduce

to each morphism in the string. The fact that ψ is a natural transformation between τX and $\mathbb{C}$ will tell us that the composition of this maps is well defined.
For $n = 0$ we have to find a morphism $\bar{\psi}_0 : X_0 \to \mathcal{N}\mathbb{C}_0$, which by definition is the same as finding a morphism between the objects of τX and $\mathbb{C}$. Therefore for any $x \in X_0$ we define $\bar{\psi}_0(x) := \psi(x)$. Moreover to find a morphism $\bar{\psi}_1 : X_1 \to \mathcal{N}\mathbb{C}_1$ is equivalent to find a morphism between the morphisms of the two categories mentioned above. Thus we can define $\bar{\psi}_1$ as ψ itself (defined on the morphisms). The condition on the composition in τX tells us that $\bar{\psi}_2$ (and so on) can be defined just on the restriction given by d_0^1 and d_1^0, i.e. on each map of the pair in $\mathcal{N}\mathbb{C}$.
This construction is functorial and so it gives us a natural transformation

$$\bar{(-)} : Cat(\tau X, \mathbb{C}) \longrightarrow sSet(X, \mathcal{N}\mathbb{C})$$

For any $\psi : \tau X \to \mathbb{C}$, using the definitions, it is easy to check that on objects, i.e. $\forall\, x \in X_0$, $\widetilde{\bar{\psi}}(x) = \bar{\psi}_0(x) = \psi(x)$ and for any arrow $f \in X_1$ it is true that $\widetilde{\bar{\psi}}(f) = \bar{\psi}_1(f) = \psi(f)$. Thus $\widetilde{(-)} \circ \bar{(-)} = \mathrm{Id}_{Cat(\tau X, \mathbb{C})}$.
On the other hand, again using the definitions, we can prove that given $\varphi : X \to \mathcal{N}\mathbb{C}$ we have that $\bar{\widetilde{\varphi}} = \varphi$. By the construction of $\bar{(-)}$ it is enough to prove the equality for the $0-$ and $1-$component of the morphism. But $\bar{\widetilde{\varphi}}_0 = \widetilde{\varphi} = \varphi_0$ (where the morphism in the middle acts on the objects) and similarly $\bar{\widetilde{\varphi}}_1 = \widetilde{\varphi} = \varphi_1$ (this time the morphism operates on arrows). Therefore we have also $\bar{(-)} \circ \widetilde{(-)} = \mathrm{Id}_{sSet(X, \mathcal{N}\mathbb{C})}$.

$\square$

Let us consider $\mathbb{C}$ a small category and $\widetilde{Id_{\mathcal{N}\mathbb{C}}} : \tau\mathcal{N}\mathbb{C} \to \mathbb{C}$. By the construction given above we can see that this morphism is an equivalence. Thus we have that $\mathcal{N}$ is fully faithful. Thanks to this we can prove the following property.

Remark 1.2.5. In *Cat* there exist all kind of colimits.

Proof Let us consider a diagram in *Cat*, by the equivalece stated before the existence of $\mathrm{colim}\mathbb{C}_i$ is equivalent to the one of $\mathrm{colim}\tau\mathcal{N}\mathbb{C}_i$. But, since in *sSet* there exist colimits, we know that it exists $\mathrm{colim}\mathcal{N}\mathbb{C}_i$. Since τ is a left adjoint we have that it preserves colimits, and therefore we get that $\mathrm{colim}\tau\mathcal{N}\mathbb{C}_i$ exists and in particular is isomoprhic to $\tau(\mathrm{colim}\mathcal{N}\mathbb{C}_i)$. [15]

$\square$

[15] [GZ67] Chapter I, §1.3 and §1.4.

By the density theorem[16] we know that any simplicial set is a colimit over the standard simplices. Since τ is a left adjoint, it preserves colimits. Therefore, for any simplicial set X, we have that

$$\tau X \cong \tau(\varinjlim_{\Delta^n \to X} \Delta^n) \cong \varinjlim_{\Delta^n \to X} \tau\Delta^n$$

But $\mathrm{Obj}(\tau\Delta^n) = \Delta^n_0 = \Delta([0], [n]) \cong [n]$ (seen as sets) and the morphism are the elements of $\Delta^n_1 = \Delta([1], [n])$. For any i, $j \in [n]$, by the definitions given before, we have that exist a unique morphism with domain i and codomain j (in $\tau\Delta^n$) if and only if $i \leq j$. Otherwise there are no morphisms from i to j. Therefore the category $\tau\Delta^n$ is equivalent to $I[n]$, or equivalently we have the commutative diagram (up to natural isomorphisms):

$$\begin{tikzcd}
\Delta \arrow[rr, "Y"] \arrow[dr, "I"'] & & sSet \arrow[dl, "\tau"] \\
& Cat &
\end{tikzcd}$$

So we can decribe τ as[17]

$$\tau X \cong \varinjlim_{\Delta^n \to X} I[n]$$

Any element in the colimit correspond to a morphism $\Delta^n \to X$ and therefore through Yoneda[18] to a unique n-simplex x of X. Thus, for any morphism of simplicial sets $\varphi : X \to Y$, the $[n]$-component of $\tau\varphi$ is the unique morphism (given by the universal property of the colimit) sending the component corresponding to $x \in X_n$ to the one of $\varphi_n(x) \in Y_n$.

1.3 Fibrations and Complexes

Definition 1.3.1. *(i) A morphism of simplicial sets $p : E \to B$ is a **fibration in the sense of Kan** if for any inclusion $i : \Lambda^n_k \hookrightarrow \Delta^n$ (for any $k \leq n$) and each commutative diagram*

$$\begin{tikzcd}
\Lambda^n_k \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
\Delta^n \arrow[r, "v"'] & B
\end{tikzcd}$$

there exist a morphism of simplicial sets $w : \Delta^n \to E$ (usually called a diagonal or a filler) such that $pw = v$ and $wi = u$.

[16] Theorem A.2.2.
[17] [JT99] §1.2 and [JT08] §1.3.
[18] Lemma A.1.1.

(ii) A simplicial set X is a ***Kan complex*** *if the unique morphism $X \to \Delta^0$ is a fibration in the sense of Kan.* [19]

Remark 1.3.2. We know that Δ^0 is a terminal object in *sSet*. Let us take a closer look at the condition to be a Kan complex. We need to have a diagonal morphism $w : \Delta^n \to X$ for any commutative diagram of the following kind:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r, "v"'] \arrow[ur, dotted, "w"] & \Delta^0 \end{tikzcd}$$

But since Δ^0 is terminal, the lower triangle commutes for any morphism from Δ^n to X. Thus to be a Kan complex is sufficient the following alternative (but equivalent) definition.

Proposition 1.3.3. ***(/Alternative definition)*** *A simplicial set X is a Kan complex if and only if (for any n and $k \leq n$) each morphism $\Lambda^n_k \to X$ can be extended to a morphism $\Delta^n \to X$. This property is usually called* ***the right lifting property*** *(RLP).*

An example of a Kan complex is the nerve of a small category which is in particular a groupoid. In fact we have the following property:

Proposition 1.3.4. *Let $\mathbb{C}$ be a small category. Then*

$$\mathcal{N}\mathbb{C} \text{ is a Kan complex} \Leftrightarrow \mathbb{C} \text{ is a groupoid}$$

Proof

($\Rightarrow$) Let $f : X \to Y$ be an arrow in $\mathbb{C}$, we want to find another morphism $g : Y \to X$ in $\mathbb{C}$ such that $fg = Id_Y$ and $gf = Id_X$.
We can define a morphism of simplicial sets $\alpha : \Lambda^2_0 \to \mathcal{N}\mathbb{C}$ in the following way:

- We set $\alpha_0 : (\Lambda^2_0)_0 = \Delta^2_0 \cong [2] \to \mathcal{N}\mathbb{C}_0 \cong \mathrm{Ob}(\mathbb{C})$ as the map sending any $i \in [2]$ to

$$\alpha_0(i) = \begin{cases} X & \text{if } i = 0,\, 2 \\ Y & \text{if } i = 1 \end{cases}$$

[19][GZ67] Chapter IV, §3.1.

– We set $\alpha_1 : (\Lambda^2_0)_1 = \{c_0,\, c_1,\, c_2\,, \delta^2_1,\, \delta^2_2\} \to \mathcal{N}\mathbb{C}_1 \cong \mathrm{Mor}(\mathbb{C})$ (where c_j denotes the constant map to j) as the morphism

$$c_0,\, c_1,\, \delta^2_1 \longmapsto Id_X$$

$$\delta^2_2 \longmapsto f$$

$$c_2 \longmapsto Id_Y$$

We recall that these informations are enough to construct a morphism into the nerve of a category (as we proved in Proposition 1.2.4). The idea is that we want to construct the following diagram in $\mathbb{C}$

$$\begin{tikzcd} & Y \arrow[dr, dotted] & \\ X \arrow[ur, "f"] \arrow[rr, "Id_X"'] & & X \end{tikzcd}$$

The dotted map will be given by the Kan condition on $\mathcal{N}\mathbb{C}$ and will give us the inverse to f. In fact, since $\mathcal{N}\mathbb{C}$ is a Kan complex we have a $\widetilde{\alpha}$ such that

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{N}\mathbb{C} \\ \Delta^2 \arrow[ur, "\widetilde{\alpha}"'] & \end{tikzcd}$$

commutes. Now, by Yoneda, we know that $\widetilde{\alpha}$ corresponds to a unique element $a \in \mathcal{N}\mathbb{C}$, i.e. to a pair of composable arrows in $\mathbb{C}$. The composition $a \circ \delta^2_0$ gives rise to a morphism $g : Y \to X$ such that $gf = Id_X$. To see this more clearly one could consider the naturality diagrams given by $\widetilde{\alpha}$ through the morphisms δ^2_i for $i = 0,\, 1,\, 2$.
Similarly we can construct a morphism $\beta : \Lambda^2_1 \to \mathcal{N}\mathbb{C}$ corresponding to the diagram

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Y \arrow[ur, dotted] \arrow[rr, "Id_Y"'] & & Y \end{tikzcd}$$

Again thanks to the Kan condition we find a morphism $g' : Y \to X$ such that $fg' = Id_y$. It is easy to prove that $g = g'$ and therefore we get that $\mathbb{C}$ is a groupoid.

($\Leftarrow$) Let us suppose that $\mathbb{C}$ is a groupoid. We have to prove that for any $n \in \mathbb{N}$, any $k \leq n$ and any map $\alpha : \Lambda^n_k \to \mathcal{N}\mathbb{C}$ there is an extension $\widehat{\alpha}$

of it to Δ^n, i.e.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\forall\,\alpha"] \arrow[d, hook] & \mathcal{N}\mathbb{C} \\ \Delta^n \arrow[ur, dotted, "\exists\,\widehat{\alpha}"'] & \end{tikzcd}$$

<u>$n = 0$</u> The statement is trivially true since $\Lambda^0_0 = \Delta^0$ by definition.

<u>$n = 1$</u> We can see that for $i = 0, 1$ we have $\Lambda^1_i \cong \Delta^0$. Therefore, by Yoneda, any map $\Lambda^1_i \to \mathcal{N}\mathbb{C}$ corresponds to a unique element of $\mathcal{N}\mathbb{C}_0 = \mathrm{Ob}(\mathbb{C})$. On the other hand, again thanks to Yoneda, to have a morphism from Δ^1 to $\mathcal{N}\mathbb{C}$ is equivalent to have an arrow in $\mathbb{C}$ (i.e. an element of $\mathcal{N}\mathbb{C}_1$). Therefore one can prove that the extension of an object X of $\mathbb{C}$ is the identity of that particular object (seeing them as morphism from Λ^1_i to $\mathcal{N}\mathbb{C}$ and from Δ^1 to $\mathcal{N}\mathbb{C}$, respectively).

<u>$n = 2$</u> The idea is that, as we developped in the first part of the proof, a morphism from Λ^2_0 to $\mathcal{N}\mathbb{C}$ is represented by a diagram of the following form:

$$\begin{tikzcd} & Y \arrow[dr, dotted] & \\ X \arrow[ur, "f"] \arrow[rr, "g"'] & & Z \end{tikzcd}$$

Therefore to extend it to a morphism $\Delta^2 \to \mathcal{N}\mathbb{C}$ we can consider $g \circ f^{-1} : Y \to Z$ (again we recall that a map $\Delta^2 \to \mathcal{N}\mathbb{C}$ is equivalent to a pair of composable arrows in $\mathbb{C}$). For Λ^2_1 and Λ^2_0 we can use a similar argument.

<u>$n > 2$</u> By the adjunction $\tau \dashv \mathcal{N}$, the problem of finding an extension for the diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{N}\mathbb{C} \\ \Delta^n \arrow[ur, dotted, "\widehat{\alpha}"'] & \end{tikzcd}$$

is equivalent to find a morphism that makes the following diagram commute

$$\begin{tikzcd} \tau\Lambda^n_k \arrow[r] \arrow[d] & \mathbb{C} \\ \tau\Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

We recall that, by Remark 1.2.3, $\tau\Lambda^n_k \cong \tau\mathrm{Sk}^2\Lambda^n_k$. Moreover, for any $n > 2$, it is clear that $\tau\mathrm{Sk}^2\Lambda^n_k \cong \tau\mathrm{Sk}^2\partial\Delta^n$. Nonetheless a

Lemma in [GZ67][20] guarantees that, for any $n > 2$, $\tau\partial\Delta^n \cong \tau\Delta^n$ and therefore there exists the extension that we needed.
Let us take a closer look to the case $n = 3$. The standard 3-simplex correspond to the following figure (in the sense explained in the previous footnote 10):

$$\begin{tikzcd}
 & 1 \arrow[ddr] \arrow[drr] & & \\
 & & & 3 \\
0 \arrow[uur] \arrow[rr] \arrow[urrr, dotted] & & 2 \arrow[ur] &
\end{tikzcd}$$

The horn Λ^3_k correspond to the boundary without the face opposite to k. Therefore we can see that the equivalence $\tau\partial\Lambda^n_k \cong \tau\Delta^n$ is strictly connected to the associative property of the composition in a category.

$\square$

Definition 1.3.5. *A set of monomorphism $\mathcal{A}$ is called* ***saturated*** *if the following properties hold:*

(i) All isomorphism belong to $\mathcal{A}$.

(ii) $\mathcal{A}$ is closed under pushouts, i.e. for any pushout diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\xi"'] & Y \arrow[d, "\eta"] \\
X' \arrow[r] & Y'
\end{tikzcd}$$

with $\xi \in \mathcal{A}$, then $\eta \in \mathcal{A}$ as well.

(iii) $\mathcal{A}$ is closed under retracts, i.e. for any commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "\xi"'] & Y \arrow[r, "v"] \arrow[d, "\eta"] & X \arrow[d, "\xi"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & X'
\end{tikzcd}$$

with $vu = Id_X$, $v'u' = Id_{X'}$ and $\eta \in \mathcal{A}$, then $\xi \in \mathcal{A}$.

[20][GZ67] Chapter II, Lemma 4.2.

(iv) *$\mathcal{A}$ is closed under arbitrary coproducts, i.e. if $g_\alpha : X_\alpha \to Y_\alpha \in \mathcal{A}$ for any $\alpha \in \Lambda$, then*

$$\coprod_{\alpha \in \Lambda} g_\alpha : \coprod_\alpha X_\alpha \to \coprod_\alpha Y_\alpha$$

is in $\mathcal{A}$.

(v) *$\mathcal{A}$ is closed under countable composition, i.e. if $f_i : X_i \to X_{i+1} \in \mathcal{A}$ for any $i \geq 1$, then*

$$X_1 \longrightarrow \varinjlim_{i \geq 1} X_i$$

is in $\mathcal{A}$ as well. [21]

Definition 1.3.6. *Let $\mathcal{B}$ a set of monomorphisms. We call* ***the saturated set generated by*** *$\mathcal{B}$ the intersection of all saturated sets containing $\mathcal{B}$.*

Definition 1.3.7. *We call* ***anodyne extension*** *any monomorphism in Δ which belongs to the saturated set generated by all the inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ (for any $n \in \mathbb{N}$ and $k \leq n$).*

Let $p : E \to B$ be a morphism in *sSet*. Let us denote with Γ the set of all monomorphism $i : K \to L$ in *sSet* such that for any commutative diagram

$$\begin{tikzcd} K \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\ L \arrow[r, "v"'] & B \end{tikzcd}$$

there is a map $w : L \to E$ such that $pw = v$ and $wi = u$. It is easy to verify that this set is saturated. Therefore we have the following lemma:

Lemma 1.3.8. ***(/Alternative definition)***

(i) *A morphism of simplicial sets $p : E \to B$ is a fibration in the sense of Kan if and only for the diagonal condition holds for any anodyne extension.*

(ii) *A simplicial set X is a Kan complex if and only if it has the right lifting property (RLP) for any anodyne extension.* [22]

[21] [GZ67] Chapter IV, §2 and [JT99] §1.4.
[22] [JT99] §1.4, Proposition 1.4.1.

Definition 1.3.9. *A morphism of simplicial set $p : E \to B$ is a* ***trivial fibration*** *if for any inclusion $i : \partial\Delta^n \hookrightarrow \Delta^n$ and any commutative diagram*

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "u"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
\Delta^n \arrow[r, "v"'] & B
\end{tikzcd}$$

exist a right lifting $w : L \to E$ (i.e. $pw = v$ and $wi = u$). [23]

Remark 1.3.10. (/Alternative definition)
Using Theorem 1.1.8 we can see that the saturated set generated by the inclusions $\partial\Delta^n \hookrightarrow \Delta^n$ is actually the class of all monomorphism. Therefore we can define a trivial fibration as a morphism with the RLP for any monomorphism in *sSet*. With this definition is clear that a trivial fibration is even a Kan fibration.

1.4 Homotopy Theory

In this section we will abstract the idea of homotopy theory that we usually have on the category of topological spaces *Top* and define it in the enviroment of *sSet*. The main reference for all the following definitions, propositions and Theorems is [JT99], especially §1.5 and §1.6.

Definition 1.4.1. *Let X be a simplicial set. The coequalizer $\Pi_0(X)$ of the diagram*

$$\begin{tikzcd}
X_1 \arrow[r, shift left, "d_1^0"] \arrow[r, shift right, "d_1^1"'] & X_0 \arrow[r, "\pi"] & \Pi_0(X)
\end{tikzcd}$$

is called ***the set of connected components of*** *X.*

Obviously this definition is functorial and gives rise to a functor Π_0 : *sSet* $\to$ *Set*, where *Set* is the category of sets.

Notation: For $i = 0, 1$, we write $\epsilon_i : \Delta^0 \to \Delta^1$ for the morphisms corresponding to the one $[0] \to [1]$ mapping $0 \mapsto i$ through Yoneda.

This definition gives rise to a relation on X_0, which corresponds to the relation "to be connected by a path" in a topological space, i.e.

$$x \sim y \Leftrightarrow \exists\ \alpha : \Delta^1 \to X \quad \text{s.t.} \quad \alpha(0) := \alpha\epsilon_0 = x \text{ and } \alpha(1) := \alpha\epsilon_1 = y$$

where here $x, y \in X_0$ are considered as morphisms $\Delta^0 \to X$. For this reason we may call a morphism $\Delta^1 \to X$ a **path in** X. Unlikely to the topological case, this relation is not always an equivalence.

[23] [JT99] §1.4, Definition 1.4.1.

Remark 1.4.2. If X is a Kan complex then the relation $\sim$ defined above is an equivalence.

Proof To see that $x \sim x$ for any $x \in X_0$ we can just consider the path

$$\Delta^1 \to \Delta^0 \xrightarrow{x} X$$

which is usually called the **constant path on** x. Now let us consider x, $y \in X_0$ such that $x \sim y$ through the path α. The idea it to consider a morphism $s : \Lambda_0^2 \to X$ such that $s\bar{\delta}_2^2 = \alpha$ and $s\bar{\delta}_1^2$ the constant path on x. Since X is a Kan complex then we find an extension t of s to Δ^2. Then $t\bar{\delta}_0^2$ is a path from y to x, and therefore $y \sim x$.
Similarly, considering Λ_1^2, we can prove even the transitivity property.

□

Remark 1.4.3. Let X and Y be two simplicial sets. By the universal property of the coequalizer we find a morphism $\Phi : \Pi_0(X \times Y) \to \Pi_0(X) \times \Pi_0(Y)$, defined for any projection by the following diagram:

$$\begin{tikzcd}
X_1 \times Y_1 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "pr_{X_1}"'] & X_0 \times Y_0 \arrow[r] \arrow[d, "pr_{X_0}"] & \Pi_0(X \times Y) \arrow[d, dotted, "\exists !"] \\
X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[r] & \Pi_0(X)
\end{tikzcd}$$

With some calculation it can be proven that Φ is actually an isomoprhism.

We recall that two maps $f, g : X \to Y$ of topological spaces are homotopic if and only if there exists a continous moprhism $h : X \times I \to Y$ such that the following diagram commutes (where $I = [0, 1]$ is an interval in the real numbers with the subspace topology):

$$\begin{tikzcd}
X \times \{0\} \cong X \arrow[d, hook] \arrow[dr, bend left, "f"] & \\
X \times I \arrow[r, "h"] & Y \\
X \times \{1\} \cong X \arrow[u, hook] \arrow[ur, bend right, "g"'] &
\end{tikzcd}$$

Replacing I with Δ^1 and the singleton with Δ^0 we obtain an analogous definition for simplicial sets.

Definition 1.4.4. *Let X and Y be two simplicial sets, and $f, g : X \to Y$ two morhpisms between them. We say that f and g are* ***homotopic*** *if there exists $h : X \times \Delta^1 \to Y$ such that the following diagram commutes:*

$$\begin{tikzcd}
X \times \Delta^0 \cong X \arrow[d, hook, "Id_X \times \epsilon_0"'] \arrow[drr, bend left, "f"] & & \\
X \times \Delta^1 \arrow[rr, "h"] & & Y \\
X \times \Delta^0 \cong X \arrow[u, hook, "Id_X \times \epsilon_1"] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

By the adjuction $- \times X \dashv (-)^X$, This definition is equivalent to say that there is a path from f to g in Y^X. Therefore, if Y^X is a Kan complex then we have that the homotopy relation between maps from X to Y is an equivalence relation. The next theorems will give us a nice condition that guarantees that Y^X would be a Kan complex.

Let $i : A \to B$ and $k : Y \to Z$ be two monomorphisms in *sSet*. The monomorphism $i \times Z : A \times Z \to B \times Z$ and $B \times k : B \times Y \to B \times Z$ induce a monomorphism $i * k : (A \times Z) \cup (B \times Y) \to B \times Z$.

Theorem 1.4.5. *(Gabriel-Zisman) If i is anodyne, then $i * k$ is anodyne too.*

Let $k : Y \to Z$ be a monomorphism and $p : E \to X$ a morphism in *sSet*. We denote the pullback of X^k and p^Y by

$$\begin{tikzcd}
(k, p) \arrow[r] \arrow[d] & E^Y \arrow[d, "p^Y"] \\
X^Z \arrow[r, "X^k"'] & X^Y
\end{tikzcd}$$

We write $k|p : E^Z \to (k, p)$ for the morphism given by the unversal property from the triple (E^Z, E^k, p^Z). Now, thanks to the adjuction $- \times Z \dashv (-)^Z$ and Theorem 1.4.5, we can prove the following result:

Theorem 1.4.6. *If $p : E \to X$ is a Kan fibration, then $k|p : E^Z \to (k, p)$ is a Kan fibration as well. Moreover if either k is anodyne or p is trivial, then $k|p$ is trivial.*

If we consider $k = Id_Y$ we get the following corollary:

Corollary 1.4.7. *If $p : E \to X$ is a Kan fibration, then for any simplicial set Y the map $p^Y : E^Y \to X^Y$ is a Kan fibration too.*

Finally we get the condition wanted to let Y^X be a Kan complex, that is:

Corollary 1.4.8. *If Y is a Kan complex, then Y^X is a Kan complex for any simplicial set X.*

Proof We know that Y is a Kan complex if and only if $Y \to \Delta^0$ is a Kan fibration. Now we notice that $(\Delta^0)^X \cong \Delta^0$, since Δ^0 is the terminal object in *sSet*. Therefore using Corollary 1.4.7 we have the thesis.

$\square$

Now we are ready to define a new category that we will denote $Ho(sSet)$ (as the *homotopy category* related to sSet). The objects of this category are all Kan complexes. For any pair of Kan complexes X and Y, we define the morphisms to be homotopy classes of maps between them, more precisely we define

$$Ho(sSet)(X, Y) \equiv [X, Y] := \Pi_0(Y^X)$$

Let X, Y and Z three Kan complexes. To define the composition map as

$$\Pi_0(Y^X) \times \Pi_0(Z^Y) \cong \Pi_0(Y^X \times Z^Y) \xrightarrow{\Pi_0(m)} \Pi_0(Z^X)$$

where the isomorphism is given by Remark 1.4.3 and m is the unique morphism from $Y^X \times Z^Y$ to Z^X determined through the adjunction by the following morphism:

$$Y^X \times Z^Y \times X \cong Z^Y \times Y^X \times X \xrightarrow{Id_{Z^Y} \times ev_Y^X} Z^Y \times Y \xrightarrow{ev_Z^Y} Z$$

Definition 1.4.9. *Let $i : A \rightarrowtail B$ be a monomorphism in sSet. We say that A is a* ***strong deformation retract of*** *B if it exists a retraction $r : B \to A$ of i (i.e. $ri = Id_A$) and a homotopy $h : B \times \Delta^1 \to B$ from Id_B to ir (i.e. $h_0 = Id_B$ and $h_1 = ir$) such that h is "stationary on A", i.e. the following triangle commutes:*

$$\begin{tikzcd}
A \times \Delta^1 \arrow[r, "i \times Id_A"] \arrow[dr, "pr_A"'] & B \times \Delta^1 \arrow[r, "h"] & B \\
& A \arrow[ur, "i"'] &
\end{tikzcd}$$

In this case we call r a ***strong deformation retraction****.*

A crucial property of strong deformation retracts is that they are stable under pullbacks on Kan fibrations. In other words the following proposition holds:

Proposition 1.4.10. *Let $p : E \to X$ be a Kan fibration and $i : A \rightarrowtail X$ a monomorphism. If A is a strong deformation retract of X, then $p^{-1}(A)$ is a strong deformation retract of E through the morphism defined by the pullback diagram*

$$\begin{tikzcd} p^{-1}(A) \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, hook, "i"'] & X \end{tikzcd}$$

Proof Let $h : X \times \Delta^1 \to X$ the strong deformation on A. The following diagrams commute

$$\begin{tikzcd} E \times \Delta^0 \arrow[rr, "\sim"] \arrow[d, hook, "Id_E \times \epsilon_0"'] & & E \arrow[r, equal] & E \arrow[d, "p"] \\ E \times \Delta^1 \arrow[rr, "p \times Id_{\Delta^1}"'] & & X \times \Delta^1 \arrow[r, "h"'] & X \end{tikzcd}$$

$$\begin{tikzcd} p^{-1}(A) \times \Delta^1 \arrow[r] \arrow[d, hook] & p^{-1}(A) \arrow[r] & E \arrow[d, "p"] \\ E \times \Delta^1 \arrow[r, "p \times Id_{\Delta^1}"'] & X \times \Delta^1 \arrow[r, "h"'] & X \end{tikzcd}$$

where in the latter square the top-left map is the projction on $p^{-1}(A)$. The first diagram commutes because $h_0 = Id_X$, instead the second one uses the fact that h restricted to $A \times \Delta^1$ is equal to $i \circ pr_A$. Therefore we get another commutative diagram:

$$\begin{tikzcd} E \times \Delta^0 \cup p^{-1}(A) \times \Delta^1 \arrow[r] \arrow[d, "j"'] & E \arrow[d, "p"] \\ E \times \Delta^1 \arrow[r] \arrow[ur, dotted, "k"] & X \end{tikzcd}$$

By Theorem 1.4.5 we know that j is an anodyne extension (since ϵ_0 is such). Thus, since p is a Kan fibration, it exists a diagonal $k : E \times \Delta^1 \to E$. The commutativity of the top triangle tells us that $k_0 = Id_E$ and that k is stationary on $p^{-1}(A)$. On the other hand, the bottom triangle shows that the image of k_1 is in $p^{-1}(A)$, since if we compose after with p we would get

$$h \circ p \times Id_{\Delta^1} \circ Id_E \times \epsilon_1 = h \circ p \times \epsilon_1 = h_1 \circ p \times Id_{\Delta^0} = i \circ r \circ p \times Id_{\Delta^0}$$

where r is the retraction associated to i. Therefore k is a strong deformation retract of E on $p^{-1}(A)$.

$\square$

1.5 Minimal Complexes

Another important notion regarding Kan complexes is the one of *minimal* complexes. To introduce them we need another definition:

Definition 1.5.1. *Let X be a simplicial set and $x, y : \Delta^n \to X$ two n-simplices such that $x_{|\partial\Delta^n} = y_{|\partial\Delta^n} = a$. We say that x is* ***homotopic modulo*** *$\partial\Delta^n$ to y (written $x \sim y$ mod $\partial\Delta^n$) if there exists an homotopy $h : \Delta^n \times \Delta^1 \to X$ from x to y such that h is "stationary on $\partial\Delta^n$, i.e. the following diagram commutes*

$$\begin{tikzcd} \partial\Delta^n \times \Delta^1 \arrow[r, "pr_{\partial\Delta^n}"] \arrow[d, hook] & \partial\Delta^n \arrow[d, "a"] \\ \Delta^n \times \Delta^1 \arrow[r, "h"'] & X \end{tikzcd}$$

Thanks to the previous results we can prove that whenever X is a Kan complex to be "homotopic modulo $\partial\Delta^n$" is an equivalence relation.

Definition 1.5.2. *Let X be a Kan complex. X is said to be* ***minimal*** *if whenever $x \sim y$ mod $\partial\Delta^n$, then $x = y$.*

One of the reasons why minimal complexes are incredibly useful is outlined by the following theorem:

Theorem 1.5.3. *Let X and Y be minimal complexes and $f : X \to Y$ an homotopy equivalence between them, then f is an isomoprhism of simplicial sets.*

Proof This theorem is a straightforward consequence of Lemma 1.6.2 of [JT99], which states that any endomorphism of a minimal complex homotopic to the identity is an isomoprhism.

□

Furthermore, the usefulness of minimal complexes can be seen through the next theorem. In a way it states that they can "approximate" Kan complexes.

Theorem 1.5.4. *Let X be a Kan complex. Then there exist a strong deformation retract X' of X which is minimal.*

Proof (Idea) We can construct X' inductively defining all its skeletons. For $\mathrm{Sk}^0(X')$ we consider one representative of each equivalence class in $\Pi_0(X)$. Then let us suppose we have defined $\mathrm{Sk}^{n-1}(X')$. To construct

$\mathrm{Sk}^n(X')$ we choose a representative in each equivalence class among those n-simplices of X whose restriction to $\partial\Delta^n$ are contained in $\mathrm{Sk}^{n-1}(X')$, taking a degenerate one wherever possible. The Lemma 1.6.1 of [JT99] guarantees that X'_n contains all degenerates simplices from X'_{n-1}, since it states that if x and y are two n-simplices such that $x_{|\partial\Delta^n} = y_{|\partial\Delta^n}$, then $x = y$.
Using Theorem 1.1.8 we can find the strong deformation retraction required.[24]

□

Remark 1.5.5. We highlight the fact that in the construction of the strong deformation retract we have to make some choices of representatives. Therefore the Axiom of Choice (AC) is crucial for this Theorem.

Now we want to generalize the definition we have just given to moprhisms and not just complexes. In other words we want to give a definition of "minimal fibration" such that, for any simplicial set X, we would have

$$X \text{ is a minimal complex} \Leftrightarrow X \to \Delta^0 \text{ is a minimal fibration} \tag{2}$$

In model categories[25] terminology we would say that minimal complexes are the fibrant object with respect to minimal fibration, just as Kan complexes are the one with respect to Kan fibrations.

Definition 1.5.6. *Let $p : E \to X$ be a morphism of simplicial sets and $e, e' : \Delta^n \to E$ two n-simplices such that $e_{|\partial\Delta^n} = e'_{|\partial\Delta^n} = a$ and $pe = pe' = b$. We say that e is **fiberwise homotopic to** e' (written $e \sim_p e'$ mod $\partial\Delta^n$) if there exists an homotopy $h : \Delta^n \times \Delta^1 \to E$ from e to e' such that $h_{|\partial\Delta^n\times\Delta^1} = a \circ pr_{\partial\Delta^n}$ and h is "fiberwise", i.e. the following diagram commutes*

$$\begin{tikzcd} \Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr_{\Delta^n}"'] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "b"'] & X \end{tikzcd}$$

Similarly as the previous proofs, we can see that if p is a Kan fibration, then the fiberwise homotopic relation is actually an equivalence relation. So now we can give the definition we were aiming at:

Definition 1.5.7. *A Kan fibration $p : E \to X$ is said to be **minimal** if whenever $e \sim_p e'$ mod $\partial\Delta^n$, then $e = e'$.*

[24] The complete proof presented here can be found in [JT99] Theorem 1.6.1, another proof can be found in [GJ99] Chapter I, Proposition 10.3.

[25] Appendix B.

It is clear that with these definitions the condition (2) holds.

Remark 1.5.8. Minimal fibrations are stable under pullback.

Proof Let us consider a pullback diagram

$$\begin{tikzcd} E' \arrow[r, "g'"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

with p a minimal fibration. We want to prove that if $e'_1 \sim_{p'} e'_2 \bmod \partial\Delta^n$, then $e'_1 = e'_2$ (as elements in E'_n). By the description pf the pullback in *sSet* it suffices to prove that $g'_n(e'_1) = g'_n(e'_2)$ and $p'_n(e'_1) = p'_n(e'_2)$.
The second condition is true by assumption, for the first one we consider the following commutative diagram

$$\begin{tikzcd} \Delta^n \times \Delta^1 \arrow[r, "h"] \arrow[d, "pr_{\Delta^n}"'] & E' \arrow[r, "g'"] \arrow[d, "p'"] & E \arrow[d, "p"] \\ \Delta^n \arrow[r, "b"'] & X' \arrow[r, "g"'] & X \end{tikzcd}$$

where h is the homotopy from e'_1 to e'_2 and $p'e_1 = p'e'_2 = b$. Therefore we have that $g'_n(e'_1) \sim_p g'_n(e'_2)$, but since p is a minimal fibration, then $g'_n(e'_1) = g'_n(e'_2)$. $\square$

Replacing the relation "$x \sim y \bmod \partial\Delta^n$" with the more general one "$e \sim_f e' \bmod \partial\Delta^n$" we get the following theorems from the previous ones. We will write "fiberwise..." when the property holds for each fiber, as stated above.

Theorem 1.5.9. *Let the following diagram be commutative*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\ & X & \end{tikzcd}$$

with p and q minimal fibrations. If f is a fiberwise homotopy equivalence, then f is an isomorphism. [26]

[26] A complete proof can be found in [GJ99] Chapter I, Lemma 10.4.

Theorem 1.5.10. *Let $p : E \to X$ be a Kan fibration. Then there exists a subcomplex $E' \subseteq E$ such that $p_{|E'}$ is a minimal fibration which is a strong fiberwise deformation retract of p.*

Theorem 1.5.11. ***(Quillen's Lemma)*** *Any Kan fibration $p : X \to Y$ can be factorize as the composition of a trivial fibration r and a minimal morphism f*

$$\begin{tikzcd} X \arrow[rr, "p"] \arrow[dr, "r"'] & & Y \\ & X' \arrow[ur, "f"'] & \end{tikzcd}$$

Proof By the last Theorem, we know that there exists $i : X' \hookrightarrow X$ a strong deformation retract of X with X' a minimal complex and pi a minimal fibration. The Proposition 1.5.4 of [JT99] tells us that any strong deformation retract is in particular an anodyne extension. Since $pi =: f$ is a minimal fibration, then it is a Kan fibration. Therefore there exist a diagonal map r of the following diagram

$$\begin{tikzcd} X' \arrow[r, equal] \arrow[d, hook, "i"'] & X' \arrow[d, "f"] \\ X \arrow[r, "p"'] \arrow[ur, dotted, "r"] & Y \end{tikzcd}$$

The lower triangle gives us the factorization of p. The only thing that we have left to prove is that r is actually a trivial fibration. The Lemma 10.11 of [GJ99] (Chapter I) states that whenever we have a Kan fibration $p : X \to Y$ together with a commutative diagram

$$\begin{tikzcd} Z \arrow[r, hook, "j"] \arrow[dr, "q"'] & X \arrow[r, "g"] \arrow[d, "p"] & Z \arrow[dl, "q"] \\ & Y & \end{tikzcd}$$

where q is a minimal fibration, $gj = Id_Z$ and jg fiberwise homotopic to Id_X, then g is a trivial fibration. Therefore it suffices to prove that ir is fiberwise homotopic to Id_X.

Let us consider $k : \partial\Delta^1 \hookrightarrow \Delta^1$ the inclusion map. Since i is anodyne then $i * k$ is anodyne too by Theorem 1.4.5. Let us define $\varphi : \partial\Delta^1 \times X \cup \Delta^1 \times X' \to X$ such that $\varphi_{|\partial\Delta^1 \times X} = (ir, Id_X)$ and $\varphi_{|\Delta^1 \times X'} = i \circ pr_{X'}$. One can easily check that this morphism makes the following diagram commutative:

$$\begin{tikzcd} \partial\Delta^1 \times X \cup \Delta^1 \times X' \arrow[r, "\varphi"] \arrow[d, hook, "i*k"'] & X \arrow[d, "p"] \\ \Delta^1 \times X \arrow[r, "p\circ pr_X"'] & Y \end{tikzcd}$$

Since $i * k$ is anodyne and p a Kan fibration we find a diagonal map $h : \Delta^1 \times X \to X$. The upper triangle tells us that h is an homotopy from ir to Id_X, the lower one (together with the upper one) guarantees that h is even fiberwise.

□

With the notions of fibration and complexes outlined in this chapter we can construct a model structure [27] on the category of *sSet*. As cofibrations we consider all monomorphism, as fibrations the Kan fibrations and as weak equivalences weak homotopy equivalences. The last ones are defined as the morphism whose geometric realization is a weak homotopy equivalence (i.e. it induces an isomorphism for any homotopy group)[28]. In particular acyclic cofibrations/fibrations would be the anodyne extensions/trivial fibrations. Moreover we can see that in *sSet* the following theorem holds.

Theorem 1.5.12. *Let the diagram below be a pullback*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & X \arrow[d, "p"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

where p is a Kan fibration and f a weak equivalence in sSet. Then f' is a weak equivalence as well.

Whenever a property of this kind is fulfilled we say that the class of weak equivalence is closed under pullback over fibrations. A model category with such characteristic is called **right proper**. On the other hand, if the class of weak equivalences is closed under pushout over cofibrations, we say that the model category is **left proper**. Whenever the two properties hold at the same time we have a **proper** model category. It can be proven that *sSet* is indeed a proper model category. [29]

[27] Appendix B
[28] [GJ99] Chapter 2, §11.
[29] [GJ99] Chapter 2, Corollary 9.6.

2 Type Theory

Every logic is a logic
over a type theory.

[Jac99] Chapter 0, §0.1

The aim of this chapter is to give an overview of Type Theory, in order to give an idea of what this topic is and to be able to understand the roots of the Univalence Axiom. For this reason we will not write down every single rule that can be found in the various theories. A more complete description can be found in the references given throughout the whole chapter.

2.1 λ-Calculus

The notions outlined in the following section can be found in [DV17] and [Sel08] §1-2.

We start explaining the basic notions of the untyped λ-calculus, principally due to Alonzo Church. This calculus is a **formal language** for logic. Therefore we start describing the syntax.

Definition 2.1.1. *Given an infinite set $\mathcal{V}$ of* ***variables*** *(denoted x, y, z, ...), we define the* ***set of λ-terms*** *in the following way:*

$$M, N := x \mid (MN) \mid (\lambda x.M)$$

We presented this definition in the so-called "Backus-Naur" form. The idea is that we define terms through the following rules:

- Any element $x \in \mathcal{V}$ is a term.
- Whenever we have two terms M and N then the *application of the function M to the argument N*, that we write (MN), is again a term.
- For any variable $x \in \mathcal{V}$ and any term M, then the λ-*abstraction* $\lambda x.M$ is a term as well.

The latter deserve a more accurate description. We can see the λ-abstraction as the operation of binding a variable x in an expression M. It means that x is bonded in M and not "free". This construction is useful to explicitly state that the letter x that we read in the expression $(\lambda x.M)$ is "local" just to this term and not others that we may find in a proof or definition. For instance the variable x is bonded in the terms $(\lambda x.x+1)$, $(\lambda x.x-y)$ but free in $(\lambda y.x-y)$. More precisely:

Definition 2.1.2. *The* ***set of free variables of a term*** M*, denoted* $FV(M)$ *is defined, with a recursion, as:*

$$\begin{array}{rcl} FV(x) & := & \{x\} \\ FV(MN) & := & FV(M) \cup FV(N) \\ FV(\lambda x.M) & := & FV(M) \setminus \{x\} \end{array}$$

Notation:

- We usually omit parentheses. For instance we would write MN or $\lambda x.M$ instead of (MN) or $(\lambda x.M)$.
- Applications associate on the left, i.e. whenever we write MNP it would mean $(MN)P$ and so on.
- The body of a λ-abstraction (the part after the dot) extends as far as it can on the right, i.e. we read $\lambda x.MN$ as $\lambda x.(MN)$ and not $(\lambda x.M)N$.
- Sometimes we could write $\lambda xyz.M$ instead of $\lambda x.\lambda y.\lambda z.M$.

Some relevant examples of λ-terms are: the identity function $\lambda x.x$; the function that given two arguments gives out the first one $\lambda x.\lambda y.x$; the function that given two arguments gives out the second one $\lambda x.\lambda y.y$.

Now we will see how to conduct the application of functions in this calculus. For instance let us consider the term $\lambda x.x+1$. We can apply it to the term 2. The operation would go in the following way:

$$(\lambda x.x+1)2 \longrightarrow 2+1 \longrightarrow 3$$

Moreover, a function can be the argument of another function. For instance we can carry out the following operation:

$$(\lambda y.y2)(\lambda x.x+1) \longrightarrow (\lambda x.x+1)2 \longrightarrow 2+1 \longrightarrow 3$$

Definition 2.1.3. *Let* x *and* y *be two variables, and* M *a term. We write* $M\{y/x\}$ *for the result of* ***renaming*** x ***as*** y*, operation that we define recursively as:*

$$\begin{array}{rcl} x\{y/x\} & := & y \\ z\{y/x\} & := & z \quad \textit{if } z \neq x \\ (MN)\{y/x\} & := & (M\{y/x\})(N\{y/x\}) \\ (\lambda x.M)\{y/x\} & := & \lambda y.(M\{y/x\}) \\ (\lambda z.M)\{y/x\} & := & \lambda z.(M\{y/x\}) \quad \textit{if } z \neq x \end{array}$$

Naturally, whenever we have two terms that are the same up to renaming of bound variables, we want them to be equivalent. For this reason we introduce the concept of α-equivalence. Firstly we give a proper definition of "equivalence relation" and "congruence" on λ-terms.

Definition 2.1.4.
- *A relation $=$ on λ term is called an* ***equivalence relation*** *if it satisfies the following rules:*

$$\textit{(refl)}\ \frac{}{M = M} \qquad \textit{(symm)}\ \frac{M = N}{N = M}$$

$$\textit{(trans)}\ \frac{M = N \qquad N = P}{M = P}$$

- *Moreover $=$ is a* ***congruence*** *if it is an equivalence relation that satisfies even:*

$$\textit{(cong)}\ \frac{M = M' \qquad N = N'}{MN = M'N'} \qquad (\xi)\ \frac{M = M'}{\lambda x.M = \lambda x.M'}$$

Definition 2.1.5. *We define the* α***-equivalence*** *to be the smallest congruence relation $=_\alpha$ on λ-terms, such that for all terms M and all variables y that do not occur in M*

$$\lambda x.M =_\alpha \lambda y.(M\{y/x\})$$

Conventions:

(i) Since we suppose $\mathcal{V}$ to be infinite, we can prove that any term is α-equivalent to another term where the names of all bound variables are distinct from each other and from any free variables. Thus, we can (and will) always consider the latter one. This convetion is called *Barendregt's variable convention.*

(ii) Since the idea of α-equivalence is just a matter of formal writing, from now on we will identify λ-terms with their equivalence class modulo α-equivalence. Therefore we may write $M = N$ instead of $M =_\alpha N$.

The last operation between terms that we need to introduce is the *substitution.* Basically we want to describe how we can replace a variable by a term.

Definition 2.1.6. *Let M and N be two terms, and x variable free in M. We define the* ***substitution of** N **for free occurrences of** x **in** M, denoted $M[N/x]$, recursively as:*

$$
\begin{array}{rcl}
x[N/x] & := & N \\
z[N/x] & := & z \quad \text{if } z \neq x \\
(MP)[N/x] & := & (M[N/x])(N[N/x]) \\
(\lambda x.M)[N/x] & := & \lambda x.M \\
(\lambda y.M)[N/x] & := & \lambda y.(M[N/x]) \quad \text{if } y \neq x \text{ and } y \notin FV(N) \\
(\lambda y.M)[N/x] & := & \lambda y'.(M\{y'/y\}[N/x]) \text{ if } y \neq x,\ y \in FV(N) \text{ and } y' \text{ fresh}
\end{array}
$$

To make this a proper definition we should specify the fresh variable we consider. This problem can be solved considering a well-ordering on the set $\mathcal{V}$ (thanks to the AC) and then explicitly define y' as the least variable that does not occur in either M or N.

Remark 2.1.7. First of all let us note that we only substitute *free* variables. That is because bound variables are "internal" and they should not affect the result of such operation. Thus, $x(\lambda xy.x)[N/x]$ is $N(\lambda xy.x)$, and not $N(\lambda xy.N)$ or $N(\lambda y.N)$.
The last two situations need to be distinguished because, for instance, we could have $M \equiv \lambda x.yx$ and $N \equiv \lambda z.xz$. Doing the substitution in a "naive" way we could get

$$M[N/y] = (\lambda x.yx)[N/y] = \lambda x.Nx = \lambda x./\lambda z.xz)x$$

But in the result x would be a bound varible, while in N was free. The issue here is that we use the same name for both the bound variable in M and the free one in N even though they are not actually the same. To avoid this confusion we use the trick to rename variables, so that the distinction is evident.

Let us consider a λ-abstraction applied to another term, i.e. a term of the form $(\lambda x.M)N$. We want to be able to *reduce* it to $M[N/x]$. In this case we would call the first term **β-redex** and the second one the **reduct**. More in general we call **β-reduction** the process of evaluating λ-terms by "plugging arguments into functions". The idea is to find inside a term a β-redex and replace it with its reduct. For instance we can reduce the term $(\lambda x.y)((\lambda z.zz)(\lambda w.w))$ in the following way:

$$
\begin{array}{rcl}
(\lambda x.y)\underline{((\lambda z.zz)(\lambda w.w))} & \to_\beta & (\lambda x.y)\underline{((\lambda w.w)(\lambda w.w))} \\
 & \to_\beta & \underline{(\lambda x.y)(\lambda w.w)} \\
 & \to_\beta & y
\end{array}
$$

Let us now give the formal definition of β-reduction and equivalence:

Definition 2.1.8. • *We define a **single-step** β**-reduction** to be the smallest relation* $\to_\beta$ *on* λ*-terms satisfying the following rule:*

$$(\beta)\ \frac{}{(\lambda x.M)N \to_\beta M[N/x]} \qquad (cong_1)\ \frac{M \to_\beta M'}{MN \to_\beta M'N}$$

$$(cong_1)\ \frac{N \to_\beta N'}{MN \to_\beta MN'} \qquad (\xi)\ \frac{M \to_\beta M'}{\lambda x.M \to_\beta \lambda x.M'}$$

We write $M \to_\beta M'$ *if and only if* M *is obtained from* M *by reducing a single* β*-redex of* M*.*

- *We define* $\twoheadrightarrow_\beta$ *to be the reflexive and transitive closure of* $\to_\beta$*, i.e. the smallest reflexive and transitive relation containing* $\to_\beta$*. Therefore* $M \twoheadrightarrow_\beta M'$ *if* M *reduces to* M' *in zero or more steps.*

- *We define the* β***-equivalence*** *as the reflexive, symmetric and transitive closure of* $\to_\beta$*.*

Definition 2.1.9. • *A* λ*-term without any* β*-redexes (e.g.* y *or* $\lambda x.x$*) is said to be in* β***-normal form****.*

- *If* M *and* M' *are terms such that* $M \twoheadrightarrow_\beta M'$*, and if* M' *is in normal form, then we say that* M ***evaluates*** *to* M'*.*

Remark 2.1.10. We saw how we can reduce the term $(\lambda x.y)((\lambda z.zz)(\lambda w.w))$ to its β-normal form y. But not all terms can be reduced to a β-normal form. For instance the term $(\lambda x.xx)(\lambda y.yy)$ always reduce to itself, and obviously it is not in a β-normal form.

We recall that in the untyped λ-calculus any term M can be regarded as a function. In fact we can always consider the term MN, for any other term N. Sometimes we could require that two terms should be equal if they define the same function. This is called the principle of **extensionality**. Formally it can be expressed as:

$$(\text{ext}_\forall)\ \frac{\forall A.MA = M'A}{M = M'}$$

Moreover, if we consider a λ calculus with the axioms (ξ), (cong) and (β), then it can be seen that $MA = M'A$ is true for all terms A if and only if $Mx = M'x$ is true for a fresh variable x[30]. Therefore we can reformulate extensionality as:

[30] The idea is that x is general and it could represent any term. In mathematics this argument is usually used during proofs, for instance whenever we consider a generic object.

$$\text{(ext)} \quad \frac{Mx = M'x, \text{ where } x \notin FV(M, M')}{M = M'}$$

On the other hand we could just ask that the so-called η-law holds, described by the following:

$$(\eta) \quad \frac{}{M = \lambda x.Mx, \text{ where } x \notin FV(M)}$$

Proposition 2.1.11. *In a λ-calculus with the presence of the axioms (ξ), (cong) and (β) extensionality and the η-law are equivalent.*

Proof

($\Rightarrow$) Let us consider a term M. We denote $M' := \lambda y.My$. By the β-reduction we have that, for any term A, $MA =_\beta M'A$. Therefore, by (ext), M and M' must be equal.

($\Leftarrow$) Let us consider two terms M and M' such that, for any $x \notin FV(M, M')$, $Mx = M'x$. Then by (ξ) we have that $\lambda x.Mx = \lambda x.M'x$. Furthermore, by ($\eta$), $M = \lambda x.Mx$ and $M' = \lambda x.M'x$, and thus the thesis.

□

Remark 2.1.12. It is important to remember that the η-law does not follow from the axioms stated before.

In a similar way as for the β case, we can define the notion of **single-step η-reduction** ($\to_\eta$). Moreover we can define the **single-step $\beta\eta$-reduction** ($\to_{\beta\eta}$) as the union of the two single-step reductions, i.e.

$$M \to_{\beta\eta} M' \Leftrightarrow M \to_\beta M' \text{ or } M \to_\eta M'$$

Then the definitions of $\twoheadrightarrow_\eta$, $\twoheadrightarrow_{\beta\eta}$, $=_\eta$ and $=_{\beta\eta}$ could be given in a natural way. We also have the evident notions of η-normal form and $\beta\eta$-normal form, etc.

2.2 Simple Type Theory

The notions outlined in the following section can be found in [Jac99] Chapter 0, Chapter 2, §2.1-2.3.

In the untyped λ-calculus we do not specify the "type" of any expression. On the ohter hand in type theory, whenever we consider an expression, we ask ourselves where this expression "lives". For example, let us consider the number 8. It can inhabit differents types, with this we mean that for instance we could consider it as an element of $\mathbb{N}$ or as an element of $\mathbb{R}$. In type theory we would write

$$8 : \mathbb{N} \quad \text{or} \quad 8 : \mathbb{R}$$

Obviously we will need to specify rules regulating inhabitation, such as

$$\frac{n : \mathbb{N} \qquad succ : \mathbb{N} \to \mathbb{N}}{succ(n) : \mathbb{N}}$$

where $\mathbb{N} \to \mathbb{N}$ indicates the type of functions from variables of type $\mathbb{N}$ to variables of the same type. We could even have a type that describes propositions in our logic, written $\textsc{Prop}$. Thus we would have

$$(\forall x : \mathbb{N}.\exists y : \mathbb{N} > succ(n)) : \textsc{Prop}$$

Nonetheless some proposition would need "contexts". For instance $x > 5 : \textsc{Prop}$ is true whenever x has the right type, such as $x : \mathbb{N}$. In this case we write

$$x : \mathbb{N} \vdash x > 5 : \textsc{Prop}$$

that should be read "in the context $x : \mathbb{N}$ we have $x > 5 : \textsc{Prop}$".

In Simpe Type Theory (STT) we start from **atomic types** (like $\mathbb{N}$, $\mathbb{R}$,...) and then we build new types starting from them and using **type constructors** such as: the exponential ($\to$), finite cartesian products (1, $\times$) and finite cartesian coproducts (0, $+$). These are regulated by some introduction and elimination rules, e.g. the projections associated to the product or the inclusions for the coproduct. We sometimes suppose the existance of a type of types, called $\textsc{Type}$ (later on will be explained in a more formal way). Generally to state that something is a type we use the judgement form [31]

$$\Gamma \vdash A \quad Type$$

that should be read as "A is a type in context Γ ". The proper definition of context will be given later on in this section. Nonetheless, for the moment we will use the notation $\sigma : \textsc{Type}$ just for convenience. In the last chapter we will give an idea of the way to construct something as "a type of type". The concept is to define a notion similar to the one of Grothendieck universe in Set Theory.

The main difference between STT and the Dependent Type Theory (DTT) is that in the first one variable terms, such as $x : \sigma$ for $\sigma : \textsc{Type}$, are not allowed to occur in another type. On the other hand in DTT we could have

[31] With "judgement" we refer to a statement in the metalanguage, e.g. the sentence "α true" for some proposition α.

something of the form $\tau(x)$: TYPE.

Under the so-called idea of **"type-as-proposition"** (that we will explain better later in this section and the following) the study of the types constructors correspond to the study of the proof theoy behind the propositional constructors as in the following table: [32]

Types	Propositions
$\rightarrow$	$\supset$
1	$\top$
$\times$	$\wedge$
0	$\perp$
$+$	$\vee$

To explain (categorically) this calculus we almost need all of them. For instance, to understand the exponential ($\rightarrow$) in a category we need binary products, since the "nature" of $\rightarrow$ can be seen through the adjunction with the cartesian product.

Moreover, we note that the untyped λ-calculus can be seen as a STT calculus with just one type Ω such that $\Omega \rightarrow \Omega = \Omega$, even though historically is preceding. In fact it would be the same as having no different types.

Before proceeding with more accurate definitions and interpretations, we need to give some important definitions. Typically a *signature* consists of a set of "basic types" (e.g. $\{\mathbb{N}, B...\}$) together with a set of typed function symbols, e.g.

$$+ : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N} \qquad succ : \mathbb{N} \longrightarrow \mathbb{N}$$

$$\wedge : B,\, B \longrightarrow B$$

A signature would be called *single-typed* if it has just one basic type, and *may-typed* otherwise. More precisely:

Definition 2.2.1. *A* ***many-typed signature*** Σ *is a pair* $(T, \mathcal{F})$ *where* T *is a set (called the* ***set of basic types****) and* $\mathcal{F} : T^* \times T \rightarrow Set$ [33] *a map which assigns*

$$T^* \times T \ni (<\sigma_1,\, ...,\, \sigma_n>,\, \sigma_{n+1}) \longmapsto \textit{functions symbols from} \; <\sigma_1,\, ...,\, \sigma_n> \textit{ to } \sigma_{n+1}$$

[32]With the symbols $\supset$, $\top$, $\wedge$, $\perp$ and $\vee$ we mean respectively the logic implication, true, conjunction, false and disjunction.

[33]With T^* we denote the set of finite sequences of elements of T.

Remark 2.2.2. The functions symbols $\mathcal{F}(\alpha)$ for different $\alpha \in T^* \times T$ may not be disjoint. For instance if $\mathbb{N}$, $\mathbb{R} \in T$ one could have

$$+ \in \mathcal{F}(<\mathbb{N}, \mathbb{N}>, \mathbb{N}) \text{ as } + : \mathbb{N}, \mathbb{N} \to \mathbb{N} \text{ the sum in } \mathbb{N}$$

and

$$+ \in \mathcal{F}(<\mathbb{R}, \mathbb{R}>, \mathbb{R}) \text{ as } + : \mathbb{R}, \mathbb{R} \to \mathbb{R} \text{ the extended sum in } \mathbb{R}$$

Notation: We denote $\mid \Sigma \mid \equiv T$ and $F : \sigma_1, ..., \sigma_n \to \sigma_{n+1}$ for a $F \in \mathcal{F}((<\sigma_1, ..., \sigma_n>, \sigma_{n+1})$.

Let us consider a infinite set *Var* $= \{v_1, v_2, ...\}$, whose elements will be called *variables*, and a many-typed signature $\Sigma = (T, \mathcal{F})$. The choice of *Var* is helpful for having "names" that we can use for the "elements" of the various types, i.e. variables.

Definition 2.2.3. *A* ***context*** *is a finite sequence of* variable declaration $(v_1 : \sigma_1, ..., v_n : \sigma_n)$ *where, for any* $i = 1, ..., n$, σ_i : TYPE.

Definition 2.2.4. *We write* $\Gamma \vdash M : \tau$ *to say that* M *is a* ***term*** *of type* τ *in context* Γ. *We can, equivalently, say that* M ***inhabits*** τ*, or just that* τ *is inhabitated (by* M*, in context* Γ*).*

For instance, in example 2.2 stated above, we would say that $succ(n)$ inhabits $\mathbb{N}$ with context $(n : \mathbb{N}, succ : \mathbb{N} \to \mathbb{N})$.

In STT, as in any theory, we need some rules to treat the sequent calculus:

Basic Rules

- **Identity** $\dfrac{}{v_1 : \sigma \vdash v_1 : \sigma}$
- **Function symbol** $\dfrac{\Gamma \vdash M_1 : \sigma_1 \ ... \ \Gamma \vdash M_n : \sigma_n}{\Gamma \vdash F(M_1, ..., M_n) : \sigma_{n+1}}$ (for any function symbol $F : \sigma_1, ..., \sigma_n \to \sigma_{n+1}$)

Structural Rules

- **Weakening** $\dfrac{v_1 : \sigma_1, ..., v_n : \sigma_n \vdash M : \tau}{v_1 : \sigma_1, ..., v_n : \sigma_n, v_{n+1} : \sigma_{n+1} \vdash M : \tau}$
- **Contraction** $\dfrac{\Gamma, v_n : \sigma, v_{n+1} : \sigma \vdash M : \tau}{\Gamma, v_n : \sigma \vdash M[v_n/v_{n+1}] : \tau}$

- **Exchange** $\dfrac{\Gamma,\, v_i : \sigma_i,\, v_{n+1} : \sigma_{i+1},\, \Delta \vdash M : \tau}{\Gamma,\, v_i : \sigma_i,\, v_{n+1} : \sigma_{i+1},\, \Delta \vdash M[v_i/v_{i+i},\, v_{i+1}/v_i] : \tau}$

Definition 2.2.5. *A sequent $\Gamma \vdash M : \sigma$ is **derivable** if there is a derivation tree consisting of the above rules. In this case we would write $\blacktriangleright\ \Gamma \vdash M : \sigma$.*

As a consequence of these rules we can obtain the **substitution** rule:

$$\frac{\Gamma,\, v_n : \sigma \vdash M : \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash M[N/v_n] : \tau}$$

In addition to this basic and structural rules we have some type formation rules. Given a many-typed signature $\Sigma = (T,\, \mathcal{F})$, we have always the following rule:

$$\frac{}{\vdash \sigma \quad \mathit{Type}}$$

for any $\sigma \in T$ (there is the empty context because in STT types are not allowed to be dependent on variables). Moreover we could add some types constructors, such as the aforementioned exponential ($\to$), finite products (1, $\times$) and finite coproducts (0, $+$).

For the first one, we need to introduce the following type-formation rule:

$$\frac{\vdash \sigma \quad \mathit{Type} \qquad \vdash \tau \quad \mathit{Type}}{\vdash \sigma \to \tau \quad \mathit{Type}}$$

Furthermore, we need to add the following rules:

- **$\to$ -Introduction** $\dfrac{\Gamma,\, v : \sigma \vdash M : \tau}{\Gamma \vdash \lambda v : \sigma.M : \sigma \to \tau}$
- **$\to$ -Elimination** $\dfrac{\Gamma \vdash M : \sigma \to \tau \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau}$

Nonetheless, we should state the rules related to β-, η- conversions to complete this theory to a λ-calculus (usually called $\lambda 1$-calculus). Anytime we would like to introduce a new types constructor, we need to add these kind of rules. A meticulous reader can find a complete description of these for example in [Jac99] Chapter 2, §2.3 (for all the types constructors described in this section).

Now let us consider the idea "proposition-as-types" in the theory constructed so far. We take a non-empty set T, whose elements will be considered as *propositional constants*. We define T_1 as the closure of T under $\to$.

Its elements can be considered as propositions of *minimal intuitionistic logic* (since they are built up from constants using just the "implication" $\to$). Let $\mathcal{A}$ be a collection of sequents $\sigma_1, ..., \sigma_n \vdash \tau$ (with $\sigma_i, \tau \in T$), which we regard as axioms. We can construct a signature $\Sigma_{\mathcal{A}}$ from $\mathcal{A}$ in the following way:

- We define $\mid \Sigma_{\mathcal{A}} \mid := T$;
- For any sequent $\sigma_1, ..., \sigma_n \vdash \tau$ in $\mathcal{A}$ we choose a function symbol $F : \sigma_1, ..., \sigma_n \to \tau$ (we can considered it as an *atomic proof* of the axiom).

It can be proven that:

$$\sigma_1, ..., \sigma_n \vdash \tau \text{ is derivable from } \mathcal{A} \Leftrightarrow \text{there is a term } M \text{ with} \\ v_1 : \sigma_1, ..., v_n : \sigma_n \vdash M : \tau \text{ in } \lambda 1(\Sigma_{\mathcal{A}})$$ [34]

The idea behind this is that the concept of "provability" in logic corresponds to the one of "inhabitation" in type theory. So that in the case of the exponential we would have that any term $M : \sigma \to \tau$ is a proof of $\sigma \supset \tau$. In other words, we can see how M tranforms each "proof" $N : \sigma$ in a "proof" $MN : \tau$, as described in the $\to$ -Elimination rule. This concept is usually know as *proposition-as-types* and *proof-as-term*, exactly for this correspondence between the proofs of a proposition and the terms of a type. The interpretation stated will extend to finite conjuctions ($\top$, $\wedge$) and finite disjunctions ($\bot$, $\vee$), thanks to finite products (1, $\times$) and finite coproducts (0, $+$) respectively.

We first expand $\lambda 1(\Sigma)$ to $\lambda 1_{\times}(\Sigma)$ adding finite products, i.e. adding the following formation rules

$$\frac{}{\vdash 1 \quad Type} \qquad\qquad \frac{\vdash \sigma \quad Type \qquad \vdash \tau \quad Type}{\vdash \sigma \times \tau \quad Type}$$

and corresponding introduction and elimination rules for tupleing (a) and projecting (b)

(a) $$\frac{}{\vdash <>: 1} \qquad\qquad \frac{\Gamma \vdash M : \sigma \qquad \Gamma \vdash N : \tau}{\Gamma \vdash < M, N >: \sigma \times \tau}$$

(b) $$\frac{\Gamma \vdash P : \sigma \times \tau}{\Gamma \vdash \pi P : \sigma} \qquad\qquad \frac{\Gamma \vdash P : \sigma \times \tau}{\Gamma \vdash \pi' P : \tau}$$

Now we add even finite coproducts, extending $\lambda 1_{\times}(\Sigma)$ to $\lambda 1_{(\times,+)}(\Sigma)$. The formation rules would be

$$\frac{}{\vdash 0 \quad Type} \qquad\qquad \frac{\vdash \sigma \quad Type \qquad \vdash \tau \quad Type}{\vdash \sigma + \tau \quad Type}$$

[34] With this we mean "in the $\lambda 1$-calculus associated to the many-typed signature $\Sigma_{\mathcal{A}}$".

while the introduction (a) and elimination (b) rules can be written as follows

(a) $$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash kM : \sigma + \tau} \qquad\qquad \frac{\Gamma \vdash N : \tau}{\Gamma \vdash k'N : \sigma + \tau}$$

(b) $$\frac{\Gamma \vdash P : \sigma + \tau \qquad \Gamma,\, x : \sigma \vdash Q : \rho \qquad \Gamma,\, y : \tau \vdash R : \rho}{\Gamma \vdash \text{unpack } P \text{ as} \, [kx \text{ in } Q,\, k'y \text{ in } R] : \rho}$$

$$\frac{}{\Gamma,\, z : 0 \vdash \{\,\} : \rho}$$

Whilst the first rules stated are clearly related to the coprojection of the coproduct, the second ones deserve a more accurate explanation. With "unpack P as[...]" we mean the following interpretation

$$\begin{cases} \text{If } P \text{ is in } \sigma \text{ then "do" } Q \text{ with } P \text{ for } x \\ \text{If } P \text{ is in } \tau \text{ then "do" } R \text{ with } P \text{ for } y \end{cases}$$

To understand properly the other elimination rule it is helpful to state the related conversion:

$$\frac{\Gamma,\, z : 0 \vdash M : \rho}{\Gamma,\, z : 0 \vdash M = \{\,\} : \rho}$$

This rule tells us that if there exists a context where the empty type 0 is inhabitated, then each term $M : \rho$ must be convertible to the empty cotuple $\{\,\}$, which (in this case) exists by the last elimination rule stated above. Thanks to this description we can see the connection between the empty type and the logical false $\perp$.

2.3 Dependent Type Theory

The notions outlined in the following section can be found in [Jac99] Chapter 0 and Chapter 10, §10.1-10.2 and [Hof97].

In STT we can build types σ from atomic types (constants) using type constructors ($\rightarrow$, $\times$, $+$, ...). In Dependent Type Theory (DTT), instead, a term variable $x : \sigma$ may occur in another type $\tau(x)$. For example, setting $Nat(n) := \{1, ..., n\}$ and $NatList(n)$ as the set of lists of natural numbers of length n, we can have

$$n : \mathbb{N} \vdash Nat(n) \quad Type \qquad \text{or} \qquad n : \mathbb{N} \vdash NatList(n) \quad Type$$

The corresponding idea in Set Theory is the one of "sets depending on sets", i.e. for I a set we can obtain a set $X = (X_i)_{i\in I}$ as a family of sets over I. In mathematics we can find plenty of examples of dependent types. For instance the n-th power of an object is such (where $n : \mathbb{N}$ is a parameter). Another explanatory example is the matrices $Mat(n, m)$ of dimension $n \times m$, which depend on two different parameters that could be in $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, ... or so on.

In particular we will introduce three important type forming operations:

1. $\Pi_{x:\sigma}\tau(x)$ the **dependent product** of $\tau(x)$ where x ranges over σ;
2. $\Sigma_{x:\sigma}\tau(x)$ the **dependent sum** of $\tau(x)$ where x ranges over σ;
3. $\mathsf{Id}_\sigma(x, x')$ the **identity type** in σ for x and x' ranging over σ.

The idea related to the latter one is that this type would be inhabitated if and only if x and x' are equal in σ.

Remark 2.3.1. In order to understand how to construct and understand them better, we look at the corresponding concepts in Set Theory: [35]

1. $\Pi_{i\in I}X_i := \{f : I \to \bigcup_{i\in I} X_i \mid \forall\, i \in I\ f(i) \in X_i\}$;
2. $\Sigma_{i\in I}X_i := \{< i,\, z >\mid\ i \in I \text{ and } z \in X_i\}$;
3. $\mathsf{Id}_I(x, x') := \begin{cases} \{*\} & \text{if } x = x' \\ \emptyset & \text{otherwise} \end{cases}$

In a more type-theoretic vision, we could say that the dependent product is the collection of functions f such that for each $a : \sigma$, then $fa : \tau[a/x]$[36]. On the other hand, with the same view, the dependent sum would be the collection of pairs $< a, b >$ such that $a : \sigma$ and $b : \tau[a/x]$.

In DTT, as well in other type theories, the definition that we used of context is not ideal. Usually we prefer another kind of context:

Definition 2.3.2.
- *We define a* ***well-formed context*** $\Gamma \equiv x_1 : \sigma_1, ..., x_n : \sigma_n$ *to be a sequence of variable declarations such that each type* σ_{i+1} *is well formed in the preceeding context, i.e. for any* $i = 1, ..., n$

$$x_1 : \sigma_1,\, ...,\, x_i : \sigma_i \vdash \sigma_{i+1} \quad \textit{Type}$$

[35] For all these definitions we consider a set I, for any $i \in I$ a set X_i and two elements $x, x' \in I$.

[36] We define fa as in the untyped λ-calculus, i.e. "the application of f to a".

- *We define a* ***substitution*** [37] *between well-formed context as a map* $f : [x_1 : \sigma_1, ..., x_n : \sigma_n] \to [y_1 : \tau_1, ..., y_m : B_m(y_1, ..., y_{m-1})]$ *represented by terms* $f_1, ..., f_m$ *such that*

$$x_1 : \sigma_1, ..., x_n : \sigma_n \vdash f_1 : \tau_1$$

$$\vdots$$

$$x_1 : \sigma_1, ..., x_n : \sigma_n \vdash f_m : \tau_m(f_1, ..., f_{m-1})$$

  Two such maps $[f_i]$, $[g_i]$ *are equal exactly if for each i:*

$$x_1 : \sigma_1, ..., x_n : \sigma_n \vdash f_i = g_i : \tau_i(f_1, ..., f_{i-1})$$ [38]

We can associate to this definition a judgement form

$$\vdash \Gamma \quad cxt$$

that clearly is derived from the judgement related to types. We notice that, given a well-formed context $\Gamma = x_1 : \sigma_1, ..., x_n : \sigma_n$, each free variable $y : \sigma_{i+1}$ must be one of the $x_1, ..., x_i$. In particular σ_1 has to be a closed type, i.e. it does not contain any term variable.

Example 2.3.3. *The context* $n : \mathbb{N}$, $l : NatList(n)$ *is well-formed. On the other hand* $n : \mathbb{N}$, $Mat(n, m)$ *is not well-formed, since* m *is not declared.*

In DTT *sequents* have one of the following forms, for some well-formed context Γ:

$$(1)\ \Gamma \vdash \sigma \quad Type \qquad\qquad (2)\ \Gamma \vdash M : \sigma$$
$$(3)\ \Gamma \vdash M = N : \sigma \qquad\qquad (4)\ \Gamma \vdash \sigma = \tau \quad Type$$

We already encountered the first two. The last ones describe the equality as terms in a type and as types, respectively. Moreover we have five basic rules for DTT:

- **Projection** $$\frac{\Gamma \vdash \sigma \quad Type}{\Gamma, x : \sigma \vdash x : \sigma}$$
- **Substitution** $$\frac{\Gamma \vdash M : \sigma \qquad \Gamma, x : \sigma, \Delta \vdash \mathcal{J}}{\Gamma, \Delta[M/x] \vdash \mathcal{J}[M/x]}$$ [39]

[37] [KL16] §1.2, Example 1.2.3.
[38] The meaning of a sequent of this kind is explained below.
[39] Since the contexts that we consider are well-formed, then Δ and $\mathcal{J}$ may contain x.

- **Contraction** $$\frac{\Gamma,\, x:\sigma,\, y:\sigma,\, \Delta \vdash \mathcal{J}}{\Gamma,\, x:\sigma,\, \Delta[x/y] \vdash \mathcal{J}[x/y]}$$

- **Weakening** $$\frac{\Gamma \vdash \sigma\ Type \qquad \Gamma \vdash \mathcal{J}}{\Gamma,\, x:\sigma \vdash \mathcal{J}}$$

- **Exchange** $$\frac{\Gamma,\, x:\sigma,\, y:\tau,\, \Delta \vdash \mathcal{J}}{\Gamma,\, y:\tau,\, x:\sigma,\, \Delta \vdash \mathcal{J}}$$ [40]

For convenience we usually assume the existence of a unit (or singleton) type 1, that has to obey the following rules:

$$\frac{}{\vdash 1 \quad Type} \qquad \frac{}{\vdash <>: 1} \qquad \frac{\Gamma \vdash M : 1}{\Gamma \vdash M =<>: 1}$$

Let us finally describe the type formers introduced before. The numbers associated to the coming rules will be the same used before on the side of the various constructors.

Formation rules:

1. $$\frac{\Gamma,\, x:\sigma \vdash \tau(x) \quad Type}{\Gamma \vdash \Pi_{x:\sigma}\tau(x) \quad Type}$$

2. $$\frac{\Gamma,\, x:\sigma \vdash \tau(x) \quad Type}{\Gamma \vdash \Sigma_{x:\sigma}\tau(x) \quad Type}$$

3. $$\frac{\Gamma \vdash \sigma \quad Type}{\Gamma,\, x:\sigma,\, x':\sigma \vdash \mathsf{Id}_\sigma(x,\, x') \quad Type}$$

Introduction and Elimination rules: [41]

1. $$\frac{\Gamma,\, x:\sigma \vdash M : \tau(x)}{\Gamma \vdash \lambda x:\sigma.M : \Pi_{x:\sigma}\tau(x)} \qquad \frac{\Gamma \vdash M : \Pi_{x:\sigma}\tau(x) \qquad \Gamma \vdash N : \sigma}{\Gamma \vdash MN : \tau[N/x]}$$

2. $$\frac{\Gamma \vdash \sigma \quad Type \qquad \Gamma,\, x:\sigma \vdash \tau(x) \quad Type}{\Gamma,\, x:\sigma,\, y:\tau(x) \vdash <x,\, y>: \Sigma_{x:\sigma}\tau(x)}$$
$$\frac{\Gamma,\, z:\Sigma_{x:\sigma}\tau(x) \vdash \rho(z)\ Type \qquad \Gamma,\, x:\sigma,\, y:\tau(x) \vdash Q : \rho[<x,\, y>/z]}{\Gamma,\, z:\Sigma_{x:\sigma}\tau(x) \vdash (\text{unpack } z \text{ as } <x,\, y> \text{ in } Q) : \rho(z)}$$

[40] For "well-form"ness we must suppose that x is not free in τ.

[41] The idea for this one is given by the "type" intepretation given before, using an approach similar to the one used for finite products and coproduct in the section about STT.

3. $\dfrac{\Gamma \vdash \sigma \quad Type}{\Gamma,\, x : \sigma \vdash r_\sigma(x) : \mathsf{Id}_\sigma(x,\, x)}$ [42]

$$\frac{\Gamma,\, x,\, x' : \sigma,\, z : \mathsf{Id}_\sigma(x,\, x') \vdash \rho\ Type \qquad \Gamma,\, x : \sigma \vdash Q : \rho[x/x',\, r_\sigma(x)/z]}{\Gamma,\, x : \sigma,\, x' : \sigma,\, z : \mathsf{Id}_\sigma(x,\, x') \vdash (Q \text{ with } x' = x \text{ via } z) : \rho}$$ [43]

Remark 2.3.4. The Remark 2.3.1 shows us that, in DTT, we can actually derive exponentials and finite products (defined exactly as in STT) from dependent products and sums, respectively. More precisely, whenever we have two types in the same context $\Gamma \vdash \sigma,\, \tau$ *Type* and a term $x : \sigma$ that does not appear in τ, then we can define: [44]

$$\sigma \times \tau := \Sigma_{x:\sigma}\tau \quad \text{and} \quad \sigma \to \tau := \Pi_{x:\sigma}\tau$$ [45]

For a term $P : \sigma \times \tau$, we define the projections (with the same notation used in the previous section) as:

$$\pi P := \text{unpack } P \text{ as } < x,\, y > \text{ in } x \text{ and } \pi' P := \text{unpack } P \text{ as } < x,\, y > \text{ in } y$$

We underline the fact that the second projection (the one regarding τ) can be define because x does not occur in τ.
In order to understand the idea behind the exponential we recall that, in a categorical setting, is usually defined as the right adjoint of the cartesian product. On the other hand, given a category $\mathbb{C}$ and a object X of it, the product in the slice category $\mathbb{C}/X$ is exactly the pullback in $\mathbb{C}$. Moreover if in the aforementioned category we have even a terminal object, then the product itself can be seen as a pullback. In the next section we will see how, in the particular enviroment of Locally Cartesian Closed Categories (LCCC), the dependent product is strictly connected to the right adjoint of the pullback functor through a map f, that we will call Π_f indeed. Thus we can see how the exponential and the dependent product are closely associated.

We end this section giving the **propositions-as-types** interpretation of the new (dependent) types introduced.
As we have already stated, the terms of the identity type $\mathsf{Id}_\sigma(x,\, x')$ give us a proof of the equality of x and x' as σ-terms.
Let us consider $P : \Pi_{x:\sigma}\tau(x)$. This term corresponds to a function $P \equiv \lambda x : \sigma.Px$ which gives *for each* $M : \sigma$ a proof of $PM : \tau[M/x]$. Therefore we can see the connection with the universal logic quantifier $\forall$.

[42] With r_σ we refer to a "reflexivity combinator", since it regards the reflexive property. It can be understood as the choice of a canonical identity 1_x for each $x : \sigma$.

[43] In the first part, due to lack of space, we wrote ρ instead of the correct $\rho(x,\, x',\, z)$.

[44] [Jac99] Chapter 10, §10.1, Example 10.1.2.

[45] Here we omit the notation $\tau(x)$ since τ does not depend on x indeed.

On the other hand, a term $R : \Sigma_{x:\sigma}\tau(x)$ corresponds to a pair $< \pi R, \pi' R > \equiv R$, where $\pi R : \sigma$ and $\pi' R : \tau[\pi R/x]$. In other words, we can find a proof of τ for the element πR, hence the strong bond with the existential logic quantifier $\exists$.

2.4 Homotopy Type Theory

The notions outlined in the following section can be found in [Awo16] §1-2.2, [nLab], [HoTT] Appendix A.1 and [KL16] Appendix A.

As the name suggests, Homotopy Type Theory (HoTT) interprets type theory in an homotopical view. More precisely types would be "spaces" and the logical constructions some constructions on these spaces with a good behaviour regarding homotopy. In particular terms $a : A$ can be understood as points $a \in A$ in a space. This idea clarifies the nature of *identity*. The type-theoretic notion of equality $a = b : A$ is translated in HoTT as the existence of a path $\gamma : a \rightsquigarrow b$ in the space A. Analogously two maps $f, g : A \to B$ can be identifued if there is a (continous) family of paths $\gamma_x : f(x) \rightsquigarrow g(x)$ in B, i.e. if they are homotopic equivalent. This way to see identification gives rise to a nice interpretation of the identity type Id_A for a type A. In fact we can consider it as the *path space* (or path object) A^I of all continous maps $I \to A$ from the unit interval. In a model category we could dipict it as an actual path object as defined in any model category. The whole idea is that, in this way, a term $\gamma : \mathsf{Id}_A(a, b)$ would really represents a path $\gamma : a \rightsquigarrow b$ in A.

Later on in this section we will describe a natural model of homotopy type theory in category theory. But first, to clarify all the concepts depicted in this chapter, it is useful to compare the various points of view connected to type theory with the following chart:

Types	Logic	Sets	Homotopy
A	proposition	set	space
$a : A$	proof	element	point
$B(x)$	predicate	family of sets	fibration
$b(x) : B(x)$	conditional proof	family of elements	section
0, 1	$\bot$, $\top$	$\emptyset$, $\{\emptyset\}$	$\emptyset$, $*$
$A + B$	$A \vee B$	disjoint union	coproduct
$A \times B$	$A \wedge B$	set of pairs	product
$A \to B$	$A \supset B$	set of functions	path space
$\Pi_{x:A}B(x)$	$\exists_{x:A}B(x)$	See Remark 2.3.1	?
$\Sigma_{x:A}B(x)$	$\forall_{x:A}B(x)$	See Remark 2.3.1	?
Id_A	equality $=$	See Remark 2.3.1	?

We will understand the interpretation of Π-, Σ- and Id-types with the following description of a natural model of homotopy type theory. All of the following notions can be found in [Awo16].

Definition 2.4.1. *Let $\mathbb{C}$ be a small category. A natural transformation $f : Y \dot{\to} X$ of presheaves on $\mathbb{C}$ is* ***representable*** *if all its fibers are representable objects, i.e. for any $C \in Ob(\mathbb{C})$ and any $x \in X(C)$ there exists an object $D \in Ob(\mathbb{C})$, a morphism $p : D \to C$ in $\mathbb{C}$ and an element $y \in Y(D)$ such that the following diagram is a pullback*

$$\begin{tikzcd} h^D \arrow[r, "y"] \arrow[d, "h^p"'] & Y \arrow[d, "f"] \\ h^C \arrow[r, "x"'] & X \end{tikzcd}$$

For the rest of the section, with an abuse of notation, we may write D instead of h^D and for any element $x \in X(C)$ we could mean both the element and the map $x : h^C \dot{\to} X$ induced by Yoneda.

We can think of $\mathbb{C}$ as a "category of context" [46], i.e. let us consider the objects of $\mathbb{C}$ as "contexts" Γ, Δ... and arrows as "substitutions" $f : \Delta \to \Gamma$. Now let us consider a representable map of presheaves $p : \widetilde{U} \dot{\to} U$. We write their elements as:

$$A \in U(\Gamma) \;\Leftrightarrow\; \Gamma \vdash A \quad \mathit{Type}$$

$$a \in \widetilde{U}(\Gamma) \;\Leftrightarrow\; \Gamma \vdash a : A$$

where $A = p \circ a$ (with the notation mentioned above). Thus we can regard to the following as:

[46] A precise definition of a *contextual category* will be given in the next chapter.

U	$\rightsquigarrow$	presheaf of types
$\widetilde{U}$	$\rightsquigarrow$	presheaf of terms
$U(\Gamma)$	$\rightsquigarrow$	set of all types in context Γ
$\widetilde{U}(\Gamma)$	$\rightsquigarrow$	set of all terms in context Γ
$p_\Gamma : \widetilde{U}(\Gamma) \to U(\Gamma)$	$\rightsquigarrow$	typing of the terms in context Γ, i.e the action to assign to each term in that context its type

The naturality of p tells us that for any substitution $f : \Delta \to \Gamma$ we have that:

$$\Gamma \vdash A \quad \mathit{Type} \Rightarrow \Delta \vdash Af \quad \mathit{Type}$$

$$\Gamma \vdash a : A \Rightarrow \Delta \vdash af : Af$$

Moreover, by functoriality, for any further susbtitution $g : \Delta' \to \Delta$ we have that:

$$(Af)g = A(f \circ g) \qquad (af)g = a(f \circ g)$$

$$A\, Id_\Gamma = A \qquad a\, Id_\Gamma = a$$

Lastly, the fact that p is representable gives the operation of "context extension". The idea is that given a type A in context Γ (i.e. $\Gamma \vdash A$ *Type*), we can find a context with both Γ and A. More precisely let us consider

$$\Gamma \vdash A \quad \mathit{Type} \Leftrightarrow A : \Gamma \to U$$ [47]

then we obtain a pullback of the following form:

$$\begin{tikzcd}
\Gamma.A \arrow[r, "q_A"] \arrow[d, "p_A"'] & \widetilde{U} \arrow[d, "p"] \\
\Gamma \arrow[r, "A"'] & U
\end{tikzcd}$$

Therefore, by the commutativity and representability of the diagram, we get $\Gamma.A \vdash q_A : Ap_A$. Moreover the square is a pullback, so for any substitution $f : \Delta \to \Gamma$ and $\Delta \vdash a : Af$ (i.e. $a : \Delta \to \widetilde{U}$ such that $pa = Af$) there exists a unique $(f,\, a) : \Delta \to \Gamma.A$ such that:

$$p_A \circ (f,\, a) = f \;\text{ and }\; q_A \circ (f,\, a) = a$$

[47] From now on we will write Γ instead of h^Γ (and same for the maps between two such objects) with an abuse of notation, but thanks to the fully faithfulness of the Yoneda embedding.

The uniqueness means that for any other substitution $g : \Delta' \to \Delta$ we have that:

$$(f, a) \circ g = (fg, ag) \text{ and } (p_A, q_A) = Id_{\Gamma.A}$$

Thanks to the interpretations given above, it makes sense to give the following definition:

Definition 2.4.2. *A representable map of presheaves $p : \widetilde{U} \to U$ on a (small) category $\mathbb{C}$ is called a **natural model of type theory**.*

Remark 2.4.3. Natural models of type theory are closed under composition, coproducts and pullbacks along any map $U' \to U$. This can easily be seen through the properties of pullbacks. The first and last one derive from the fact that composition of pullbacks is again a pullback[48]. The second one is obtained from the universal property of coproducts (in particular the characterization of maps into a coproduct).

In order to understand the formulations of Π-, Σ- and Id-types in natural models we need to recall some preliminary definitions.

Definition 2.4.4. *A category $\mathbb{C}$ is said to be Locally Cartesian Closed (LCCC) if for any object x the slice category $\mathbb{C}/x$ is cartesian closed.*

Remark 2.4.5. Usually another condition that is required to be LCCC is to have a terminal object. In this case a LCCC would be even cartesian closed and with all finite limits. In fact, setting $*$ as the terminal object in $\mathbb{C}$, we have a canonical equivalence $\mathbb{C}/* \cong \mathbb{C}$ and in any category to have finite limits is equivalent to have a terminal object and pullbacks, which are exactly the cartisian products in a slice category.

Proposition 2.4.6. ***(/Alternative definition)*** *A category $\mathbb{C}$ is LCCC if and only if it has pullbacks (and a terminal object if required in the definition) such that each base change $f^* : \mathbb{C}/Y \to \mathbb{C}/X$ has a right adjoint Π_f, usually called **dependent product**.* [49]

Notation: In a LCCC, for any map $f : X \to Y$, we have two adjoint pairs namely

$$\begin{tikzcd} \mathbb{C}/Y \arrow[r, "f^*"] & \mathbb{C}/X \arrow[l, bend right, "\Sigma_f"'] \arrow[l, bend left, "\Pi_f"] \end{tikzcd}$$

where $\Sigma_f(W \xrightarrow{g} X) := f \circ g$ and $\Sigma_f \dashv f^* \dashv \Pi_f$.

[48]To be precised the last one it true since if we have two square such that the right one is a pullback and the "composition" square is a pullback, then even the square on the left is such.

[49][nLab] §2, Propositions 2.1 and 2.3.

Remark 2.4.7. Since for any morphism $f : X \to Y$ in a category $\mathbb{C}$ we have that ${}^{(\mathbb{C}/Y)}/_{f} \cong \mathbb{C}/X$, then any slice category of a LCCC is again LCCC.

Proposition 2.4.8. *If $\mathbb{C}$ is a LCCC with a terminal object $*$, then for any object X in $\mathbb{C}$ the functor $X \times - : \mathbb{C} \to \mathbb{C}/X$ (defined as a pullback functor identifying $\mathbb{C}$ with $\mathbb{C}/*$) preserves finite products and exponentials (up to isomorphism).* [50]

Lemma 2.4.9. *Let $\mathbb{C}$ be a (small) category. Then $\widehat{\mathbb{C}}$ is LCCC.*

Proof Let us consider $F \in \mathrm{Ob}(\widehat{\mathbb{C}})$, if F is representable then it is clear that

$$\widehat{C}/F \cong \widehat{C}/h^X \cong \widehat{\mathbb{C}/X}$$

and therefore is catersian closed, since any presheaf category is such. But actually, for any presheaf F over $\mathbb{C}$ we can prove that

$$\widehat{C}/F \cong \widehat{el(F)}$$ [51]

Thus any slice category is cartesian closed.

□

Remark 2.4.10. The equivalence stated in the last lemma tells us that for any set B we have $Set^B \cong Set/B$ [52]. We note that in the slice category the dependent sum Σ_f, for any map of sets $f : B' \to B$, is *strictly associative*, whilst the dependent product is not. On the other hand in the category of functors Π_f can be seen as a composition of functors, hence it is strictly associative, while the Σ_f could be not strict.
Therefore we can see how this equivalence is crucial for the so-called *coherence problem*, i.e. the requirement of some construction to be *strictly functorial*.

Notation: Let $\mathcal{E}$ be a LCCC with a terminal object $*$. For any morphism $f : B \to A$ in $\mathcal{E}$ we can define a **polynomial endofunctor** $P_f : \mathcal{E} \to \mathcal{E}$ in the following way:

$$P_f(X) \equiv \Sigma_{a:A} X^{B_a} := \Sigma_A \Pi_f B^*(X)$$

where $B^* : \mathcal{E} \to \mathcal{E}/B$ indicates the pullback along $B \to *$ (through the usual equivalence $\mathcal{E}/* \cong \mathcal{E}$) and Σ_A is the forgetful functor (or equivalently the composition with $A \to *$).

[50] [nLab] §2, Proposition 2.5.

[51] The category $el(F)$ is defined in the proof of Theorem A.2.2. A proof of this equivalence can be found in [Awo10] Chapter 9, §9.7, Lemma 9.23.

[52] Where with Set^B we mean the category of functors from B (seen as a discrete category) and Set or, equivalently, the category of family of sets $(X_b)_{b \in B}$ over B.

Lemma 2.4.11. *There is a natural bijection between maps $g : Y \to \Sigma_{a:A} X^{B_a}$ and pairs of maps $(g_1 : Y \to A,\ g_2 : Y \times_A B \to X)$ as shown in the following diagram:* [53]

$$\begin{tikzcd} X & Y \times_A B \arrow[l, "g_2"'] \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ & Y \arrow[r, "g_1"'] & A \end{tikzcd}$$

Proof Let us consider a map $g : Y \to \Sigma_{a:A} X^{B_a} = \Sigma_A \Pi_f B^*(X)$. We define g_1 as the composition of

$$Y \xrightarrow{g} \Sigma_A \Pi_f B^*(X) \xrightarrow{\pi_1} A$$

where π_1 is given by the fact that $\Pi_f B^*(X)$ is in $\mathcal{E}/A$ and Σ_A is a forgetful functor. Moreover g becomes a morphism in $\mathcal{E}/A$, since the following triangle commutes:

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "g_1"'] & & \Sigma_A \Pi_f B^*(X) \arrow[dl, "\pi_1"] \\ & A & \end{tikzcd}$$

Nonetheless, we note that, as objects in $\mathcal{E}/A$, $\Sigma_A \Pi_f B^*(X) = \Pi_f B^*(X) \cong \Pi_f f^* A^*(X) = (A^*(X))^f$ (exponential in $\mathcal{E}/A$ given by the LCCC structure). Therefore, by the adjunction and the fact that the cartesian product in $\mathcal{E}/A$ is the pullback, we get an equivalent commutative triangle

$$\begin{tikzcd} Y \times_A B \arrow[rr, "\widetilde{g}"] \arrow[dr, "g_1 \times f"'] & & A^*(X) \arrow[dl, "\pi_1"] \\ & A & \end{tikzcd}$$

Now we define g_2 as the composition

$$Y \times_A B \xrightarrow{\widetilde{g}} A^*(X) = A \times X \xrightarrow{pr_X} X$$

The mapping $g \mapsto (g_1, g_2)$ can be proven to be natural and bijective.

$\square$

Let us go back to the case where $f = p : \widetilde{U} \to U$ a natural model. We denote the polynomial functor associated $P \equiv P_p$ and $P(X) \equiv \Sigma_{A:U} X^A$. By the last lemma we have a bijective (natural) correspondence between

[53] [Awo16] §2, Lemma 2.1.

morphism $(A, B) : \Gamma \to \Sigma_{A:U} U^A$ and pairs of maps A and B as in the following diagram

$$\begin{tikzcd} U & \Gamma.A \arrow[l, "B"'] \arrow[r] \arrow[d] & \widetilde{U} \arrow[d, "p"] \\ & \Gamma \arrow[r, "A"'] & U \end{tikzcd}$$

and therefore, through the interpretation given at the start of the section, to pairs of sequents

$$\Gamma \vdash A \quad \mathit{Type} \quad \text{and} \quad \Gamma.A \vdash B \quad \mathit{Type}$$

In other words we can say that, as U classifies types in context $\Gamma \vdash A$ *Type*, $P(U)$ classifies types in an extended context $\Gamma.A \vdash B$ *Type*. More precisely there is a commutative diagram of the following form

$$\begin{tikzcd} & U & \\ \Gamma.A \arrow[ur, "B"] \arrow[r] \arrow[d] & \cdot \arrow[u] \arrow[r] \arrow[d] & \widetilde{U} \arrow[d, "p"] \\ \Gamma \arrow[r] \arrow[rr, bend right, "A"'] & \Sigma_{A:U} U^A \arrow[r] & U \end{tikzcd}$$

where both squares are pullbacks.

Proposition 2.4.12. ***(Dependent Products)*** *Let $P(X) = \Sigma_{A:U} X^A$ be the polynomial functor associated to a natural model $p : \widetilde{U} \to U$. Then the type-theoretic rules for (extensional) dependent products are modelled by maps λ and Π making the following diagram a pullback:* [54]

$$\begin{tikzcd} P(\widetilde{U}) \arrow[r, "\lambda"] \arrow[d, "P(p)"'] & \widetilde{U} \arrow[d, "p"] \\ P(U) \arrow[r, "\Pi"'] & U \end{tikzcd} \tag{3}$$

Proof Let us recall the Π-formation rule, written in a slightly different (but equivalent) way than the one in the paragraph before:

$$\frac{\Gamma \vdash A \ \mathit{Type} \qquad \Gamma.A \vdash B \ \mathit{Type}}{\Gamma \vdash \Pi_{a:A} B \quad \mathit{Type}}$$

[54] [Awo16] §2.1, Prop 2.4.

By the characterization of morphism into $P(U)$, we can see that $\Pi : P(U) \to U$ represents this particular type-theoretic rule. In fact whenever we have two types A and B, in the contexts stated above, we have a type in context Γ that factors through Π (seeing it as a map $\Gamma \to U$). On the other hand $\widetilde{U}$ is related to terms of types, so in a similar way we can see that $P(\widetilde{U})$ classifies pairs

$$\Gamma \vdash A \quad \mathit{Type} \quad \text{and} \quad \Gamma.A \vdash b : B$$

Therefore $\lambda : P(\widetilde{U}) \to \widetilde{U}$ models the Π-introduction rule, that we can see as:

$$\frac{\Gamma \vdash A \ \mathit{Type} \qquad \Gamma.A \vdash b : B}{\Gamma \vdash \lambda_A b : \Pi_{a:A} B}$$

Here we should read $\Gamma.A$ as $a : A$ and $\lambda_A b$ as $\lambda a : A.b$, in order to see the connection with the formulation of Π-intro given in the previous section. Now we suspend for a moment the proof of the proposition to make an important remark regarding the λ-substitution.

Remark 2.4.13. How to interpret substitution: We have already seen how to have a term $\Gamma \vdash a : A$, for a type A, is the same to have a commutative diagram

$$\begin{tikzcd} & \widetilde{U} \arrow[d, "p"] \\ \Gamma \arrow[ur, "a"] \arrow[r, "A"'] & U \end{tikzcd}$$

Therefore, by the universal property of the pullback we find a unique morphism $(1, a) : \Gamma \to \Gamma.A$ such that the following diagram commutes:

$$\begin{tikzcd} \Gamma \arrow[dr, dotted, "{(1,a)}"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "Id_\Gamma"'] & & \\ & \Gamma.A \arrow[r] \arrow[d] & \widetilde{U} \arrow[d, "p"] \\ & \Gamma \arrow[r, "A"'] & U \end{tikzcd}$$

Therefore if we want to compute $\lambda a : A.b$ (written as $\lambda_A b$ above), where

$$\begin{tikzcd} & \widetilde{U} \arrow[d, "p"] \\ \Gamma.A \arrow[ur, "b"] \arrow[r, "B"'] & U \end{tikzcd}$$

we just set $B[a] := B \circ (1,\, a)$ and $b[a] := b \circ (1,\, a)$. In particular we have that

$$\Gamma \vdash B[a]\ Type$$
$$\Gamma \vdash b[a] : B[a]$$

It can be proven that the diagram (3) is a pullback if and only if the Π-elimination rule and β and η computation ones hold [55]. For instance let us see why, if the diagram is a pullback, then the elimination rule holds.
Let us consider $\Gamma \vdash a : A$ and $\Gamma \vdash f : \Pi_{a:A} B$. Then we have the following commmutative diagram (induced by the universal property of the pullback):

$$\begin{tikzcd}
\Gamma \arrow[dr, dotted, "{(A, \widetilde{f})}"] \arrow[drr, bend left, "a"] \arrow[ddr, bend right, "{(A, B)}"'] & & \\
& P(\widetilde{U}) \arrow[r, "\lambda"] \arrow[d, "P(p)"'] & \widetilde{U} \arrow[d, "p"] \\
& P(U) \arrow[r, "\Pi"'] & U
\end{tikzcd}$$

where the map $(A,\, \widetilde{f})$ is defined by the pullback. By the classifying property of $P(\widetilde{U})$ it corresponds to a unique $\Gamma.A \vdash \widetilde{f} : B$ such that $p \circ \widetilde{f} = B$. Now we set $f(a) := \widetilde{f}[a] \equiv \widetilde{f} \circ (1,\, a)$. Therefore we have the Π-elimination rule:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash f : \Pi_{a:A} B}{\Gamma \vdash f(a) : B[a]}$$

□

Let us now consider dependent sums. First of all we recall that, in order to define them, we need to define the concept of "pair". For now let us denote with $< a,\, b >$ a pair of terms. Now we can write the Σ-introduction rule in the following way (with the notation introduce in this section):

$$\frac{\Gamma \vdash A\ Type \qquad \Gamma.A \vdash B\ Type \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : \Sigma_A B}{\Gamma \vdash < a,\, b >: \Sigma_A B}$$

Since Σ-formation has the same premises of the Π-formation we can use again a morphism of the kind $P(U) \to U$, but we have to change the second one. In particular it would need to classify the premises written above (in the introduction rule). We can find an object that can accomplish this:

[55] [Awo16] §2.1, Corollary 2.5.

$$\Sigma_{A:U}\Sigma_{B:U^A}\Sigma_{a:A}B(a) \cong \Sigma_{(A,\,B):P(U)}\Sigma_{a:A}B(a)$$ [56]

Just as we found the map $P(U) \to U$, we can find a natural projection $\pi : \Sigma_{(A,\,B):P(U)}\Sigma_{a:A}B(a) \to P(U)$. In particular we can prove that we have a natural bijection between the morphisms making the following diagram commutative

$$\begin{tikzcd}
 & \Sigma_{A:U}\Sigma_{B:U^A}\Sigma_{a:A}B(a) \arrow[d, "\pi"] \\
\Gamma \arrow[ur, dotted] \arrow[r, "{(A,\,B)}"'] & \Sigma_{A:U}U^A
\end{tikzcd}$$

and the data required in the Σ-introduction rule. Moreover one could prove the following proposition, that gives us the wanted Σ-structure.

Proposition 2.4.14. ***(Dependent Sums)*** *Let $P(X) = \Sigma_{A:U}X^A$ be the polynomial functor associated to a natural model $p : \widetilde{U} \to U$. Then the type-theoretic rules for (extensional) dependent sums are modelled by maps pair and Σ making the following diagram a pullback:* [57]

$$\begin{tikzcd}
\Sigma_{A:U}\Sigma_{B:U^A}\Sigma_{a:A}B(a) \arrow[r, "pair"] \arrow[d, "\pi"'] & \widetilde{U} \arrow[d, "p"] \\
P(U) \arrow[r, "\Sigma"'] & U
\end{tikzcd}$$

Lastly we need to interpret (extensional) identities. We recall the type-theoretic Id-formation and introduction rules:

$$\frac{\Gamma \vdash A \ Type \qquad \Gamma \vdash a : A \qquad \Gamma \vdash a' : A}{\Gamma \vdash \mathsf{Id}_A(a,\, a') \ Type} \qquad\qquad \frac{\Gamma \vdash a : A}{\Gamma \vdash r_\sigma(a) : \mathsf{Id}_A(a,\, a)}$$

It is natural to consider the diagonal map $\delta : \widetilde{U} \times_U \widetilde{U} \to \widetilde{U}$, given by the universal property of the pullback through the following diagram:

$$\begin{tikzcd}
\widetilde{U} \arrow[dr, dotted, "\delta"] \arrow[drr, bend left, "Id_{\widetilde{U}}"] \arrow[ddr, bend right, "Id_{\widetilde{U}}"'] & & \\
 & \widetilde{U} \times_U \widetilde{U} \arrow[r, "t"'] \arrow[d] & \widetilde{U} \arrow[d, "p"] \\
 & \widetilde{U} \arrow[r, "p"'] & U
\end{tikzcd}$$

[56]For a complete description one can look [Awo16] §2.2.
[57][Awo16] §2.2, Prop 2.8.

Proposition 2.4.15. ***(Identity Types)*** *For a natural model $p : \widetilde{U} \to U$ the type-theoretic rules for (extensional) identity types are modelled by maps r_σ and Id making the following diagram a pullback:* [58]

$$\begin{tikzcd} \widetilde{U} \arrow[r, "r_\sigma"] \arrow[d, "\delta"'] & \widetilde{U} \arrow[d, "p"] \\ \widetilde{U} \times_U \widetilde{U} \arrow[r, "\mathsf{Id}"'] & U \end{tikzcd}$$

Putting together all the the three main proposition just stated we can assert a theorem that characterize some models of Martin-Löf Type Theory:

Theorem 2.4.16. *A natural model of (extensional) Martin-Löf Type Theory with dependent products, sums and identity types is given by a small category $\mathbb{C}$ equipped with a representable map of presheaves $p : \widetilde{U} \to U$ together with maps*

$$\Pi,\ \lambda,\ \Sigma,\ \mathit{pair},\ \mathsf{Id},\ r_\sigma$$

as in the propositions 2.4.12, 2.4.14, 2.4.15.

[58] [Awo16] §2.3, Prop 2.11.

3 A Univalent Model

In the following chapter we will explain the **Univalence Axiom** for Homotopy Type Theory, conceived by Vladimir Voevodsky. The idea of this axiom is to state that "identity" and "equivalence" are equivalent

$$(A = B) \simeq (A \simeq B)$$

More precisely, we consider a universe U in a type theory. This roughly means that we consider a type that characterize types themselves as terms $A : U$. As any type, U has an identity type Id_U, which express the identity relation $A = B$ between types. This gives rise to the type of identifications of A with B called $\mathsf{Id}_U(A,\, B)$. On the other hand we can construct even the type of equivalences between them, that we write $\mathsf{HIso}(A,\, B)$. Obviously we want the identities to be equivalences, and thus we find a morphism

$$\mathsf{Id}_U(A,\, B) \longrightarrow \mathsf{HIso}(A,\, B)$$

The univalence Axiom states that this map is itself an equivalence. [59]

Furthermore we will show how, starting from Kan complexes, we can find a model of Per Martin-Löf dependent type theory where this important axiom actually holds. In order to be able to show that we will need some definition and the construction of a "universe" in the model of Kan complexes.

Notation: For any category $\mathbb{C}$ and any object $X \in \mathrm{Ob}\,\mathbb{C}$, in this chapter we will write 1_X for the identity morphism of X, to avoid possible confusion with the type-theoretic notion of identity.

3.1 Categorical Setting

We start giving the natural structure on a category in order to have a possible interpretation of a given type theory.

Definition 3.1.1. *A **contextual ategory** $\mathbb{C}$ consists of the following data:*

1. *a category $\mathbb{C}$;*
2. *a grading of objects as $Ob\,\mathbb{C} = \amalg_{n:\mathbb{N}}\, Ob_n\,\mathbb{C}$;*
3. *an object $1 \in Ob\,\mathbb{C}$ such that is a terminal object in $\mathbb{C}$ and it is the unique object in $Ob_0\,\mathbb{C}$;*

[59] [HoTT] Introduction.

4. *maps* $ft_n : Ob_{n+1}\,\mathbb{C} \to Ob_n\,\mathbb{C}$*;*

5. *for each* $X \in Ob_{n+1}\,\mathbb{C}$*, a map* $p_X : X \to ft_n(X)$ *(called the* ***canonical projection*** *from* X*);*

6. *for each* $X \in Ob_{n+1}\,\mathbb{C}$ *and* $f : Y \to ft(X)$*, an object* $f^*(X)$ *together with a map* $q(f, X) : f^*(X) \to X$ *such that* $ft(f^*X) = Y$ *and the following diagram is a pullback*

$$\begin{tikzcd} f^*X \arrow[r, "q(f,X)"] \arrow[d, "p_{f^*X}"'] & X \arrow[d, "p_X"] \\ Y \arrow[r, "f"'] & ft(X) \end{tikzcd}$$

Moreover we required that these canonical pullbacks are strictly functorial: i.e. for any $X \in Ob_{n+1}\,\mathbb{C}$ *we want* $1^*_{ft\,X}X = X$ *and* $q(1_{ft\,X}, X) = 1_X$*; and for* $X \in Ob_{n+1}\,\mathbb{C}$*,* $f : Y \to ft(X)$ *and* $g : Z \to Y$ *we have that* $(fg)^*(X) = g^*(f^*(X))$ *and* $q(fg, X) = q(f, X)q(g, f^*X)$*.* [60]

Remark 3.1.2. In order to make the definiton above clearer we can see a fundamental example of contextual category. Given any type theory $\mathbf{T}$ we can construct a contextual category $\mathbb{C}(\mathbf{T})$ [61] from it, in the following way:

$\mathrm{Ob}_n(\mathbb{C}(\mathbf{T}))$:= well-formed contexts of the form $\Gamma = x_1 : A_1, ..., x_n : A_n$ (up to definitional equality and renaming of free variables)

In this enviroment we would define the ft'n as:

$$ft_n[x_1 : A_1, ..., x_{n+1} : A_{n+1}] := [x_1 : A_1, ..., x_n : A_n]$$

With this view, the type terms $\Gamma \vdash t : A$ of $\mathbf{T}$ may be recovered from $\mathbb{C}(\mathbf{T})$ as sections of the canonical projection

$$p : [\Gamma, x : A] \longrightarrow \Gamma(= ft[\Gamma, x : A]\,)$$

In any contextual category we can define the notions of Π-, Σ-, Id-type structures [62], even though they do not always exist. On the other hand, for instance, if we consider the contextual category created from a type theory with those structures then even the category has got them. In particular the category $\mathbb{C}(\mathbf{T})$ is initial among all contextual categories carrying the same structures determined by the logical rules of $\mathbf{T}$. [63]

[60] [KL16] §1.2, Definition 1.2.1.
[61] A complete description can be found in [KL16] §1.2, Example 1.2.3.
[62] [KL16] §1.2, Definition 1.2.4 and Example 1.2.5.
[63] [KL16] §1.2, Theorem 1.2.9.

Definition 3.1.3. *A* ***model*** *of depedent type theory with any selection of logical rules is a contextual category equipped with the structure corresponding to the chosen rules.* [64]

A crucial problem in the construction of a model of type theory is the **coherence problem** (in this case the requirement to be *strictly* functorial regards pullbacks) and for logical structure to commute *strictly* with it. In fact, usually, we have just a commutative property up to isomophism. In order to solve this problem we introduce the notion of *universes.*

Definition 3.1.4. *Let $\mathbb{C}$ be a category. A* ***universe*** *in $\mathbb{C}$ is the data of an object U in the category together with a morphism $p : \widetilde{U} \to U$ and a choice of a pullback square*

$$\begin{tikzcd} (X;\, f) \arrow[r, "Q(f)"] \arrow[d, "P_{(X,f)}"'] & \widetilde{U} \arrow[d, "p"] \\ X \arrow[r, "f"'] & U \end{tikzcd}$$

for any map $f : X \to U$. [65]

The idea is that the map p represents the generic family of types over the universe U, as seen in section 2.4.

Notation: Given $f : Y \to X$ we write $\ulcorner f \urcorner$ (or sometimes $\ulcorner Y \urcorner$ if the morphism is clear) for a map $\ulcorner f \urcorner : X \to U$ such that $f \cong P_{(X, \ulcorner f \urcorner)}$. For $f_1 : X \to U$ and $f_2 : (X;\, f_1) \to U$ we write $(X;\, f_1,\, f_2) := ((X;\, f_1);\, f_2)$, and so on for any finite sequence of maps.

Definition 3.1.5. *Given a category $\mathbb{C}$ with a universe $p : \widetilde{U} \to U$ and a terminal object 1, we define* ***the contextual category associated to the universe*** *U as the category $\mathbb{C}_U$ defined as follows:* [66]

1. *$Ob_n\, \mathbb{C}_U := \{(f_1,\, ...,\, f_n) \mid f_i : (1; f : 1,\, ...,\, f_{i-1}) \to U \text{ for } 1 \leq i \leq n\}$;*
2. *$\mathbb{C}_U((f_1,\, ...,\, f_n),\, (g_1,\, ...,\, g_m)) := \mathbb{C}((1;\, f_1,\, ...,\, f_n),\, (1;\, g_1,\, ...,\, g_m))$;*
3. *$1_{\mathbb{C}_U} := ()$ the empty sequence;*
4. *$ft_n\, (f_1,\, ...,\, f_{n+1}) := (f_1,\, ...,\, f_n)$;*

[64] [KL16] §1.2, Definition 1.2.10.
[65] [KL16] §1.3, Definition 1.3.1.
[66] [KL16] §1.3, Definition 1.3.2.

5. *the projection $p_{(f_1,\, ...,\, f_{n+1})}$ is the map $P_{(X,\, f_{n+1})}$ given by the universe structure;*

6. *given $(f_1, \, ..., \, f_{n+1})$ and a map $\alpha : (g_1, \, ..., \, g_m) \to (f_1, \, ..., \, f_{n+1})$ in $\mathbb{C}_U$, the canonical pullback $\alpha^*(f_1, \, ..., \, f_{n+1})$ is determined by $(g_1, \, ..., \, g_m, \, f_{n+1} \circ \alpha)$, with projection induced by $Q(f_{n+1}\alpha)$, i.e.*

$$
\begin{tikzcd}
(1;\, g_1, \, ..., \, g_m, \, f_{n+1} \circ \alpha) \arrow[r, "\exists!"'] \arrow[d] \arrow[rr, bend left, "Q(f_{n+1}\alpha)"] & (1;\, f_1, \, ..., \, f_{n+1}) \arrow[r, "Q(f_{n+1})"'] \arrow[d] & \widetilde{U} \arrow[d, "p"] \\
(1;\, g_1, \, ..., \, g_m) \arrow[r, "\alpha"'] & (1;\, f_1, \, ..., \, f_n) \arrow[r, "f_{n+1}"'] & U
\end{tikzcd}
$$

where both squares are pullbacks. [67]

Remark 3.1.6. We see straight away the similarities between the sequences $(f_1, \, ..., \, f_n)$ and the well-formed context in a type theory. In fact we notice how each f_i is constructed, through a pullback, from the previous ones. With this view the choice of ft_n is clear, since it has to represent the "forgetful" operation in well-formed context (i.e. to drop the last declaration).

Moreover if a category $\mathbb{C}$ with a universe U is locally cartesian closed (LCCC), then we can give the right definitions of Π-, Σ-, Id-structures (as a particular case of the one discussed in section 2.4). This is possible mainly because LCCC are the right enviroment to talk about objects of "U-context". These structures induce the respective type structures on $\mathbb{C}_U$. [68]

3.2 The Simplicial Model

The aim of this section is to construct (for any regular cardinal number α) a Kan fibration $p_\alpha : \widetilde{U_\alpha} \to U_\alpha$ weakly universal among Kan fibration with α-small fibers. These morphisms will be crucial in the construction of the model of type theory on *sSet*.

Definition 3.2.1. *An infinite cardinal number α is* ***regular*** *if no set of cardinality α is the union of fewer than α sets of cardinality less than α.*

Definition 3.2.2.
- *A* ***well-ordered morphism*** *of simplicial sets consists of an ordinary map $f : Y \to X$ in sSet together with a function assigning to each $x \in X_n$ a well-oredering on the fiber $Y_x := f_n^{-1}(x) \subseteq Y_n$.*

[67]The left square is a pullback because the right one and the composition square are such.

[68][KL16] §1.4, Theorem 1.4.15.

- *An **isomorphism of well-ordered morphisms** $f : Y \to X$ and $f' : Y' \to X$ is an isomorphism $\psi : Y \to Y'$ of simplicial sets that makes the following triangle commutative*

$$\begin{tikzcd} Y \arrow[rr, "\psi"] \arrow[dr, "f"'] & & Y' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

and preserves the well-ordering on all the fibers. [69]

We will now state a classical result about well-ordered sets (proven by transfinite induction[70]) with an important consequence in the simplicial sets enviroment.

Proposition 3.2.3. *There exists at most one isomorphism between two well-ordered sets. Therefore there exists at most one isomorphism between two well-ordered moprhisms over a common base.*

From now on we will fix a regular cardinal and we will call it α.

Definition 3.2.4. *A morphism of simplicial sets $f : Y \to X$ is α**-small** if for any n and any $x \in X_n$, the fiber Y_x has cardinality $\mid Y_x \mid < \alpha$.*

We define a particular presheaf $\mathbf{W}_\alpha : sSet^{op} \to Set$ on *sSet* in the following way:

- For any simplicial set X we define $\mathbf{W}_\alpha(X)$ as the set of isomorphism classes [71] of α-small, well-ordered morphism $Y \to X$.
- For any morphism $f : X' \to X$ in *sSet* we set $\mathbf{W}_\alpha(f) := f^*$ as the pullback action.

Precomposing with the Yoneda embedding we obtain a simplicial set that we call W_α

$$\begin{tikzcd} \Delta^{op} \arrow[r, "Y^{op}"] \arrow[rr, bend right, "W_\alpha"'] & sSet^{op} \arrow[r, "\mathbf{W}_\alpha"] & Set \end{tikzcd}$$

[69] [KL16] §2.1, Definition 2.1.1.

[70] A proof can be found in P.H. Halmos, *Naive Set Theory*, Springer, 1974.

[71] The class of all well-ordered morphisms isomorphic to a given one is actually a proper class, but here we use the subclass of such morphisms of *minimal rank*, which is a set. For more references one can check [KL16] §2.1, Footnote 4.

Lemma 3.2.5. *The functor $\boldsymbol{W}_\alpha$ is represented by W_α.*[72]

Proof For any $n \in \mathbb{N}$ by the Yoneda Lemma we have that

$$sSet(\Delta^n, W_\alpha) \cong (W_\alpha)_n = \mathbf{W}_\alpha(\Delta^n)$$

Moreover, thanks to Proposition 3.2.3, we can prove that $\mathbf{W}_\alpha$ preserve all limits (i.e. $\mathbf{W}_\alpha(\mathrm{colim}_i X_i) \cong \lim_i \mathbf{W}_\alpha(X_i)$)[73]. By the density theorem we have that any simplicial set X is a colimit over some standard simplices. Therefore, from the aforementioned considerations, we get

$$sSet(X, W_\alpha) \cong \mathbf{W}_\alpha(X)$$

for any simplicial set X.

□

Notation: Any α-small and well-ordered morphism $f : Y \to X$ correspond to an element of $\mathbf{W}_\alpha(X) \cong sSet(X, W_\alpha)$. We write $\ulcorner f \urcorner : X \to W_\alpha$ for the unique map that comes from f. This notation will be clearer later, when we will introduce the map that makes W_α universal with respect to a particular kind of morphisms.

Let us consider $w_\alpha : \widetilde{W_\alpha} \to W_\alpha$ as a representative of the unique isomorphism class in $\mathbf{W}_\alpha(W_\alpha)$ corresponding to $1_{W_\alpha} \in sSet(W_\alpha, W_\alpha)$[74]. Now, given any α-small and well-ordered morphism $f : Y \to X$, we can consider the following commutative square

$$\begin{tikzcd}
\mathbf{W}_\alpha(W_\alpha) \arrow[r, "\sim"] \arrow[d, "\mathbf{W}_\alpha(\ulcorner f \urcorner) = (\ulcorner f \urcorner)^*"'] & sSet(W_\alpha, W_\alpha) \arrow[d, "- \circ \ulcorner f \urcorner"] \\
\mathbf{W}_\alpha(X) \arrow[r, "\sim"] & sSet(X, W_\alpha)
\end{tikzcd}$$

Therefore we can see how we can express f in a unique way as a pullback of the form

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "f"'] & \widetilde{W_\alpha} \arrow[d, "w_\alpha"] \\
X \arrow[r, "\ulcorner f \urcorner"'] & W_\alpha
\end{tikzcd}$$

Therefore we obtain the following proposition and corollary:

[72] [KL16] §2.1, Lemma 2.1.5.

[73] [KL16] §2.1, Lemma 2.1.4.

[74] Explicitly one could prove that $(\widetilde{W_\alpha})_n$ is formed by classes of pairs $(f : Y \to \Delta^n,\ s \in f_n^{-1}(1_{[n]}))$.

Proposition 3.2.6. *The map $w_\alpha : \widetilde{W_\alpha} \to W_\alpha$ is universal for α-small well-ordered morphism in sSet, i.e. any such map can be express in a unique way as a pullback of w_α.*

Corollary 3.2.7. *The map $w_\alpha : \widetilde{W_\alpha} \to W_\alpha$ is weakly universal for α-small morphism in sSet, i.e. any such map can be express, not necessarily in a unique way, as a pullback of w_α.*

Proof It follows directly from the axiom of choice, that allows us to well-order each fiber, and the previous proposition. Since the choice of the well-ordering is not unique we have a weak universality.

□

The idea behind all of this construction is that the contextual category that we want to find, will be required to have Kan Complexes as objects and Kan fibrations as projections. Therefore we want to find a universe that is actually inside this category. In order to do that we reduce to a subobject of $\mathbf{W}(/W_\alpha)$ which is a Kan complex and has a Kan fibration as related map. The need for Kan complexes derives from the Id structure. In fact, in the HoTT interpretation, they are strongly connected to path objects. As we have seen in section 1.5, they "well-behave" when the objects considered are Kan complexes.
Moreover, in [Shu15] Michael Shulman showed how the most relevant features of a model category (for the relative type-theoretic notions) are exactly the fibrant objects and fibrations.

Definition 3.2.8.

- *We define $\boldsymbol{U}_\alpha \subseteq \boldsymbol{W}_\alpha$ (respectively $U_\alpha \subseteq W_\alpha$) as the subojects consinsting of (isomorphism classes of) α-small well-ordered Kan fibration.*
- *We define $p_\alpha : \widetilde{U_\alpha} \to U_\alpha$ as the following pullback:* [75]

$$\begin{tikzcd} \widetilde{U_\alpha} \arrow[r] \arrow[d, "p_\alpha"'] & \widetilde{W_\alpha} \arrow[d, "w_\alpha"] \\ U_\alpha \arrow[r, hook] & W_\alpha \end{tikzcd}$$

[75][KL16] §2.1, Definition 2.1.9.

Lemma 3.2.9. *The morphism $p_\alpha : \widetilde{U_\alpha} \to U_\alpha$ is a Kan fibration.* [76]

Proof Let us consider a commutative diagram of the following form:

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & \widetilde{U_\alpha} \arrow[d, "p_\alpha"] \\ \Delta^n \arrow[r] & U_\alpha \end{tikzcd}$$

The bottom map, by definition of the n-simplices of U_α, correspond to a pullback

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "x"'] & \widetilde{U_\alpha} \arrow[d, "p_\alpha"] \\ \Delta^n \arrow[r, "\ulcorner x \urcorner"'] & U_\alpha \end{tikzcd}$$

where x is a fibration. By the universal property of the pullback the first diagram gives us a commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, "x"] \\ \Delta^n \arrow[r, equal] & \Delta^n \end{tikzcd}$$

Now, since x is a Kan fibration we find a diagonal map for the latter diagram. Thus, composing with h we find the diagonal for the first one, proving that p_α is a Kan fibration. □

Lemma 3.2.10. *Let $f : Y \to X$ be a α-small well-ordered map of simplicial sets. Then:* [77]

$$f \text{ is a Kan fibration} \Leftrightarrow \ulcorner f \urcorner : X \to W_\alpha \text{ factors through } U_\alpha$$

Proof

(⇒) Since f is a Kan fibration, and they are stable under pullback (easy to check using the universal property of pullbacks), for any $x : \Delta^n \to X$ we have that x^*f is a Kan fibration too. Therefore we obtain the following diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "x^*f"'] & Y \arrow[r] \arrow[d, "f"] & \widetilde{W_\alpha} \arrow[d, "w_\alpha"] \\ \Delta^n \arrow[r, "x"'] & X \arrow[r, "\ulcorner f \urcorner"'] & W_\alpha \end{tikzcd}$$

[76] [KL16] §2.1, Lemma 2.1.10.
[77] [KL16] §2.1, Lemma 2.1.11.

where both squares are pullbacks. Therefore we can see that $\ulcorner f \urcorner_n(x) = x^* f \in (U_\alpha)_n$ (seeing x as an element of X_n through the Yoneda Lemma).

($\Leftarrow$) If $\ulcorner f \urcorner$ factors through U_α we can find the following commutative diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] & \widetilde{U_\alpha} \arrow[r] \arrow[d, "p_\alpha"] & \widetilde{W_\alpha} \arrow[d] \\ X \arrow[r, "\ulcorner f \urcorner"'] & U_\alpha \arrow[r, hook] & W_\alpha \end{tikzcd}$$

Since the square on the right and the "composition" square are pullbacks, then even the one on the left is such. Therefore, because p_α is a Kan fibration, even f is a Kan fibration.

□

From the previous proposition regarding $\mathbf{W}_\alpha$ and W_α, and the last Lemma we can deduce the following important corollary:

Corollary 3.2.11. *The functor $\boldsymbol{U}_\alpha$ is represented by U_α. Therefore the Kan fibration $p_\alpha : \widetilde{U_\alpha} \to U_\alpha$ is strictly universal for α-small well-ordered Kan fibrations, and weakly universal for α-small Kan fibrations.* [78]

This shows that $p_\alpha : \widetilde{U_\alpha} \to U_\alpha$ can be chosen as a universe in *sSet*. Since we want to focus on the relative contextual category, the choice of the pullbacks is not important. In fact we know that given a category $\mathbb{C}$ with a universe U and a terminal object, the contextual category associated to it is well-defined (up to canonical isomorphism) just by $\mathbb{C}$ and U[79].

Remark 3.2.12. Let us consider two cardinals $\beta < \alpha$. Obviously, by definition, we have that $U_\beta \subseteq U_\alpha$. Furthermore, since p_β is β small, it is even α-small. Nonetheless it is also a weel-ordered Kan fibration. Therefore it correspond to a pullback diagram of the following kind:

$$\begin{tikzcd} \widetilde{U_\beta} \arrow[r] \arrow[d, "p_\beta"'] & \widetilde{U_\alpha} \arrow[d, "p_\alpha"] \\ U_\beta \arrow[r, hook] & U_\alpha \end{tikzcd}$$

But the enviroment that we want to consider is the category with objects Kan complexes. Thus it is essential to prove that U_α is a Kan complexes.

[78] [KL16] §2.1, Corollary 2.1.11.

[79] [KL16] §1.3, Proposition 1.3.3.

It would trivially follows that even $\widetilde{U_\alpha}$ is such, since the composition of Kan fibrations is still a Kan fibration. In order to achieve this result we need a couple of preliminary lemmas, that will be useful even in the following sections.

Lemma 3.2.13. *For any cofibration in sSet (i.e. a monomorphism) $v : Y \to Y'$ and any trivial Kan fibration $p : X \to Y$, then:* [80]

(i) There exist a trivial fibration $p' : X' \to Y'$ and a pullback

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r, "v"'] & Y' \end{tikzcd}$$

(ii) Moreover, if p is α-small, then p' can be chosen to be α-small as well.

Proof

(i) First of all we note that *sSet* is LCCC (by Lemma 2.4.9), so we have the following adjoint pairs

$$\Sigma_v \dashv v^* \dashv \Pi_v : sSet/Y \to sSet/Y'$$

where, in particular, Σ_v is the composition with v and v^* is the pullback action. Since v is a monomorphism, for any $g : W \to Y$ the following diagram is a pullback:

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, equal] & Y \arrow[d, "v"] \\ W \arrow[r, "vg"'] & Y' \end{tikzcd}$$

Therefore $v^*\Sigma_v \cong Id_{sSet/Y}$. Moreover, by the transpose argument[81], it follows that $v^*\Pi_v \cong Id_{sSet/Y}$ as well. By the characterization of acyclic fibration in a model category[82] we have that if a left adjoint preserves cofibrations, then its right adjoint preserves acyclic fibrations. In our setting this means that, since v^* preserves cofibrations (because monormorphisms are stable under pullback), then Π_v preserves trivial fibrations. Thus we can define $p' := \Pi_v(p)$.

[80][Cis16] §2, Proposition 2.17 and [KL16] §2.2, Lemma 2.2.4 Part 3.
[81]Proposition A.2.3
[82]Proposition B.2.1.

(ii) For any n-simplices $y' : \Delta^n \to Y'$ of W' we have that:

$$(p')^{-1}(y') \cong sSet/Y'(y', \Pi_v p) \cong sSet/Y(v^*y', p)$$

Since pullbacks preserve monomorphisms, we have that $v^*y' \subseteq \Delta^n$. By the characterization of non-degenerate simplices for a simplicial set[83] we can see that Δ^n has only finitely many of them, and therefore the same holds for v^*y'. Thus $sSet/Y(v^*y', p)$ injects into a finite product of fibers of p (one for each non-degenerate simplices of v^*y'). Since α is regular, then $\mid (p')^{-1}(y') \mid < \alpha$.

□

Lemma 3.2.14. *For any minimal fibration $p : X \to Y$ and any anodyne extension $v : Y \to Y'$ there exist a minimal fibration $p' : X' \to Y'$ and a pullback:* [84]

$$\begin{tikzcd} X \arrow[r, "v'"] \arrow[d, "p"'] & X' \arrow[d, "p'"] \\ Y \arrow[r, "v"'] & Y' \end{tikzcd}$$

Proof By the model structure on *sSet* and the Quillen's Lemma[85] we can find a commutative diagram

$$\begin{tikzcd} X \arrow[r, hook, "u"] \arrow[d, "p"'] & X'' \arrow[r, "q"] \arrow[dr, "p''"'] & X' \arrow[d, "p'"] \\ Y \arrow[rr, "v"'] & & Y' \end{tikzcd}$$

where $p''u$ is a fibration-acyclic cofibration decomposition in the model structure (i.e. p'' is a Kan fibration and u is an anodyne extension) and $p'q$ is the decomposition given by Quillen's Lemma (i.e. q a trivial fibration and p' a minimal fibration). Let us consider the following pullback square:

$$\begin{tikzcd} Y \times_{Y'} X' \arrow[r, "\eta"] \arrow[d, "\psi"'] & X' \arrow[d, "p'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

By Remark 1.5.8 we have that ψ is a minimal fibration. By the universal property we can find a map $\varphi : X \to Y \times_{Y'} X'$ such that, in particular,

[83]Proposition 1.1.4
[84][Cis16] §2, Proposition 2.20.
[85]Theorem 1.5.11

$qu = \eta\varphi$. Since *sSet* is right proper[86], then η is a weak equivalence. Therefore, by the 2-out-of-3 condition in a model category, φ is a weak equivalence (qu is such because is a composition of weak equivalences). But $\psi\varphi = p$ with ψ and p minimal fibrations, thus φ is an isomorphism. Therefore the first square drawn is a pullback.

□

Now we are ready to prove the main theorem.

Theorem 3.2.15. *U_α is a Kan complex.* [87]

Proof For some $n \geq 0$ and $0 \leq k \leq n$, let us consider a diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\ulcorner q \urcorner"] \arrow[d, hook] & U_\alpha \\ \Delta^n \arrow[ur, dotted, "\exists ? \ulcorner q' \urcorner"'] & \end{tikzcd}$$

By the characterization of morphisms into U_α, to find an extension $\ulcorner q' \urcorner$ of $\ulcorner q \urcorner$ is equivalent to find, for a α-small well-ordered Kan fibration $q : Y \to \Lambda^n_k$, a pullback of the following form

$$\begin{tikzcd} Y \arrow[r, dotted] \arrow[d, "q"'] & Y' \arrow[d, dotted, "q'"] \\ \Lambda^n_k \arrow[r, hook] & \Delta^n \end{tikzcd}$$

with q' a α-small well-ordered Kan fibration. In fact this would correspond to two consecutive pullbacks

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "q"'] & Y' \arrow[r] \arrow[d, "q'"'] & \widetilde{U_\alpha} \arrow[d, "p_\alpha"] \\ \Lambda^n_k \arrow[r, hook] \arrow[rr, bend right, "\ulcorner q \urcorner"'] & \Delta^n \arrow[r, "\ulcorner q' \urcorner"] & U_\alpha \end{tikzcd}$$

Combining the two previous Lemmas and using Quillen's Lemma, we can prove that we can find a pullback of the form required with q' a Kan fibra-

[86] Theorem 1.5.12

[87] [KL16] §2.2., Theorem 2.2.1.

tion.[88] So let us denote the diagram that we can find as

$$\begin{tikzcd}
Y \arrow[r, "s"] \arrow[d, "q_t"] \arrow[dd, bend right, "q"'] & Y' \arrow[d, "q'_t"'] \arrow[dd, bend left, "q'"] \\
W \arrow[r, hook, "u"] \arrow[d, "q_m"] & W' \arrow[d, "q'_m"'] \\
\Lambda^n_k \arrow[r, hook, "v"'] & \Delta^n
\end{tikzcd}$$

where both squares are pullback, $q_m q_t$ is a Quillen's decomposition of q (i.e. q_m a minimal fibration and q_t a trivial one), and we define $q' := q'_m q'_t$. First of all we notice that, since q is α-small, then both q_t and q_m are such, in fact:

- For any simplices w of W we have $q_t^{-1}(w) \subseteq (q_m q_t)^{-1}(q_m(w)) = q^{-1}(q_m(w))$. Therefore $\mid q_t^{-1}(w) \mid < \mid q^{-1}(q_m(w)) \mid$ and the latter one is strictly less than α since q is α-small.

- Since q_t is a trivial fibration, in particular it is surjective. Therefore, for any $y \in (\Lambda^n_k)_r$, every element of $q_m^{-1}(y)$ is of the form $q_t(x)$ for a $x \in X_r$ such that $q(x) = y$. Thus $q_m^{-1}(y)$ is a quotient of $q^{-1}(y)$, and so $\mid q_m^{-1}(y) \mid < \mid q^{-1}(y) \mid < \alpha$.

We recall that, since we just proved that q_t is α-small, we can choose q'_t to be such as well. Moreovere a lemma in [KL16][89] guarantees that in this particular case even q'_m can be choosen to be α-small too.
Let us now consider a $y' \in Y'_r$, then we have that

$$(q')^{-1}(y') = (q'_m \circ q'_t)^{-1}(y') = \bigcup_{w' \in (q'_m)^{-1}(y')} (q'_t)^{-1}(w')$$

Since both q'_m and q'_t are α-small and α is regular, $\mid (q')^{-1}(y') \mid < \alpha$. Therefore q' is α-small.

We have left to prove that q' is even well-oredered. But by the Axiom of Choice (AC) we can extend the well-ordering of q to q', hence the thesis.

□

[88] We recall that pullbacks preserves monomorphisms and in the model structure consider on *sSet* cofibrations are exactly the monomorphisms.
[89] [KL16] §2.2, Lemma 2.2.3.

3.3 The Univalence Axiom

The notions outlined in the following section can be found in [KL16] §3.1-3.2.

We start giving the definition of the Type-theoretic Univalence Axiom. First of all we recall that, under the idea of "type-as-proposition", the Π- and Σ-types give us an interpretation of the logic quantifiers $\forall$ and $\exists$ respectively. With them we can recover the type of functions between two types $(A \to B)$ and the product of two types $(A \times B)$. The latter one, always under "type-as-proposition", can be understood as the logical conjunction $\wedge$. This note will make the next definitions clearer.

Definition 3.3.1. *Let $f : A \to B$ be a function in some context Γ (i.e. $\Gamma \vdash f : A \to B$ in some dependent type theory), we define:*

- *a* ***left homotopy inverse for*** *f some g derived as $\Gamma \vdash g : B \to A$ with "a homotopy $gf \simeq 1_A$", i.e. we define more formally the* ***type of left homotopy inverses of*** *f as*

$$\Gamma \vdash \textsf{LInv(f)} := \Sigma_{g:B\to A}\Pi_{x:A}\ \textsf{Id}_A(gf(x),\, x) \quad \textit{Type}$$

- *the* ***type of right homotopy inverses of*** *f analogously*

$$\Gamma \vdash \textsf{RInv(f)} := \Sigma_{g:B\to A}\Pi_{x:A}\ \textsf{Id}_A(fg(x),\, x) \quad \textit{Type}$$

- *We say that f is a* ***homotopy isomorphism*** *(or simply a* ***h-isomorphism****) if it has both a left and a right homotopy inverses, i.e. if and only if the following type is inhabitated:*

$$\Gamma \vdash \textsf{isHIso(f)} := \textsf{LInv(f)} \times \textsf{RInv(f)} \quad \textit{Type}$$

- *For any types A and B, we can finally define the* ***type of h-isomorphisms from*** *A* ***to*** *B as follows*

$$\Gamma \vdash \textsf{HIso}(A,\, B) := \Sigma_{f:A\to B}\textsf{isHIso(f)}$$

Remark 3.3.2. Here we used the notion of homotopy isomorphism and not the one of homotopy equivalence. The difference is that we do not require that the left and right inverse have to be the same map. When we will state the Univalence Axiom in the simplicial setting, it will be important to remember this particular choice.

For any type B the canonical identity $1_B : B \to B$ is obviously an h-isomophism. Let us now consider a type A and a family of types over A

$$x : A \vdash B(x) \quad Type$$

By the Id-elimination rule we can deduce

$$x : A,\, y : A,\, u : \mathsf{Id}_A(x,\, y) \vdash w_{x,\,y,\,u} : \mathsf{HIso}(B(x),\, B(y))$$

or equivalently (through the $\to$-rules)

$$x : A,\, y : A \vdash w_{x,y} : \mathsf{Id}_A(x,\, y) \to \mathsf{HIso}(B(x),\, B(y))$$

The idea is that given any identity between two terms in A we obtain an h-isomophism between $B(x)$ and $B(y)$. Moreover this construction gives rise to a "map" from the type of identities between the two terms and the one of h-isomoprhisms between the respective types in the family.

Definition 3.3.3. *A family $B(x)$ of types over a type A is called **univalent** if for each $x,\, y : A$ we have that $w_{x,y}$ (using the notation given above) is an h-isomoprhism, i.e.*

$$\vdash \mathit{isUnivalent}(x : A.B(x)) := \Pi_{x,\,y:A}\mathit{isHIso}(w_{x,\,y})$$

We recall that we work in a type theory with universes, i.e. a way to classify types inside a type. In particular whenever we have a universe U we have a canonical family El over the type U as the following rules briefly describe:

$$\frac{}{\vdash U \quad Type} \qquad\qquad \frac{}{x : U \vdash El(x)\ Type}$$

Therefore we can give the definiton of the Univalence Axiom as:

Axiom 3.3.4. ***(Univalence Axiom)*** *Given a type-theoretic universe U, the canonical family El of types over U is univalent.*

Informally we can interpret this axiom in the following way: equalities in the universe correspond to equivalences between types. In particular it states that the language can never distinguish between equivalent types, since all concepts has to respect propositional equality.

Now we want to define the Univalence Axiom in the simplicial setting.

Notation: We know that *sSet* is LCCC (as a particular case of $\widehat{\mathbb{C}}$ with $\mathbb{C}$ a small category). Thus, for any simplicial set B and two objects E_1 and

E_2 in the slice category over B, there exist an exponential in *sSet*$/B$ from E_1 to E_2, we denote it as

$$\rho_{E_1,\,E_2} : \mathbf{Hom}_B(E_1,\, E_2) \longrightarrow B$$

It can be proved that this particular morphism is a Kan fibration. [90]

Remark 3.3.5. Any map $g : X \to \mathbf{Hom}_B(E_1,\, E_2)$ correspond in a unique way, through the adjunction, to a map $\widehat{g} : X \times_{sSet/B} E_1 \to E_2$. Let us define $f := \rho_{E_1,\,E_2} \circ g$. Then $X \times_{sSet/B} E_1 \cong f^* E_1$. Therefore the map g corresponds, in a natural bijective way, to a pair of maps

$$(f : X \to B,\, f^*E_1 \to f^*E_2)$$

where the second map is induced by the universal property of the pullback by $\widehat{g}$ and $f^*E_1 \to X$ (defined in the relative pullback). Finally by Yoneda we get that:

$$\mathbf{Hom}_B(E_1,\, E_2)_n \cong \{(\Delta^n \xrightarrow{b} B,\, b^*E_1 \xrightarrow{u} b^*E_2)\}$$

The aim is to define the same idea as the "equivalences" in Type Theory. In order to do that we want to specify a particular subobject $\mathbf{Eq}_B(E_1,\, E_2) \subseteq \mathbf{Hom}_B(E_1,\, E_2)$. The idea will be to use the description given in the previous remark. In order to have a good definition we need a couple of lemmas:

Lemma 3.3.6. *Let $f : E_1 \to E_2$ be a weak equivalence of simplicial sets. Then for any map $g : B' \to B$ in sSet, the morphism $g^*E_1 \to g^*E_2$ induced by f is again a weak equivalence.* [91]

Lemma 3.3.7. *Let $f : E_1 \to E_2$ be a morphism in sSet$/B$. If for any n-simplex $b : \Delta^n \to B$ of B the map $f_b : b^*E_1 \to b^*E_2$ induced on the pullbacks is a weak equivalence of simplicial sets, then f itself is a weak equivalence.*[92]

Now we can give the definition we were looking for:

Definition 3.3.8. *We define the simplicial set $\boldsymbol{Eq}_B(E_1,\, E_2)$ as the subobject of $\boldsymbol{Hom}_B(E_1,\, E_2)$ determined by the n-simplices:*

$$\boldsymbol{Eq}_B(E_1,\, E_2)_n := \{(\Delta^n \xrightarrow{b} B,\, b^*E_1 \xrightarrow{w} b^*E_2) \mid w \textit{ is a weak equivalence}\}$$

[90] [KL16] §3.2, Lemma 3.2.2.
[91] [KL16] §3.2, Lemma 3.2.3.
[92] [KL16] §3.2, Lemma 3.2.4.

Let us see why the previous lemmas imply that the subobject above is actually well define. Let us set $X := \mathbf{Eq}_B(E_1, E_2) \subseteq \mathbf{Hom}_B(E_1, E_2) =: Y$. We need to verify that, for any morphism $f : [n] \to [m]$ in Δ, $Y(f)_{|X_m}(X_m) \subseteq X_n$, so that we can define $X(f)$ as the restriction of $Y(f)$. Let us denote $\widehat{f} : \Delta^n \to \Delta^m$, then $Y(f) : Y_m \to Y_n$ is defined as

$$\Delta^m \xrightarrow{y} Y \quad \longmapsto \quad \Delta^n \xrightarrow{\widehat{f}} \Delta^m \xrightarrow{y} Y$$

more precisely

$$(\Delta^n \xrightarrow{b} B,\, b^*E_1 \xrightarrow{w} b^*E_2) \quad \longmapsto (b \circ \widehat{f},\, (b \circ \widehat{f})^*E_1 \xrightarrow{\widehat{w}} (b \circ \widehat{f})^*E_2)$$

If we consider the image of X_m (i.e. setting w a weak equivalence) then, by Lemma 3.3.6, even $\widehat{w}$ is such, since it is the map induced by w on

$$\widehat{f}^*(b^*E_1) \cong (b \circ \widehat{f})^*E_1 \xrightarrow{\widehat{w}} (b \circ \widehat{f})^*E_2 \cong \widehat{f}^*(b^*E_2)$$

Moreover Lemma 3.3.7 implies that a map $X \to \mathbf{Hom}_B(E_1, E_2)$ is a weak equivalence if and only if it factors through $\mathbf{Eq}_B(E_1, E_2)$ [93]. This means that morphism $X \to \mathbf{Eq}_B(E_1, E_2)$ corresponds to pairs of map

$$(f : X \to B,\, w : f^*E_1 \to f^*E_2)$$

where w is a weak equivalence. Moreover we can even prove that $\mathbf{Eq}_B(E_1, E_2) \to B$ is a Kan fibration [94].

Now we have all the tools we need to define univalence in the enviroment of simplicial sets. Let $p : E \to B$ be a Kan fibration. We denote π_1 and π_2 the projections of $B \times B$. Then, since Kan fibrations are stable under pullback, we have that both maps $\pi_1^*E \to B \times B$ and $\pi_2^*E \to B \times B$ are Kan fibrations as well. Therefore we can define

$$\mathbf{Eq}(E) := \mathbf{Eq}_{B\times B}(\pi_1^*E,\, \pi_2^*E)$$

$$\mathbf{Eq}(E)_n = \{(b_1,\, b_2,\, b_1^*E \xrightarrow{w} b_2^*E)\}$$

By the observation given above we have that any map $X \to \mathbf{Eq}(E)$ corresponds to

$$f_1,\, f_2 : X \to B \text{ and a weak equivalence } u : f_1^*E \to f_2^*E$$

[93][KL16] §3.2, Corollary 3.2.6.
[94][KL16] §3.2, Corollary 3.2.8.

We define $\delta_B : B \to \mathbf{Eq}(E)$ as the unique map corresponding to $f_1 \equiv f_2 := 1_B$ and $u := 1_E$. More precisely we have that $\delta_E(b) = (b, b, 1_{E_b})$ (where $E_b \equiv b^*E$). This morphism has two retractions, for $i = 1, 2$:

$$r_i : \mathbf{Eq}(E) \to B \times B \xrightarrow{\pi_i} B \quad (b_1, b_2, w) \mapsto b_i$$

Remark 3.3.9. If B is a Kan complex, then both projections π_i's are Kan fibrations, as pullback of Kan fibrations

$$\begin{tikzcd} B \times B \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & B \arrow[d] \\ B \arrow[r] & \Delta^0 \end{tikzcd}$$

Therefore, if this is the case, since the first map displayed in the definition of r_i is always a Kan fibration, then even r_i is a Kan fibration. This will be useful to prove that the univalence axiom holds in the simplicial model.

Definition 3.3.10. *A Kan fibration $p : E \to B$ is* ***univalent*** *if δ_E is a weak equivalence.* [95]

Remark 3.3.11. Since δ_E has a retraction, then it is a monomorphism. In particular, for the model structure considered on *sSet*, is a cofibration. Therefore we have that

$$p \text{ is univalent} \Leftrightarrow \delta_E \text{ is an anodyne extension}$$

since anodyne extensions represents acyclic cofibrations in *sSet*. In the vocabulary of model categories we would say that: p is univalent if and only if $\mathbf{Eq}(E)$ is a path object of B.

One could prove that the type-theoretic definition of univalence given at the start of this section, iis equivalent to the one just stated in the simplicial enviroment. A meticolous reader can find a complete explanation in [KL16] §3.3.

[95] [KL16] §3.2, Definition 3.2.10.

3.4 Simplicial Univalence

We start stating the theorem that allows us to talk about a model of Martin-Löf Type Theory inside simplicial sets:

Theorem 3.4.1. *Let α be an inaccesible cardinal*[96]*, then U_α carries Π-, Σ-, Id- and other type-theoretic structures.* [97]
Moreover, if $\beta < \alpha$ is also inaccessible, then U_β gives an internal universe in U_α closed under all the constructors. [98]
Again, if $\beta < \alpha$ are two inaccessible cardinals, then there exists a model of Martin-Löf Type Theory in $sSet_{U_\alpha}$ with a universe (given by U_β) closed under the logical constructors. [99]

Finally we will prove that in this particular model the univalence axiom holds, i.e.:

Theorem 3.4.2. *The Kan fibration $p_\alpha : \widetilde{U_\alpha} \to U_\alpha$ is univalent.* [100]

Proof Since U_α is a Kan complex (Theorem 3.2.15) $r_2 : \mathbf{Eq}(\widetilde{U_\alpha}) \to U_\alpha$ is a Kan fibration (Remark 3.3.9). So, if we show that r_2 is a trivial fibration[101], then $\delta_{\widetilde{U_\alpha}}$ would be a weak equivalence too (by the 2-out-of-3 property of model categories, since $r_2 \circ \delta_{\widetilde{U_\alpha}} = 1_{U_\alpha}$ and r_2 would be such). Thus for any cofibration $i : A \hookrightarrow B$ and any commutative diagram

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, hook, "i"'] & \mathbf{Eq}(\widetilde{U_\alpha}) \arrow[d, "r_2"] \\ B \arrow[r, "\ulcorner\varphi\urcorner"'] & U_\alpha \end{tikzcd} \tag{4}$$

we have to find a filler $B \dashrightarrow \mathbf{Eq}(\widetilde{U_\alpha})$. By the characterization we gave in the previous section, such a map would correspond to two maps $\ulcorner\bar{p_1}\urcorner, \ulcorner\bar{p_2}\urcorner : B \to U_\alpha$ and a weak equivalence $\bar{w} : \ulcorner\bar{p_1}\urcorner^* B \to \ulcorner\bar{p_2}\urcorner^* B$. Moreover, by the

[96] I.e. infinite, regular and strong limit; *strongly inaccesible* in some literature.
[97] All the logical contructors that we refer to in all the following theorems can be found in [KL16] Section A.2.
[98] This is possible thanks to remark 3.2.12.
[99] [KL16] §2.3, Theorem 2.3.4 and Corollary 2.3.5.
[100] The proof that we will outline puts together the one of [KL16] §3.4, Theorem 3.4.1 and the one of [Cis16] §2, Proposition 2.18.
[101] In the case of *sSet* it would be an acyclic fibration.

description of U_α this data is equivalent to two pullbacks, for $i = 1, 2$,

$$\begin{tikzcd}
E_i \arrow[r] \arrow[d, "\bar{p_i}"'] & \widetilde{U_\alpha} \arrow[d, "p_\alpha"] \\
B \arrow[r, "\ulcorner \bar{p_i} \urcorner"'] & U_\alpha
\end{tikzcd}$$

with $\bar{p_i}$ two α-small well-ordered Kan fibration, together with a weak equivalence $\bar{w}$.

So we can see that the commutative diagram (4) correspond to the following commutative diagram:

$$\begin{tikzcd}
E_1 \arrow[dr, "w"] \arrow[ddr, "p_1"'] & & \\
& E_2 \arrow[r, "v_2"] \arrow[d, "p_2"] & \bar{E}_2 \arrow[d, "\bar{p}_2"] \\
& A \arrow[r, hook] & B
\end{tikzcd}$$

where the triangle on the left is induced by ψ through the characterization of maps into $\mathbf{Eq}(\widetilde{U_\alpha})$, the map φ is given by $\ulcorner \varphi \urcorner$ and the square is a pullback by the commutativity of (4). Moreover p_1 and p_2 are α-small well-ordered Kan fibrations over A, w a weak equivalence between them and $\bar{E}_2$ an extension of E_2 to an α-small well-ordered Kan fibration over B. Therefore to find the filler required is the same as to complete the diagram to

$$\begin{tikzcd}
E_1 \arrow[rr, dashed, "v_1"] \arrow[dr] \arrow[ddr] & & \bar{E}_1 \arrow[dr, dashed, "\bar{w}"] \arrow[ddr, dashed, "\bar{p}_1"'] & \\
& E_2 \arrow[rr] \arrow[d] & & \bar{E}_2 \arrow[d] \\
& A \arrow[rr, hook] & & B
\end{tikzcd}$$

where $\bar{w}$ is a weak equivalence, $\bar{p_1}$ is a α-small well-ordered Kan fibration and the square on the back is a pullback. As in all other proofs regarding this kind of Kan fibrations, it is enough to find a α-small Kan fibration, then using the Axiom of Choice the well-ordering will follow.
By Quillen's Lemma (Theorem 1.5.11), we can consider a minimal fibration-

trivial fibration factorization of $\bar{p_2}$:

$$\begin{tikzcd}
\bar{E}_2 \arrow[rr, "\bar{p_2}"] \arrow[dr, "\bar{r_2}"'] & & B \\
& \bar{S} \arrow[ur, "\bar{q}"'] &
\end{tikzcd}$$

We define S as the pullback of $\bar{q}$ along i

$$\begin{tikzcd}
S \arrow[r, "k"] \arrow[d, "q"'] & \bar{S} \arrow[d, "\bar{q}"] \\
A \arrow[r, hook, "i"'] & B
\end{tikzcd} \tag{5}$$

Since $\bar{q}$ is a minimal fibration, then q is such too (Remark 1.5.8). Let us now consider the commutative square given by the universal property of the pullback (5):

$$\begin{tikzcd}
E_2 \arrow[r, "v_2"] \arrow[d, "r_2:="'] & \bar{E}_2 \arrow[d, "\bar{r_2}"] \\
S \arrow[r, "k"'] & \bar{S}
\end{tikzcd}$$

This is a pullback because (5) and the composition diagram (see below) are pullbacks

$$\begin{tikzcd}
E_2 \arrow[r, "v_2"] \arrow[d, "p_2=qr_2"'] & \bar{E}_2 \arrow[d, "\bar{p_2}=\bar{q}\bar{r_2}"] \\
A \arrow[r, hook, "i"'] & B
\end{tikzcd}$$

Moreovere, since $\bar{r_2}$ is a trivial fibration, then r_2 is such as well.

CLAIM: $r_1 := r_2 \circ w$ is a trivial fibration.

Let us consider a strong deformation retract $u : T \hookrightarrow E_1$ such that $p_1 \circ u$ is a minimal fibration (it is possible by Theorem 1.5.10). Then we have a commutative diagram:

$$\begin{tikzcd}
T \arrow[rr, "r_2wu"] \arrow[dr, "p_1u"'] & & S \arrow[dl, "q"] \\
& A &
\end{tikzcd}$$

We have that r_2wu is a weak equivalence, since it is a composition of such, between minimal fibrations. Therefore, by Theorem 1.5.9, is an isomorphism.

Therefore $r_1 = r_2 w$ is a retract of u, and so (through the same argument used in the proof of Quillen's Lemma) r_1 is a trivial fibration.

We obtain the following diagram, with both squares pullbacks:

$$\begin{tikzcd}
E_1 \arrow[r] \arrow[dr, "r_1"'] & E_2 \arrow[r, "v_2"] \arrow[d, "r_2"] & \bar{E_2} \arrow[d, "\bar{r_2}"] \\
 & S \arrow[r, "k"'] \arrow[d, "q"'] & \bar{S} \arrow[d, "\bar{q}"] \\
 & A \arrow[r, "i"'] & B
\end{tikzcd}$$

with r_1 a trivial fibration and k a cofibration. Moreover, since $p_1 = qr_1$ is α-small, then r_1 is the same[102]. Therefore, by Lemma 3.2.13, there exists a pullback of the following form

$$\begin{tikzcd}
E_1 \arrow[r, "v_1"] \arrow[d, "r_1"'] & \bar{E_1} \arrow[d, "\bar{r_1}"] \\
S \arrow[r, "k"'] & \bar{S}
\end{tikzcd} \tag{6}$$

where $\bar{r_1}$ is an α-small trivial fibration as well. In particular v_1 is a monomorphism, since k is such and these kind of maps are stable under pullback. Thus we can find a diagonal filler $\bar{w} : \bar{E_1} \to \bar{E_2}$ of the following commutative diagram:

$$\begin{tikzcd}
E_1 \arrow[r, "w"] \arrow[d, "v_1"'] & E_2 \arrow[r, "v_2"] & \bar{E_2} \arrow[d, "\bar{r_2}"] \\
\bar{E_1} \arrow[urr, dotted, "\bar{w}"] \arrow[rr, "\bar{r_1}"'] & & S
\end{tikzcd}$$

Since $\bar{r_1}$ and $\bar{r_2}$ are weak equivalences, then by the 2-out-of-3 property of model categories, even $\bar{w}$ is a weak equivalence. Composing the pullback (6) with the other pullback (5) we obtain a new pullback and therefore a diagram of the form required:

$$\begin{tikzcd}
E_1 \arrow[rr, "v_1"] \arrow[dr, "w"] \arrow[ddr, "p_1"'] & & \bar{E_1} \arrow[dr, "\bar{w}"] \arrow[ddr, "\bar{p_1}"'] & \\
 & E_2 \arrow[rr, "v_2"] \arrow[d, "p_2"'] & & \bar{E_2} \arrow[d, "\bar{p_2}"] \\
 & A \arrow[rr, hook, "i"'] & & B
\end{tikzcd}$$

---

102 Same argument used in Theorem 3.2.15.

We have left to prove that $\bar{p_1}$ is actually α-small. But, since $\bar{p_2} = \bar{q}\bar{r_2}$ is α-small and $\bar{r_1}$ is surjective (since it is a fibration), then $\bar{q}$ is α-small too. Therefore $\bar{p_1} = \bar{q}\bar{r_1}$ is α-small, since it is the composition of α-small morphisms with α regular. [103]

□

Finally, putting together all the results of the last sections, we obtain the theorem that gives the name to this master thesis:

Theorem 3.4.3. *Let $\beta < \alpha$ be inaccesisible cardinals. Then there is a model of Martin-Löf Type Theory in sSet_{U_α} with a universe (given by U_β) closed under the logical constructors and satisfying the Univalence Axiom.*
Therefore, the Univalence Axiom is consistent with Martin-Löf Type Theory, assuming the existence of two inaccessible cardinals.

[103]Same argument used in Theorem 3.2.15.

A Yoneda Lemma

A.1 The Statement

In this appendix we will recall the Yoneda Lemma together with some important consequences.

Let $\mathbb{C}$ be a locally small category. We write $\mathbb{C}(A, X)$ for the set of morphism in $\mathbb{C}$ between the two objects A and X of $\mathbb{C}$. With **Cat** we mean the category of categories.
For any object A of $\mathbb{C}$ we denote with $h_A : \mathbb{C} \to Set$ the representable functor defined as follow

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] & \mathbb{C}(A, X) \arrow[d, "f\circ -"] \\ Y \arrow[r, mapsto] & \mathbb{C}(A, Y) \end{tikzcd}$$

On the other hand we will write $h^A : \mathbb{C}^{op} \to Set$ for the contravariant functor represented by A, defined as $X \mapsto \mathbb{C}(A, X)$.

Lemma A.1.1. ***(Yoneda)***

- *For any covariant functor $F : \mathbb{C} \to Set$ there is a bijection*

  $$n.t.(h_A, F) \cong F(A)$$

  where the first set is the set of all natural transformations between the two functors. Moreover the bijection is natural both in A and in F.

- *For any contravariant functor $G : \mathbb{C}^{op} \to Set$ there is a bijection*

  $$n.t.(h^A, G) \cong G(A)$$

  Moreover the bijection is natural both in A and in G.

Now let us introduce the so called *Yoneda Embedding*. We will denote it as $Y' : \mathbb{C} \to \mathbf{Cat}(\mathbb{C}, Set)$ and it is defined in the following way:

$$\begin{tikzcd} C \arrow[r, mapsto] \arrow[d, "f"'] & h_C \\ C' \arrow[r, mapsto] & h_{C'} \arrow[u, "-\circ f"'] \end{tikzcd}$$

Corollary A.1.2. *The Yoneda embedding is full and faithful. Thus in particular the object that represents a representable functor is unique up to isomorphism.*

Proof The first part is actually equivalent to the Yoneda Lemma itself. Because Y' is full and faithful if and only if for any $C, C' \in \mathrm{Ob}(\mathbb{C})$ we have

$$Y' : \mathbb{C}(C, C') \xrightarrow{\sim} \text{n.t.}(h_{C'}, h_C)$$

But by definition $\mathbb{C}(C, C') = h_C(C')$, thus we get the equivalence.
To show the second part we recall that any functor which is full and faithful reflects isomorphism. Therefore it is easy to see the uniqueness condition (up to isomoprhism).

$\square$

Obviously the same corollary stated in the contravariant setting is still true. In that case we would have a slightly different "Yoneda Embedding", which we will denote $Y : \mathbb{C} \to \widehat{\mathbb{C}}$ (where $\widehat{\mathbb{C}}$ stands for the category of contravariant functors from $\mathbb{C}$ to *Set*):

$$\begin{tikzcd} C \arrow[r, mapsto] \arrow[d, "f"'] & h^C \\ C' \arrow[r, mapsto] & h^{C'} \arrow[u, "f\circ-"'] \end{tikzcd}$$

A.2 Important Consequences

Proposition A.2.1. *Let $\mathbb{C}$ be a small category with terminal object $*$. Then we have that*

$$colim_{\mathbb{C}} Y := colim_{X \in Ob(\mathbb{C})} Y(X) \cong Y(*)$$

Proof Let us denote $\{*\} : \mathbb{C}^{op} \to Set$ the constant functor to the singleton set. First of all we can see straight away that $Y(*) = \{*\}$, since for any $X \in \mathrm{Ob}(\mathbb{C})$ we have that $Y(*)(X) = h^*(X) = \mathbb{C}(X, *)$ which is a singleton by the definition of terminal object.
Moreover for any $F : \mathbb{C}^{op} \to Set \in \mathrm{Ob}(\widehat{\mathbb{C}})$ we have that exist a unique natural transformation $\xi : F \dot{\to} \{*\}$ defined componentwise as the constant map from $F(X)$ to the singleton set. Thus $\{*\}$ is the terminal object in $\widehat{\mathbb{C}}$. Let us denote ξ^X the unique natural transformation that exists from $Y(X)$ to $\{*\}$. So for any $f : X \to X'$ morphism in $\mathbb{C}$ we have the following commutative diagram

$$\begin{tikzcd} Y(X) \arrow[rr, "f\circ -"] \arrow[dr, "\xi^X"'] & & Y(X') \arrow[dl, "\xi^{X'}"] \\ & \{*\} & \end{tikzcd}$$

Now we have left to prove that $\{*\}$ satisfies the universal property of the colimit. In order to do that we consider a functor $S \in \mathrm{Ob}(\widehat{\mathbb{C}})$ together with natural trasformations $s^X : Y(X) \dot{\to} S$ for any object X in $\mathbb{C}$ such that for any morphism $f : X' \to X \quad s^X \circ (f \circ -) = s^{X'}$.

WTS: $\exists!\, \eta : \{*\} \dot{\to} S$ such that $\forall X \quad \eta \circ \xi^X = s^X$

($\exists$) For any object X we define $\eta_X : \{*\} \to S(X)$ as the map sending the unique element of the singleton to $s^X_X(Id_X)$. We need to prove that η is actually a natural transformation, i.e. for any $f : X' \to X$ we need to prove that the following diagram commutes

$$\begin{tikzcd} \{*\} \arrow[r, "\eta_X"] \arrow[d, equal] & S(X) \arrow[d, "S(f)"] \\ \{*\} \arrow[r, "\eta_{X'}"'] & S(X') \end{tikzcd}$$

Therefore we have to prove that $S(f)(s^X_X(Id_X)) = s^{X'}_{X'}(Id_{X'})$. But since s is a natural transformation we have that $S(f)(s^X_X(Id_X)) = s^X_{X'}(f)$. Moreover $s^X_{X'}(f) = (s^X \circ (f \circ -))_{X'}(Id_X)$ which is equal to $s^{X'}_{X'}(Id_{X'})$ by the way we have choosen the s^X's.
Nevertheless for any object X the equality $\eta \circ \xi^X = s^X$ holds. Explicity for any $f : X' \to X$ we know that $s^X_{X'}(f) = s^{X'}_{X'}(Id_{X'}) = \eta_{X'} \circ \xi^X_{X'}(f)$. Thus we get the existence of the natural transformation that we wanted.

(!) The equality stated above gives us the uniqueness, since any η has to be such that $\eta_X(*) = \eta_X \xi^X_X(Id_X) = s^X_X(Id_X)$.

$\square$

Theorem A.2.2. ***(Density Theorem)*** *Any functor $F : \mathbb{C}^{op} \to Set$ can be represented (in a canonical way) as a colimit of a diagram of representale functors h^C for objects C in $\mathbb{C}$.*[104]

Proof First of all we have to define the category for the diagram (for the colimit). We define the so-called "category of elements of F"[105] as $el(F)$:

- The objects are the couples $(C,\, x)$ where C is an object of $\mathbb{C}$ and $x \in F(C)$.
- We define a morphism from $(C,\, x)$ to $(C',\, x')$ as a morphism $f : C \to C'$ in $\mathbb{C}$ such that $F(f)(x') = x$. The composition is defined in the obvious way.

Let us introduce the diagram $M : el(F) \to \widehat{\mathbb{C}}$ defined as:

$$\begin{tikzcd}
(C,\, x) \arrow[r, mapsto] \arrow[d, "f"'] & h^C \arrow[d, "f\circ -"] \\
(C',\, x') \arrow[r, mapsto] & h^{C'}
\end{tikzcd}$$

CLAIM: $F \cong \mathrm{colim} M$

For any element $(C,\, x)$ we find a natural tranformation $y^x : h^C \dot{\to} F$, namely the one corresponding to x through the Yoneda isomorphism. It is easy to prove that we obtain a cocone of the diagram with vertex F.

Now we have left to prove the universal property. Therefore we consider another cocone $\{\eta^x : h^C \to L\}_{(C,\, x)}$ and we want to find a unique morphism $\vartheta : F \to L$ as shown in the following diagram

$$\begin{tikzcd}
h^C \arrow[rr, "f\circ -"] \arrow[dr, "y^x"'] \arrow[ddr, bend right, "\eta^x"'] & & h^{C'} \arrow[dl, "y^{x'}"] \arrow[ddl, bend left, "\eta^{x'}"] \\
& F \arrow[d, dotted, "\exists!\,\vartheta"] & \\
& L &
\end{tikzcd}$$

But by Yoneda each η^x correspond to an element $z \in L(C)$ and thus $\eta^x = y^z$ (where the last one is the natural transformation defined by Yoneda). Since

[104] [Mac78] Chapter III, §7, Theorem 1.
[105] [Rie11] §3, page 5.

L is a cocone we have that $L(f)(z') = z$.

We define $\theta_C : F(C) \to L(C)$ as $x \mapsto z$, where z is the element found by Yoneda starting from the component of the cocone L corresponding to the element (C, x) of $el(\mathbb{C})$. To prove that ϑ is natural let us consider a morphism $f : C' \to C$ and let us denote $x' := F(f)(x)$. Then f is a morphism in $el(\mathbb{C})$ and so $z' = L(f)(z)$ too. We need to prove that the following diagram commutes:

$$\begin{tikzcd} F(C) \arrow[r, "\vartheta_C"] \arrow[d, "F(f)"'] & L(C) \arrow[d, "L(f)"] \\ F(C') \arrow[r, "\vartheta_{C'}"'] & L(C') \end{tikzcd}$$

But $L(f) \circ \vartheta_C(x) = L(f)(z) = z'$ and $\vartheta_{C'} \circ F(f)(x) = \vartheta_{C'}(x') = z'$ since the Yoneda isomorphism is natural. Always for Yoneda the natural transformation found is the unique making all the triangles commute. Thus the claim.

$\square$

Proposition A.2.3. ***(Transpose Argument)*** *Given a pair of adjunctions $F_i \dashv G_i : \mathbb{D} \to \mathbb{C}$ for $i = 0, 1$, then we have a bijective correspondence between natural transformations $F_0 \dot{\to} F_1$ and the ones $G_1 \dot{\to} G_0$. Moreover:*

$$F_0 \cong F_1 \iff G_1 \cong G_0$$

Proof A natural transformation $\varphi : F_0 \dot{\to} F_1$ is the data of a collection of morphism $\varphi_X : F_0(X) \to F_1(X)$ for any object X in $\mathbb{C}$ natural in X. Since the Yoneda Embedding is fully faithful, this collection correspond to a collection (still natural in X) of natural transformation

$$h^{F_1(X)} \dot{\to} h^{F_0(X)}$$

By definition this means to have, for any $X \in \mathrm{Ob}(\mathbb{C})$ and $Y \in \mathrm{Ob}(\mathbb{D})$, maps (natural in X and Y)

$$\mathbb{D}(F_1(X), Y) \longrightarrow \mathbb{D}(F_0(X), Y)$$

Through the adjunctions $F_i \dashv G_i$ we can see that the data of such maps is equivalent to

$$\mathbb{C}(X, G_1(Y)) \longrightarrow \mathbb{C}(X, G_0(Y))$$

again natural both in X and Y. By definition this gives rise to a natural transformation

$$h_{G_1(X)} \dot{\to} h_{G_0(X)}$$

and therefore, applying another time Yoneda, we get a collection of maps $G_1(Y) \to G_0(Y)$, for any object Y of $\mathbb{D}$, natural in Y. Thus we obtain the unique natural transformation $G_1 \dot{\to} G_0$ required.

Since the Yoneda embedding reflects isomorphism we have even the second property.

$\square$

Corollary A.2.4. *If we have $F \dashv G \dashv H : \mathbb{C} \to \mathbb{D}$, then:*

$$GH \cong Id_{\mathbb{C}} \;\Leftrightarrow\; GF \cong Id_{\mathbb{C}}$$

Thus H is fully faithful if and only if F is such.

Proof We just apply the previous proposition with $F_0 := Id_{\mathbb{C}}$, $F_1 := GF$, $G_0 := Id_{\mathbb{C}}$ and $G_1 := GH$. For the second part we just recall that a right adjoint is fully faithful if and only if the adjunction counit is an isomorphism.

$\square$

B Model Categories

All the definitions and propositions stated in the following appendix can be found in [DS95] in §3 and successive. Anoother reference can be [GJ99] Chapter II, §1.

B.1 Main Definitions

Definition B.1.1. *Let $\mathbb{C}$ be a category and $f : X \to X'$ and $g : Y \to Y'$ two morphism in $\mathbb{C}$. We say that f is a* ***retract*** *of g if there exists a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] \arrow[rr, bend left, "Id_X"] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"'] \arrow[rr, bend right, "Id_{X'}"'] & Y' \arrow[r, "r'"'] & X' \end{tikzcd}$$

Definition B.1.2. *A category $\mathbb{C}$ is a* ***model category*** *if it has three classes of morphism*

(i) Weak equivalences, usually denoted by $\xrightarrow{\sim}$

(ii) Fibrations, usually denoted by $\twoheadrightarrow$

(iii) Cofibrations, usually denoted by $\hookrightarrow$

closed under composition and such that for any object X in $\mathbb{C}$ the identity map Id_X is in all of them. The fibrations(/cofibrations) that are even weak equivalences are called ***acyclic fibrations(/cofibrations)****. Furthermore it is required that the following axioms hold:*

<u>*MC1*</u> *$\mathbb{C}$ has finite limits and finite colimits.*

<u>*MC2*</u> *(2-out-of-3) Given two composable maps f, g in $\mathbb{C}$, if two maps between $\{f, g, g \circ f\}$ are weak equivalences, then the last one it is such.*

<u>*MC3*</u> *If f is a retract of g and g is a weak equivalence, a fibration or a cofibration, then f is a weak equivalence, a fibration or a cofibration, respectively.*

MC4 For any commutative diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ C \arrow[r, "g"'] & Y \end{tikzcd}$$

If either i is a cofibration and p an acyclic fibration, or i an acyclic cofibration and p a fibration, then there exists a diagonal map $h : B \to X$, i.e. a map in $\mathbb{C}$ such that $ph = g$ and $hi = f$.

MC5 For any morphism $f : A \to B$ in $\mathbb{C}$ there are two factorizations

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "i"'] & & B \\ & X \arrow[ur, "p"'] & \end{tikzcd}$$

where

(a) i is a cofibration and p is an acyclic cofibration.

(b) i is an acyclic cofibration and p is a cofibration.

Remark B.1.3. It is quite straight forward to prove that if $\mathbb{C}$ is a model category then we can obtain a model structure on the dual category $\mathbb{C}^{op}$ as well. In this case we should choose as cofibrations the fibrations in $\mathbb{C}$ and as fibrations the cofibrations in $\mathbb{C}$.

Notation: By *MC1* we know that in any model category $\mathbb{C}$ there are both an initial object and a terminal one. We denote them $\emptyset$ and $*$ respectively.

Definition B.1.4. *An object X of a model category $\mathbb{C}$ is called:*

- ***Fibrant object*** *if the unique morphism $X \twoheadrightarrow *$ is a fibration.*
- ***Cofibrant object*** *if the unique morphism $\emptyset \hookrightarrow X$ is a cofibration.*

Definition B.1.5. • *An object $A \wedge I$ of a model category $\mathbb{C}$ is a* ***cylinder for*** *A if there is a factorization*

$$\begin{tikzcd} A \amalg A \arrow[rr, "1_A+1_A"] \arrow[dr, "i"'] & & A \\ & A \wedge I \arrow[ur, "\varphi"', "\sim"] & \end{tikzcd}$$

where φ is a weak equivalence.

- *It is a* ***good cylinder*** *if i is moreover a cofibration.*
- *It is a* ***very good cylinder*** *if i is a cofibration and φ a fibration.*

Definition B.1.6. • *An object X^I of a model category $\mathbb{C}$ is a* ***path object for*** *X if there is a factorization*

$$\begin{tikzcd} X \arrow[rr, "(1_X,1_X)"] \arrow[dr, "\psi"', "\sim"] & & X \times X \\ & X^I \arrow[ur, "p"'] & \end{tikzcd}$$

where ψ is a weak equivalence.

- *It is a* ***good path*** *if p is moreover a fibration.*
- *It is a* ***very good path*** *if p is a fibration and ψ a cofibration.*

B.2 Some Properties

The following proposition tells us that actually, in a model category, we can recover fibrations (/cofibrations) just from the classes of cofibrations(/fibrations) and weak equivalences.

Proposition B.2.1. *In a model category $\mathbb{C}$ we have that:*

(i) The cofibrations in $\mathbb{C}$ are the maps which have the LLP (left lifting property) with respect to acyclic fibrations.

(ii) The acyclic cofibrations in $\mathbb{C}$ are the maps which have the LLP (left lifting property) with respect to fibrations.

(iii) The fibrations in $\mathbb{C}$ are the maps which have the RLP (right lifting property) with respect to acyclic cofibrations.

(iv) The acyclic fibrations in $\mathbb{C}$ are the maps which have the RLP (right lifting property) with respect to cofibrations.

Proof The proof can be found in [DS95], Proposition 3.13.

□

Proposition B.2.2. *In a model category $\mathbb{C}$ we have that:*

(i) Fibrations and acyclic fibrations are stable under pullback.

(ii) Cofibrations and acyclic cofibrations are stable under pushout.

Proof The idea is to use the proposition above, a complete proof can be found in [DS95], Proposition 3.14.

□

In the enviroment of model categories is possible to introduce the concept of left and right homotopy between maps (through the definition of cylinder and path object). These relations turn out to be equivalence relations if the starting object is cofibrant or the ending object is cofibrant, respectively. In this way we can construct a new category $Ho(\mathbb{C})$ called the **homotopy category** associated to the model category $\mathbb{C}$. [106]

Lemma B.2.3. *Let $f : A \to X$ be a map in a model category $\mathbb{C}$. If A and X are both fibrant and cofibrant, then:* [107]

$$f \text{ is a weak equivalence} \Leftrightarrow f \text{ has an homotopy inverse}$$

Another way to get the homotopy category is through the categorical notion of localization.

Definition B.2.4. *Let $\mathbb{E}$ be any category and $\mathcal{W}$ a class of morphism in $\mathbb{E}$. A category $\mathbb{D}$ together with a functor $F : \mathbb{E} \to \mathbb{D}$ is said to be a* ***localization of*** *$\mathbb{E}$ with respect to $\mathcal{W}$ if:*

- *For any morphism $f \in \mathcal{W}$, $F(f)$ is an isomorphism in $\mathbb{D}$.*
- *(Universal Property) For any other pair $(\mathbb{D}', G)$ such that G is a functor from $\mathbb{E}$ to $\mathbb{D}'$ sending any morphism in $\mathcal{W}$ to an isomophism, there exists a unique functor $G' : \mathbb{D} \to \mathbb{D}'$ such that $G' \circ F = G$.*

$$\begin{tikzcd} \mathbb{E} \arrow[r, "F"] \arrow[dr, "\forall\, G"'] & \mathbb{D} \arrow[d, dotted, "\exists!\, G'"] \\ & \mathbb{D}' \end{tikzcd}$$

[106] [DS95] §4 and §5.
[107] [DS95] §4, Lemma 4.24.

Definition B.2.5. *Let $\mathbb{C}$ be a model category. We define* ***the homotopy category*** $Ho(\mathbb{C})$ *associated to* $\mathbb{C}$ *as the localization of* $\mathbb{C}$ *with respect to the class of weak equivalences.* [108]

[108] [DS95] §6, Theorem 6.2.

References

[Awo10] Steve Awodey, *Category Theory*, Oxford University Press, 2006.

[Awo16] ———, *Natural models of homotopy type theory*, Cambridge University Press, 2016.

[Cis16] Denis-Charles Cisinski, *Univalent universes for elegant models of homotopy types*, arXiv:1406.0058, 2016.

[Cis18] ———, *Higher Categories and Homotopical Algebra*, Cambridge University Press, 2018.

[DV17] Mariangiola Dezani-Caincaglini and Betti Venneri, *Lambda Calcolo*, APhEx (Analytical and Philosofical Explanation), n°16, June 2017.

[DS95] W.G. Dwyer and J. Spalinski, *Homotopy theories and model categories*, North-Holland, 1995.

[GJ99] Paul G. Goerss and John F. Jardine, *Simplicial Homotopy Theory*, Birkh a user, Basel, 1999.

[GZ67] P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer-Verlag New York, Inc., New York, 1967.

[Hof97] Martin Hofmann, *Syntax and semantics of dependent types*, Semantics and logics of computation (Cambridge, 1995), Publ. Newton Ist., vol. 14, Cambridge Univ. Press, Cambdrige, 1997.

[HoTT] HoTT group, *Homotopy Type Theory: Univalent Foundations of Mathematics*, The Univalent Foundations Program , 2013.

[Jac99] Bart Jacobs, *Categorical Logic and Type Theory*, Elsevier Science, Amsterdam and New York, 1999.

[JT99] André Joyal and Myles Tierney, *An Introduction to Simplicial Homotopy Theory*, Preliminary Version, August 5, 1999.

[JT08] ———, *Notes on simplicial homotopy theory*, <`http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern47.pdf`>, 2008.

[KL16] Chris Kapulkin and Peter Lefanu Lumsdaine, *The simplicial model of univalent foundations (after Voevodsky)*, arXiv:1211.2851, 2016.

[Mac78] Saunders Mac Lane, *Categories for the Working Mathematician*, Springer, 1978.

[nLab] nLab authors, *locally cartesian closed categories*, <`https://ncatlab.org/nlab/show/locally+cartesian+closed+category#definition`>, February 28, 2018.

[Rie11] Emily Riehl, *A Leisurely Introduction to Simplicial Sets* (notes), <`http://www.math.jhu.edu/~eriehl/ssets.pdf`>, 2011.

[Sel08] Peter Selinger, *Lecture notes on the lambda calculus*, arXiv:0804.3434, 2008.

[Shu15] Michael Shulman, *Univalence for Inverse Diagrams and Homotopy Canonicity*, Math. Structures Comput. Sci. **25** (2015), n0. 5, 1203-1277, arxiv:1203.3253.

Ich habe die Arbeit selbständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und bisher keiner anderen Prüfungsbehörde vorgelegt. Außerdem bestätige ich hiermit, dass die vorgelegten Druckexemplare und die vorgelegte elektronische Version der Arbeit identisch sind und dass ich von den in §26 Abs. 6 vorgesehenen Rechtsfolgen Kenntnis habe.