ALGANT MASTER'S THESIS

ON NEARLY HOLOMORPHIC MODULAR FORMS

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"When kings are building, carters have work to do."

Introduction

"There's a saying attributed to Eichler that
there are five fundamental operations of arithmetic:
addition, subtraction, multiplication, division, and modular forms."

Andrew Wiles

Modular forms have a captivating history with thousands of applications across the sciences. Specifically, they are[1] analytic maps on the complex upper half plane $\mathbf{H}$ satisfying certain modular transformation properties under the action of a congruence subgroup of $\Gamma(1)$. Although this definition might seem purely analytic, this does not turn out to be the case.

Indeed, the word *modular* refers to the moduli space of complex tori $\mathbf{C}/\Omega$, for given lattices Ω. We consider a modular function F as a map assigning a complex number to each lattice, such that F takes the same values on lattices which define isomorphic tori. Since for any lattice we can find a basis given by $\{\tau, 1\}$ for some $\tau \in \mathbf{H}$, the map F is completely determined by its value $F(\mathbf{Z}\tau + \mathbf{Z}) =: f(\tau)$, specifying a function f on $\mathbf{H}$. As the space of all oriented lattices is given by $\Gamma(1)\backslash \mathrm{GL}^+$, the function f should satisfy the modular property $f(\frac{a\tau+b}{c\tau+d}) = f(\tau)$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, that is, f is invariant under the action of $\Gamma(1)$ on $\mathbf{H}$. However, the maps of our interest are usually holomorphic and satisfy a specific growth condition at infinity. Therefore, as the above modular invariance property turns out to be restrictive to do interesting arithmetic, we consider a slightly different definition of the function F, giving rise to a more general class of functions: for any lattices Ω_1, Ω_2 defining isomorphic complex tori, we request that the function F satisfies $F(\Omega_1) = \lambda^k F(\Omega_2)$, for a non negative fixed integer k, called the weight. As a result, the modular property of f becomes

$$f(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

The consequence is the analytic definition of a holomorphic modular form. It is a holomorphic map $f : \mathbf{H} \to \mathbf{C}$ satisfying the modular invariance property and a growth condition. There are at least two reasons for which modular forms are relevant: they appear naturally in a variety of areas in mathematics, and their space $\mathcal{M}_k(\Gamma, \mathbf{C})$ is finite dimensional. Regarding the latter, for the full modular group $\Gamma(1)$, the finite dimensional of the $\mathbf{C}$−vector

[1]The literature is wide. For a first glance at this field we refer to [Diamond, 2000] [Lang, 1976], [Lang, 1987], [Miyake, 1989] and [Schoeneberg, 1974].

space $\mathcal{M}_k(\Gamma(1), \mathbf{C})$ is quite trivial. However, for arbitrary congruence subgroups of $\Gamma(1)$ having only regular cusps, the result requires some more effort. One possible way to reach the outcome is presenting modular forms as sections of a bundle over a compact Riemann surface, and then applying Riemann-Roch theorem. As for the former, the modular invariance property together with the holomorphicity condition, leads to a Fourier expansion of any holomorphic modular forms. Furthermore, quoting Zagier [Bruinier, 2008], *"the fact that all of these different objects land in the little space $\mathcal{M}_k(\Gamma, \mathbf{C})$ forces the existence of relations amongst their coefficients".* As it turns out, these Fourier coefficients encode arithmetically interesting information, such as sums over divisors of integers or numbers of solutions to Diophantine equations.

The history of modular forms dates back to the first half of the nineteenth century, the era of Jacobi and Eisenstein, and began with elliptic functions, which are doubly periodic meromorphic complex maps. Partly because of Hitler and the war, modular forms were ignored by most mathematicians for about 30 years after the thirties, with the exception of Eichler, Maass and Petersson. After this interruption, they eventually came back into fashion thanks to Taniyama and Shimura and their well known conjecture, which asserts that the L−function of any elliptic curve over $\mathbf{Q}$ comes from a modular form. The proof of this conjecture is due to Andrew Wiles, and was crucial to prove Fermat's Last Theorem in the nineties.

Throughout this time, many generalizations of the classical notion of modular forms were defined. Amongst them all, we discuss nearly holomorphic modular forms.
In a collection of influential papers[2] Shimura introduced and extensively studied this notion, which was independently described by Kaneko and Zagier, in [Kaneko, 1995]. Shimura's definition is of an analytic nature, though he also proved different algebraicity results. Roughly speaking, a nearly holomorphic modular form is a polynomial of functions over the ring of all holomorphic maps, transforming like a modular form of fixed weight with respect to some congruence subgroup of $\Gamma(1)$. The classical examples of nearly holomorphic modular forms are provided by special values of Eisenstein series, which link this arithmetic theory to the theory of L−functions.

The objective of this paper is to address the theory of nearly holomorphic modular forms from their analytic description to their geometric characterization as global sections of an algebraic vector bundle, which arises from the De Rham bundle related to universal elliptic curves over the compact modular curve $X(\Gamma)$. The main reference is [Urban, 2013], who introduces in this article a sheaf-theoretic formulation of Shimura's theory of modular forms, which later leads him to define and study some basic properties of nearly overconvergent modular forms.

This thesis consists of four chapters. The first one introduces the notations and the

[2] [Shimura, 1986], [Shimura, 1987], [Shimura, 1987b], [Shimura, 1990], [Shimura, 1994], [Shimura, 2003].

geometric background needed throughout this work, namely involving the theory of bundles. The second part recalls the setting of holomorphic modular forms, and then deals with nearly holomorphic modular forms presenting Shimura's analytic definitions and Eisenstein series as examples. The third part constructs the vector bundles which play a fundamental role in the geometric representation of nearly holomorphic modular forms, deeply focusing on the De Rham bundle. The fourth and last one aims to prove the following isomorphism $\Gamma(X(\Gamma), \mathcal{H}_k^r(X(\Gamma))) = \mathbf{N}_k^r(\Gamma, \mathbf{C})$, after enlarging the whole picture to the compactification of the modular curve $Y(\Gamma)$.

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The creation of this work has left me floundered innumerable times. I have been convinced that *the perfect solution* does not exist. Although there may be an excellent solution to a given problem, I believe that there is always room to find a better one. On account for that, during the drafting period I was always seeking better words, ideas, constructions, proofs, and approaches, and after having found them, the existence of even better solutions was doubtless to me. After several iterations of writing and deleting, the outcome of this work in this current form is my best *better* one, and this final version would not have been achievable without the constant, both personal and academic, support of several people.

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M.M.

Contents

§1. Geometric Background

The first chapter aims to present the geometric notions needed throughout this work. As the final goal is to characterize nearly holomorphic modular forms as sections of a specific bundle, we start introducing the theory of bundles. Importance is given to the special class of associated vector bundles and their sections, since the fundamental vector bundle over $Y(\Gamma)$ presented in Chapter 3 is of this form. For a quick introduction to bundles we refer to [Sontz, 2010].

1.1 Principal and Fibre Bundles

In this section we define principal and fibre bundles and recall some constructions we need later. While fibre bundles came from the 1930's, a period in which topology developed some of its important tools; principal bundles were introduced by Serre in 1958. Bundles are used to enlarge the notion of topological product: spaces that globally are not products but locally are.

Principal G–bundles

In the following, let G be a complex Lie group and let M be a complex manifold.

Definition 1.1.1. A G-bundle over M is a complex manifold T with a free right action of G and a morphism $\pi : T \to M$ such that π induces an isomorphism of complex manifold $\pi : T/G \to M$.

T is usually referred as the total space, M as the base space and π as the structure map, while for every $m \in M$ the submanifold $\pi^{-1}(m) \subset T$ is called the fibre over m. We obtain that the action of G preserves fibres, that is $t \in \pi^{-1}(m)$ implies $tg \in \pi^{-1}(m)$ for all $g \in G$. Therefore the Lie group action on the total space T acts transitively on the fibres.
The easiest example is given by $\mathrm{pr}_1 : M \times G \to M$ with canonical right action given by multiplication in G in the second entry. It is called the trivial G–bundle.

Definition 1.1.2. A morphism of G–bundles $\pi_T : T \to M$ and $\pi_L : L \to M$ is given by a G–equivariant map $f : T \to L$

$$\begin{array}{ccc} T \times G & \longrightarrow & T \\ {\scriptstyle f\times \mathrm{Id}}\downarrow & & \downarrow{\scriptstyle f} \\ L \times G & \longrightarrow & L \end{array}$$

such that $\pi_L \circ f = \pi_T$.
f is an isomorphism if it has right and left inverse.

The above data are now subjected to the condition of local triviality, giving rise to the construction of principal $G-$bundles.

Definition 1.1.3. A principal $G-$bundle over M is a $G-$bundle $\pi : T \to M$ such that T is locally trivial, i.e. for all $m \in M$ there exists $U \ni m$ an open neighbourhood with an isomorphism of $G-$bundle $\Phi : \pi^{-1}(U) \xrightarrow{\sim} U \times G$, called trivialization.

We point out that a morphism of trivial bundle $f : M \times G \to M \times G$ is of the form $f(m, g) = (m, h(m)g)$ for $h : M \to G$ holomorphic.
Indeed let $h(m) := \mathrm{pr}_2 \circ f(m, 1_G)$. Then

$$f(m, g) = f(m, 1_G)g = (m, h(m))g = (m, h(m)g).$$

Let $\pi : T \to M$ be a principal $G-$bundle, $(U_i)_i$ be a covering of M such that T can be trivialized on it, let $\Phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times G$ be the trivializations and define $U_{ij} := U_i \cap U_j$. Then the morphisms of trivial bundles

$$\Phi_i \circ \Phi_j^{-1} : U_{ij} \times G \to U_{ij} \times G$$

are, for what we have remarked before, of the form $(u, g) \mapsto (u, \varphi_{ij}(u)g)$, for the holomorphic maps $\varphi_{ij} : U_{ij} \to G$.

Definition 1.1.4. The holomorphic maps $\varphi_{ij} : U_{ij} \to G$ are called transition functions and they take value in what is called the structure group of the bundle.

<u>Examples of principal $G-$bundles</u>

$\pi : \mathbf{C}^{n+1} \smallsetminus \{0\} \to \mathbf{P}^n(\mathbf{C})$ is a principal $\mathbf{C}^\times-$bundle.
Indeed, for the canonical open $U_i := \{(z_0 : ... : z_n) \in \mathbf{P}^n(\mathbf{C}) : z_i \neq 0\} \subseteq \mathbf{P}^n(\mathbf{C})$ the preimage under π is given by $\pi^{-1}(U_i) := \{(z_0, ..., z_n) \in \mathbf{C}^{n+1} \smallsetminus \{0\} : z_i \neq 0\}$. Therefore we easily get the desired isomorphism $\pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbf{C}^\times$ given by $(z_0, ..., z_n) \mapsto ((\frac{z_0}{z_i} : ... : 1 : ... : \frac{z_n}{z_i}), z_i)$ with inverse $(\omega_0\lambda, ..., \omega_n\lambda) \leftarrow\!\shortmid ((\omega_0 : ... : \omega_n), \lambda)$.
Consider the complex upper half plane

$$\mathbf{H} := \{\tau = x + iy \in \mathbf{C} : \ y > 0\},$$

and, for $\Im(\cdot)$ the imaginary part of a complex number,

$$\mathrm{GL}^+ := \left\{ \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \in (\mathbf{C} \smallsetminus \{0\})^2 : \Im\left(\frac{\omega_1}{\omega_2}\right) > 0 \right\},$$

the set of all oriented $\mathbf{R}$ basis in $\mathbf{C}$. The restriction of the previous bundle for $n = 1$ to GL^+ is again a principal $\mathbf{C}^\times-$bundle denoted by

$$P : \mathrm{GL}^+ \to \mathbf{H},$$

which maps $(\omega_1, \omega_2) \mapsto \frac{\omega_1}{\omega_2}$. Moreover it is isomorphic to the trivial $\mathbf{C}^\times-$bundle, $\mathrm{GL}^+ \xrightarrow{\sim} \mathbf{H} \times \mathbf{C}^\times$ where $(\omega_1, \omega_2) \mapsto \frac{\omega_1}{\omega_2}$ and $(\tau\omega_2, \omega_2) \leftarrow\!\shortmid (\tau, \omega_2)$.

Remark Recall that a right action by G corresponds precisely to the same action by the opposite group[1], where composition works in the reverse order. Indeed, suppose G acts on a set X on the right and on the left. By the right action convention, acting by g and then by h is equivalent to acting by gh; while by the left action convention the action of gh is equivalent by acting by h first and then by g. In the definition of $G-$bundle we assume the action of G on the total space T to be right by convention. Indeed, we want the action to commute with transition functions, and these are usually assumed to act on the left. Therefore we point out that left and right is mostly just a matter of convention.

Examples for the left case

Recall that the group $\mathrm{GL}_2(\mathbf{R})^+$ acts on the complex upper half plane $\mathbf{H}$ through the left action defined by

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is easy to see that the action is well defined as the denominator is non-zero and $\mathbf{H}$ is map to itself as

$$\Im(\gamma\tau) = \frac{\Im(\tau)}{|c\tau + d|^2}.$$

Let $q : \mathbf{H} \to \Gamma\backslash\mathbf{H}$ be the projection on the discrete subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbf{Z})$, such that the left action of Γ on $\mathbf{H}$ has no fixed point. Then $q : \mathbf{H} \to \Gamma\backslash\mathbf{H}$ is a principal left $\Gamma-$bundle. Indeed, as the stabilizer of each point is trivial, then for each $\tau \in \mathbf{H}$ there exists an open neighbourhood $U \ni \tau$ such that $\gamma(U) \cap U = \varnothing$ for all non trivial $\gamma \in \Gamma$. This result is not trivial, and it is a consequence of the fact that the action of Γ on $\mathbf{H}$ is properly discontinuous. For this reason we recall the following.

Proposition 1.1.5. Let G be a locally compact group, $K \subseteq G$ a compact subgroup and let the quotient G/K be Hausdorff. Further let $\Gamma \subseteq G$ be a discrete subgroup. For each $x \in G/K$ there exists a neighbourhood $U \ni x$ such that

$$\{\gamma \in \Gamma : \gamma(U) \cap U \neq \varnothing\} = \mathrm{Stab}_\Gamma(x).$$

PROOF. As Γ is discrete, by the characterization of discrete subgroups, for each $x \in G/K$ we get a neighbourhood U_x such that the intersection $\gamma(U_x) \cap U_x$ is not empty for only finitely many $\gamma \in \Gamma$. The same property still holds if we replace U_x with a smaller neighbourhood containing x. We want to find a neighbourhood $U \subseteq U_x$ of x, such that $\gamma(U)$ meets U if and only if $\gamma(x) = x$, that is if γ is in the stabilizer of[2] x. Let $\gamma_1, ..., \gamma_n \in \Gamma$ be the elements satisfying $\gamma_i(U_x) \cap U_x \neq \varnothing$. For any other $V \subseteq U_x$ we can have $\gamma(V) \cap V \neq \varnothing$ only if $\gamma = \gamma_i$ for some $i \in \{1, ..., n\}$. Therefore it suffices to show that for each i such that

[1]The opposite group of a group is the group with the same underlying elements set endowed with a new multiplication corresponding to the old multiplication with the order of elements reversed.

[2]We remark that if the action is also free then $\gamma(U)$ is disjoint from U for all non trivial $\gamma \in \Gamma$.

$\gamma_i(x) \in U_x \setminus \{x\}$ there exists $U_i \subseteq U_x$ such that $\gamma_i(U_i) \cap U_i = \varnothing$, so that we define U to be the intersection of these finitely many U_i for which $\gamma_i(x) \neq x$. As the space is Hausdorff, when $\gamma_i(x) \in U_x \setminus \{x\}$, there exist disjoint neighbourhoods V_i and $\tilde{V}_i$ around x and $\gamma_i(x)$ respectively. Choose $W_i \subseteq U_x$ around x such that $\gamma_i(W_i) \subseteq \tilde{V}_i$ and define $U_i := W_i \cap V_i$. By construction U_i is disjoint from $\tilde{V}_i$ and $\gamma_i(U_i) \subseteq \tilde{V}_i$, whence $\gamma_i(U_i) \cap U_i = \varnothing$. Therefore the neighbourhood $U := \cap_i U_i$ of x has the desired property. □

Note that we can apply proposition 1.1.5 as $\mathbf{H} \cong \mathrm{SL}_2(\mathbf{R})/\,\mathrm{SO}(2)$. Further, as $\Gamma\backslash\mathbf{H}$ has the quotient topology, then $q(U)$ is an open neighbourhood of $[\tau] \in \Gamma\backslash\mathbf{H}$ and by definition of the projection, $q^{-1}(q(U)) = \sqcup_{\gamma\in\Gamma}\gamma(U)$. The trivializations are then given by $\Gamma \times q(U) \xrightarrow{\sim} q^{-1}(q(U))$ where $(\gamma, [\tau]) \mapsto \gamma\tau$.

Fibre bundles

Let F be a manifold.

Definition 1.1.6. A holomorphic map $\pi : T \to M$ is a fibre bundle with typical fibre F if there exist a covering $(U_i)_i$ of M and trivializations $\Phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times F$ such that $\pi = \mathrm{pr}_1 \circ \Phi_i$.

We now put more structure on the fibres to get vector bundles. Vector bundles are the most important special class of bundles with given structure group, where the fibre is a vector space and the structure group is a group of linear automorphism of the vector space. The construction supports the idea of a collection of vector spaces parametrized by another space (which is in our setting a manifold) M: to each point $m \in M$ we attach a vector space in such a way that these vector spaces glue together to form another space, namely T.

Definition 1.1.7. A fibre bundle is a complex vector bundle if the typical fibre is $\mathbf{C}^n$ and if for all $m \in M$ the isomorphisms $\Phi_i \circ \Phi_j^{-1} : \{m\} \times \mathbf{C}^n \xrightarrow{\sim} \{m\} \times \mathbf{C}^n$ are $\mathbf{C}$–linear.

The idea is that on the overlaps of two open subsets there exists a continuous map to GL_n. Further, differently from the principal bundles situation, there is in general no Lie group action on the total space acting freely on the fibres. Of course all fibres are Lie groups, but this does not imply that there exists a Lie group that acts on the whole space and such that its restriction to each fibre has a simply transitive action.
We call n the rank of the bundle, and for $n = 1$ the complex vector bundle is called a line bundle.
Let $\pi_T : T \to M$ and $\pi_L : L \to M$ be vector bundles with transition functions Φ_i and Ψ_i respectively.

Definition 1.1.8. A homomorphism of vector bundles $f : T \to L$ is a holomorphic map such that $\pi_L \circ f = \pi_T$ and $\Psi_i \circ f \circ \Phi_i^{-1} : \{m\} \times \mathbf{C}^n \to \{m\} \times \mathbf{C}^n$ is $\mathbf{C}$–linear.

Let $\pi : T \to M$ be a vector bundle with $\Phi_i : \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbf{C}^n$. Similarly as before, the maps $\Phi_i \circ \Phi_j^{-1} : U_{ij} \times \mathbf{C}^n \to U_{ij} \times \mathbf{C}^n$ are of the form $(u, v) \mapsto (u, \varphi_{ij}(u)v)$. But this time the holomorphic transition functions are given by $\varphi_{ij} : U_{ij} \to \mathrm{GL}_n(\mathbf{C})$. Further, while in principal G−bundle the fibre coincides with the structure group G, in the case of vector bundles, the fibre $\mathbf{C}^n$ is not the structure group, $\mathrm{GL}_n(\mathbf{C})$.

Examples of vector bundles

$\pi : \mathcal{O}(-1) \to \mathbf{P}^n(\mathbf{C})$ is a line bundle, called the tautological line bundle over $\mathbf{P}^n(\mathbf{C})$. Indeed $\mathcal{O}(-1) := \{((z_0 : ... : z_n), (\lambda z_0, ..., \lambda z_n)) \in \mathbf{P}^n(\mathbf{C}) \times \mathbf{C}^{n+1} : \lambda \in \mathbf{C}\}$ assigns a one dimensional vector space to each point $(z_0 : ... : z_n) \in \mathbf{P}^n(\mathbf{C})$; as $(z_0 : ... : z_n)$ defines a complex line in $\mathbf{C}^{n+1}$ passing through the origin and $(\lambda z_0, ..., \lambda z_n)$ is a point on this line. The map π corresponds to the projection on the first component.

The associated vector bundle

There exists a canonical way to obtain a fibre bundle with specific fibre starting from a given principal bundle. Indeed let $\pi : T \to M$ be a principal G−bundle and let F be a complex manifold with a left G−action $\sigma : G \times F \to F$. Let G acts on the right on $T \times F$ by

$$(t, f)g := (tg, g^{-1}f).$$

Definition 1.1.9. The associated fibre bundle $\pi_F : E(T, F, \sigma) \to M$ with typical fibre F is the quotient $T \times^G F := (T \times F)/G$, with $\pi_F[(t, f)] := \pi(t)$.

It is easy to see that the above construction actually gives rise to a fibre bundle. Indeed, let $\Phi : \pi^{-1}(U) \xrightarrow{\sim} U \times G$ be the trivialization of T then by definition of π_F, we get the isomorphism $\Phi \times \mathrm{Id} : \pi_F^{-1}(U) = (\pi^{-1}(U) \times F)/G \xrightarrow{\sim} (U \times G \times F)/G$ where the right action of G on $U \times G \times F$ is described as $(u, h, f)g = (u, hg, g^{-1}f)$. Further $\Theta : (U \times G \times F)/G \xrightarrow{\sim} U \times F$ where $[(u, g, f)] \mapsto (u, gf)$ and $[(u, 1_G, f)] \leftarrow (u, f)$. Therefore $\Theta \circ \Phi \times \mathrm{Id}$ gives the desired trivializations.

Definition 1.1.10. In case $F = \mathbf{C}^n$ and $\sigma : G \to \mathrm{GL}_n(\mathbf{C})$ is a morphism of complex Lie groups[3], where the action of $\mathrm{GL}_n(\mathbf{C})$ on $\mathbf{C}^n$ is canonically given by left matrix multiplication, $E(T, F, \sigma)$ defines a vector bundle.

Examples of the associated fibre bundle

Consider the principal $\mathbf{C}^\times$−bundle $\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n(\mathbf{C})$, presented before as an instance of principal bundle. Let $\sigma^k : \mathbf{C}^\times \to \mathrm{GL}_1(\mathbf{C})$ be the representation given by $\lambda \mapsto \lambda^k$ and let $\overline{\sigma}^k : \mathbf{C}^\times \to \mathrm{GL}_1(\mathbf{C})$ be the representation given by $\lambda \mapsto \overline{\lambda}^k$. We point out that usually we require the representation to be a morphism of complex Lie groups, but in this case $\overline{\sigma}^k$ is not holomorphic. The associated line bundle $E(\mathbf{C}^{n+1} \setminus \{0\}, \mathbf{C}, \sigma^k)$ is denoted by $\mathcal{O}(-k)$. Note that for $k = 1$ we get the vector bundle presented in the previous section.

[3] Recall that, as usual, every representation $\sigma : G \to \mathrm{GL}_n(\mathbf{C})$ defines an action $G \times \mathbf{C}^n \to \mathbf{C}^n$ given by $(g, v) \mapsto \sigma(g)v$.

Further, for $n = 1$, we consider again the restriction of $\pi : \mathbf{C}^2 \setminus \{0\} \to \mathbf{P}^1(\mathbf{C})$ to GL^+, namely $P : \mathrm{GL}^+ \to \mathbf{H}$. The associated vector bundles are

$$\underline{\omega}^k_{\mathbf{H}} := E(\mathrm{GL}^+, \mathbf{C}, \sigma^k) = (\mathrm{GL}^+ \times \mathbf{C})/\mathbf{C}^\times$$

and the non-holomorphic bundle

$$\underline{\overline{\omega}}^k_{\mathbf{H}} := E(\mathrm{GL}^+, \mathbf{C}, \overline{\sigma}^k) = (\mathrm{GL}^+ \times \mathbf{C})/\mathbf{C}^\times,$$

which correspond to the restriction of $E(\mathbf{C}^2 \setminus \{0\}, \mathbf{C}, \sigma^k)$ and $E(\mathbf{C}^2 \setminus \{0\}, \mathbf{C}, \overline{\sigma}^k)$ to GL^+ respectively.

<u>Summary with the main bundles</u>

Amongst the bundles seen above, the following diagram summarizes the ones more relevant for our work. Again, the action of Γ on $\mathbf{H}$ is assumed to be free; note that this is indeed the case for $N \geq 3$ for the discrete subgroup of $\mathrm{SL}_2(\mathbf{Z})$:

$$\Gamma(N) := \ker\left(\mathrm{SL}_2(\mathbf{Z}) \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})\right) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z}) : a \equiv d \equiv 1 \bmod N,\ b \equiv c \equiv 0 \bmod N,\right\},$$

where we define $\Gamma(1) := \mathrm{SL}_2(\mathbf{Z})$ the full modular group[4].

$$\begin{tikzcd} \mathrm{GL}^+ \arrow[r, "P"] \arrow[d, "Q"'] & \mathbf{H} \arrow[d, "q"] \\ \Gamma\backslash \mathrm{GL}^+ \arrow[r, "p"'] & \Gamma\backslash\mathbf{H} \end{tikzcd}$$

P and q are the structure maps defining a principal $\mathbf{C}^\times$−bundle and a principal left Γ−bundle respectively; the action of Γ on GL^+ is given by the standard matrix multiplication and so p and Q are the structure maps defining again a principal $\mathbf{C}^\times$−bundle and a principal left Γ−bundle respectively.

1.2 Sections of Bundles

This section deals with some basic notions of sections and their properties. A section of a bundle might be considered as a right inverse of the structure map π defining the given bundle, endowed with some additional structures. Sections can be seen as generalized functions on our manifold M.
Let $\pi : T \to M$ be a principal or fibre bundles.

Definition 1.2.1. A (global) section of $\pi : T \to M$ is a holomorphic map $s : M \to T$ such that $\pi \circ s = \mathrm{id}_M$.

[4]The name comes from the fact that the elements of $\Gamma(1)\backslash\mathbf{H}$ are moduli, that is parameters, for the isomorphism classes of elliptic curves over $\mathbf{C}$.

In other words, a section is a holomorphic assignment to each point in the base M of a point in the fibre over it. For instance, when T is a vector bundle, a section assigns to each point of the base a vector from the attached vector space in a holomorphic way.
When a map with the same properties is defined only on an open subset of M, then it is said to be a local section. One of the main goal in studying sections, is to prove whether or not global sections do exist. The idea is that the space might be too twisted to admit global sections. For instance, a principal $G-$bundle admits a global section if and only if it is isomorphic to the trivial bundle. On the other hand, every vector bundle admits at least one global section, namely the zero section, which maps every element of the base to the zero element of the vector space lying over that point.
In the following, when referring to sections, global sections are meant.
We define sections also for complex manifolds not endowed with the structure of a bundle.

Definition 1.2.2. Let T, V be complex manifolds with right $G-$action, and let $f : T \to V$ be a $G-$equivariant map. Then

$$\Gamma(V,T) := \{s : V \to T : f \circ s = \mathrm{Id}_V,\ s \text{ holomorphic}\}$$

is the set of holomorphic sections, and

$$\Gamma(V,T)^G := \{s \in \Gamma(V,T) : s(vg) = s(v)g \ \forall g \in G\},$$

is the set of $G-$equivariant sections.

The following proposition defines a correspondence between sections on the total spaces and sections on the bases respectively.

Proposition 1.2.3. Let $\pi : T \to M$ and $\eta : V \to N$ be principal $G-$bundles and let $f : T \to V$ be a $G-$equivariant map, such that the following diagram commutes

$$\begin{tikzcd} T \arrow[r, "\pi"] \arrow[d, "f"'] & M \cong T/G \arrow[d, "\tilde{f}"] \\ V \arrow[r, "\eta"'] & N \cong V/G. \end{tikzcd}$$

Then $\Gamma(V,T)^G \cong \Gamma(N,M)$.

PROOF. Define a map $\Gamma(V,T)^G \to \Gamma(N,M)$ by sending $s \mapsto \tilde{s}$, where $\tilde{s}(n) := \pi(s(v))$ for $\eta(v) = n$. The map is well defined as $v = vg \in N$ and $\pi(s(vg)) = \pi(s(v)g) = \pi(s(v))$, also $\tilde{f} \circ \tilde{s}(n) = \tilde{f}(\pi(s(v))) = \eta(v) = n$ and s is holomorphic as it is defined as composition of holomorphic maps. The inverse map is given by assigning $\tilde{s} \to s$ where $s(v) = (\tilde{s}(\eta(v)), v) \in M \times_N V \cong T$. The last isomorphism is given by $\varphi : T \to V \times_N M = \{(v,m) \in V \times M : \tilde{f}(m) = \eta(v)\}$ where $t \mapsto (f(t), \pi(t))$ with inverse $tg \leftarrow (v,m)$ for $\pi(tg) = m$ and $f(tg) = v$. As $\tilde{f}(\tilde{s}(\eta(v))) = \eta(v)$, the map is well defined and as before is holomorphic by construction. Further $s(vg) = (\tilde{s}(\eta(vg)), vg) = (\tilde{s}(\eta(v))g, vg) = s(v)g$, and $f \circ \varphi^{-1} \circ s(v) = f \circ \varphi^{-1}(\tilde{s}(\eta(v)), v) = v$. □

The above proposition is useful to have an explicit characterization of the sections of the associated vector bundle.

Proposition 1.2.4. Let $\pi : T \to M$ be a principal G–bundle and $\sigma : G \to \mathrm{GL}(F)$ be a representation for a vector space F. Then

$$\Gamma(M, E(P, F, \sigma)) \cong \mathrm{Mor}(T, F)^G,$$

for $\mathrm{Mor}(T, F)^G := \{f : T \to F : f \text{ is holomorphic, } f(tg) = \sigma(g^{-1})f(t)\}$.

PROOF. Applying proposition 1.2.3 to the following diagram

$$\begin{tikzcd} T \times F \arrow[r, "\pi_E"] \arrow[d, "\mathrm{pr}_1"'] & E(T, F, \sigma) \arrow[d, "\tilde{\mathrm{pr}}_1"] \\ T \arrow[r, "\pi"'] & M \cong T/G, \end{tikzcd}$$

we obtain $\Gamma(M, E(T, F, \sigma)) \cong \Gamma(T, T \times F)^G$, and $\Gamma(T, T \times F)^G \cong \mathrm{Mor}(T, F)^G$ where we map $s \mapsto \mathrm{pr}_2 \circ s$ and $\{t \mapsto (t, f(t))\} \leftarrow f$. $\square$

§2. Nearly Holomorphic Modular Forms

We want to introduce and discuss the notion of nearly holomorphic modular forms, the objects that we will characterize in the main result. This chapter is divided into two sections. First, we give a quick review of the classical concept of modular forms. This allows us to set up the appropriate setting that leads to the definition of nearly holomorphic modular forms. Next, we switch to a more computational part, in which we present some concrete examples of nearly holomorphic modular forms obtained by applying Maass-Shimura operator to a specific class of functions.

2.1 Definitions

Starting from the first half of the nineteenth century, many generalizations of the classical notion of modular forms have been given, together with their applications in several area of mathematics. Regardless all of these generalizations, there exist some ingredients in common: an upper half space, a congruence subgroup acting on this upper half space, a functional equation, and a Fourier expansion. As all of these constituents are essential for rigorous definitions, we devote the next pages to outline these tools.
The main reference for this section is [Diamond, 2000], from where the notation is taken. Moreover we will use some definitions and notations introduced in the previous chapter.

Upper half space

The fundamental space is the complex upper half plane $\mathbf{H}$.

Congruence subgroup

Definition 2.1.1. A congruence subgroup of $\Gamma(1)$ is a subgroup $\Gamma \subseteq \Gamma(1)$ such that $\Gamma(N) \subseteq \Gamma$ for some N. The smallest N satisfying this condition is called the level of the congruence subgroup.

Let Γ be a fixed congruence subgroup. The left action of Γ on $\mathbf{H}$ comes from the one defined previously, while naming few examples of principal bundles. It is given by

$$\gamma\tau := \frac{a\tau + b}{c\tau + d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Functional equation

For a fixed integer k and for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, we define the weight-$k$ operator $[\gamma]_k$ on the set of functions $f : \mathbf{H} \to \mathbf{C}$, by

$$f[\gamma]_k(\tau) := (c\tau + d)^{-k} f(\gamma\tau).$$

Definition 2.1.2. For a fixed integer k, a meromorphic function $f : \mathbf{H} \to \mathbf{C}$ is weakly modular of weight k with respect to Γ if

$$f[\gamma]_k = f,$$

for all $\gamma \in \Gamma$.

We underline that while weak modularity does not make a function f fully Γ–invariant, at least $f(\tau)$ and $f(\gamma\tau)$ always have the same zeroes and poles, as the factor $(c\tau + d)$ on $\mathbf{H}$ has neither. Hence its zeros and poles are Γ–invariant as sets. Also, if a function is weakly modular of weight k with respect to some set of matrices then f is weakly modular of weight k with respect to the group of matrices the set generates.

Fourier expansion

Holomorphic modular forms are weakly modular functions that satisfy a holomorphy condition described through a Fourier expansion. We start by dealing with the case when Γ is the full modular group $\Gamma(1)$ and then switching to the case of smaller subgroups.
We recall that the full modular group is generated by the matrices

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, if $f[\gamma] = f$ for all $\gamma \in \Gamma(1)$, in particular the relation holds for the translation matrix T, whence $f(\tau + 1) = f(\tau)$, that is f is $\mathbf{Z}$–periodic. Let $D := \{q \in \mathbf{C} : |q| < 1\}$ be the complex unit disk and let $D' := D \setminus \{0\}$ be the punctured complex unit disk. Also, consider the $\mathbf{Z}$–periodic holomorphic map $\tau \mapsto e^{2\pi i\tau} := q$ mapping $\mathbf{H} \to D'$. Then the function $g : D' \to \mathbf{C}$ which maps $q \mapsto f(\log(q)/(2\pi i))$ is well defined and $f(\tau) = g(e^{2\pi i\tau})$. Furthermore, if f is holomorphic on the complex upper half plane then g is holomorphic on the punctured disk. Therefore g admits the Laurent expansion

$$g(q) := \sum_{n \in \mathbf{Z}} a_n q^n.$$

As $|q| = e^{-2\pi\Im(\tau)}$, then for $\Im(\tau) \to \infty$, $q \to 0$. We think of ∞ as a point lying far in the imaginary direction on the complex upper half plane. The idea is that for $y > 1$, the image in $\Gamma(1)\backslash\mathbf{H}$ of the part of $\mathbf{H}$ lying above the line $\Im(\tau) = y$ can be identified via q with the punctured disk $0 < q < e^{-2\pi y}$. The resulting Fourier expansion of f tells us that f is not only a well defined function on this punctured disk, but extends holomorphically to the point $q = 0$. This is the reason for which we will introduce later the compactification $\Gamma(1)\backslash(\mathbf{H} \cup \{\infty\})$, where the point ∞ corresponds to $q = 0$, with q as a local parameter.

Definition 2.1.3. A weakly modular function f with respect to $\Gamma(1)$ is holomorphic at ∞ if g extends holomorphically to the removed point $q = 0$, that is the Laurent expansion sums over $n \in \mathbf{N}$.

Therefore f has a Fourier expansion[1]

$$f(\tau) := \sum_{n=0}^{\infty} a_n(f) q^n,$$

for $q = e^{2\pi i \tau}$.

It is relevant to remark that as $q \to 0$ if and only if $\Im(\tau) \to \infty$, consequently it is sufficient to show that $f(\tau)$ is bounded as $\Im(\tau) \to \infty$ to say that a weakly modular function $f : \mathbf{H} \to \mathbf{C}$ with respect to $\Gamma(1)$, is holomorphic at ∞.

We now enlarge the picture by making the subgroup smaller. By definition any congruence subgroup contains $\Gamma(N)$ for some N, therefore it contains a translation matrix of the form $T_h := \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}$ for some positive integer h, hence each weakly modular function $f : \mathbf{H} \to \mathbf{C}$ is $h\mathbf{Z}$−periodic. So, similarly as in the previous case, there exists a corresponding function $g : D' \to \mathbf{C}$, such that $f(\tau) = g(q_h)$ where this time $q_h := e^{2\pi i \tau / h}$. Again, if f is holomorphic on $\mathbf{H}$ then g is holomorphic on D' and therefore it admits a Laurent expansion.

Definition 2.1.4. A weakly modular function f with respect to Γ is holomorphic at ∞ if g extends holomorphically to the removed point $q = 0$, that is the Laurent expansion sums over $n \in \mathbf{N}$.

Hence f has a Fourier expansion, which is this time given by

$$f(\tau) := \sum_{n=0}^{\infty} a_n(f) q_h^n,$$

for $q_h = e^{2\pi i \tau / h}$.

We require holomorphic modular forms to be holomorphic not only on $\mathbf{H}$ but also at limit points. For a congruence subgroup Γ we adjoin to the complex upper half plane not only ∞ but also other elements of $\mathbf{P}^1(\mathbf{Q})$, identified under the Γ−action. Indeed recall that

$$\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty := (1:0)\} = \mathbf{H}^{\pm} \cup \mathbf{R} \cup \{\infty\} = \mathbf{H}^{\pm} \cup \mathbf{P}^1(\mathbf{R}),$$

and $\mathbf{P}^1(\mathbf{Q}) \subseteq \mathbf{P}^1(\mathbf{R})$. Further there exists an action of Γ on $\mathbf{P}^1(\mathbf{C})$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z_0 : z_1) = (a z_0 + b z_1 : c z_0 + d z_1),$$

which can be restricted to $\mathbf{P}^1(\mathbf{Q})$ and which matches the action of Γ on $\mathbf{H}$.

Definition 2.1.5. A Γ−equivalence class of points in $\mathbf{P}^1(\mathbf{Q})$ is called a cusp of Γ.

[1] It is this Fourier expansion which is responsible for the enormous importance of modular forms, as it pops up that there are many examples of modular forms f for which the Fourier coefficients $a_n(f)$ are numbers that are of extreme interest in other areas of mathematics.

Proposition 2.1.6. For the full modular group $\Gamma(1)$, all elements of $\mathbf{P}^1(\mathbf{Q})$ are $\Gamma(1)-$equivalent to ∞.

PROOF. Let $(z_0 : z_1) \in \mathbf{P}^1(\mathbf{Q})$. We may assume $z_0, z_1 \in \mathbf{Z}$ coprime, by clearing the denominator and dividing by their gcd. By the completion lemma, there exist $b, d \in \mathbf{Z}$ such that $\begin{pmatrix} z_0 & b \\ z_1 & d \end{pmatrix} \in \Gamma(1)$. Then $\begin{pmatrix} z_0 & b \\ z_1 & d \end{pmatrix}(1:0) = (z_0 : z_1)$. □

The above proposition tells us that $\Gamma(1)$ has only one cusp, namely ∞. But when Γ is a smaller group then fewer points are $\Gamma-$equivalent and so Γ will have other cusps as well. We require a holomorphic modular function with respect to a congruence subgroup Γ to be holomorphic at the cusps.
Following the above proof, it is clear that for each $s \in \mathbf{P}^1(\mathbf{Q})$ there exists a matrix $\alpha \in \Gamma(1)$ such that $s = \alpha \cdot \infty$. Consequently being holomorphy at the cusp s is defined in terms of being holomorphy at ∞ through the weight$-k$ operator $[\alpha]_k$. Indeed if $f : \mathbf{H} \to \mathbf{C}$ is weakly modular of weight k with respect to Γ and holomorphic on $\mathbf{H}$, then $f[\alpha]_k$ is again holomorphic on $\mathbf{H}$, but now it is weakly modular with respect to[2] $\alpha^{-1}\Gamma\alpha$, and for the above discussion, the notion of its being holomorphy at ∞ is clear.
Let Γ be a congruence subgroup.

Definition 2.1.7. A holomorphic modular form of weight k with respect to Γ is a holomorphic function $f : \mathbf{H} \to \mathbf{C}$ such that

(i) $f[\gamma]_k = f$, for all $\gamma \in \Gamma$;

(ii) the functions $f[\alpha]_k$ are holomorphic at ∞ for all $\alpha \in \Gamma(1)$.

The space of holomorphic modular form of weight k and level Γ is usually denoted by $\mathcal{M}_k(\Gamma, \mathbf{C})$.
The zero function on $\mathbf{H}$ is a holomorphic modular form of every weight and every constant function on $\mathbf{H}$ is a holomorphic modular form of weight 0. Non-trivial examples of holomorphic modular forms are given by the Eisenstein series for even $k > 2$.
The generalized notion of holomorphic modular forms appealing us is given by nearly holomorphic modular forms: they are polynomials in $1/\Im(\tau)$ with coefficients that are holomorphic functions of τ, satisfying certain modular transformation and holomorphic properties.

Definition 2.1.8. Let $k, r \in \mathbf{Z}_{\geq 0}$, and $f : \mathbf{H} \to \mathbf{C}$. f is said to be a nearly holomorphic modular form of weight k and order $\leq r$ for a congruence subgroup Γ, if the following hold

(i) f is smooth;

(ii) $f[\gamma]_k = f$ for all $\gamma \in \Gamma$;

(iii) there exist holomorphic functions $f_0, ..., f_r$ on $\mathbf{H}$ such that

$$f(\tau) = \sum_{j=0}^{r} \frac{f_j(\tau)}{\Im(\tau)^j};$$

[2] As $f[\alpha]_k[\alpha^{-1}\gamma\alpha]_k = f[\gamma]_k[\alpha]_k = f[\alpha]_k$ for all $\gamma \in \Gamma$.

(iv) f has finite limit at the cusps.

The last property means that for each $\alpha \in \Gamma(1)$ we have

$$f[\alpha]_k(\tau) = \sum_{j=0}^{r} \frac{1}{\Im(\tau)^j} \sum_{n=0}^{\infty} a_{\alpha,j,n} e^{2\pi i n\tau/N_\alpha},$$

with $a_{\alpha,j,n} \in \mathbf{C}$ and a positive integer N_α. Indeed, this property generalizes the second property defining holomorphic modular forms. However, in this current situation, the function f is only smooth and not holomorphic, therefore it makes no sense the request of a special kind of Fourier expansion for f itself, on the other hand we can have a Fourier expansion for the holomorphic maps f_j.

The space of nearly holomorphic modular forms of weight k, order $\leq r$ and level Γ is denoted by $\mathbf{N}_k^r(\Gamma, \mathbf{C})$. It is clear that for $r = 0$ we obtain the space of holomorphic modular forms $\mathcal{M}_k(\Gamma, \mathbf{C})$.

2.2 More than Examples

The purpose of the second part of this chapter is to study explicit examples of nearly holomorphic modular forms using the action of the Maass-Shimura raising differential operator on holomorphic modular forms or generalized ones. We therefore start by introducing this operator, showing it preserves modularity at the expense of holomorphicity, and that when applying to nearly holomorphic modular form of weight k and degree $\leq r$ the result again turns out to be a nearly holomorphic modular form, but this time of weight $k+2$ and degree increased by 1. In the last part, we run some explicit computations involving Eisenstein series.

Maass-Shimura raising operator

As usual, Γ will be a congruence subgroup.

Definition 2.2.1. The Maass-Shimura raising differential operator on nearly holomorphic modular forms of weight k is the operator

$$\delta_k := \frac{1}{2\pi i}\left(\partial_\tau + \frac{k}{2i\Im(\tau)}\right).$$

Proposition 2.2.2. The Maass-Shimura raising differential operator δ_k takes value in

$$\delta_k : \mathbf{N}_k^r(\Gamma, \mathbf{C}) \to \mathbf{N}_{k+2}^{r+1}(\Gamma, \mathbf{C}).$$

PROOF. We need to show that if $f \in \mathbf{N}_k^r(\Gamma, \mathbf{C})$ then $\delta_k f \in \mathbf{N}_{k+2}^{r+1}(\Gamma, \mathbf{C})$. Clearly the resulting function $\delta_k f$ is again smooth (by definition of the operator), has finite limit at the cusps (as the resulting function preserves that the q−expansion sums over non negative integers), and it can be written as a polynomial in $1/\Im(\tau)$ with holomorphic coefficients. While it is clear by the presence of the factor $\frac{k}{2i\Im(\tau)}$ that $\delta_k f$ has degree increased by 1, it is not straightforward to see that $\delta_k f$ satisfies the modularity functional equation of weight $k+2$.

Hence it remains to prove that $\delta_k f[\gamma]_{k+2} = \delta_k f$ for all $\gamma \in \Gamma$. As $f \in \mathbf{N}_k^r(\Gamma, \mathbf{C})$, then $f[\gamma]_k = f$ for all $\gamma \in \Gamma$, therefore the statement is equivalent to show that

$$\delta_k f[\gamma]_{k+2} = \delta_k(f[\gamma]_k).$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the left hand side is given by[3]

$$\begin{aligned}\delta_k f[\gamma]_{k+2} &= \frac{1}{2\pi i}\left(\partial_\tau f + \frac{k}{2i\Im(\tau)} f\right)[\gamma]_{k+2} \\ &= \frac{1}{2\pi i}\left((c\tau+d)^{-k-2}\partial_\tau f(\gamma\tau) + \frac{k}{2i\Im(\gamma\tau)} f(\gamma\tau)(c\tau+d)^{-k-2}\right) \\ &= \frac{1}{2\pi i}\left((c\tau+d)^{-k-2}\partial_\tau f(\gamma\tau) + \frac{k}{2i\Im(\tau)} f(\gamma\tau)|c\tau+d|^2(c\tau+d)^{-k-2}\right).\end{aligned}$$

For $\tau = x + iy$, the right hand side is given by

$$\begin{aligned}\delta_k(f[\gamma]_k) &= \delta_k(f(\gamma\tau)(c\tau+d)^{-k}) \\ &= \frac{1}{2\pi i}\left(\partial_\tau + \frac{k}{2iy}\right)(f(\gamma\tau)(c\tau+d)^{-k}) \\ &= \frac{1}{2\pi i}\left(\partial_\tau f(\gamma\tau)\frac{1}{(c\tau+d)^2}(c\tau+d)^{-k} + f(\gamma\tau)(-ck)(c\tau+d)^{-k-1} + \frac{k}{2iy} f(\gamma\tau)(c\tau+d)^{-k}\right) \\ &= \frac{1}{2\pi i}\left(\partial_\tau f(\gamma\tau)(c\tau+d)^{-k-2} + \frac{k}{2iy} f(\gamma\tau)(c\tau+d)^{-k-1}(-2ciy + cx + ciy + d)\right) \\ &= \frac{1}{2\pi i}\left(\partial_\tau f(\gamma\tau)(c\tau+d)^{-k-2} + \frac{k}{2iy} f(\gamma\tau)|c\tau+d|^2(c\tau+d)^{-k-2}\right),\end{aligned}$$

which proves the equivalence and concludes the proof. □

As the application of Maass-Shimura operator increases the weight of 2, it makes sense to define the following: $\delta_k^0 := \mathrm{Id}$ and $\delta_k^2 := \delta_{k+2} \circ \delta_k$, and in general for $r > 1$

$$\delta_k^r := \delta_{k+2r-2} \circ \ldots \circ \delta_{k+2} \circ \delta_k.$$

Furthermore, as the application of this differential operator increases the degree by 1, it is clear that we never obtain holomorphic modular forms but only nearly holomorphic ones. Also, we point out that not all nearly holomorphic modular forms are the results of Maass-Shimura operator applied to some modular forms of some kind. A nice counterexample is given in the following section in which we discuss some nearly holomorphic Eisenstein series.

[3] Note that $\Im(\gamma\tau) = \frac{\Im(\tau)}{|c\tau+d|^2}$.

Eisenstein series

To produce some examples of nearly holomorphic modular forms we apply Maass-Shimura operator to the well known holomorphic modular forms given by the normalized Eisenstein series in their analytic expressions

$$E_k(\tau) := \frac{1}{2} \sum_{\substack{(c,d)\in \mathbf{Z}^2 \smallsetminus (0,0) \\ (c,d)=1}} \frac{1}{(c\tau + d)^k},$$

for even $k > 2$.

The resulting series, the non holomorphic Eisenstein series, are not only examples of nearly holomorphic modular forms: they lead naturally to related subjects that are appealing in their own right, such as zeta and L−functions, Bernoulli numbers, Fourier analysis, theta functions, and Mellin transformation. Non holomorphic Eisenstein series came to fashion thanks to the work of Hecke and Shimura, by augmenting the series with an extra parameter.

A first exposure to Eisenstein series can be found in Chapter 1 but mostly in Chapter 4 of [Diamond, 2000], while a more exciting reading is given in the article 83b by [Shimura, 2003].

It is easier to divide the computation in two steps: in the former we apply the operator only to the addend $\frac{1}{(c\tau+d)^k}$, while in the latter we deal with the whole series.

For the moment we only state the following computational proposition.

Proposition 2.2.3.

$$\delta_k\left(\frac{1}{(c\tau + d)^k}\right) = \frac{k}{(2i)^2 \pi y} \cdot \frac{c\overline{\tau} + d}{(c\tau + d)^{k+1}},$$

and

$$\delta_k^r\left(\frac{1}{(c\tau + d)^k}\right) = \frac{(k)_r}{(2i)^{2r} (\pi y)^r} \cdot \frac{(c\overline{\tau} + d)^r}{(c\tau + d)^{k+r}},$$

for $r > 1$, where $(k)_r = k(k+1) \cdot \ldots \cdot (k + r - 1)$.

It is then a straightforward computation to obtain

$$\delta_k^r(E_k(\tau)) = \frac{1}{2} \cdot \frac{(k)_r}{(2i)^{2r} (\pi y)^r} \cdot \sum_{\substack{(c,d)\in \mathbf{Z}^2 \smallsetminus (0,0) \\ (c,d)=1}} \frac{(c\overline{\tau} + d)^r}{(c\tau + d)^{k+r}},$$

for even $k > 2$ and for $r > 1$.

We point out that the series found are related[4] to the following generalized Eisenstein series

$$G_k^v(\tau, s) := \sum_{\substack{(c,d)\in \mathbf{Z}^2 \smallsetminus (0,0) \\ (c,d)\equiv v \mod N}} \frac{y^s}{(c\tau + d)^k |c\tau + d|^{2s}},$$

for N a positive integer, v a vector in $\mathbf{Z}^2$, and k any integer[5].

[4] Recall that for a complex number x we have $|x|^2 = x\overline{x}$.

[5] Note that here we do not have the condition $(c,d) = 1$. This condition is due to the fact that E_k it is obtained from the non normalized Eisenstein series defined over a lattice.

PROOF. The first result follows in the following way

$$
\begin{aligned}
\delta_k\left(\frac{1}{(c\tau+d)^k}\right) &= \frac{1}{2\pi i}\left(\partial_\tau + \frac{k}{2iy}\right)\left(\frac{1}{(c\tau+d)^k}\right)\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(c\tau+d)^k}\left(\frac{-c}{c\tau+d}+\frac{1}{2iy}\right)\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(c\tau+d)^{k+1}2iy}\cdot(-2ciy+cx+ciy+d)\\
&= \frac{k}{(2i)^2\pi y}\cdot\frac{c\overline{\tau}+d}{(c\tau+d)^{k+1}}.
\end{aligned}
$$

The second one requires a bit more of computation and follows by induction on $r > 1$. Therefore we start by computing δ_k^2.

$$
\begin{aligned}
\delta_k^2\left(\frac{1}{(c\tau+d)^k}\right) &= \delta_{k+2}\circ\delta_k\left(\frac{1}{(c\tau+d)^k}\right)\\
&= \delta_{k+2}\left(\frac{k}{(2i)^2\pi y}\cdot\frac{c\overline{\tau}+d}{(c\tau+d)^{k+1}}\right)\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\left(\partial_\tau+\frac{k+2}{2iy}\right)\left(\frac{1}{y}\cdot\frac{c\overline{\tau}+d}{(c\tau+d)^{k+1}}\right)\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\left(\partial_\tau\left(\frac{1}{y}\cdot\frac{c\overline{\tau}+d}{(c\tau+d)^{k+1}}\right)+\frac{k+2}{2iy}\left(\frac{1}{y}\cdot\frac{c\overline{\tau}+d}{(c\tau+d)^{k+1}}\right)\right)\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\left(\frac{c\overline{\tau}+d}{2y^2(c\tau+d)^{k+2}}\cdot(i(c\tau+d)-2yc(k+1))+\frac{k+2}{2iy}\left(\frac{1}{y}\cdot\frac{c\overline{\tau}+d}{(c\tau+d)^{k+1}}\right)\right)\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\cdot\frac{c\overline{\tau}+d}{2y^2(c\tau+d)^{k+2}}\cdot((k+2)(-i)(c\tau+d)+i(c\tau+d)-2yc(k+1))\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\cdot\frac{c\overline{\tau}+d}{2y^2(c\tau+d)^{k+2}}\cdot((-i)(c\tau+d)(k+1)-2yc(k+1))\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\cdot\frac{c\overline{\tau}+d}{2y^2(c\tau+d)^{k+2}}\cdot(-ki(c\tau+d)-i(c\tau+d)-2yc(k+1))\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\cdot\frac{c\overline{\tau}+d}{2y^2(c\tau+d)^{k+2}}\cdot(-kicx+kcy-kid-2yck-icx+cy-id-2yc)\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\cdot\frac{c\overline{\tau}+d}{2y^2(c\tau+d)^{k+2}}\cdot(-ki(cx-icy+d)-i(cx-ciy+d))\\
&= \frac{1}{2\pi i}\cdot\frac{k}{(2i)^2\pi}\cdot\frac{c\overline{\tau}+d}{2y^2(c\tau+d)^{k+2}}\cdot(-i)(k+1)(c\overline{\tau}+d)\\
&= \frac{1}{2\pi i^2}\cdot\frac{k(k+1)}{(2i)^2\pi}\cdot\frac{(c\overline{\tau}+d)^2}{2y^2(c\tau+d)^{k+2}}\\
&= \frac{(k)_2}{(2i)^4(\pi y)^2}\cdot\frac{(c\overline{\tau}+d)^2}{(c\tau+d)^{k+2}}.
\end{aligned}
$$

Next, suppose the statement is true for $r-1$. We compute δ_k^r as $\delta_{k+2r-2} \circ \delta_k^{r-1}$.

$$\begin{aligned}
\delta_k^r\left(\frac{1}{(c\tau+d)^k}\right) &= \delta_{k+2r-2}\left(\frac{(k)_{r-1}}{(2i)^{2(r-1)}(\pi y)^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{(c\tau+d)^{k+r-1}}\right) \\
&= \frac{1}{2\pi i} \cdot \frac{(k)_{r-1}}{(2i)^{2(r-1)}\pi^{r-1}}\left(\partial_\tau + \frac{k+2r-2}{2iy}\right)\left(\frac{1}{y^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{(c\tau+d)^{k+r-1}}\right) \\
&= \frac{1}{2\pi i} \cdot \frac{(k)_{r-1}}{(2i)^{2(r-1)}\pi^{r-1}}\left(\partial_\tau\left(\frac{1}{y^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{(c\tau+d)^{k+r-1}}\right)+\right. \\
&\left.+\frac{k+2r-2}{2iy}\left(\frac{1}{y^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{(c\tau+d)^{k+r-1}}\right)\right) \\
&= \frac{1}{2\pi i} \cdot \frac{(k)_{r-1}}{(2i)^{2(r-1)}\pi^{r-1}}\left(\frac{(c\overline{\tau}+d)^{r-1}}{2y^r(c\tau+d)^{k+r}} \cdot ((c\tau+d)(r-1)i - 2yc(k+r-1))+\right. \\
&\left.+\frac{k+2r-2}{2iy} \cdot \frac{1}{y^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{(c\tau+d)^{k+r-1}}\right) \\
&= \frac{1}{2\pi i} \cdot \frac{(k)_{r-1}}{(2i)^{2(r-1)}\pi^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{2iy^r(c\tau+d)^{k+r}} \cdot (-(c\tau+d)(r-1) - 2iyc(k+r-1)+ \\
&+(k+2r-2)(c\tau+d)) \\
&= \frac{1}{2\pi i} \cdot \frac{(k)_{r-1}}{(2i)^{2(r-1)}\pi^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{2iy^r(c\tau+d)^{k+r}} \cdot ((c\tau+d)(k+r-1) - 2iyc(k+r-1)) \\
&= \frac{1}{2\pi i} \cdot \frac{(k)_{r-1}}{(2i)^{2(r-1)}\pi^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{2iy^r(c\tau+d)^{k+r}} \cdot (k+r-1)(cx+ciy+d-2ciy) \\
&= \frac{1}{2\pi i} \cdot \frac{(k)_{r-1}}{(2i)^{2(r-1)}\pi^{r-1}} \cdot \frac{(c\overline{\tau}+d)^{r-1}}{2iy^r(c\tau+d)^{k+r}} \cdot (k+r-1)(c\overline{\tau}+d) \\
&= \frac{(k)_r}{(2i)^{2r}(\pi y)^r} \cdot \frac{(c\overline{\tau}+d)^r}{(c\tau+d)^{k+r}}.
\end{aligned}$$

Which completes the proof. □

The Eisenstein series $\hat{G}_2$

In the above section we have considered the normalized Eisenstein series of even weight for $k > 2$. Indeed the series are absolutely convergent for $k > 2$ and they define holomorphic modular forms of weight k. For $k = 2$ we still get a holomorphic function, which is however not modular. It is the non-normalized Eisenstein series

$$G_2(\tau) := \frac{1}{(2\pi i)^2}\sum_{c\in\mathbf{Z}}\left(\sum_{d\in\mathbf{Z}\smallsetminus\{0\}}\frac{1}{(c\tau+d)^2}\right).$$

Notice that, apart from some normalization factors, it is given by a similar expression as before, if we carry out the summation over c first and then over d. As we have a non-absolute convergence series, the point is that now we can not interchange the order of the summation which results in loosing modularity. Namely, as said before, to check modularity we would only need to verify if the functional equations are satisfied for the two generators S and T of $\Gamma(1)$, as we would like G_2 to be an element of $\mathcal{M}_2(\Gamma(1), \mathbf{C})$. But while it is clear from the definition of G_2 that $G_2[T]_2 = G_2$ is invariant under translation, $G_2[S]_2$ turns out to equals $G_2 - 1/(2\pi i\tau)$, whence modularity is lost.

However the problem might be solved if we add the extra factor $1/(4\pi\Im(\tau))$, indeed

$$\hat{G}_2(\tau) := G_2(\tau) + \frac{1}{4\pi\Im(\tau)}$$

is $\Gamma(1)-$invariant and defines therefore an element[6] of $\mathcal{N}_2^1(\Gamma(1), \mathbf{C})$. The statement is not trivial: as for holomorphicity of the addend G_2, the Fourier expansion of the Eisenstein series G_k converges rapidly and defines a holomorphic function of τ also for $k = 2$, while the modular invariance can be proved in several ways, but we recall the one due to Hecke, which involves a slightly modification of the sum, given by

$$G_{2,\varepsilon}(\tau) := \frac{1}{2} \sum_{(c,d)\in\mathbf{Z}^2\setminus(0,0)} \frac{1}{(c\tau+d)^2|c\tau+d|^{2\varepsilon}},$$

for $\tau \in \mathbf{H}$ and $\varepsilon > 0$. The idea is to prove that $G_{2,\varepsilon}$ converges absolutely, has some modular properties and it admits a finite limit for $\varepsilon \to 0$.
If $\hat{G}_2$ were the result of the application of Maass-Shimura operator to some modular form f, then this function would come from a modular form of 0 degree and 0 weight, which forces f to be an element in $\mathcal{M}_0(\Gamma(1), \mathbf{C})$. However, an easy consequence of the Weight Formula shows that $\mathcal{M}_0(\Gamma(1), \mathbf{C}) \cong \mathbf{C}\cdot 1$ is given by constant functions, and there does not exist a constant function such that it transforms into $\hat{G}_2$ via the action of Maass-Shimura.
The Fourier expansion of the normalized and non holomorphic Eisenstein series E_2 is given by

$$E_2(\tau) := 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^n + \frac{1}{4\pi y} \text{ for } \sigma_1(n) := \sum_{d|n, d>0} d.$$

Further, this series appears in a specific decomposition as the only obstruction to a structure theorem, as stated in the following proposition, which can be found in [Urban, 2013] and [Rosso, 2014].

Proposition 2.2.4. Let $f \in \mathcal{N}_k^r(\Gamma, \mathbf{C})$ and suppose that $(k, r) \neq (2, 1)$. If k is odd, or k is even and $k > 2 + 2r$ then[7] there exist $g_i \in \mathcal{M}_{k-2i}(\Gamma, \mathbf{C})$ for $i = 0, ..., r$ such that

$$f(\tau) = \sum_{i=0}^{r} \delta_{k-2i}^i g_i(\tau).$$

If k is even and $2 \leq k < 2 + 2r$ then there exist $g_i \in \mathcal{M}_{k-2i}(\Gamma, \mathbf{C})$ for $i = 0, ..., r-1$ and $c \in \mathbf{C}^\times$ such that

$$f(\tau) = \sum_{i=0}^{r-1} \delta_{k-2i}^i g_i(\tau) + c\delta_2^{(k-2)/2} E_2(\tau).$$

The result is due to Shimura and states that all nearly holomorphic modular forms are obtained by applying differential operators on classical holomorphic modular forms or on E_2, giving a direct sum decomposition. Interesting examples of nearly holomorphic modular forms are therefore given by applying Maass-Shimura operator to[8] $\hat{G}_2$.

[6]For more details about this series, we mention section 2 of Chapter 3 in [Schoeneberg, 1974].
[7]The hypothesis $k > 2 + 2r$ implies that we never reach down to weight 2.
[8]We choose this representation of this Eisenstein series has it resembles the previous case.

$$
\begin{aligned}
\delta_2(\hat{G}_2) &= \frac{1}{2\pi i}\left(\partial_\tau + \frac{1}{iy}\right)(\hat{G}_2)\\
&= \frac{1}{2\pi i}\left(\partial_\tau + \frac{1}{iy}\right)\left(\frac{1}{(2\pi i)^2}\sum_{c\in\mathbf{Z}}\sum_{d\in\mathbf{Z}\setminus\{0\}}\frac{1}{(c\tau+d)^2} + \frac{1}{4\pi y}\right)\\
&= \frac{1}{2\pi i}\left(\frac{1}{(2\pi i)^2}\sum_{c\in\mathbf{Z}}\sum_{d\in\mathbf{Z}\setminus\{0\}}\frac{(-2c)}{(c\tau+d)^2} + \frac{i}{8\pi y^2} + \frac{1}{iy}\sum_{c\in\mathbf{Z}}\sum_{d\in\mathbf{Z}\setminus\{0\}}\frac{1}{(c\tau+d)^2} + \frac{1}{iy}\cdot\frac{1}{4\pi y}\right)\\
&= \frac{1}{2\pi i}\left(\frac{1}{(2\pi i)^2 iy}\sum_{c\in\mathbf{Z}}\sum_{d\in\mathbf{Z}\setminus\{0\}}\frac{1}{(c\tau+d)^2}\cdot\left(\frac{-2ciy+cx+ciy+d}{c\tau+d}\right) - \frac{i}{8\pi y^2}\right)\\
&= \frac{1}{2\pi i}\left(\frac{1}{(2\pi i)^2 iy}\sum_{c\in\mathbf{Z}}\sum_{d\in\mathbf{Z}\setminus\{0\}}\frac{c\overline{\tau}+d}{(c\tau+d)^3} - \frac{i}{8\pi y^2}\right)\\
&= \frac{1}{2\pi i}\cdot\frac{1}{iy}\left(\frac{1}{(2\pi i)^2}\sum_{c\in\mathbf{Z}}\sum_{d\in\mathbf{Z}\setminus\{0\}}\frac{c\overline{\tau}+d}{(c\tau+d)^3} + \frac{1}{8\pi y}\right).
\end{aligned}
$$

§3. More on Vector Bundles

The goal of this chapter is to exploit the tools defined in Chapter 1 to fashion the specific vector bundle involved in the final geometric result. To obtain the bundle of our interest, $\mathcal{H}_k^r(X(\Gamma))$, which will be defined in 4.2.5, we start by introducing the fundamental vector bundle $\underline{\omega}^k$ on $Y(\Gamma)$. We then give an algebraic interpretation of $\mathcal{H}^1_{dR}$, whose Hodge decomposition will be seen to be strictly related to the previous bundle. Lastly, we give a quick recollection of the structure of symmetric bundle of a bundle. In addition, for each of these bundles we will present some properties of the corresponding sections.

As usual, Γ will be a congruence subgroup acting freely on $\mathbf{H}$. As mentioned in Chapter 1, a general bundle $\pi : T \to M$ requires a fixed manifold M, over which it is defined. For our purpose, we are interested in the Riemann surface

$$Y(\Gamma) := \Gamma\backslash\mathbf{H},$$

called the modular curve. As there exist bijections between modular curves and moduli spaces, which are equivalence classes of elliptic curves enhanced with some torsion data, we can see each modular curve as an algebraic variety whose points classify isomorphism classes of other algebraic varieties of some fixed type. We are also interested in the slightly larger space $X(\Gamma)$ which is a compactification of $Y(\Gamma)$.

Definition 3.0.5. Let $X(\Gamma) := \Gamma\backslash\mathbf{H}^*$ where $\mathbf{H}^* := \mathbf{H} \cup \mathbf{P}^1(\mathbf{Q})$ is the complex upper half plane together with the cusps.

3.1 Fundamental Vector Bundles

Recall the two associated line bundles $\underline{\omega}^k_{\mathbf{H}}$ and $\overline{\underline{\omega}}^k_{\mathbf{H}}$ previously defined while naming few examples of bundles in Chapter 1. Considering $p : \Gamma\backslash\operatorname{GL}^+ \to Y(\Gamma)$ instead of $P : \operatorname{GL}^+ \to \mathbf{H}$, we get the following bundles over $Y(\Gamma)$ with typical fibre $\mathbf{C}$:

- $\underline{\omega}^k := E(\Gamma\backslash\operatorname{GL}^+, \mathbf{C}, \sigma^k) = (\Gamma\backslash\operatorname{GL}^+ \times \mathbf{C})/\mathbf{C}^\times$;
- $\overline{\underline{\omega}}^k := E(\Gamma\backslash\operatorname{GL}^+, \mathbf{C}, \overline{\sigma}^k) = (\Gamma\backslash\operatorname{GL}^+ \times \mathbf{C})/\mathbf{C}^\times$.

To simplify the notation we write $\underline{\omega}$ and $\overline{\underline{\omega}}$ for $k = 1$. We now see how weakly modular functions of weight k come into play as sections of the fundamental vector bundle $\underline{\omega}^k$. We will present how to get the same results in two ways: the former is based on a technical proposition stated in Chapter 1, while in the latter we give an explicit expression of the sections of the pullback over $\mathbf{H}$ of the bundle $\underline{\omega}^k$, corresponding to $\underline{\omega}^k_{\mathbf{H}}$.

First approach

Recalling proposition 1.2.4, which identifies the sections of the associated bundle with a set of G−invariant functions, we represent weakly modular functions as sections of the fundamental vector bundle $\underline{\omega}^k$.

Proposition 3.1.1. There exists a bijection between sections of the fundamental vector bundle $\underline{\omega}^k$ over $Y(\Gamma)$ and the set of weakly modular functions of weight k with respect to Γ

$$\Gamma(Y(\Gamma), \underline{\omega}^k) \cong \{f : \mathbf{H} \to \mathbf{C} : f[\gamma]_k = f \;\; \forall \gamma \in \Gamma\}.$$

PROOF. By proposition 1.2.4

$$\Gamma(Y(\Gamma), \underline{\omega}^k) \cong \mathrm{Mor}(\Gamma \backslash \mathrm{GL}^+, \mathbf{C})^{\mathbf{C}^\times},$$

where by definition

$$\mathrm{Mor}(\Gamma \backslash \mathrm{GL}^+, \mathbf{C})^{\mathbf{C}^\times} := \left\{ F : \Gamma \backslash \mathrm{GL}^+ \to \mathbf{C} : F \text{ is holomorphic, } F\begin{pmatrix} \omega_1 \lambda \\ \omega_2 \lambda \end{pmatrix} = \lambda^{-k} F\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right\}.$$

It remains to show that the last set corresponds to the set of weakly modular functions of weight k with respect to Γ. Recall that a function $F : \mathrm{GL}^+ \to \mathbf{C}$, satisfying $F\binom{\omega_1 \lambda}{\omega_2 \lambda} = \lambda^{-k} F\binom{\omega_1}{\omega_2}$ for all $\lambda \in \mathbf{C}^\times$, is said to be of weight k. Moreover each $F : \mathrm{GL}^+ \to \mathbf{C}$ defines a function $f : \mathbf{H} \to \mathbf{C}$ via $f(\tau) := F\binom{2\pi i \tau}{2\pi i}$. Last, if a function F of weight k is invariant under Γ, that is there exists a well defined function $F : \Gamma \backslash \mathrm{GL}^+ \to \mathbf{C}$, then the corresponding map f is weakly modular of weight k. Indeed for all $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$\begin{aligned} f(\tau) &= F\begin{pmatrix} 2\pi i \tau \\ 2\pi i \end{pmatrix} \\ &= (F \circ \gamma)\begin{pmatrix} 2\pi i \tau \\ 2\pi i \end{pmatrix} \\ &= F\begin{pmatrix} 2\pi i (a\tau + b) \\ 2\pi i (c\tau + d) \end{pmatrix} \\ &= (c\tau + d)^{-k} F\begin{pmatrix} 2\pi i \gamma\tau \\ 2\pi i \end{pmatrix} \\ &= (c\tau + d)^{-k} f(\gamma\tau), \end{aligned}$$

which concludes the proof that Γ−invariant functions of weight k on GL^+ correspond to weakly modular functions of weight k with respect to Γ. □

Second approach

Let $\underline{\omega}_{\mathbf{H}}^k := E(\mathrm{GL}^+, \mathbf{C}, \sigma^k) = (\mathrm{GL}^+ \times \mathbf{C})/\mathbf{C}^\times$, the vector bundle corresponding to the pullback of $\underline{\omega}^k$ through $q : \mathbf{H} \to Y(\Gamma)$. The intention is now to construct sections for these bundle, as they are deeply related to the ones of $\underline{\omega}^k$. We start by showing the relationship and then give an explicit characterization.

Proposition 3.1.2. The sections of the fundamental vector bundle over $Y(\Gamma)$ correspond to the Γ-invariant sections of $\underline{\omega}_{\mathbf{H}}^k$

$$\Gamma(Y(\Gamma), \underline{\omega}^k) \cong \Gamma(\mathbf{H}, \underline{\omega}_{\mathbf{H}}^k)^\Gamma.$$

PROOF. It suffices to apply proposition 1.2.3 to the following commutative diagram

$$\begin{tikzcd} \underline{\omega}_{\mathbf{H}}^k \arrow[r, "Q"] \arrow[d] & \underline{\omega}^k \arrow[d] \\ \mathbf{H} \arrow[r, "q"'] & Y(\mathbf{H}). \end{tikzcd}$$

□

We now give an explicit description of the elements of $\Gamma(\mathbf{H}, \underline{\omega}_{\mathbf{H}}^k)^\Gamma$.

Definition 3.1.3. The section $\omega^k \in \Gamma(\mathbf{H}, \underline{\omega}_{\mathbf{H}}^k)$ is given by

$$\omega^k(\tau) := \left[\begin{pmatrix} \tau \\ 1 \end{pmatrix}, 1\right].$$

Proposition 3.1.4. Each section of $\underline{\omega}_{\mathbf{H}}^k$ is of the form[1] $f \cdot \omega^k$ for some $f : \mathbf{H} \to \mathbf{C}$ holomorphic.

PROOF. Note that the bundle $\underline{\omega}_{\mathbf{H}}^k$ is isomorphic to $\mathbf{H} \times \mathbf{C}$ via $\varphi : \mathbf{H} \times \mathbf{C} \xrightarrow{\sim} \underline{\omega}_{\mathbf{H}}^k$ given by

$$(\tau, z) \mapsto z \cdot \omega^k(\tau) = \left[\begin{pmatrix} \tau \\ 1 \end{pmatrix}, z\right]$$

with inverse

$$\left(\frac{\omega_1}{\omega_2}, \omega_2^k z\right) \leftarrow\!\shortmid \left[\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}, z\right].$$

As sections of $\mathbf{H} \times \mathbf{C}$ are given by $\tau \mapsto (\tau, f(\tau))$ for some $f : \mathbf{H} \to \mathbf{C}$ holomorphic, we get the sections of the desired bundle as

$$\mathbf{H} \to \mathbf{H} \times \mathbf{C} \cong \underline{\omega}_{\mathbf{H}}^k$$

$$\tau \mapsto (\tau, f(\tau)) \mapsto f(\tau) \cdot \omega^k(\tau) = \left[\begin{pmatrix} \tau \\ 1 \end{pmatrix}, f(\tau)\right].$$

□

By proposition 3.1.2, we are only interested in the Γ-invariant sections of $\underline{\omega}_{\mathbf{H}}^k$.

[1] It is meant component wise product.

Proposition 3.1.5. $f \cdot \omega^k$ is Γ−invariant if and only if f is a weakly modular function of weight k with respect to Γ, that is $\gamma f \omega^k(\tau) = f\omega^k(\gamma\tau)$.

PROOF. For each $\gamma \in \Gamma$ we need to show that

$$\left[\gamma\begin{pmatrix}\tau\\1\end{pmatrix}, f(\tau)\right] = \left[\begin{pmatrix}\gamma\tau\\1\end{pmatrix}, f(\gamma\tau)\right]$$

if and only if $f[\gamma]_k = f$. For $\gamma = \begin{pmatrix}a & b\\c & d\end{pmatrix}$, we have

$$\begin{aligned}\left[\gamma\begin{pmatrix}\tau\\1\end{pmatrix}, f(\tau)\right] &= \left[\begin{pmatrix}a\tau+b\\c\tau+d\end{pmatrix}, f(\tau)\right]\\ &= \left[(c\tau+d)\begin{pmatrix}\gamma\tau\\1\end{pmatrix}, f(\tau)\right]\\ &= \left[\begin{pmatrix}\gamma\tau\\1\end{pmatrix}, (c\tau+d)^k f(\tau)\right],\end{aligned}$$

and $(c\tau+d)^k f(\tau) = f(\gamma\tau)$ if and only if f is a weakly modular form of weight k with respect to Γ. □

Combining the last results we obtain

$$f \cdot \omega^k \in \Gamma(\mathbf{H}, \underline{\omega}_{\mathbf{H}}^k)^\Gamma \cong \Gamma(Y(\Gamma), \underline{\omega}^k) \Leftrightarrow f[\gamma]_k = f \;\; \forall\gamma \in \Gamma.$$

3.2 De Rham Bundle

We present the De Rham vector bundle through a purely algebraic construction, using the associated vector bundle. For this aim, consider the principal left Γ−bundle $q : \mathbf{H} \to Y(\Gamma)$ and the vector space $\mathbf{C}^2$. We define a right action $\rho : \mathbf{C}^2 \times \Gamma \to \mathbf{C}^2$ given by inverse matrix multiplication $v \cdot \gamma^{-1}$, for $v \in \mathbf{C}^2$ and $\gamma \in \Gamma$. Whence Γ acts on the left on $\mathbf{H} \times \mathbf{C}^2$ by

$$\gamma(\tau, v) = (\gamma\tau, (\rho(\gamma))^{-1}v) = (\gamma\tau, \gamma v).$$

The associated rank 2 vector bundle on $Y(\Gamma)$ is therefore given by

$$\mathcal{H}^1_{dR} \coloneqq E(\mathbf{H}, \mathbf{C}^2, \rho) = \Gamma\backslash(\mathbf{H} \times \mathbf{C}^2).$$

The structure of the De Rham vector bundle can be studied through its Hodge decomposition, which relates $\mathcal{H}^1_{dR}$ to the fundamental vector bundles $\underline{\omega}$ and $\overline{\underline{\omega}}$.

Hodge decomposition

We proceed in the following way, by producing an explicit isomorphism at the $\mathbf{H}$ level and then proving it is Γ−invariant. The result will be the decomposition

$$\underline{\omega} \oplus \overline{\underline{\omega}} \cong \mathcal{H}^1_{dR}.$$

Proposition 3.2.1. The map

$$\varphi : q^*(\underline{\omega} \oplus \overline{\underline{\omega}}) = (\mathrm{GL}^+ \times \mathbf{C} \oplus \mathbf{C})/\mathbf{C}^\times \to \mathbf{H} \times \mathbf{C}^2 = q^*\mathcal{H}^1_{dR}$$

defined by

$$\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, z, w\right] \mapsto \left(\frac{\omega_1}{\omega_2}, \begin{pmatrix}\omega_1 & \overline{\omega}_1\\ \omega_2 & \overline{\omega}_2\end{pmatrix}\begin{pmatrix}z\\ w\end{pmatrix}\right) = \left(\frac{\omega_1}{\omega_2}, \begin{pmatrix}\omega_1 z + \overline{\omega}_1 w\\ \omega_2 z + \overline{\omega}_2 w\end{pmatrix}\right)$$

is a $\Gamma-$equivariant isomorphism of vector bundles, which induces the Hodge decomposition

$$\underline{\omega} \oplus \overline{\underline{\omega}} \cong \mathcal{H}^1_{dR}.$$

PROOF. Clearly the map is well defined as for each $\lambda \in \mathbf{C}^\times$ it holds

$$\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, z, w\right] = \left[\begin{pmatrix}\omega_1\lambda\\ \omega_2\lambda\end{pmatrix}, \lambda^{-1}z, \overline{\lambda}^{-1}w\right] \mapsto \left(\frac{\omega_1}{\omega_2}, \begin{pmatrix}\omega_1\lambda & \overline{\omega}_1\overline{\lambda}\\ \omega_2\lambda & \overline{\omega}_2\overline{\lambda}\end{pmatrix}\begin{pmatrix}\lambda^{-1}z\\ \overline{\lambda}^{-1}w\end{pmatrix}\right) = \left(\frac{\omega_1}{\omega_2}, \begin{pmatrix}\omega_1 z + \overline{\omega}_1 w\\ \omega_2 z + \overline{\omega}_2 w\end{pmatrix}\right).$$

Further, it is $\Gamma-$ equivariant as the following diagram, in which the horizontal arrows are given by the actions of Γ on $q^*(\underline{\omega} \oplus \overline{\underline{\omega}})$ and on $q^*(\mathcal{H}^1_{dR})$ respectively, commutes.

$$\begin{tikzcd}
\Gamma \times q^*(\underline{\omega} \oplus \overline{\underline{\omega}}) \arrow[r] \arrow[d, "\mathrm{Id} \times \varphi"'] & q^*(\underline{\omega} \oplus \overline{\underline{\omega}}) \arrow[d, "\varphi"] \\
\Gamma \times q^*(\mathcal{H}^1_{dR}) \arrow[r] & q^*(\mathcal{H}^1_{dR})
\end{tikzcd}$$

Indeed For $\gamma = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma$, we have

$$\begin{tikzcd}
\left(\gamma, \left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, z, w\right]\right) \arrow[r] \arrow[d] & \left[\gamma\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, z, w\right] \arrow[d] \\
\left(\gamma, \left(\frac{\omega_1}{\omega_2}, \begin{pmatrix}\omega_1 z + \overline{\omega}_1 w\\ \omega_2 z + \overline{\omega}_2 w\end{pmatrix}\right)\right) \arrow[r] & \left(\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}, \begin{pmatrix}a\omega_1 z + a\overline{\omega}_1 w + b\omega_2 z + b\overline{\omega}_2 w\\ c\omega_1 z + c\overline{\omega}_1 w + d\omega_2 z + d\overline{\omega}_2 w\end{pmatrix}\right).
\end{tikzcd}$$

Last, it is an isomorphism of vector bundles as the following diagram commutes.

$$\begin{tikzcd}
q^*(\underline{\omega} \oplus \overline{\underline{\omega}}) \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & q^*(\mathcal{H}^1_{dR}) \arrow[dl, "\pi_2"] \\
\mathbf{H} &
\end{tikzcd}$$

Indeed

$$\begin{tikzcd}
\left[\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}, z, w\right] \arrow[r, "\varphi"] \arrow[d, "\pi_1"'] & \left(\frac{\omega_1}{\omega_2}, \begin{pmatrix}\omega_1 z + \overline{\omega}_1 w\\ \omega_2 z + \overline{\omega}_2 w\end{pmatrix}\right) \arrow[dl, "\pi_2"] \\
\frac{\omega_1}{\omega_2}. &
\end{tikzcd}$$

where $\pi_1 : q^*(\underline{\omega} \oplus \overline{\underline{\omega}}) \to \mathbf{H}$ is defined via $P : \mathrm{GL}^+ \to \mathbf{H}$ and $\pi_2 : q^*(\mathcal{H}^1_{dR}) \to \mathbf{H}$ is the projection into the first component. $\square$

Different bases

We present two different bases for the set of sections of the De Rham bundle: a combination of them will play a fundamental role in the characterization theorem in Chapter 4. The first one is given by the Hodge decomposition together with the previous results on the sections of $\underline{\omega}_{\mathbf{H}}^{k}$, the second one involves the universal elliptic curves. These two approaches allow to compare differential forms and paths on the space of the De Rham bundle, setting the basis for the theory of periods.

First basis

The Hodge decomposition $\underline{\omega} \oplus \overline{\underline{\omega}} \cong \mathcal{H}^1_{dR}$ together with proposition 3.1.4, gives us the basis $\mathcal{C}^\infty_{\mathbf{H}}\omega \oplus \mathcal{C}^\infty_{\mathbf{H}}\overline{\omega}$ for the set of sections of $q^*\mathcal{H}^1_{dR} \otimes \mathcal{C}^\infty_{\mathbf{H}} = \mathbf{H} \times \mathbf{C}^2 \otimes \mathcal{C}^\infty_{\mathbf{H}}$.

Second basis

We start with a short introduction in the theory of periods[2], and then switch to the construction of the universal elliptic curve. The result will be a new basis for the set of sections of our bundle. For what follows, we assume some knowledge in algebraic geometry, and we refer to [Hartshorne, 1977], [Görtz, 2010] and [Mumford, 1970] for further details. We only recall that under some assumptions, the right derived functors are just the sheaf cohomology of $\mathcal{F}$ on X: $R^n f_*\mathcal{F} = H^n(X, \mathcal{F})$, for $f : X \to S$, a topological space X and a sheaf $\mathcal{F}$.
Let E_τ be an elliptic curve over $\mathbf{C}$. Is it a well known result that there exists an isomorphism of Riemann surfaces between E_τ and the specific torus $\mathbf{C}/\Omega_\tau$, for the lattice $\Omega_\tau := \mathbf{Z}\tau \oplus \mathbf{Z}$ spanned by the complex numbers $\{\tau, 1\}$. The coordinate chart $z : \mathbf{C} \to \mathbf{C}$ over $\mathbf{C}$ defines the holomorphic 1-form dz, which is an element in $\Gamma(\mathbf{C}, \Omega^1(\mathbf{C}))$ for the dual bundle $\Omega^1(\mathbf{C})$ of the tangent bundle. This holomorphic form dz descends to the quotient $\mathbf{C}/\Omega_\tau \cong \mathrm{E}_\tau$ and defines a holomorphic 1-form[3] ω on E_τ, which generates $\Omega^1(\mathrm{E}_\tau)$. Indeed each point $x \in \mathrm{E}_\tau$ has a tangent space which is a copy of $\mathbf{C}$, and the standard basis ∂z can be taken by translation. Its dual basis consists of the coordinate differential $d_x z$, which still makes sense as a basis[4] of $(T_x \mathrm{E}_\tau)^*$. Further, $H_1(\mathrm{E}_\tau, \mathbf{Z}) \cong \mathbf{Z}^2$, as we can consider as generators the loops around the two copies of $\mathbf{S}^1$. We fix a basis γ_1, γ_2 for the group $H_1(\mathrm{E}_\tau, \mathbf{Z})$ and consider

$$(z_1, z_2) = \left(\int_{\gamma_1} \omega, \int_{\gamma_2} \omega \right) \in \mathbf{C}^2,$$

which defines a point $p = (z_1 : z_2) \in \mathbf{P}^1(\mathbf{C})$. The De Rham cohomology $H^1_{dR}(\mathrm{E}_\tau)$ corresponds to $H^1(\mathrm{E}_\tau, \mathbf{R})$ and it is given by the exterior algebra on dx and dy, where $z = x + iy$, whence it is also generates by $\omega = dz$ and $\overline{\omega} = d\overline{z}$. The map which assigns $\gamma \mapsto \int_\gamma \omega$ turns out to be an isomorphism $H^1(\mathrm{E}_\tau, \mathbf{R}) \to \mathbf{C}$, so the point p lies in $\mathbf{P}^1(\mathbf{C}) \smallsetminus \mathbf{P}^1(\mathbf{R})$. So far the

[2] The idea is the integration along loops of 1-forms.

[3] Note that there exists indeed a unique 1-form up to scale, otherwise if ω' is another, then their quotient would be a holomorphic function.

[4] It is the standard notation for the dual of the tangent space of E_τ at the point x.

point p, called the period point, relies on the choice of the isomorphism $H_1(\mathrm{E}_\tau, \mathbf{Z}) \cong \mathbf{Z}^2$. If we choose another basis, say $\gamma_1' = a\gamma_1 + b\gamma_2$ and $\gamma_2' = c\gamma_1 + d\gamma_2$, then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ defines an element in $\mathrm{GL}_2(\mathbf{Z})$ and the point p becomes $(az_1 + bz_2 : cz_1 + dz_2)$. We have seen in Chapter 2 that $\mathrm{GL}_2(\mathbf{Z})$ acts on $\mathbf{P}^1(\mathbf{C})$ by the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$. We call period the orbit of p under the action of $\mathrm{GL}_2(\mathbf{Z})$. Thus the period is an element in $\mathrm{GL}_2(\mathbf{Z})\backslash\mathbf{P}^1(\mathbf{C})$, which corresponds[5] to $\Gamma(1)\backslash\mathbf{H}$. Moreover we can choose a basis (γ_1, γ_2) such that $(\tau, 1) = (\int_{\gamma_1}\omega, \int_{\gamma_2}\omega)$ for $\tau \in \mathbf{H}$, so that the period[6] is given by $\Gamma(1)\tau$. Starting from this orbit is now possible to construct the isomorphism classes of our elliptic curve, by $\mathbf{C}/(\mathbf{Z}\tilde{\tau} + \mathbf{Z})$ for $\tilde{\tau} \in \Gamma(1)\tau$.
Let $f : \varepsilon \to S$ be a smooth family of elliptic curves over some variety S, endowed with a holomorphic section, so that we can identify all fibres with complex tori, and let $R^1 f_* \mathbf{Z}$ be the right derived functor of the locally constant sheaf $\mathbf{Z}$ of abelian groups. From the previous remark, $R^1 f_*\mathbf{Z}$ corresponds to $H^1(\varepsilon_s, \mathbf{Z})$, therefore its stalk at s, $(R^1 f_*\mathbf{Z})_s$ is given by the fibre $H^1(\varepsilon_s, \mathbf{Z})$. Locally, so over some sufficiently small open set U, we can choose a basis of $R^1 f_*\mathbf{Z}$ to define the period maps $U \to \mathbf{P}^1(\mathbf{C}) \smallsetminus \mathbf{P}^1(\mathbf{R})$ and $U \to \Gamma(1)\backslash\mathbf{H}$, if we fix a basis or we consider the orbit respectively. This general procedure will now be applied for the specific case in which $S = Y(\Gamma)$. The universal elliptic curve[7] ε is given by

$$\varepsilon := \Gamma\backslash\Big(\mathbf{Z}^2\backslash(\mathbf{C}\times\mathbf{H})\Big),$$

for the left actions:

- $\mathbf{Z}^2$ acts on $\mathbf{C}\times\mathbf{H}$ by $(m,n).(z,\tau) := (z + m\tau + n, \tau)$,
- Γ acts on $\mathbf{Z}^2\backslash(\mathbf{C}\times\mathbf{H})$ by $\gamma[(z,\tau)] := \left[\left(\frac{z}{c\tau+d}, \gamma\tau\right)\right]$, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

The projection

$$\pi_\varepsilon : \varepsilon \to Y(\Gamma)$$

defines the universal family over $Y(\Gamma)$. We point out that the map π_ε is induced by $\mathbf{Z}^2\backslash(\mathbf{C}\times\mathbf{H}) \to \mathbf{H}$ which is given by $[(z,\tau)] \mapsto \tau$. It is clear that the fibre over τ of the latter map corresponds to $\mathrm{E}_\tau \times \{\tau\} \cong \mathrm{E}_\tau$, therefore the fibre over $[\tau]$ of the former is given by isomorphism class of E_τ, which makes clear why the map $\varepsilon \to Y(\Gamma)$ defines a family of elliptic curves over $Y(\Gamma)$. Moreover, for $\tau \in \mathbf{H}$ we can associate the elliptic curve E_τ and for $\gamma \in \Gamma$ as before, the isomorphism $\mathrm{E}_{\gamma\tau} \to \mathrm{E}_\tau$ is induced by multiplication by $(c\tau + d)$, as

$$\mathrm{E}_{\gamma\tau} = \mathbf{C}\Big/\left(\frac{1}{c\tau+d}\Omega_\tau\right) = (c\tau+d)\,\mathrm{E}_\tau\,.$$

[5]Indeed, note that $\mathbf{P}^1(\mathbf{C}) \smallsetminus \mathbf{P}^1(\mathbf{R})$ corresponds to the complex upper and lower half plane, and the latter is the image of the former via a matrix of $\mathrm{GL}_2(\mathbf{Z})$ with determinant -1.

[6]The idea is that the generators of Ω_τ are exactly the periods of ω.

[7]It is also possible to define ε as $(\Gamma \ltimes \mathbf{Z}^2)\backslash(\mathbf{C}\times\mathbf{H})$, where the action is defined by

$$(c\tau+d)^{-1}\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{pmatrix}\begin{pmatrix} \tau \\ 1 \\ z \end{pmatrix}.$$

Therefore the definition of the action of Γ on $\mathbf{Z}^2\backslash(\mathbf{C}\times\mathbf{H})$ allows to identify corresponding elements

$$[(z,\tau)] = \left[\left(\frac{z}{c\tau+d},\gamma\tau\right)\right].$$

By construction $q^*\mathcal{H}^1_{dR} = \mathbf{H}\times\mathbf{C}^2$ and for what we have recalled before, $R^1\pi_{\varepsilon*}\mathbf{Z} = H^1(\varepsilon,\mathbf{Z})$, therefore its stalk of its pullback over $\mathbf{H}$ at τ corresponds to $H^1((\mathbf{Z}^2\backslash(\mathbf{C}\times\mathbf{H}))_\tau,\mathbf{Z})$ which is, by the previous argument, $H^1(\mathrm{E}_\tau,\mathbf{Z})$. Again from what we have said before, we can find a basis of $H^1(\mathrm{E}_\tau,\mathbf{Z})$ such that the corresponding period is given by $(\tau,1)$ and their corresponding dual loops define the paths $\tau^\vee,1^\vee$, which generates $H^1(\mathrm{E}_\tau,\mathbf{Z})$.
In this way we have two natural bases for $\mathcal{H}^1_{dR}$: one given by $\{\omega,\overline{\omega}\}$, where $\overline{\omega}$ is not holomorphic, the other one is holomorphic and given by $\{\tau^\vee,1^\vee\}$.
The above argument can be summarized by

$$q^*\mathcal{H}^1_{dR} = q^*R^1\pi_{\overline{\varepsilon}*}\mathbf{Z}\otimes\mathcal{O}_\mathbf{H} = \mathrm{Hom}(q^*R_1\pi_{\overline{\varepsilon}*}\mathbf{Z},\mathbf{Z})\otimes\mathcal{O}_\mathbf{H} = \mathrm{Hom}(q^*R_1\pi_{\overline{\varepsilon}*}\mathbf{Z},\mathcal{O}_\mathbf{H}) = \mathcal{O}_\mathbf{H}\tau^\vee\oplus\mathcal{O}_\mathbf{H}1^\vee.$$

Last, it follows that

$$\begin{pmatrix}\omega\\ \overline{\omega}\end{pmatrix} = \begin{pmatrix}\tau & 1\\ \overline{\tau} & 1\end{pmatrix}\begin{pmatrix}\tau^\vee\\ 1^\vee\end{pmatrix}.$$

Action of Γ on the set of sections

We now proceed with a quick computation regarding the action of Γ on the set of sections, which will be used in the last theorem in Chapter 4. Let $\gamma := \begin{pmatrix}a & b\\ c & d\end{pmatrix} \in \Gamma$. The action of Γ on the first basis of the sections is given by the pullbacks

$$\begin{cases}\gamma^*\omega = \frac{1}{c\tau+d}\omega\\ \gamma^*\overline{\omega} = \frac{1}{c\overline{\tau}+d}\overline{\omega}\end{cases}.$$

Whence

$$\begin{cases}\gamma^*\omega = \gamma^*(\tau\tau^\vee+1^\vee) = \frac{1}{c\tau+d}(\tau\tau^\vee+1^\vee)\\ \gamma^*\overline{\omega} = \gamma^*(\overline{\tau}\tau^\vee+1^\vee) = \frac{1}{c\overline{\tau}+d}(\overline{\tau}\tau^\vee+1^\vee),\end{cases}$$

which implies

$$\begin{cases}\gamma^*1^\vee = \frac{\overline{\tau}\tau^\vee+1^\vee}{c\overline{\tau}+d} - (\gamma\overline{\tau})\gamma^*\tau^\vee\\ (\gamma\tau-\gamma\overline{\tau})\gamma^*\tau^\vee = \frac{(d\tau^\vee-c1^\vee)(\tau-\overline{\tau})}{(c\tau+d)(c\overline{\tau}+d)}\end{cases}.$$

From the second equation we get, as $ad-bc=1$,

$$\gamma^*\tau^\vee = d\tau^\vee - c1^\vee.$$

Recalling that $\omega = \tau\tau^\vee+1^\vee$, we can rewrite the last equation as

$$\gamma^*\tau^\vee = -c\omega + (c\tau+d)\tau^\vee,$$

and we obtain

$$\gamma^*\begin{pmatrix}\omega\\ \tau^\vee\end{pmatrix} = \begin{pmatrix}(c\tau+d)^{-1} & 0\\ -c & (c\tau+d)\end{pmatrix}\begin{pmatrix}\omega\\ \tau^\vee\end{pmatrix}.$$

The last equation will imply that ω and $\tau^\vee$, which are both holomorphic sections, constitute the right basis to consider as they are both Γ_∞–invariant.

§4. $\mathbf{N}_k^r(\Gamma, \mathbf{C})$ as Sections

In this last chapter we discuss the main geometric result of this work, which corresponds to proposition 2.2.3 of [Urban, 2013]: nearly holomorphic modular forms corresponds to the set of sections of a purely arithmetic bundle defined over $X(\Gamma)$. This characterization allows us to link these algebraic objects to a geometric definition, where we specify the local conditions on $X(\Gamma)$ so that they match the properties at the cusps. An analogous result is known for holomorphic modular forms. In this case, the further application of Riemann Roch[1], justifiable as $X(\Gamma)$ is a compact Riemann surface, leads to an explicit formula for the finite dimension of $\mathcal{M}_k(\Gamma, \mathbf{C})$, which lays foundation for the existence of a mechanical procedure to prove any given identity amongst holomorphic modular forms.
The first part of this chapter is devoted to introduce $X(\Gamma)$ as a Riemann surface and then enlarge the picture by extending the bundles to this manifold. In the second part we introduce the bundle $\mathcal{H}_k^r(X(\Gamma))$, after recalling the extended Kodaira-Spencer isomorphism. In the third and last part we finally prove the geometric theorem. The essential reference for this unit, which was additionally the starting point of this entire work, is [Urban, 2013]. The result is also presented with a more algebraic geometry flavour in the articles of [Rosso, 2014] and [Liu, 2015].

4.1 Enlarging the Picture to $X(\Gamma)$

The bundles discussed so far are defined over the non-compact modular curve $Y(\Gamma)$. This section will adjoin cusps with appropriate local coordinate charts to the modular curve $Y(\Gamma)$, completing it to a compact Riemann surface denoted $X(\Gamma)$ and extending the bundles over this compactification. The Hodge decomposition and proposition 2.1.6 will allow us to reduce this extension problem to the study of only one cusp point, namely ∞, and only one bundle, namely $\underline{\omega}$. We start by introducing irregular and regular cusps, then we treat the extension of $\underline{\omega}$ to ∞, and last we generalize the result to the other bundles.

Denote by $\Gamma_c := \mathrm{Stab}_\Gamma(c)$ the stabilizer of the cups c, and recall that the stabilizer of the conjugate is given by $\Gamma_{\gamma c} = \gamma \Gamma_c \gamma^{-1}$.

Proposition 4.1.1. The stabilizer of $\infty = (1:0)$ in $\Gamma(1)$ is given by

$$\Gamma(1)_\infty := \{\pm T^m : m \in \mathbf{Z}\}.$$

[1] We refer to [Diamond, 2000] and [Forster, 1981].

PROOF. As
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(1:0) = (a:c),$$
then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)_\infty$ if and only if[2] $c = 0$. □

By proposition 2.1.6, for each cusp c there exists $\alpha \in \Gamma(1)$ such that $\alpha c = \infty$. Whence $\alpha\Gamma_c\alpha^{-1} = \Gamma_\infty \subseteq \Gamma(1)_\infty$ and therefore there exists a positive integer h_c, called the width of the cusp, such that $\pm T_{h_c}$ generates the stabilizer of αc in Γ. The cusp c is said to be regular if T_{h_c} generates the stabilizer, irregular otherwise.
From now on, Γ will be a congruence subgroup acting freely on $\mathbf{H}$ and with only regular cusps[3]. To extend the bundle $\underline{\omega}$ to the compactification of $Y(\Gamma)$, we need to consider the structure of $X(\Gamma)$ as a complex manifold. Therefore we recall the fundamental system of open neighbourhoods defined on $X(\Gamma)$ and the corresponding coordinate charts around cusps.

$X(\Gamma)$ as a Riemann surface

Topology on $X(\Gamma)$

The upper half plane $\mathbf{H}$ inherits the Euclidean topology as a subspace of $\mathbf{C}$, and the natural projection $q : \mathbf{H} \to Y(\Gamma)$ gives $Y(\Gamma)$ the quotient topology. The key to putting coordinate charts on $Y(\Gamma)$ is the idea that any two points in $\mathbf{H}$ have neighbourhoods small enough that every Γ transformation, taking one point away from the other, also takes its neighbourhood away from the other's. We wish to reach the same construction of a Riemann surface on $X(\Gamma)$, starting from realizing $\mathbf{H}^*$ as a topological space. As the topology on $\mathbf{H}^*$ consisting of its intersection with canonical open complex disks contains too many cusps in each neighbourhood to make the quotient $X(\Gamma)$ Hausdorff, we proceed as follows. For $\tau \in \mathbf{H}$ the open neighbourhoods are the canonical ones we had previously. For ∞, the fundamental system of neighbourhoods consists of the part of $\mathbf{H}$ lying above some horizontal line, namely for $n > 0$

- $U(n) := \{\tau \in \mathbf{H} : \Im(\tau) > n\}$;
- $U(n)^* := U(n) \cup \{\infty\}$.

For each other cusp $c \in \mathbf{P}^1(\mathbf{Q})$ we take for $n > 0$, $\alpha^{-1}(U(n)^*)$, where $\alpha \in \Gamma(1)$ is such that $\alpha c = \infty$, which exists by proposition 2.1.6. Note that $\alpha^{-1}(U(n)^*) = \{\tau \in \mathbf{H} : \Im(\alpha\tau) > n\}$ and for $\alpha := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the cusp c can be written as $c = (-\frac{d}{c} : 1)$. Also, for $\tau := x + iy$, the set

[2]Note that $c = 0$ and determinant equals 1 imply $a = d = \pm 1$.
[3]Note that this is indeed the situation for $\Gamma(N)$ for $N \geq 3$. Further, we point out that in our case, the existence of irregular cusps would make the isomorphism in 4.1.3 not well-defined.

$\mathfrak{I}(\alpha\tau) > n$ corresponds to a circle of radius $1/(2nc^2)$ tangent to the real axis at $x = -d/c$, indeed

$$\begin{aligned}
\mathfrak{I}(\alpha\tau) &> n \\
\frac{\mathfrak{I}\tau}{|c\tau + d|^2} &> n \\
\frac{y}{(cx+d)^2 + (cy)^2} &> n \\
\frac{1}{(2nc^2)^2} &> \left(x + \frac{d}{c}\right)^2 + \left(y - \frac{1}{2nc^2}\right)^2.
\end{aligned}$$

We then give $X(\Gamma)$ the quotient topology.

Charts around cusps on $X(\Gamma)$

In the same situation as before, $\alpha c = \infty$, therefore α transforms each open neighbourhood of c of our fundamental system into an open neighbourhood of ∞. This reduces the study of charts defined in some neighbourhood of some cusp into the study of charts defined in a neighbourhood of ∞. Indeed, let $U := \alpha^{-1}(U(2))$ and $U^* := \alpha^{-1}(U(2)^*)$ be open neighbourhoods of the cusp c. The projection $q^* : \mathbf{H}^* \to \Gamma\backslash\mathbf{H}^*$ sends $U \mapsto \Gamma_c\backslash U$, as $\Gamma_c = \{\gamma \in \Gamma : \gamma(U^*) \cap U^* \neq \varnothing\}$, and α induces a homeomorphism which sends the neighbourhood of c into a neighbourhood of ∞

$$\alpha : \Gamma_c\backslash U \to \Gamma_{\alpha c}\backslash U(2).$$

We consider the h_s–periodic wrapping map $e : U(2) \to \mathbf{C}^\times$ which maps $\tau \mapsto e^{\frac{2\pi i \tau}{h_s}}$, where h_s is the width of the cusp. The map factors through

$$\begin{tikzcd}
U(2) \arrow[r, "e"] \arrow[d, "q^*"'] & \mathbf{C}^\times \\
\Gamma_{\alpha c}\backslash U(2) \arrow[ur, "\tilde{e}"'] &
\end{tikzcd}$$

Let $\varphi := \tilde{e} \circ \alpha : q^*(U^*) \to \Gamma_{\alpha c}\backslash U(2)^* \to V \subseteq \mathbf{C}$ where $\infty \mapsto 0$ and $V := \tilde{e}(\Gamma_{\alpha c}\backslash U(2)^*)$. We take $(q^*(U^*), \varphi)$ as a chart around c. Roughly speaking, the transformation α straightens neighbourhoods of c by making identified points differ by a horizontal set, and then the map $\tilde{e}$ wraps the upper half plane into a cylinder which becomes a disk with ∞ at its center.

Proposition 4.1.2. $X(\Gamma)$ is compact.

PROOF. We want to find a compact subset of $\mathbf{H}^*$ which maps surjectively to $X(\Gamma)$. Recall $F := \{\tau \in \mathbf{H} : -\frac{1}{2} < \mathcal{R}(\tau) \le \frac{1}{2}, |\tau| \ge 1; -\frac{1}{2} < \mathcal{R}(\tau) \le 0, |\tau| > 1\}$ the fundamental domain[4] for $\Gamma(1)$, and define $F^* := F \cup \{\infty\}$. Let $\gamma_1, ..., \gamma_n$ be coset representatives for $\Gamma\backslash\Gamma(1)$. Then $F_\Gamma^* := \cup_{i=1}^n \gamma_i(F^*)$ satisfies $\mathbf{H}^* = \cup_{\gamma\in\Gamma}\gamma(F_\Gamma^*)$. The projection $\mathbf{H}^* \to X(\Gamma)$ restricted to F_Γ^* is therefore still surjective, and by construction F_Γ^* is compact. □

[4]A fundamental domain for $\Gamma(1)$ is a subset of $\mathbf{H}$ such that every orbit of $\Gamma(1)$ has exactly one element in F.

Extensions

Extension of $\underline{\omega}$

Proposition 4.1.3. There exists a canonical extension of $\underline{\omega}$ to $X(\Gamma)$ such that for all cusps c, the local section $\omega \in \Gamma(\Gamma_c \backslash U, \underline{\omega})$ extends to a non-vanishing section $\omega \in \Gamma(\Gamma_c \backslash U^*, \underline{\omega})$.

PROOF. As for each cusp c there exists $\alpha \in \Gamma(1)$ such that $\alpha c = \infty$, using the isomorphism

$$\alpha : \Gamma_c \backslash \mathrm{GL}^+ \to \Gamma_{\alpha c} \backslash \mathrm{GL}^+,$$

we are allowed to study only the case $c = \infty$.
For the cusp ∞ we consider the open neighbourhood $U(n)$ in $\mathbf{H}^*$, for a fixed positive integer n. The principal bundle defining the extended associated vector bundle $\underline{\omega}$ on $X(\Gamma)$ is locally given[5] by $\Gamma_\infty \backslash \mathrm{GL}^+ \to \Gamma_\infty \backslash U(n) \subseteq X(\Gamma)$. Further, for what we have said before, the local coordinate chart around ∞ is given by the wrapping map $\varphi := e^{2\pi i/h_\infty} : \Gamma_\infty \backslash U(n) \to D \smallsetminus \{0\}$. On the other hand we have the isomorphism $\Gamma_\infty \backslash \mathrm{GL}^+ \to D \smallsetminus \{0\} \times \mathbf{C}^\times$ defined by

$$\left[\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right] \mapsto \left(e^{\frac{2\pi i}{h_\infty} \frac{\omega_1}{\omega_2}}, \omega_2 \right),$$

which is well defined as long a Γ has no irregular cusps, which is true by assumption. To adjoin the cusp ∞ we therefore extends $\Gamma_\infty \backslash \mathrm{GL}^+ \cong D \smallsetminus \{0\} \times \mathbf{C}^\times$ to $D \times \mathbf{C}$, by sending $\infty \mapsto 0$. The local section $\omega : \Gamma_\infty \backslash U(n) \to \underline{\omega} = (\Gamma_\infty \backslash \mathrm{GL}^+ \times \mathbf{C})/\mathbf{C}^\times = (D \times \mathbf{C} \times \mathbf{C})/\mathbf{C}^\times$ is now defined by $q \mapsto [(q, 1, 1)]$, for q the local coordinate at ∞. □

Further extensions

The extension of $\underline{\overline{\omega}}$ is done in a similar way, after recalling that it differs from $\underline{\omega}$ only by the representation $\overline{\sigma}$, which makes it non-holomorphic. Namely we can define in the same way the extended bundle[6] $\underline{\overline{\omega}} = (D \times \mathbf{C} \times \mathbf{C})/\mathbf{C}^\times$ over $X(\Gamma)$. As for the De Rham bundle, the extension is done considering the Hodge decomposition together with the extension of the fundamental vector bundles, $\underline{\omega} \oplus \underline{\overline{\omega}} \cong \mathcal{H}^1_{dR}$.

4.2 $\mathcal{H}^r_k(X(\Gamma))$

Definition 4.2.1. With the previous notations, considering all the bundles over $Y(\Gamma)$, we define on $Y(\Gamma)$

$$\mathcal{H}^r_k := \underline{\omega}^{k-r} \otimes \mathrm{Sym}^r(\mathcal{H}^1_{dR}).$$

To extend this bundle to $X(\Gamma)$, we first recall the Kodaira-Spencer isomorphism on $Y(\Gamma)$, and after extending the cotangent bundle on $X(\Gamma)$, we proceed by presenting the KS isomorphism on $X(\Gamma)$.

Lemma 4.2.2. The global section $\omega^{-2} \otimes d\tau$ of $\underline{\omega}^{-2} \otimes \Omega^1_{Y(\Gamma)}$ induces the isomorphism

$$KS : \underline{\omega}^2 \to \Omega^1_{Y(\Gamma)},$$

which sends $(2\pi i)^2 \omega^2 \mapsto \frac{dq}{q}$, for $q = e^{2\pi i \tau}$.

[5] Recall the principal $\mathbf{C}^\times$–bundle $p : \Gamma \backslash \mathrm{GL}^+ \to Y(\Gamma)$.
[6] The difference here lies in the action of $\mathbf{C}^\times$ on $\mathbf{C}$ in the third component.

The Kodaira-Spencer isomorphism gives an interpretation of the cotangent bundle of $Y(\Gamma)$ in terms of the fundamental bundle $\underline{\omega}^2$. It can be thought as a measurement of the deformation of a family of manifolds.

Definition 4.2.3. Let C_Γ be the set of cusps[7] of Γ and[8] $C(\Gamma) := \sum_{c \in C_\Gamma}(c) \in \mathrm{Div}(X(\Gamma))$. Let $L(C(\Gamma))$ be the associated line bundle. We define the cotangent bundle with logarithm poles at the cusps as

$$\Omega^1_{X(\Gamma)}(C(\Gamma)) := \Omega^1_{X(\Gamma)} \otimes L(C(\Gamma)).$$

We point out that the augmentation of the line bundle $L(C(\Gamma))$ is done only to take care of the cusp points, indeed its restriction to $Y(\Gamma)$ corresponds to the trivial bundle.

Proposition 4.2.4. The KS isomorphism on $Y(\Gamma)$ extends to $X(\Gamma)$

$$KS : \underline{\omega}^2 \to \Omega^1_{X(\Gamma)}(C(\Gamma)).$$

PROOF. Let s be a holomorphic section of $\Gamma(X(\Gamma), L(C(\Gamma)))$ such that $div(s) = C(\Gamma)$. Then the non vanishing section $\omega^{-2} \otimes d\tau \otimes s \in \Gamma(Y(\Gamma), \underline{\omega}^{-2} \otimes \Omega^1_{X(\Gamma)}(C(\Gamma))|_{Y(\Gamma)})$ extends to a non vanishing global section of $\underline{\omega}^{-2} \otimes \Omega^1_{X(\Gamma)}(C(\Gamma))$. As usual, we just need to treat the cusp ∞. At that point we have the before mentioned chart given by $e^{2\pi i/h_\infty}$. As the local coordinate around ∞ is given by q^{1/h_∞}, for $q = e^{2\pi i \tau}$, then $\frac{2\pi i}{h_\infty} d\tau$ corresponds to $\frac{dq^{1/h_\infty}}{q^{1/h_\infty}}$, which has a pole of first order at q^{1/h_∞}. At this point intervenes $L(C(\Gamma))$: as $div(s) = (\infty)$ around ∞, s is given, locally around ∞, by gq^{1/h_∞}, which takes care of the pole. □

Definition 4.2.5. With the previous notations, and considering the extended version of the aforementioned bundles, we define on $X(\Gamma)$

$$\mathcal{H}^r_k(X(\Gamma)) := \underline{\omega}^{k-r} \otimes \mathrm{Sym}^r(\mathcal{H}^1_{dR}).$$

4.3 Geometric Characterization

We are finally able to prove the following.

Theorem 4.3.1. There exists an isomorphism between the sections of the bundle $\mathcal{H}^r_k(X(\Gamma))$ on $X(\Gamma)$ and the set $\mathbf{N}^r_k(\Gamma, \mathbf{C})$ of nearly holomorphic modular forms of weight k and order $\leq r$,

$$\Gamma(X(\Gamma), \mathcal{H}^r_k(X(\Gamma))) \cong \mathbf{N}^r_k(\Gamma, \mathbf{C}).$$

PROOF. We start by showing that each section gives rise to a nearly holomorphic modular forms of $\mathbf{N}^r_k(\Gamma, \mathbf{C})$. Using the result of Chapter 3 section 2, we recall that $\{\tau, 1\}$ form a basis of $H_1(\mathrm{E}_\tau, \mathbf{Z})$, after the identification of $\mathbf{Z}^2$ with Ω_τ. Over $\mathbf{H}$ we therefore have the

[7] Note that they constitute a set of finitely many isolated points, as they correspond to $\Gamma\backslash\Gamma(1)/\Gamma(1)_\infty$.
[8] We refer to [Diamond, 2000] for the theory of divisors.

global basis $\{\tau^\vee, 1^\vee\}$ for the bundle $q^*\mathcal{H}^1_{dR}$, defined as $\tau^\vee(a\tau+b) = a$ and $1^\vee(a\tau+b) = b$. Also, after switching from holomorphicity to smoothness, the Hodge decomposition gives us the basis $\{\omega, \overline{\omega}\}$ for $q^*\mathcal{H}^1_{dR} \otimes \mathcal{C}^\infty_{\mathbf{H}}$. Point being, neither $\{\tau^\vee, 1^\vee\}$ nor $\{\omega, \overline{\omega}\}$ gives rise to an element of $q^*\mathcal{H}^r_k$. Indeed $\{\tau^\vee, 1^\vee\}$ is holomorphic, but while $\tau^\vee$ is Γ_∞−invariant and therefore it extends to $X(\Gamma)$, $1^\vee$ is not, as we have seen at the end of Chapter 3. On the other hand $\overline{\omega}$ is not holomorphic. However, $\{\omega, \tau^\vee\}$ does define an element of $q^*\mathcal{H}^r_k$ as both element define holomorphic sections which extend to $X(\Gamma)$.
Let η be a section of $\mathcal{H}^r_k(X(\Gamma))$, and denote again by η its restriction to $Y(\Gamma)$, so that $\eta \in \Gamma(X(\Gamma)|_{Y(\Gamma)}, \mathcal{H}^r_k)$. Its pull-back $q^*\eta$ on $\mathbf{H}$, is now an element of $\Gamma(\mathbf{H}, q^*\mathcal{H}^r_k)$. From what we have said, it can be written as

$$q^*\eta(\tau) = \sum_{l=0}^{r} f_l(\tau)\omega^{k-l} \otimes \tau^{\vee l},$$

for some $f_l : \mathbf{H} \to \mathbf{C}$ holomorphic functions. As

$$\begin{pmatrix}\omega\\ \overline{\omega}\end{pmatrix} = \begin{pmatrix}\tau & 1\\ \overline{\tau} & 1\end{pmatrix}\begin{pmatrix}\tau^\vee\\ 1^\vee\end{pmatrix},$$

it follows that $\tau^\vee = \frac{\omega-\overline{\omega}}{\tau-\overline{\tau}} = \frac{\omega-\overline{\omega}}{2iy}$ for $\tau = x+iy$. Newton's binomial implies that

$$\tau^{\vee l} = \frac{1}{(2iy)^l}\sum_{j=0}^{l}(-1)^j\binom{l}{j}\omega^{l-j}\otimes\overline{\omega}^j.$$

Whence

$$q^*\eta(\tau) = \sum_{l=0}^{r}\frac{f_l(\tau)}{(2iy)^l}\sum_{j=0}^{l}(-1)^j\binom{l}{j}\omega^{k-j}\otimes\overline{\omega}^j \in \Gamma(\mathbf{H}, q^*\mathcal{H}^r_k).$$

Note that for each l we have to consider $j=0$, which gives rise to the addend $\frac{f_l(\tau)}{(2iy)^l}\omega^k$. Whence the $(k,0)$−component, corresponding to the coefficient of ω^k, is given by

$$f(\tau) := \sum_{l=0}^{r}\frac{f_l(\tau)}{(2iy)^l},$$

which defines an element in $\mathbf{N}^r_k(\Gamma, \mathbf{C})$. Indeed, properties (i) and (iii) of the definition are trivially satisfied. For the Γ−invariant, we want to show that for each l and $\gamma \in \Gamma$,

$$\frac{f_l(\gamma\tau)}{(2iy)^l}(c\tau+d)^{-k+l}(c\overline{\tau}+d)^l = \frac{f_l(\tau)}{(2iy)^l}$$

holds[9]. But this property is already satisfied, as $q^*\eta$ comes from a Γ−invariant section. Indeed the action of Γ on the basis $\{\omega, \tau^\vee\}$ is given by

$$\gamma^*\begin{pmatrix}\omega\\ \tau^\vee\end{pmatrix} = \begin{pmatrix}(c\tau+d)^{-1} & 0\\ -c & (c\tau+d)\end{pmatrix}\begin{pmatrix}\omega\\ \tau^\vee\end{pmatrix},$$

as computed in Chapter 3. This result, together with

$$\begin{pmatrix}\omega\\ -\frac{\overline{\omega}}{2iy}\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\frac{1}{2iy} & 1\end{pmatrix}\begin{pmatrix}\omega\\ \tau^\vee\end{pmatrix},$$

[9] We have used the usual formula for $\Im(\gamma\tau)$.

determines the action

$$\gamma^*\begin{pmatrix}\omega\\ -\frac{\overline{\omega}}{2iy}\end{pmatrix}=\begin{pmatrix}(c\tau+d)^{-1} & 0\\ 0 & (c\tau+d)\end{pmatrix}\begin{pmatrix}\omega\\ -\frac{\overline{\omega}}{2iy}\end{pmatrix},$$

from which we obtain the factors $(c\tau+d)^{-k+l}$ and $(c\overline{\tau}+d)^l$, and consequently, modularity. It remains to show that f has finite limit at the cusps. As usual, it suffices to study the case for the cusp ∞. We first rewrite the bundle as $\underline{\omega}^{k-r-2}\otimes\Omega^1_{X(\Gamma)}(C(\Gamma))\otimes\mathrm{Sym}^r(\mathcal{H}^1_{dR})$, using the Kodaira-Spencer isomorphism. For each $l\in\{0,\dots,r\}$, locally around ∞, each addend is of the form $f_l(\tau)\omega^{k-l-2}\otimes\frac{dq^{1/h_\infty}}{q^{1/h_\infty}}\otimes s\otimes\tau^{\vee l}$. As by assumption η defines a non vanishing section over $\mathcal{H}^r_k(X(\Gamma))$, f_l has to be holomorphic in q^{1/h_∞}, therefore its $q-$expansion[10] has not negative term. As this holds for each l, the function f turns out to have finite limit at the cusp ∞. The other cusps are treated in the same way.
Next, suppose we have a nearly holomorphic modular form

$$f(\tau):=\sum_{l=0}^r\frac{f_l(\tau)}{y^l}\in\mathbf{N}^r_k(\Gamma,\mathbf{C}).$$

We construct a section such that f corresponds to its projection on the $(k,0)-$component. Let

$$q^*\eta(\tau):=\sum_{l=0}^r(2i)^lf_l(\tau)\omega^{k-l}\otimes\tau^{\vee l}.$$

As f satisfies the modularity property with respect to Γ, the above section is $\Gamma-$invariant, and therefore it defines an element η in $\Gamma(Y(\Gamma),\mathcal{H}^r_k)$. Further, as f has finite limit at the cusps, the $q-$expansion of f_l's have not negative term, therefore η defines a section of $\mathcal{H}^r_k(X(\Gamma))$. Using the same argument as before, we rewrite η as

$$\sum_{l=0}^r(2i)^l\frac{f_l(\tau)}{(2iy)^l}\sum_{j=0}^l(-1)^j\binom{l}{j}\omega^{k-j}\otimes\overline{\omega}^j=\sum_{l=0}^r\frac{f_l(\tau)}{y^l}\sum_{j=0}^l(-1)^j\binom{l}{j}\omega^{k-j}\otimes\overline{\omega}^j,$$

therefore its projection is given by $\sum_{l=0}^r\frac{f_l(\tau)}{y^l}=f(\tau)$.
We point out that the last map is well defined as the projection on the $(k,0)-$component is injective. Indeed, following the first part of the proof, consider the projection on the coefficient of the $(k,0)-$component. For what has been proved in this first part, the projection $\pi_{(k,0)}$ defines a map from the sections of the bundle $\mathcal{H}^r_k$ to the space of nearly holomorphic modular forms. Consider two sections, such that $q^*\eta$ and $q^*\xi$ define elements in $\Gamma(\mathbf{H},q^*\mathcal{H}^r_k)$, and suppose that $\pi_{(k,0)}(q^*\eta)=\pi_{(k,0)}(q^*\xi)$. We want to show η and ξ define the same element.
As sections over $\mathbf{H}$, they admit a representation as

$$q^*\eta(\tau)=\sum_{l=0}^rf_l(\tau)\omega^{k-l}\otimes\tau^{\vee l}$$

and

$$q^*\xi(\tau)=\sum_{l=0}^rg_l(\tau)\omega^{k-l}\otimes\tau^{\vee l},$$

[10] We mean the one appearing in the definition of a nearly holomorphic modular form having finite limit at the cusps.

for some holomorphic functions $f_l, g_l : \mathbf{H} \to \mathbf{C}$, which we want to prove are equals $f_l = g_l$ for each $l = 0, ..., r$. From the previous computation

$$\pi_{(k,0)}(q^*\eta)(\tau) = \sum_{l=0}^{r} \frac{f_l(\tau)}{(2iy)^l} \text{ and } \pi_{(k,0)}(q^*\xi)(\tau) = \sum_{l=0}^{r} \frac{g_l(\tau)}{(2iy)^l},$$

and by assumption

$$\sum_{l=0}^{r} \frac{f_l(\tau)}{(2iy)^l} = \sum_{l=0}^{r} \frac{g_l(\tau)}{(2iy)^l} \text{ that is } \sum_{l=0}^{r} \frac{(f_l - g_l)(\tau)}{(2iy)^l} = 0.$$

As $f_l - g_l$ is holomorphic for each j, the problem is now reduced to prove that if

$$s(\tau) := \sum_{l=0}^{r} \frac{s_l(\tau)}{y^l} \equiv 0$$

for holomorphic s_l then all the s_l are zero. Indeed, if this is the case, then all $f_l - g_l$ are zero and so $\eta = \xi$. It remains to prove the reduced problem. The idea is to apply the differential operator $\frac{ir}{2y} + \partial_{\overline{\tau}}$ and using induction on r. For $r = 0$ nothing needs to be proved, as $s = s_0 \equiv 0$. Suppose it is true for $r - 1$ and let $s := \sum_{l=0}^{r} \frac{s_l}{y^l} \equiv 0$. Then $(\frac{ir}{2y} + \partial_{\overline{\tau}})s = 0$.

$$\begin{aligned}
0 &= \left(\frac{ir}{2y} + \partial_{\overline{\tau}}\right) \sum_{l=0}^{r} \frac{s_l}{y^l} \\
&= \sum_{l=0}^{r} \frac{irs_l}{2y^{l+1}} - \sum_{l=1}^{r} \frac{ils_l}{2y^{l+1}} \\
&= \frac{irs_0}{2y} + \sum_{l=1}^{r-1} \frac{i(r-l)s_l}{2y^{l+1}} \\
&= \frac{i}{y}\left(\frac{rs_0}{2} + \sum_{l=1}^{r-1} \frac{(r-l)s_l}{2y^l}\right) \\
&= \frac{i}{y}\left(\sum_{l=0}^{r-1} \frac{(r-l)s_l}{2y^l}\right)
\end{aligned}$$

By induction hypothesis $\sum_{l=0}^{r-1} \frac{(r-l)s_l}{2y^l} = 0$ implies that all the s_l are zero, concluding the proof. $\square$

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