On Noether's Normalization Lemma for projective schemes

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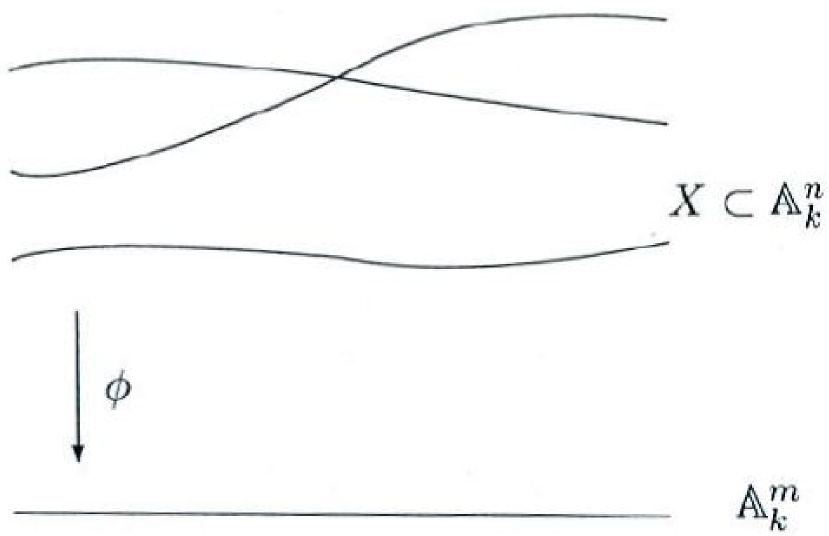
Preface

Noether's normalization lemma is a well known result in commutative algebra, due to the German mathematician Emmy Noether (1882-1935). It states that every commutative algebra of finite type A over a field K contains a subring of polynomials $K[X_1, ..., X_d]$ and it is finite over it. This purely algebraic result has a very fascinating geometric meaning: any affine algebraic variety over a field is a finite cover of an affine space. Morally speaking this means that some very common geometric objects can be pushed (or projected) without too much overlapping over another simpler (the simpliest possible we can say) geometric object of the same dimension.



There exists a graded version of Noether's normalization lemma, whose geometric expression is simular to the one stated above: any projective variety is a finite cover of a projective space. Furthermore, it is possible to impose some nice conditions on the covers.

The aim of this master thesis is to first study the case of projective varieties over a field of the normalization lemma. The second step would be to study the case of projective schemes over some particular classes of rings.
In Chapter 1 we introduce a foundamental tool to tackle these topics, that is to say sheaves of modules, aiming to a particular class of them: ample invertible sheaves. We are also going to provide a very essential list of results for a special sheaf of module, the sheaf of differentials, which plays a central role in the theory of étale covers.
Chapter 2 is about the Noether's normalization lemma for projective schemes over a field k. After some remarks on the case of a field of characteristic zero, we largely talk about the case of a field of positive characteristic, reviewing Noether normalization lemma and imposing some conditions on projective morphisms, such as étaleness and particular behaviours on closed subschemes.
Finally, Chapter 3 is devoted to the case of projective schemes over ans affine scheme S, with the property that every finite scheme over it has a torsion Picard group.

References and aknowledgments

I started this work basing on aknowledgments achieved during the course of Introduction to Algebraic Geometry, held by professor Q. Liu in the first semester of the current academic year here in Bordeaux. Hence I am considering the basics of scheme theory, specially projective shemes, as prerequisites. I am going to try to be as punctual as possible in giving references about every part which will not be exaustively treated, but, for the cases where I will not result enough precise, I refer to classical texts on these topics, such as [H] and [L] in particular. Furthermore I recommend the *Stacks Project* (`http:stacks.math.columbia.edu/`), which in more than one occasion helped me in orienteering in the extent of Algebraic Geometry. For this thesis the theory of schemes morphisms will be fundamental: I am going to treat projective, affine, proper, separated, of finite type, unramified, étale and, of course, finite morphisms, whose definitions and properties can be found in the references above mentioned.

Rewording Fermat, I have discovered truly marvellous people, whom this margin is too narrow to contain, but in particular this work would not have been possible without the extraordinary help of my advisor, Professor Qing Liu: I gratefully thank him for his kind availability to listen and answer to my uncountable doubts and curiosities.

Contents

Chapter 1

Preliminaries

The theory of sheaves had been developped in about 15 years during the first half of last century. The intuitive idea, which is going to be founding for the sheme theory, is that some geometric objects may carry some algebraic structures intrisecally connected to their geometric nature. It is the case of, as an example, rings of functions defined over open subsets of a underlying topological space. In this chapter we are considering richer algebraic structures, such as modules and algebras over a ring. This will let us enlarge the number of tools for investigating the nature of our geometric objects. Indeed, considering sheaves over a scheme other than the structure one, provides us more flexible tools. Especially important it will be the notion of quasi-coherent sheaves (introduced by Serre), which have some powerful properties, having a local structure easy to study. This chapter will mainly follow the work of [L] (Chapter 5 Section 1) and [H] (Chapter II Section 5).

1.1 Sheaves of Modules

1.1.1 Definition and first examples

Definition 1.1.1. Let $(X, \mathcal{O}_X)$ be a ringed topological space. A *sheaf of $\mathcal{O}_X$-modules* (or simply an *$\mathcal{O}_X$-module*) is a sheaf $\mathcal{F}$ on X, such that for every open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and for each inclusion of open sets $V \subseteq U$, the restriction map $\mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ is compatible with the module structure via the ring homomorphism $\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(V)$; i.e. for every $a \in \mathcal{O}_X(U)$ and every $f \in \mathcal{F}(U)$, we have $(af)|_V = a|_V f|_V$.
In an obvious way a *morphism of $\mathcal{O}_X$-modules* $\mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves such that, on each open subset $U \subseteq X$, it defines a homomorphism of $\mathcal{O}_X(U)$-modules $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$.

A trivial example of $\mathcal{O}_X$-module is the structure sheaf $\mathcal{O}_X$ itself. Another example we already know is the sheaf of ideals $\mathcal{I}$, for which every open $U \subseteq X$ corresponds to an ideal $\mathcal{I}(U) \subseteq \mathcal{O}_X(U)$ (which is an $\mathcal{O}_X(U)$-module).
Starting with two $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$, we can use the usual operations on modules over a ring to construct other $\mathcal{O}_X$-modules. For example we define the *tensor product* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf[1]

$$U \longmapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

In the same way we can define direct sum of $\mathcal{O}_X$-modules, kernel and cokernel of morphisms of $\mathcal{O}_X$-modules. Moreover, the *tensor algebra*, *exterior algebra* and *symmetric algebra* of $\mathcal{F}$ are defined in the same way. For example, the k-th exterior power $\bigwedge^k \mathcal{F}$ is the sheaf associated to the presheaf

$$U \longmapsto \bigwedge_{\mathcal{O}_X(U)}^{k} \mathcal{F}(U).$$

If $\mathcal{F}$ is locally free of rank n (i.e. there exists an open covering for which $\mathcal{F}|_U$ is isomorphic to a direct sum of copies of $\mathcal{O}_X|_U$), then $\bigwedge^n \mathcal{F}$ is called the *determinant line bundle* of $\mathcal{F}$, denoted by $\det(\mathcal{F})$.

Now we are going to present other more structured examples of $\mathcal{O}_X$-modules.
Example 1.1.2. *(Inverse image of a sheaf of modules)* Let $f : X \longrightarrow Y$ be a morphism of schemes. The homomorphism $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ induces a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$. Given an $\mathcal{O}_Y$-module $\mathcal{G}$ we define on the ringed topological space $(X, f^{-1}\mathcal{O}_Y)$ the tensor product of $f^{-1}\mathcal{O}_Y$-modules

$$f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X,$$

which can be seen as $\mathcal{O}_X$-module via multiplication on the right. It is also called *pull-back* of $\mathcal{G}$ (notice that $f^*\mathcal{O}_Y = f^{-1}\mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \cong \mathcal{O}_X$).
Example 1.1.3. *(Quasi-coherent[2] sheaf on an affine scheme)* Let $X = \operatorname{Spec} A$ be an affine scheme and let M be an A-module. For any principal open subset $D(f) \subseteq X$ define the presheaf $\widetilde{M}$ as

$$\widetilde{M}(D(f)) := M_f$$

with restriction maps given by the canonical morphism of localization. It can be easily proved (follow [L] Proposition 2.3.1 with very few adaptations) that

[1] Restriction maps are induced by the universal property of tensor product of modules.
[2] The expression quasi-coherent will be clearified later and the reason we use it in these cases too (Proposition 1.1.9).

(i) this indeed defines a sheaf (in particular an $\mathcal{O}_X$-module),

(ii) $\widetilde{M}_\mathfrak{p} = M_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$, and

(iii) $\widetilde{M}(X) = M$ (this easily follows from the definition and the fact that $X = D(1)$ and $M_1 = M$).

Example 1.1.4. *(Quasi-coherent sheaf on a projective sheme)* Given a graded ring $B = \oplus_{n\geq 0} B_n$, let $X = \operatorname{Proj} B$ be a projective scheme and let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a graded B-module (i.e. $B_m M_n \subseteq M_{m+n}$ for every $m \geq 0$ and every $n \in \mathbb{Z}$). For any non-nilpotent homogeneous $f \in B_+$, define

$$M_{(f)} = \left\{ \frac{m}{f^d} \in M_f : m \in M_{d\deg(f)} \right\},$$

i.e. the sets of elements of M_f of degree 0. It is a $B_{(f)}$-module (defined replacing M by B) in a natural way. We define a sheaf $\widetilde{M}$ on X by glueing togheter[3] the sheaves $\widetilde{(M_{(f)})}$ defined, as in the previous example, on the affine scheme $D_+(f) = \operatorname{Spec} B_{(f)}$ for any principal open subset $D_+(f) \subseteq X$. It can be proved as before that $\widetilde{M}_\mathfrak{p} = M_{(\mathfrak{p})}$, for every $\mathfrak{p} \in \operatorname{Proj} A$, where $M_{(\mathfrak{p})}$ is the set of elements degree 0 of $M_\mathfrak{p}$.

This last two examples show us more clearly the connection between modules over a ring and sheaves on a ringed space.

1.1.2 Quasi-coherent sheaves

Among the $\mathcal{O}_X$-modules a central role is played by the quasi-coherent sheaves. These are sheaves which are in some sense closely linked to the geometric properties of the underlying space. In particular they have a local presentation, as the following definition can suggest.

Definition 1.1.5. Let $(X, \mathcal{O}_X)$ be a ringed topological space and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. $\mathcal{F}$ is said to be *quasi-coherent* if for every $x \in X$, there exists an open neighbourhood $U \subseteq X$ of x and an exact sequence of $\mathcal{O}_X$-modules

$$\mathcal{O}_X^{(J)}|_U \longrightarrow \mathcal{O}_X^{(I)}|_U \longrightarrow \mathcal{F}|_U \longrightarrow 0 ,$$

for some sets I, J.
We say that $\mathcal{F}$ is *finitely generated* if for every $x \in X$ there exists an open neighbourhood $U \subseteq X$, an integer $n \geq 1$ and a surjective homomorphism $\mathcal{O}_X^n|_U \longrightarrow \mathcal{F}|_U$.

[3]see [L] Exercise 2.2.8.

A trivial example of quasi-coherent sheaf is the structural sheaf $\mathcal{O}_X$. Some authors prefer to highlight the analogy between sheaves of $\mathcal{O}_X$-modules and modules over a ring, defining a quasi-coherent sheaf as a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, which is isomorphic to a sheaf of the form $\widetilde{M_i}$ (like in Example 1.1.3) on the open sets of some affine covering $X = \bigcup_i U_i$. By considering this definition, it follows immediately that the sheaves defined in Example 1.1.3 and 1.1.4 are quasi-coherent. We will show that these definitions are equivalent.

Proposition 1.1.6. *Let $X = \operatorname{Spec} A$ be an affine scheme, we have that*[4]

(a) given a family of A-modules $\{M_i\}_i$, then $\widetilde{\oplus_i M_i} \cong \oplus_i \widetilde{M_i}$;

(b) a sequence of A-modules $L \longrightarrow M \longrightarrow N$ is exact if and only if the sequence of $\mathcal{O}_X$-modules $\widetilde{L} \longrightarrow \widetilde{M} \longrightarrow \widetilde{N}$ is exact;

(c) for any A-module M, the sheaf $\widetilde{M}$ is quasi-coherent;

(d) given two A-modules M, N then there is a canonical isomorphism $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$.

Proof. For every principal open subset $D(f) \subseteq X$, by definition, we have $\widetilde{\oplus_i M_i}(D(f)) = (\oplus_i M_i)_f = \oplus_i (M_i)_f = \oplus_i \widetilde{M_i}(D(f))$; so we get (a). Let us suppose $L \longrightarrow M \longrightarrow N$ exact. Then for every $\mathfrak{p} \in \operatorname{Spec} A$, $L_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow N_\mathfrak{p}$ is exact ($A_\mathfrak{p}$ is flat), but this sequence corresponds to $\widetilde{L}_\mathfrak{p} \longrightarrow \widetilde{M}_\mathfrak{p} \longrightarrow \widetilde{N}_\mathfrak{p}$ (Example 1.1.3(ii)), so it means that $\widetilde{L} \longrightarrow \widetilde{M} \longrightarrow \widetilde{N}$ is exact. On the other hand, consider the exact sequence $\widetilde{L} \xrightarrow{\alpha} \widetilde{M} \xrightarrow{\beta} \widetilde{N}$. By Example 1.1.3(ii)(iii), for every $\mathfrak{p} \in \operatorname{Spec} A$ we have the following commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\alpha_X} & M & \xrightarrow{\beta_X} & N \\ \downarrow & & \downarrow & & \downarrow \\ L_\mathfrak{p} & \xrightarrow{\alpha_\mathfrak{p}} & M_\mathfrak{p} & \xrightarrow{\beta_\mathfrak{p}} & N_\mathfrak{p} \end{array}$$

where the lower horizontal sequence is exact, since is corresponds to $\widetilde{L}_\mathfrak{p} \longrightarrow \widetilde{M}_\mathfrak{p} \longrightarrow \widetilde{N}_\mathfrak{p}$ and we are supposing the sequence of $\mathcal{O}_X$-modules exact. This implies that, for all $\mathfrak{p} \in \operatorname{Spec} A$[5],

$$\left(\frac{\ker \beta_X}{\mathrm{Im}\alpha_X}\right)_\mathfrak{p} = \frac{(\ker \beta_X)_\mathfrak{p}}{(\mathrm{Im}\alpha_X)_\mathfrak{p}} = \frac{\ker \beta_\mathfrak{p}}{\mathrm{Im}\alpha_\mathfrak{p}} = 0.$$

[4]Note that, since X is affine, we are considering the construction of Example 1.1.3: not to confuse the two cases.

[5]Recalling that $A_\mathfrak{p}$ is flat, we have that, since $0 \longrightarrow \ker \beta_X \longrightarrow M \xrightarrow{\beta_X} N$ is exact, then $0 \longrightarrow (\ker \beta_X)_\mathfrak{p} \longrightarrow M_\mathfrak{p} \xrightarrow{\beta_\mathfrak{p}} N_\mathfrak{p}$ is exact as well, thus $(\ker \beta_X)_\mathfrak{p} = \ker \beta_\mathfrak{p}$. The same argument works for $(\mathrm{Im}\alpha_X)_\mathfrak{p} = \mathrm{Im}\alpha_\mathfrak{p}$.

Therefore $\dfrac{\ker \beta_X}{\mathrm{Im}\alpha_X} = 0$ in M, i.e. $L \longrightarrow M \longrightarrow N$ is exact and we have (b).
For (c) we notice that for every A-module M we have an exact sequence

$$K \longrightarrow L \longrightarrow M \longrightarrow 0,$$

where K, L are free A-modules, then by (a),(b) we get the statement needed. Finally we have that

$$\begin{aligned}\widetilde{M \otimes_A} N(D(f)) &= (M \otimes_A N) \otimes_A A_f \\ &\cong (M \otimes_A A_f) \otimes_{A_f} (N \otimes_A A_f) \\ &= \widetilde{M}(D(f)) \otimes_{\mathcal{O}_X(D(f))} \widetilde{N}(D(f)) \\ &= (\widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N})(D(f)),\end{aligned}$$

where the last equality comes from our definition of tensor product of sheaves of modules. This is an isomorphism of $\mathcal{O}_X(D(f))$-modules compatible with restriction maps. Since the $D(f)$ form a base for the topology of X, the latter isomorphism induces the isomorphism of $\mathcal{O}_X$-modules stated by (d). $\square$

Proposition 1.1.7. *Given a noetherian or separated and quasi-compact*[6] *scheme X, let $\mathcal{F}$ be a quasi-coherent sheaf on it. Then for any $f \in \mathcal{O}_X(X)$, the canonical homomorphism*

$$\mathcal{F}(X)_f = \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(X)_f \longrightarrow \mathcal{F}(X_f),$$

where $X_f := \{x \in X : f_x \in \mathcal{O}^*_{X,x}\}$, is an isomorphism.

We are going to prove a more general form of this Proposition later, for its proof see [L] Proposition 5.1.6.
Finally we come to the proof of the theorem stated before, which shows the equivalence of the two definition of quasi-coherent sheaf we gave.

Theorem 1.1.8. *Let X be a scheme and $\mathcal{F}$ an $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasi-coherent if and only if for every affine open subset $U \subseteq X$, we have $\mathcal{F}|_U \cong \widetilde{\mathcal{F}(U)}$.*

Proof. Suppose $\mathcal{F}$ quasi-coherent. Let $U \subseteq X$ be an affine open set, then by the previous result, for any $f \in \mathcal{O}_X(U)$, $\mathcal{F}(U)_f \cong \mathcal{F}(D(f))$. Hence $\widetilde{\mathcal{F}(U)}(D(f)) = \mathcal{F}(U)_f \cong \mathcal{F}(D(f)) = \mathcal{F}|_U(D(f))$ and, as before, this implies $\mathcal{F}|_U \cong \widetilde{\mathcal{F}(U)}$. Conversely, Proposition 1.1.6(c) tells us that $\mathcal{F}$ is quasi-coherent on open affines of X; but the definition is given on open neighbourhoods of points of X, so $\mathcal{F}$ is quasi-coherent on X. $\square$

[6] e.g. affine shemes.

Remark 1.1.9. Given $\mathcal{F}, \mathcal{G}$ quasi-coherent sheaves on X, this last result, togheter with Proposition 1.1.6(d), implies that for every open affine subset $U \subseteq X$,

$$(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U) = \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

Further, we can also say that

$$\begin{aligned}
(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x &= ((\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})|_U)_x \\
&= (\mathcal{F} \widetilde{\otimes_{\mathcal{O}_X} \mathcal{G})(U)})_x \\
&= (\mathcal{F}(U) \widetilde{\otimes_{\mathcal{O}_X(U)}} \mathcal{G}(U))_x \\
&= (\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_{X,x} \\
&= (\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_{X,x}) \otimes_{\mathcal{O}_{X,x}} (\mathcal{G}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_{X,x}) \\
&= \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x.
\end{aligned}$$

Actually this result can be proved in general for any $\mathcal{O}_X$-module (see [B]II Section 6 Proposition 7), in particular if we consider any $\mathcal{O}_X$-module $\mathcal{G}$ as in Example 1.1.2, recalling the fact that $(f^{-1}\mathcal{G})_x = \mathcal{G}_{f(x)}$ (see [L] Subsection 2.2.1 page 37), then

$$(f^*\mathcal{G})_x \cong \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}.$$

We continue providing some useful examples of quasi-coherent sheaves on a projective scheme.

Example 1.1.10. *(Twisting sheaves)* Given a ring A and a graded A-algebra B, for any $n \in \mathbb{Z}$ let $B(n)$ be the graded B-module defined by $B(n)_d = B_{n+d}$ (we call it a *twist* of B). Let $X = \operatorname{Proj} B$, we define the $\mathcal{O}_X$-module $\mathcal{O}_X(n) := \widetilde{B(n)}$ (so it follows the construction of Example 1.1.4, do not confuse this with the affine case). We call $\mathcal{O}_X(1)$ the *twisting sheaf of Serre.*

Remark 1.1.11. This last example plays a central role in the theory of projective scheme. We remark some results about it.

(i) For any homogeneous element $f \in B$ of degree 1, we have $B(n)_{(f)} = f^n B_{(f)}$. Thus on the open affine subset $D_+(f) \subseteq X$, we have $\mathcal{O}_X(n)|_{D_+(f)} = f^n \mathcal{O}_X|_{D_+(f)}$.

(ii) By Proposition 1.1.6(d), $\mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{O}_X(m) = \mathcal{O}_X(m+n)$.

Another important result is provided by the following lemma

Lemma 1.1.12. *Let $B = A[T_0, ..., T_d]$, consider the setting provided by the previous example, so $X = \mathbb{P}^d_S$. Then*

$$\mathcal{O}_X(n)(X) = \begin{cases} B_n & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases} .$$

In particular $\bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)(X) = B$.

Proof. We may suppose $d \geq 1$. Global sections of $\mathcal{O}_X(n)(X)$ correspond to the local sections in $\mathcal{O}_X(n)(D_+(T_i))$ which coincide on the intersections. By (ii) of the last remark, $\mathcal{O}_X(n)(D_+(T_i)) = T_i^n\mathcal{O}_X(D_+(T_i)) = T_i^n A\left[\frac{T_0}{T_i}, ..., \frac{T_d}{T_i}\right]$, further we have $\mathcal{O}_X(n)(D_+(T_i) \cap D_+(T_j)) = \mathcal{O}_X(n)(D_+(T_iT_j))$. Thus we can consider $f \in \mathcal{O}_X(n)(X)$ as an element of $A[T_0, ..., T_d, T_0^{-1}, ..., T_d^{-1}]$. For $i > 0$, we have $f \in \mathcal{O}_X(n)(D_+(T_i)) = T_i^n A\left[\frac{T_0}{T_i}, ..., \frac{T_d}{T_i}\right]$, which implies that T_0 does not figure as denominator for f. On the other hand $f \in T_0^n\mathcal{O}_X(D_+(T_0))$, thus if $n < 0$ the only possible element of $T_0^n A\left[\frac{T_1}{T_0}, ..., \frac{T_d}{T_0}\right]$ without T_0 at the denominator is $f = 0$; otherwise if $n \geq 0$, then $f = \sum_{i=1}^{d}\sum_{k\geq 0} a_{i,k}T_i^kT_0^{n-k}$ has so T_0 at the denominator if and only if $k \leq n$, i.e. it is of the form $f = \sum_{i=1}^{d}\sum_{\substack{h,k\geq 0\\ h+k=n}} a_{i,k}T_i^hT_0^k$, that is it belongs to B_n. Conversely any element of B_n is an element of $\mathcal{O}_X(n)(D_+(T_i))$ which coincides on the intersections $\mathcal{O}_X(n)(D_+(T_iT_j)) \subseteq A[T_0, ..., T_d, T_0^{-1}, ..., T_d^{-1}]$. $\square$

1.2 Invertible Sheaves

Invertible sheaves represent a particular class of quasi-coherent sheaves of modules, which in some sense generalizes the structure sheaf of a scheme. In particular we are going to see how they provide us a way to construct a morphism from a scheme X to a projective space (Proposition 1.2.7). Among the invertible sheaves, very-ample and ample sheaf will be our main tool to ask more properties to this kind of morphisms in Chapter 2. Further, invertible sheaves provide a possible definition of a ubiquitous object in mathematic: the group of Picard.

1.2.1 Definition and first properties

Definition 1.2.1. Given a scheme X, an $\mathcal{O}_X$-module $\mathcal{L}$ is *invertible* is for all $x \in X$ there exists an open neighbourhood $U \subseteq X$ of x and an isomorphism of $\mathcal{O}_U$-modules

$$\mathcal{O}_X|_U \xrightarrow{\sim} \mathcal{L}|_U.$$

In particular an invertible sheaf is quasi-coherent. An example of invertible sheaf is provided by the sheaves $\mathcal{O}_X(n)$ of 1.1.10, by remark 1.1.11(i). Given an invertible sheaf $\mathcal{L}$ and a global section $\sigma \in \mathcal{L}(X)$, we put

$$X_\sigma := \{x \in X : \mathcal{L}_x = \sigma_x \mathcal{O}_{X,x}\}.$$

This is an open subset of X.
The following lemma turns out to be a foundamental tool for our next pourposes.

Lemma 1.2.2. *Let X be a noetherian scheme, $\mathcal{F}$ be a quasi-coherent sheaf on it and $\mathcal{L}$ an invertible sheaf. Given $f \in \mathcal{F}(X)$ and $\sigma \in \mathcal{L}(X)$, then*

(a) if $f|_{X_\sigma} = 0$, then there exists an integer $n > 0$ such that $f \otimes \sigma^{\otimes n} = 0$ in $\mathcal{F} \otimes \mathcal{L}^{\otimes n}(X)$.

(b) given $g \in \mathcal{F}(X_\sigma)$, then there exists an integer $n_0 > 0$ such that for all $n \geq n_0$, $g \otimes (\sigma|_{X_s})^{\otimes n}$ lifts to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

Remark 1.2.3. We notice that this lemma generalizes Proposition 1.1.7. Indeed if $\mathcal{L} = \mathcal{O}_X$, then $X_\sigma = \{x \in X : \sigma_x \in \mathcal{O}_{X,x}^*\}$ and the canonical morphism

$$\mathcal{F}(X)_f = \mathcal{F}(X) \otimes_{\mathcal{O}_X(X)} \mathcal{O}_X(X)_f \longrightarrow \mathcal{F}(X_f)$$

is injective by (a) and surjective by (b), so it is an isomorphism.

Proof. Let $X = \bigcup_{i=1}^{r}$ be an open affine covering of X such that $\mathcal{L}|_{X_i} \cong \mathcal{O}_X|_{X_i}$ for all $i = 1, ..., r$. In particular for each i we can find $e_i \in \mathcal{L}(X_i)$ such that $\mathcal{L}|_{X_i} = e_i \mathcal{O}_X|_{X_i}$. Then $\sigma|_{X_i} = h_i e_i$, for some $h_i \in \mathcal{O}_X(X_i)$, and

$$\begin{aligned} X_\sigma \cap X_i &= \{x \in X_i : \mathcal{L}_x = (e_i)_x \mathcal{O}_{X,x} = (h_i)_x (e_i)_x \mathcal{O}_{X,x} = \sigma_x \mathcal{O}_{X,x}\} \\ &= \{x \in X_i = \operatorname{Spec} A : \mathcal{O}_{X,x} = (h_i)_x \mathcal{O}_{X,x}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A : h_i \in A_\mathfrak{p}^*\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A : h_i \notin \mathfrak{p}\} \end{aligned}$$

so $X_\sigma \cap X_i \subseteq X_i$ is the principal open $D(h_i)$.
Now suppose $f|_{X_\sigma} = 0$, then $f|_{X_\sigma \cap X_i} = 0$, but this means that $f|_{X_i} = 0$ in A_{h_i}, i.e. $f|_{X_i} h_i^n = 0$ for some integer $n \geq 0$. Since X is noetherian and we have a finite number of affine opens, we can choose n such that $f|_{X_i} h_i^n = 0$ for all $i = 1, ..., r$. Let us consider the isomorphism[7]

$$\mathcal{O}_{X_i} \otimes_{\mathcal{O}_{X_i}} ... \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i} = \mathcal{O}_{X_i} \xrightarrow{\cdot(e_i \otimes ... \otimes e_i)} \mathcal{L}|_{X_i} \otimes ... \otimes \mathcal{L}|_{X_i} =: \mathcal{L}|_{X_i}^{\otimes n},$$

which induces another isomorphism

$$\mathcal{F}|_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{O}_{X_i} \xrightarrow{\varphi_{i,n}} \mathcal{F}|_{X_i} \otimes_{\mathcal{O}_{X_i}} \mathcal{L}|_{X_i}^{\otimes n}.$$

On X_i we have

$$\begin{array}{ccc} \mathcal{F}|_{X_i}(X_i) & \xrightarrow{(\varphi_{i,n})_{X_i}} & \mathcal{F}|_{X_i}(X_i) \otimes_{\mathcal{O}_{X_i}(X_i)} \mathcal{L}|_{X_i}^{\otimes n}(X_i) \\ f|_{X_i} h_i^n & \longmapsto & (f|_{X_i} h_i^n) \otimes e_i \otimes ... \otimes e_i \end{array}$$

and

$$\begin{aligned} (f|_{X_i} h_i^n) \otimes e_i \otimes ... \otimes e_i &= f|_{X_i} \otimes (h_i e_i) \otimes ... \otimes (h_i e_i) \\ &= f|_{X_i} \otimes \sigma|_{X_i} \otimes ... \otimes \sigma|_{X_i} \\ &= f|_{X_i} \otimes \sigma|_{X_i}^{\otimes n}. \end{aligned}$$

Now, since $f|_{X_i} h_i^n = 0$, then $(f \otimes \sigma^{\otimes n})|_{X_i} = f|_{X_i} \otimes \sigma|_{X_i}^{\otimes n} = 0$. But since this holds for all $i = 1, ..., r$, by the axioms of sheaves $f \otimes \sigma^{\otimes n} = 0$ in $\mathcal{F}(X) \otimes \mathcal{L}^{\otimes n}(X)$. So we have (a).
Now consider $g \in \mathcal{F}(X_\sigma)$. We have $g|_{X_\sigma \cap X_i} \in \mathcal{F}(X_\sigma \cap X_i) = \mathcal{F}|_{X_i}(D(h_i)) = (\mathcal{F}|_{X_i})_{h_i}$, so we can write

$$g|_{X_\sigma \cap X_i} = \frac{f_i|_{X_\sigma \cap X_i}}{h_i^m|_{X_\sigma \cap X_i}} \quad \text{with } f_i \in \mathcal{F}(X_i) \text{ and } m \geq 0.$$

As before we can choose a representation such that m is the same for all $i = 1, ..., r$. Using the notation of the first part, we put $t_i := \varphi_{i,m}(f_i) \in (\mathcal{F} \otimes \mathcal{L}^{\otimes m})(X_i)$. Then we have

$$\begin{aligned} t_i|_{X_\sigma \cap X_i} &= (\varphi_{i,m})_{X_\sigma \cap X_i}(f_i) \\ &= f_i|_{X_\sigma \cap X_i} \otimes e_i \otimes ... \otimes e_i \\ &= g|_{X_\sigma \cap X_i} h_i^m \otimes e_i \otimes ... \otimes e_i \\ &= g|_{X_\sigma \cap X_i} \otimes s|_{X_\sigma \cap X_i}^{\otimes m}. \end{aligned}$$

[7] Hereafter we are going to indicate it as $\cdot e_i^{\otimes n}$.

This means that on some open sets t_i does not depend by f_i, in particular $t_i|_{X_\sigma \cap X_i \cap X_j} - t_j|_{X_\sigma \cap X_i \cap X_j} = 0$ for all i, j. Then, by (a) applied to the scheme $X_i \cap X_j$, there exists $p \geq 0$ such that $(t_i|_{X_i \cap X_j} - t_j|_{X_i \cap X_j}) \otimes s|^{\otimes p}_{X_i \cap X_j} = 0$ in $((\mathcal{F} \otimes \mathcal{L}^{\otimes m}) \otimes \mathcal{L}^{\otimes p})(X_i \cap X_j)$ (Notice that the equality still holds for integers greather than p). This means that for all $n \geq m + p$, the sections $t_i \otimes \sigma|^{\otimes(n-m)}_{X_i} \in (\mathcal{F} \otimes \mathcal{L}^{\otimes n})(X_i)$ coincide on $X_i \cap X_j$. Then, by the axioms of sheaves, there exists a section $t \in (\mathcal{F} \otimes \mathcal{L}^{\otimes n})(X)$, whose restriction to X_i is $t_i \otimes \sigma|^{\otimes(n-m)}_{X_i}$. Then

$$\begin{aligned}
(t|_{X_\sigma})|_{X_\sigma \cap X_i} &= t|_{X_\sigma \cap X_i} \\
&= (t|_{X_i})|_{X_\sigma \cap X_i} \\
&= (t_i \otimes \sigma|^{\otimes(n-m)}_{X_i})|_{X_\sigma \cap X_i} \\
&= t_i|_{X_\sigma \cap X_i} \otimes \sigma|^{\otimes(n-m)}_{X_\sigma \cap X_i} \\
&= g|_{X_\sigma \cap X_i} \otimes s|^{\otimes m}_{X_\sigma \cap X_i} \otimes \sigma|^{\otimes(n-m)}_{X_\sigma \cap X_i} \\
&= g|_{X_\sigma \cap X_i} \otimes s|^{\otimes n}_{X_\sigma \cap X_i} \\
&= (g \otimes \sigma|^{\otimes n}_{X\sigma})|_{X_\sigma \cap X_i},
\end{aligned}$$

so, by the axioms of sheaves on the open covering $X_\sigma = \bigcup_i (X_\sigma \cap X_i)$, we have $t|_{X_\sigma} = g \otimes \sigma|^{\otimes n}_{X\sigma}$, thus we have (b). □

This result assures that for a quasi-coherent sheaf $\mathcal{F}$, once we found a section over an open set of the form X_σ, we can extend it to a global section modulo passing to a sheaf which keeps some affinities with the starting one (consider, for example, the case in which $\mathcal{F}$ equals $\mathcal{L}$ itself). Now we are going to introduce an important property, shared by all the quasi-coherent sheaves over a scheme.

Definition 1.2.4. Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is *generated by its global sections* if there exist a family of global sections $\{\sigma_i \in \mathcal{F}(X)\}_{i \in I}$, such that $\mathcal{F}_x$ is generated as $\mathcal{O}_{X,x}$-module by the images of the σ_i, for all $x \in X$.

Remark 1.2.5. Note that the definition is equivalent to the fact that there exists a surjective morphism of sheaves $\mathcal{O}_X^{(I)} \longrightarrow \mathcal{F}$ (defined indeed by the generating sections). In particular this means that every quasi-coherent sheaf over a scheme X is generated by its global sections.

Remark 1.2.6. For an invertible sheaf $\mathcal{L}$, it follows immediately that $X = \bigcup_{i=1}^d X_{\sigma_i}$ if and only if $\mathcal{L}$ is generated by $\sigma_0, ..., \sigma_d$. Indeed, since an invertible sheaf is locally of rank 1, the two conditions are both equivalent to the fact that for all $x \in X$ there exists $i = 1, ..., d$ such that $\mathcal{L}_x = (\sigma_i)_x \cdot \mathcal{O}_{X,x}$

An important morphism. The next result shows the relation between morphism to a projective space and invertible sheaves: it will be fundamental for our pourposes. Before starting we point out the following fact. Let $\mathcal{L}$ be an invertible sheaf on X. Let $\sigma \in \mathcal{L}(X)$. Since, for every $x \in X_\sigma$, by definition σ_x is a basis of $\mathcal{L}_x$ over $\mathcal{O}_{X,x}$, the multiplication by σ induces an isomorphism

$$\mathcal{O}_X|_{X_\sigma} \xrightarrow{\cdot\sigma} \sigma \cdot \mathcal{O}_X|_{X_\sigma} = \mathcal{L}(X_\sigma).$$

In particular, for $\tau \in \mathcal{L}(X_\sigma)$ we can write, without ambiguity, τ/σ as an element of $\mathcal{O}_X(X_\sigma)$.

Proposition 1.2.7. *Let A be a commutative ring and X be a scheme over it, considering the projective space $\mathbb{P}^d_A =$Proj$A[T_0, ..., T_d]$,*

(a) given a morphism of A-schemes $f : X \longrightarrow \mathbb{P}^d_A$, then $f^*\mathcal{O}_{\mathbb{P}^d_A}(1)$ is an invertible sheaf over X generated by $d+1$ global sections;

(b) given any invertible sheaf $\mathcal{L}$ on X generated by $d+1$ global sections $\sigma_0, ..., \sigma_d \in \mathcal{L}(X)$, then there exists an A-morphism $f : X \longrightarrow \mathbb{P}^d_A$, such that $\mathcal{L} \cong f^*\mathcal{O}_{\mathbb{P}^d_A}(1)$.

Proof. Let us prove the first point. We put $P = \mathbb{P}^d_A$. The twisting sheaf of Serre $\mathcal{O}_P(1)$ is generated by the global sections $T_0, ..., T_d$ (see Lemma 1.1.12).

These sections canonically induce global sections of $f^*\mathcal{O}_P(1)$ via the canonical morphism

$$\mathcal{O}_P(1) \longrightarrow f^*\mathcal{O}_P(1) = f^{-1}\mathcal{O}_P(1) \otimes_{f^{-1}\mathcal{O}_P} \mathcal{O}_X.$$

Being $\sigma_0, ..., \sigma_d \in f^*\mathcal{O}_P(1)$ such global sections, for $x \in X$ and $y = f(x) \in \mathbb{P}^d_A$ we have

$$\begin{aligned}
(f^*\mathcal{O}_P(1))_x &= \mathcal{O}_P(1)_y \otimes_{\mathcal{O}_{P,y}} \mathcal{O}_{X,x} \\
&= \sum_i (T_i)_y \mathcal{O}_{P,y} \otimes_{\mathcal{O}_{P,y}} \mathcal{O}_{X,x} \\
&= \sum_i (\sigma_i)_x \mathcal{O}_{X,x},
\end{aligned}$$

i.e. $f^*\mathcal{O}_P(1)$ is invertible[8] generated by the global sections $\sigma_1, ..., \sigma_d$.
For part (b), we know by remark 1.2.6, that the open subsets X_{σ_i} cover X. For $i = 0, ..., d$ define the morphism $f_i : X_{\sigma_i} \longrightarrow D_+(T_i)$ corresponding to

$$\begin{array}{ccc} \mathcal{O}_P(D_+(T_i)) & \longrightarrow & \mathcal{O}_X(X_{\sigma_i}) \\ T_j/T_i & \longmapsto & \sigma_j/\sigma_i \end{array},$$

which is well defined by what we said at the beginning of this paragraph. It remains to show that the morphisms f_i glue to a morphism $f : X \longrightarrow P$, but this is easy since on $D_+(T_iT_j)$ we have that f_i is defined by

$$\frac{T_hT_k}{T_iT_j} \longmapsto \frac{\sigma_h\sigma_i^{-1}\sigma_k\sigma_i^{-1}}{\sigma_j\sigma_i^{-1}} = \frac{\sigma_h\sigma_k}{\sigma_i\sigma_j},$$

while f_j is defined by

$$\frac{T_hT_k}{T_iT_j} \longmapsto \frac{\sigma_h\sigma_j^{-1}\sigma_k\sigma_j^{-1}}{\sigma_i\sigma_j^{-1}} = \frac{\sigma_h\sigma_k}{\sigma_i\sigma_j}.$$

Further we have by remark 1.1.9 that

$$\begin{aligned} (f^*\mathcal{O}_P(1))_x &\cong \mathcal{O}_P(1)_{f(x)} \otimes_{\mathcal{O}_{P,f(x)}} \mathcal{O}_{X,x} \\ &\cong (T_i)_{f(x)}\mathcal{O}_{P,f(x)} \otimes_{\mathcal{O}_{P,f(x)}} \mathcal{O}_{X,x} \\ &\cong (\sigma_i)_x\mathcal{O}_{X,x} \\ &\cong \mathcal{L}_x, \end{aligned}$$

which implies $f^*\mathcal{O}_P(1) \cong \mathcal{L}$. □

Remark 1.2.8. Note that the latter morphism behaves like

$$X \ni x \longmapsto (\sigma_0(x) : ... : \sigma_d(x)) \in \mathbb{P}^d_A.$$

1.2.2 Very ample and ample sheaves

Definition 1.2.9. Let $f : X \longrightarrow \mathrm{Spec} A$ be a scheme over a ring A and let $i : X \longrightarrow \mathbb{P}^d_A$ be an immersion. The sheaf $\mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}^d_A}(1)$ is called a *very ample sheaf* (relative to f). In general we put $\mathcal{O}_X(n) := i^*\mathcal{O}_{\mathbb{P}^d_A}(n)$.
Given a quasi-coherent sheaf $\mathcal{F}$, we denote $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$.

[8]Recalling remark 1.1.9, given a morphism $f : X \longrightarrow Y$ and an invertible sheaf $\mathcal{L}$ on Y, then for all $x \in X$,

$$(f^*\mathcal{L})_x \cong \mathcal{L}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} = e_x \cdot \mathcal{O}_{Y,f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x} \cong (f^*e)_x \cdot \mathcal{O}_{X,x},$$

i.e. $(f^*\mathcal{L})$ is invertible.

The sheaf $\mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}^d_A}(1)$ is invertible (see the proof of Proposition 1.2.7) and depends on i. By Proposition 1.2.7, the case Y =Spec$A$ is the same thing as saying that $\mathcal{O}_X(1)$ admits a set of global sections such that the corrisponding morphism $X \longrightarrow \mathbb{P}^d_A$ is an immersion.
A more flexible notion than that of a very ample sheaf is the following.

Definition 1.2.10. Let X be a quasi-compact scheme. An invertible sheaf $\mathcal{L}$ is *ample* if for any finitely generated quasi-coherent sheaf $\mathcal{F}$ on X, there existes an integer $n_0 \geq 1$ such that for every $n \geq n_0$, $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections.

We underline the fact that, while in the definition of very ample sheaf the base scheme plays an important role (it defines the projective space where X immerges), the definition of ample sheaf is somehow an "absolute" notion. As the terminology may suggest, on projective schemes over a ring, every very ample sheaf is ample.

Theorem 1.2.11. *Let X be a projective scheme over a ring A. Then for any finitely generated quasi-coherent sheaf $\mathcal{F}$, there exists an integer $n_0 \geq 0$ such that $\mathcal{F}(n)$ is generated by its global sections for every $n \geq n_0$.*

Proof. It suffices to prove the statement for $X = \mathbb{P}^d_A$ (see [L] Theorem 1.27). Consider $U_i = D_+(T_i)$, then $\mathcal{F}(U_i)$ is generated by a finite number of elements[9] σ_{ij}, $j = 1, ..., m$, where m can be taken the same for all $i = 1, ..., d$. We have that $\sigma_{ij} \in \mathcal{F}(X_{T_i})$ $(U_i = X_{T_i})$, where $T_i \in \mathcal{O}_X(1)(X)$ (see Lemma 1.1.12) is a global section of an invertible sheaf on X. Then by Lemma 1.2.2(b) there exists $n_0 \geq 0$ such that $\sigma_{ij} \otimes T_i^n$ is the restriction of a global section of $(\mathcal{F} \otimes \mathcal{O}_X(n))(X) = \mathcal{F}(n)(X)$, for all $n \geq n_0$. In particular $\mathcal{F}(n)(U_i) = (\mathcal{F} \otimes \mathcal{O}_X(n))(U_i) = \mathcal{F}(U_i) \otimes T_i^n \mathcal{O}_X(U_i)$, therefore on U_i the sheaf $\mathcal{F}(n)$ is generated by $\{\sigma_{ij} \otimes T_i^n\}_j$ and this immediately implies that $\mathcal{F}(n)$ is generated by the global sections we found. $\square$

On the other hand, an ample sheaf always provides a very ample sheaf.

[9] Cover U_i with a finite number of principal open subset V_k such that there exist an exact sequence $\mathcal{O}_X^{n_k}|_{V_k} \longrightarrow \mathcal{F}_{V_k}$, given by the definition of finitely generated sheaf. By Proposition 1.1.6, $\mathcal{O}_X(V_k)^{n_k} \longrightarrow \mathcal{F}(V_k)$ is exact. Thus $\mathcal{F}(V_k)$ is finitely generated as $\mathcal{O}_X$-module. Note that $\mathcal{F}(V_k) = \mathcal{F}(U_i) \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(V_k)$, by Proposition 1.1.7, so we can find a finitely generated submodule M of $\mathcal{F}(U_i)$, such that $\mathcal{F}(V_k) = M \otimes_{\mathcal{O}_X(U_i)} \mathcal{O}_X(V_k)$ for every k. The sequence $\widetilde{M} \longrightarrow \mathcal{F}|_{U_i} \longrightarrow 0$ is exact, since it is exact on every V_k. Therefore $M \longrightarrow \mathcal{F}(U_k)$ is surjective and we are done.

Proposition 1.2.12. *Let $f : X \longrightarrow \operatorname{Spec} A$ be a morphism of finite type and suppose X Noetherian. Given an ample sheaf $\mathcal{L}$ on X, then there exists an $m \geq 1$, such that $\mathcal{L}^{\otimes m}$ is very ample for f.*

Remark 1.2.13. Let $\mathcal{L}$ be an invertible sheaf over a scheme X and consider a global section $\sigma \in \mathcal{L}(X)$. Then $X_\sigma = X_{\sigma^{\otimes n}}$, since $\sigma_x \mathcal{O}_{X,x} = \mathcal{L}_x$ if and only if $(\sigma^{\otimes n})_x \mathcal{O}_{X,x} = \mathcal{L}_x^{\otimes n}$.

Proof. First of all, we show that for any point $x \in X$ there exists an integer $n = n(x)$ and a global section $\sigma \in \mathcal{L}^{\otimes n}(X)$ such that X_σ is an affine neighbourhood of x. Consider an open affine neighbourhood of x such that $\mathcal{L}|_U \cong \mathcal{O}_X|_U$. So $X - U$ is closed. Let $\mathcal{J}$ be the sheaf of ideals definining it, i.e. $X - U = \mathcal{V}(\mathcal{J})$. For any $y \in X$, $\mathcal{L}_y$ is free, hence flat, so for any $n \geq 1$ $(\mathcal{J} \otimes \mathcal{L}^{\otimes n})_y = \mathcal{J}_y \otimes \mathcal{L}_y^{\otimes n} \cong \mathcal{J}_y \mathcal{L}_y^{\otimes n} = (\mathcal{J}\mathcal{L}^{\otimes n})_y \subseteq \mathcal{L}_y^{\otimes n}$, i.e. $\mathcal{J} \otimes \mathcal{L}^{\otimes n}$ can be identified with $\mathcal{J}\mathcal{L}^{\otimes n} \subseteq \mathcal{L}^{\otimes n}$. Since $\mathcal{L}$ is ample we can choose n such that $\mathcal{J} \otimes \mathcal{L}^{\otimes n} = \mathcal{J}\mathcal{L}^{\otimes n}$ is generated by its global sections. So that there exists $\sigma \in (\mathcal{J}\mathcal{L}^{\otimes n})(X) \subseteq \mathcal{L}^{\otimes n}(X)$ such that σ_x is base of $\mathcal{J}_x \mathcal{L}_x^{\otimes n} =$[10]$\mathcal{L}^{\otimes n}$ and in particular $x \in X_\sigma \subseteq$ [11]U. Let us write $\mathcal{L}|_U = e \cdot \mathcal{O}_U$, so that $\sigma|_U = eh$ for some $h \in \mathcal{O}_X(U)$. Then, as in Lemma 1.2.2, $X_\sigma = D_U(h)$, principal open in U affine. Thus X_σ is affine $(= \operatorname{Spec} \mathcal{O}_X(U)_h)$.
For now on, we won't use the fact that $\mathcal{L}$ is ample.
Since X is quasi-compact, it can be covered by a finite number of open subsets of the form X_{σ_i}, with $\sigma_i \in \mathcal{L}^{\otimes n}(X)$ and n can be choosen to be the same for all i. The fact that $f : X \longrightarrow \operatorname{Spec} A$ is of finite type means that on affine schemes (like $\operatorname{Spec}A$) and on every affine open subset of $f^{-1}(\operatorname{Spec}A)$ (like X_{σ_i}) the morphism

$$\mathcal{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) = A \longrightarrow \mathcal{O}_X(X_{\sigma_i})$$

in a ring homomorphism of finite type, i.e. $\mathcal{O}_X(X_{\sigma_i}) = A[f_{i1}, ..., f_{im_i}]$, for a finite number of $f_{ij} \in \mathcal{O}_X(X_{\sigma_i})$. Since $f_{ij} \in \mathcal{O}_X(X_{\sigma_i})$, where $\sigma_i \in \mathcal{L}^{\otimes n}(X)$ ($\mathcal{L}^{\otimes n}$ invertible and $\mathcal{O}_X$ quasi-coherent), by Lemma 1.2.2, there exists $n_0 \geq 1$ such that for every $r \geq n_0$ the section $f_{ij} \otimes \sigma_i^{\otimes r}$ lifts to a global section $\sigma_{ij} \in (\mathcal{O}_X \otimes \mathcal{L}^{\otimes rn})(X) = \mathcal{L}^{\otimes rn}(X)$. We can choose a common r for all i, j.
Now, since $\{X_{\sigma_i}\}_i$ cover X, then $\{\sigma_i^{\otimes r}\}_i$ generate[12] $\mathcal{L}^{\otimes rn}(X)$ and the same holds for $\{\sigma_i^{\otimes r}, \sigma_{ij}\}_{i,j} \subset \mathcal{L}^{\otimes rn}(X)$. With the latter set of global sections, let us define the morphism

$$\pi : X \longrightarrow P := \operatorname{Proj} A[S_i, S_{ij}]_{i,j}$$

[10] $x \notin \mathcal{V}(\mathcal{J}) = X - U$, so $\mathcal{J}_x = \mathcal{O}_{X,x}$

[11] If $y \in \mathcal{V}(\mathcal{J})$, then $\sigma_y \in \mathcal{J}_y \mathcal{L}_y^{\otimes n} \neq \mathcal{L}_y^{\otimes n}$; hence $\sigma_y \mathcal{O}_{X,y} \subseteq \mathcal{J}_y \mathcal{L}_y^{\otimes n} \neq \mathcal{L}_y^{\otimes n}$, i.e. $y \notin X_\sigma$

[12] see remark 1.2.13 and 1.2.6

as in Proposition 1.2.7. Put $U_i := D_+(S_i)$, so that $X_{\sigma_i} = \pi^{-1}(U_i)$ and the morphism

$$\mathcal{O}_P(U_i) = A\left[\frac{S_j}{S_i}, \frac{S_{ij}}{S_i}\right] \longrightarrow \mathcal{O}_X(X_{\sigma_i}) = A[f_{ij}]$$

is surjective, because $S_{ij}/S_i \longmapsto f_{ij}$, since, on X_{σ_i}, $\dfrac{\sigma_{ij}}{\sigma_i^{\otimes r}} = \dfrac{f_{ij} \otimes \sigma_i^{\otimes r}}{\sigma_i^{\otimes r}} = f_{ij}$. Therefore π induces a closed immersion from X to $U := \cup_i U_i$. Hence π is an immersion and, again by Proposition 1.2.7, $\mathcal{L}^{\otimes rn}$ is the inverse image of the twisting sheaf of Serre, hence it is very ample. □

1.2.3 Picard Group

Consider the following proposition.

Proposition 1.2.14. *Let X be any scheme.*

(a) If $\mathcal{L}, \mathcal{L}'$ are invertible sheaves on X, so it is $\mathcal{L} \otimes \mathcal{L}'$.

(b) If $\mathcal{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathcal{L}^{-1}$ on X, such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Proof. The first statement comes from the fact that locally $(\mathcal{L} \otimes \mathcal{L}')_x = \mathcal{L}_x \otimes \mathcal{L}'_x = e \cdot \mathcal{O}_{X,x} \otimes f \cdot \mathcal{O}_{X,x} \cong ef \cdot \mathcal{O}_{X,x}$. For (b), take $\mathcal{L}^{-1}$ to be the so called *dual sheaf* $\mathrm{Hom}(\mathcal{L}, \mathcal{O}_X)$ defined from the presheaf

$$U \longmapsto \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{L}(U), \mathcal{O}_X(U)).$$

Then we have the canonical morphism

$$\mathcal{L} \otimes \mathrm{Hom}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{O}_X,$$

which locally gives the homomorphism $\mathcal{L}_x \otimes \mathrm{Hom}(\mathcal{L}_x, \mathcal{O}_{X,x}) \cong \mathcal{O}_{X,x}$ given by $e_x \otimes (f : \mathcal{L}_x \to \mathcal{O}_{X,x}) \longmapsto f(e_x) \in \mathcal{O}_{X,x}$. □

This allows us to give the following definition.

Definition 1.2.15. For any scheme X the *Picard group* of X, Pic$X$, is the group of isomorphism classes of invertible sheaves on X endowed with the operation $\otimes$, which makes it a group in virtue of the previous proposition.

1.3 Sheaves of differentials

Definition 1.3.1. Given a ring A, Let B be an A-algebra and M a B-module. An *A-derivation of B into M* is an A-linear map $d : B \longrightarrow M$ such that for any $b_1, b_2 \in B$ is verified the Leibniz rule

$$d(b_1 b_2) = b_1 db_2 + b_2 db1.$$

We denote the set of derivations by $\mathrm{Der}_A(B, M)$.

It comes out that $\mathrm{Der}_A(B, M)$ is an A-module.

Remark 1.3.2. An immediate consequence of the Leibniz rule is that $da = 0$, for every $a \in A$. Indeed, since d is A-linear, $da = ad(1)$, thus we can prove the statement for $a = 1$. Now

$$\begin{aligned} d(1) &= d(1 \cdot 1) \\ &= d(1) \cdot 1 + 1 \cdot d(1) \\ &= 2 \cdot d(1), \end{aligned}$$

implies that $d(1) = 0$.

Definition 1.3.3. Let B be an A-algebra. The *module of relative differential forms of B over A* is a B-module $\Omega^1_{B/A}$ endowed with an A-derivation $d : B \longrightarrow \Omega^1_{B/A}$ having the following universal property: for any B-module M and any A-derivation $d' : B \longrightarrow M$, there exists a unique homomorphism of B-modules $\phi : \Omega^1_{B/A} \longrightarrow M$ such that $d' = \phi \circ d$.

Consider the free B-module F generated by the symbols db for all $b \in B$ and take the quotient on the submodule generated by all the expressions of the form $d(b + b') - db - db'$, $d(bb') - bdb' - b'db$ for $b, b' \in B$ and da for $a \in A$. Calling such quotient $\Omega^1_{B/A}$ and defining $d : B \longrightarrow \Omega^1_{B/A}$ by $b \longmapsto db$, it comes out that the pair $(\Omega^1_{B/A}, d)$ satisties the above definition. Thus we see that the module of relative differential forms exists. Furthermore it is unique up to isomorphism. Indeed if we suppose there exists another pair (D, d') satisfying the above definition, then the universal property gives us two unique morphisms $\phi : \Omega^1_{B/A} \longrightarrow D$ and $\phi' : D \longrightarrow \Omega^1_{B/A}$, such that $\phi \circ d = d'$ and $\phi' \circ d' = d$. But again we have two unique morphisms f, f' such that $f \circ d = d$ and $f' \circ d' = d'$, but both $\phi' \circ \phi$ and the identity map satisfy the first equation and both $\phi \circ \phi'$ and the identity map satisty the second equation. By the uniqueness of f and f' it must be $\phi' \circ \phi = 1_{\Omega^1_{B/A}}$ and $\phi \circ \phi' = 1_D$, therefore ϕ and ϕ' are isomorphisms, one inverse of the other.

Remark 1.3.4. For any B-module M, we have the map

$$\begin{array}{ccc} \mathrm{Hom}_B(\Omega^1_{B/A}, M) & \longrightarrow & \mathrm{Der}_A(B, M) \\ \phi & \longmapsto & \phi \circ d \end{array}.$$

We note the fact that, by the universal property of the module of differential forms, this map is an isomorphism of A-modules.

We need to prove some properties of $\Omega^1_{B/A}$, in particular, consider the following remark.

Remark 1.3.5. Given an homomorphism of A-algebras $f : B \longrightarrow C$, we introduce two canonical morphisms of C-modules.

$$\begin{array}{rccc} \alpha : & \Omega^1_{B/A} \otimes_B C & \longrightarrow & \Omega^1_{C/A} \\ & db \otimes c & \longmapsto & c \cdot df(b) \end{array}$$

and

$$\begin{array}{rccc} \beta : & \Omega^1_{C/A} & \longrightarrow & \Omega^1_{C/B} \\ & dc & \longmapsto & dc \end{array},$$

where we underline the fact that dc is considered in two different class of equivalence, in particular $\beta(df(b)) = 0$.

Proposition 1.3.6. *Let B be an A-algebra.*

(a) For any A-algebra A', let $B' = B \otimes_A A'$, then there exists a canonical isomorphism of B'-modules $\Omega^1_{B'A'/A'} \cong \Omega^1_{B/A} \otimes_B B'$.

(b) Given $f : B \longrightarrow C$, homomorphism of A-algebras and α, β as above, then we have an exact sequence

$$\Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \xrightarrow{\beta} \Omega^1_{C/B} \longrightarrow 0.$$

(c) Given a multiplicative subset S of B, $S^{-1}\Omega^1_{B/A} \cong \Omega^1_{S^{-1}B/A}$.

(d) If C is the quotient of B by an ideal I, then we have an exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega^1_{B/A} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \longrightarrow 0,$$

where for any $b \in I$, $\delta(b + I^2) := db \otimes 1$.

See the proof in [L] Proposition 6.1.8.

We now carry the definition of the module of differentials to schemes and thus define the sheaf of relative differentials.

Proposition 1.3.7. *Let $f : X \longrightarrow Y$ be a morphism of schemes. Then there exists a unique quasi-coherent sheaf $\Omega^1_{X/Y}$ on X, such that for any affine open subset $V \subseteq Y$ and $U \subseteq f^{-1}(V)$ and for any $x \in U$ we have*

$$\Omega^1_{X/Y}|_U \cong \widetilde{\Omega^1_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)}} \text{ and } (\Omega^1_{X/Y})_x \cong \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}.$$

The proof can be found in [L] Proposition 6.1.17.

This result allow us to write the following definition.

Definition 1.3.8. Let $f : X \longrightarrow Y$ be a morphism of schemes. The quasi-coherent sheaf $\Omega^1_{X/Y}$ is called the *sheaf of relative differentials of degree 1 of X over Y.*

We conclude with the following proposition, which translates Proposition 1.3.6 in the setting of morphism of schemes.

Proposition 1.3.9. *Let $f : X \longrightarrow Y$ be a morphism of schemes.*

(a) For any Y-scheme Y', consider the projection $p : X \times_Y Y' \longrightarrow X$, then there exists a canonical isomorphism of $\mathcal{O}_{X'}$-modules $\Omega^1_{X\times_Y Y'/Y'} \cong p^*\Omega^1_{X/Y}$.

(b) Given $Y \longrightarrow Z$, morphism of schemes, then we have an exact sequence

$$f^*\Omega^1_{Y/Z} \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0.$$

(c) Given an open subset $U \subseteq X$, then $\Omega^1_{X/Y}|_U \cong \Omega^1_{U/Y}$ and in particular, for any $x \in X$ we have $(\Omega^1_{X/Y})_x \cong \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$.

(d) If $Z \subseteq X$ is a closed subscheme, defined by a quasi-coherent sheaf of ideals $\mathcal{I}$, then we have an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \xrightarrow{\delta} \Omega^1_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_Z \longrightarrow \Omega^1_{Z/Y} \longrightarrow 0.$$

1.4 Projective space bundles

A definition of *sheaf of $\mathcal{O}_X$-algebras* comes naturally from the definition on $\mathcal{O}_X$-modules asking that on the open sets the sheaf defines an algebra. Sheaves of algebra provides a way to construct some particular classes of schemes. In this section we present a generalization of the Proj construction, replacing the ring S with a sheaf of algebras $\mathcal{S}$. The result is a scheme which might be thought of as a fribration of Proj's of rings.

Given a Noetherian scheme X, let $\mathcal{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras. It is a sheaf with a direct sum decomposition

$$\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}_d,$$

where each $\mathcal{S}_d$ is an $\mathcal{O}_X$-module such that for every open subset $U \subseteq X$, $\mathcal{S}(U)$ is an $\mathcal{O}_X(U)$-algebra and the resulting direct sum decomposition

$$\mathcal{S}(U) = \bigoplus_{d \geq 0} \mathcal{S}_d(U)$$

is a grading of this algebra as a ring. We assume furthermore that $\mathcal{S}$ is a quasi-coherent sheaf, $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_1$ is a coherent sheaf and that $\mathcal{S}$ is locally generated by $\mathcal{S}_1$ as $\mathcal{O}_X$-algebra.

For each open affine subset $U \subseteq X$ we can consider Proj$\mathcal{S}(U)$ and the canonical map π_U :Proj$\mathcal{S}(U) \longrightarrow U$, corresponding to the inclusion of $\mathcal{O}_X(U)$ into $\mathcal{S}(U)$. It can be shown that these data can be glued togheter over each intersection of two open affines, definining a scheme that we indicate as Proj$(\mathcal{S})$ togheter with a morphism π :Proj$(\mathcal{S}) \longrightarrow S$ such that for each open affine $U \subseteq X$, $\pi^{-1}(U) \cong$Proj$\mathcal{S}(U)$.
Furthermore the invertible sheaves $\mathcal{O}_X(1)$ on each Proj$\mathcal{S}(U)$ are compatible under this construction (here we use the hypotesis on $\mathcal{S}_1$), so they glue togheter to give an invertible sheaf on Proj$(\mathcal{S})$, which we indicate as $\mathcal{O}_X(1)$.

We can consider a particular graded $\mathcal{O}_X$-algebra. Let $\mathcal{E}$ be a locally free quasi-coherent sheaf on a scheme X. Consider the symmetric algebra of $\mathcal{E}$ $\mathcal{S} := \bigoplus_{d>0} S^d(\mathcal{E})$ defined in the first section.
Then $\mathcal{S}$ is a sheaf of graded $\mathcal{O}_X$-algebra satisfying the assumpions we did for the previous construction.

Definition 1.4.1. Let X be a Noetherian scheme and $\mathcal{E}$ be a locally free quasi-coherent sheaf on it. Let $\mathcal{S}$ be a symmetric $\mathcal{O}_X$-algebra as before, we define the *projective space bundle* associated to $\mathcal{E}$ as $\mathbb{P}(\mathcal{E})$:=Proj$\mathcal{S}$. As such, it comes with a projection morphism $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow X$ and an invertible sheaf $\mathcal{O}(1)$.

We remark that, since $\mathcal{E}$ is locally free (suppose of rank $n+1$), if we take an open affine cover of X, such that, when restricted to each open U, $\mathcal{E}$ is free over $\mathcal{O}_X(U)$, then

$$\mathbb{P}(\mathcal{E})|_{\pi^{-1}(U)} \cong \mathbb{P}^n_U.$$

Example 1.4.2. We have in particular that the symmetric product of $\mathcal{O}_S^{r+1}$ equals the algebra of polynomials $\mathcal{O}_S[T_0, ..., T_r]$. So that

$$\mathbb{P}(\mathcal{O}_S^{r+1}) = \mathbb{P}_S^r.$$

Chapter 2

Projective schemes over a field...

Recall what the Normalization Lemma says for projective varieties.

Theorem. *Every projective variety X of dimension d over a field k is a finite covering of a projective space $\mathbb{P}^d_k$.*

In this chapter we want to develop such result asking that the finite morphism $X \longrightarrow \mathbb{P}^d_k$ statisfies a condition of étaleness over an affine space $\mathbb{A}^d_k$ contained in $\mathbb{P}^d_k$.

2.1 ... of characteristic 0

In characteristic 0 the condition above is really hard to have. Indeed it can be proved that for $\operatorname{char} k = 0$, the affine space $\mathbb{A}^d_k$ is simply connected. Thus every étale cover of the affine space is trivial, c'est-à-dire it is a disjoint union of copies of $\mathbb{A}^d_k$. So only in few cases we have this kind of result.

2.2 ... of positive characteristic

For this part we are going to present and develop the results of [K]. Given a projective scheme X on a field k of positive characteristic, we are going to prove first a projective version of Noether's normalization lemma, by using the theory of invertible sheaves presented in Chapter 1. Then we add some properties to the morphism $X \longrightarrow \mathbb{P}^n_k$, such as being étale away from an hyperplane of the projective space and controlling the behaviour of some subschemes in X. We underline what we did for Proposition 2.2.8: clarifying Lemma 6 of [K], we found out the need of adding an hypothesis of separability for the residue field of some points of X, in order to avoid some contradictions in the original result.

2.2.1 Review of Noether normalization lemma

Firstly we point out the following fact

Remark 2.2.1. Given an invertible ample sheaf $\mathcal{L}$ over a projective scheme X over a field k and a global section $\sigma \in \mathcal{L}(X)$, we have that

$$X_\sigma := \{x \in X : \sigma_x \mathcal{O}_{X,x} = \mathcal{L}_x\}$$

is affine. Indeed, since $\mathcal{L}$ is ample, there exists an integer $m_0 > 0$, such that $\mathcal{L}^{\otimes m_0}$ is very ample and in particular $\mathcal{L}^{\otimes m_0}(X) \cong \mathcal{O}_X(1)(X)$ (see Proposition 1.2.12). Writing $X = \mathrm{Proj}\dfrac{k[T_0, ..., T_N]}{I}$ for some homogeneous ideal I, by Serre vanishing theorem[1] there exists n_0 such that the morphism

$$\mathcal{O}_{\mathbb{P}^N_k}(n_0)(\mathbb{P}^N_k) \longrightarrow \mathcal{O}_X(n_0)(X)$$

is surjective. Hence $\sigma^{\otimes m_0 n_0} \in (\mathcal{L}^{\otimes m_0})^{\otimes n_0}(X) = \mathcal{O}_X(n_0)(X)$ can be seen as an homogeneous polynomial f_0 (of degree n_0). Now it's easy to see (see remark 1.2.13) that $X_\sigma = X_{\sigma^{\otimes m_0 n_0}} = \mathcal{D}_+(f_0)$, which is affine ([L] Proposition 2.3.38).

Lemma 2.2.2. *Let X be a projective scheme over a field k and $\mathcal{L}$ be an ample invertible sheaf on it. If $\mathcal{L}$ is generated by global sections $\sigma_0, ..., \sigma_d$, then we have a finite morphism $X \longrightarrow \mathbb{P}^d_k$.*

Proof. By Proposition 1.2.7 we have that the global sections induce $f : X \longrightarrow \mathbb{P}^d_k$, which behaves as

$$\begin{array}{ccc} X_{\sigma_i} \subseteq X & \longrightarrow & U_i := \mathrm{Spec} k[\frac{T_0}{T_i}, ..., \frac{T_d}{T_i}] \subseteq \mathbb{P}^d_k \\ \sigma_j/\sigma_i & \longleftarrow & T_j/T_i, \end{array} .$$

[1] It states that given a coherent sheaf $\mathcal{F}$ on a projective $S-$scheme X, where S is affine and Noetherian, there exists a positive integer n_0 such that for all $n \geq n_0$ the cohomology group $H^1(X, \mathcal{F}(n))$ is trivial (see [L] Theorem 5.3.2 and chapter 5 for more general considerations on Cech Cohomology). Now if we put $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^N_k}$ to be the sheaf of ideals defining X, for any $n > 0$ we have the canonical short exact sequence

$$0 \longrightarrow \mathcal{J}(n) \longrightarrow \mathcal{O}_{\mathbb{P}^N_k}(n) \longrightarrow \mathcal{O}_X(n) \longrightarrow 0 ,$$

whose associated long exact sequence in cohomology produce the following

$$\mathcal{O}_{\mathbb{P}^N_k}(n_0)(\mathbb{P}^N_k) \longrightarrow \mathcal{O}_X(n_0)(X) \longrightarrow H^1(\mathbb{P}^N_k, \mathcal{J}(n_0)) ,$$

and, for n_0 sufficiently large, $H^1(\mathbb{P}^N_k, \mathcal{J}(n_0)) = 0$.

So we just need to prove that it is finite. Since by remark 2.2.1 all the $f^{-1}(D_+(T_i)) = X_{\sigma_i}$ are affine and cover X, f is an affine morphism. Further $\mathbb{P}^d_k \longrightarrow k$ is projective, hence separated ([L] Corollary 3.3.10), and composed with f gives $X \longrightarrow k$, which is projective by hypotesis. This means (see [L] Corollary 3.3.32(e)) that f is projective. Being projective, hence proper by [L] Theorem 3.3.30, and affine, the morphism $X \longrightarrow \mathbb{P}^d_k$ is finite ([L] Lemma 3.3.17). □

2.2.2 Some avoidance lemmas

Now we need some avoidance lemmas. This is a classical problem in algebraic geometry and it provides a nice example of a geometric problem solved in its algebraic aspect. The idea is the following: given a closed subset D of a projective variety and a finite number of points S not contained in D, we want to find an hypersurface containing D, but not meeting any of the points. If the projective variety corresponds to a graded ring B, the closed subset is an homogeneus ideal $J \subseteq B$ not containing B_+ and we have a finite number of prime homogeneus ideals (the points of S) $\mathfrak{p}_1, ..., \mathfrak{p}_s$, such that $J \not\subset \mathfrak{p}_i$, for all i. Then we have to prove that there exists a homogeneus element $\sigma \in J$ (of some degree $l > 0$), such that $\sigma \notin \mathfrak{p}_i$ for all i, so that the hypersurface defined by the zeroes of σ contains D and avoids the points of S.
To prove this result we may assume there are no inclusions among the $\mathfrak{p}_i$ and consider firstly the case of just one point, where the statement is obvious. Assuming the result holds for $r-1$ points, we can pick $\sigma \in J$ homogeneus of positive degree such that $\sigma \notin \mathfrak{p}_i$ for all $i = 1, ..., r-1$. Assume $\sigma \in \mathfrak{p}_r$ (otherwise we are done). If $J\mathfrak{p}_1...\mathfrak{p}_{r-1} \subset \mathfrak{p}_r$, since we are assuming $\mathfrak{p}_r$ to be prime not containing any of the other $\mathfrak{p}_i$, we have $J \subset \mathfrak{p}_r$, which is a contradiction. So we can pick $\tau \in J\mathfrak{p}_1...\mathfrak{p}_{r-1}$, not in $\mathfrak{p}_r$. Then $\sigma^{\deg(\tau)} + \tau^{\deg(\sigma)}$ satisfies the statement.

We are going to consider a more "modern" form of the lemma, but not so distant from the previous intuitive description.

Lemma 2.2.3. *Let X be a projective scheme over a field k and $\mathcal{L}$ an ample invertible sheaf on it. Given a closed subscheme $D \subseteq X$ and another one $S \subseteq X$ of dimension*[2] *0, such that their intersection is empty, then there exists an integer $l > 0$ and a global section of $\mathcal{L}^{\otimes l}$ vanishing*[3] *along D, but without any zeroes on S.*

[2]Recall that a Noetherian scheme of dimension 0 is discrete, in particular it is finite.

[3]Given a global section $\sigma \in \mathcal{L}(X)$, the zero locus $\mathcal{V}(\sigma)$ is given by those points $x \in X$

Proof. As in remark 2.2.1, we can choose a power of $\mathcal{L}$ such that it is very ample on X =Proj$B$, where $B = \dfrac{k[T_0, ..., T_N]}{I}$. Therefore we have an homogeneus ideal $J \subseteq B$ and a finite family of primes $\mathfrak{p}_1, ..., \mathfrak{p}_m$, corresponding to D and the points of S, and we have that $J \not\subset \mathfrak{p}_i$ for all i. As we proved before, we can find an homogeneus polynomial $\sigma \in B$ of degree $l > 0$, such that $\sigma \in J$ and $\sigma \neq \mathfrak{p}_i$ for all i, i.e. $\mathcal{V}(\sigma) \supset D$ and $\mathcal{V}(\sigma) \cap S = \emptyset$. Now, by Lemma 1.1.12, the homogeneus polynomial σ corresponds to a global section of some power of $\mathcal{L}$ and we are done. □

Now we are considering a sort of "dual" of the previous problem: to find some hypersurfaces with no common intersections with a given closed subset, but all passing throught a finite set of points. We also add some new elements to the problem as the reader can note.

Lemma 2.2.4. *Let X be a projective scheme of dimension n over a field k and $\mathcal{L}$ an ample invertible sheaf on it. Let $S \subset X$ be a subscheme of dimension* $\dim S = 0$ *and be $D \subseteq X$ a closed subscheme not meeting S. Given $D_1, ..., D_m$, $0 \leq m \leq n$, closed subschemes of codimension 1, suppose they are such that for any nonempty subset $T \subseteq \{1, ..., m\}$ the set $D \cap \bigcap_{t \in T} D_t$ has codimension at least $\#T$ in D. Then there exists an integer $l > 0$ such that there exist $\sigma_1, ..., \sigma_n \in \mathcal{L}^{\otimes l}$ global sections with no common zero on D, vanishing on S and such that σ_i vanishes along D_i for $i = 1, ..., m$.*

Remark 2.2.5. The hypothesis on the D_i means that given any intersection of some of them in D, its intersection with any other D_i gives a subset of codimensions increased at least by one.

Proof. Consider the following statement:

For $i = 1, ..., j$ there exist integers $l_i > 0$ and sections $\sigma'_i \in \mathcal{L}^{\otimes l_i}(X)$, such that[4]

(a) σ'_i vanishes on S for any i;

(b) if $i \leq \min\{j, m\}$, then σ'_i vanishes along D_i;

(c) for any subset $T \subseteq \{j+1, ..., m\}$, the subset

$$Y_{j,T} = D \cap \bigcap_{i=1}^{j} \mathcal{V}(\sigma'_i) \cap \bigcap_{t \in T} D_t$$

has codimension in D at least $j + \#T$.

for which $\sigma_x \in \mathfrak{m}_x \mathcal{L}_x$.

We want to prove it by induction on j. First suppose it is true for $j > 0$. With the σ_i' as above, let $Z_j \subseteq D - D_{j+1}$[5] be a zero-dimensional subscheme such that it meets each irreducible components of $Y_{j,T'}$ of codimension $j+\#T'$ in D, for each subset $T' \subseteq \{j+2,...,m\}$. We notice that none of this components is actually contained in $D \cap D_{j+1}$, indeed we have that

$$D \cap D_{j+1} \cap \bigcap_{i=1}^{j} \mathcal{V}(\sigma_i') \cap \bigcap_{\substack{t \in T' \\ T' \subseteq \{j+2,...,m\}}} D_t$$

has codimensions in D at least $j + \#T' + 1$ by (c) (the $+1$ is given by the fact that actually we enlarge T by one element each time). Now we apply Lemma 2.2.3 to find a section $\sigma_{j+1}' \in \mathcal{L}^{\otimes l_{j+1}}(X)$, for a suitable $l_{j+1} > 0$, such that

(i) it vanishes along $S \cup D_{j+1}$ and

(ii) it has no zeroes on Z_j.

The first point tells us that conditions (a), (b) are satisfied also for the case $j+1$. For condition (c) we can notice that

$$D \cap \bigcap_{i=1}^{j+1} \mathcal{V}(\sigma_i') \cap \left(\bigcap_{\substack{t \in T \\ T \subseteq \{j+2,...,m\}}} D_t \right) = \underbrace{D \cap \bigcap_{i=1}^{j} \mathcal{V}(\sigma_i') \cap \left(\bigcap_{\substack{t \in T \\ T \subseteq \{j+2,...,m\}}} D_t \right)}_{\text{codim} \geq j + \#T} \cup \mathcal{V}(\sigma_{j+1}')$$

and σ_{j+1}' does not vanish on the components of codimension $j + \#T$ by (ii), so the intersection must have codimension at least $j + 1\#T$, which satisfies condition (c) for the case $j+1$.

For $j = 1$ we apply the same strategy. Note that, in this case, our hypotesis on the D_i substitutes (c) in assuring that we can construct $Z_0 \subseteq D$.

By induction, conditions (a),(b),(c) can be satisfied for the case $j = n$. Let l be a common multiple of the l_i, then since raising a section to some power does not affect its zeroes, we can consider $\sigma_i = (\sigma_i')^{\otimes l \backslash l_i} \in \mathcal{L}^{\otimes l}(X)$, which satisfy the thesis. □

[5] as before, for indices greather than m we do not consider any subvariety D_t

2.2.3 Étale condition on finite morphism

Recall the last result of the previous section. If in particular we consider the zero locus of a global section σ as closed subset $D \subseteq X$, the latter result, togheter with Lemma 2.2.2, allows us to construct a finite morphism from X to a projective space. We want to request some more properties to this morphism, such as étaleness and the possibility to control the image of some closed subscheme. We recall the definition of unramified and étale morphism.

Definition 2.2.6. Let $f : X \longrightarrow Y$ be a morphism of finite type of locally noetherian schemes. Let $x \in X$ and $y = f(x) \in Y$. We say that f is *unramified* at x if

- the homomorphism $\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{X,x}$ verifies $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$ and
- the (finite) extension of residue fields $k(y) \to k(x)$ is separable.

We say that f is *étale* at x if it is unramified and flat[6] at x.

We also need the definition of transverse intersection.

Definition 2.2.7. Let X be an affine scheme of dimension n. Given some hypersurfaces $\mathcal{V}(f_1), ..., \mathcal{V}(f_m)$, $m \leq n$, we say that they intersect *transversely* at a point $x \in X$ if there existe $f_{m+1}, ...f_n \in \mathcal{O}_X(X)$ such that $f_1, ..., f_n$ form a *complete system of parametres*, c'est-à-dire their images in the cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a base. In the case of a general scheme we have to check the definition on an open affine subset containing x.

We notice that the latter condition implies that every $\mathcal{V}(f_i)$ is regular at x. In particular they are irreducible in a neighbourhood of x.

We recall the fact that a certain property is said to be *geometrically* if it holds for the base change to the algebraic closure of k.

Proposition 2.2.8. *Let X be a geometrically reduced[7] projective scheme over a field k of (pure) dimension n and $\mathcal{L}$ an ample invertible sheaf on it. Let $S \subseteq X$ be a subscheme of dimension $dimS = 0$, whose points are smooth and with residue field separable over k. Consider a global section $\sigma \in \mathcal{L}(X)$ whose zero locus D has no intersections with S. Then for some integer $l > 0$ there exist $\delta_1, ..., \delta_n \in \mathcal{L}^{\otimes l}(X)$, such that*

[6] Recall that $f : X \longrightarrow Y$ is flat at x if $f_x^\# : \mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_{X,x}$ is a flat homomorphism. See [L] Section 4.3.1 for more details.

[7] This hypothesis is needed for having each irreducible component generically smooth, which is a necessary condition for the existence of a unramified morphism, see remark 2.2.11 for more details.

(a) $\sigma^{\otimes l}, \delta_1, ..., \delta_n$ *has no common intersection*

(b) the finite morphism $g : X \longrightarrow \mathbb{P}^n_k$*, induced by the sections of the previous point, sends* S *in rational points and it is unramified on* S*.*

Furthermore, given $D_1, ..., D_m$ *subscheme of codimension* 1*, such that any intersection among them in* D *produces a subset of codimension at least the number of* D_i *considered (as in Lemma 2.2.4), if they have transversal intersection at each point of* S*, then we can also ensure that, for* $i = 1, ..., m$*,* δ_i *vanishes on* D_i*.*

Remark 2.2.9. Recalling the definition of an unramified morphism of schemes, we have that for such an g as in (b), for any $s \in S$ we have a finite separable extensions of residue fields

$$k(g(s)) \longrightarrow k(s).$$

So the hypothesis on the residue fields for the points of S is necessary (in particular we are going to construct a morphism such that $k(g(s)) = k$).

Remark 2.2.10. Furthermore if $k(s)$ is separatble over $k(g(s))$, proving that a morphism g is unramified on the point s needs only to check that $\mathfrak{m}_s = \mathfrak{m}_{g(s)}\mathcal{O}_{X,s}$. By the Nakayama lemma, this is equivalent to check that

$$\mathfrak{m}_s = \mathfrak{m}_{g(s)}\mathcal{O}_{X,s} + \mathfrak{m}_s^2$$

i.e. the canonical map

$$\frac{\mathfrak{m}_{g(s)}}{\mathfrak{m}_{g(s)}^2} \otimes_{k(g(s))} k(s) \longrightarrow \frac{\mathfrak{m}_s}{\mathfrak{m}_s^2}$$

is surjective (actually we can prove it is an isomorphism since we are considering schemes of the same dimension and we are considering smooth points).

Proof. Since $\mathcal{L}$ is ample, by remark 2.2.1, we have that $U := X - D$ is affine. Each divisors corresponds to a sheaf of ideals on X, in particular $D_i = \mathcal{V}(I_i) \subset U$, for some ideal $I_i \subseteq \mathcal{O}_X(U)$. This means that we can find $f_i \in \mathcal{O}_X(U)$ vanishing on D_i for each $i = 1, ..., m$. By definition of transversal intersection, for each $s \in S$ we can choose $f_{s,i} = f_i$ for $i = 1, ..., m$, such that there exist $f_{s,m+1}, ..., f_{s,n} \in \mathcal{O}_X(U)$ and $f_{s,1}, ..., f_{s,m}, f_{s,m+1}, ..., f_{s,n}$ form a complete system of parameters, i.e. their images in $\dfrac{\mathfrak{m}_s}{\mathfrak{m}_s^2}$ form a $k(s)$-base

(recall that s is smooth, so the cotangent space has dimension n).
Now consider the surjective morphism

$$\mathcal{O}_X(U) \longrightarrow \frac{\mathcal{O}_X(U)}{\mathfrak{m}_s^2}.$$

In particular, given two distinct points $s, t \in S$ we have that $\mathfrak{m}_s^2 + \mathfrak{m}_t^2 = \mathcal{O}_X(U)$, since $\mathcal{V}(\mathfrak{m}_s^2 + \mathfrak{m}_t^2) = \mathcal{V}(\mathfrak{m}_s^2) \cap \mathcal{V}(\mathfrak{m}_t^2) = \mathcal{V}(\mathfrak{m}_s) \cap \mathcal{V}(\mathfrak{m}_t) = \{s\} \cap \{t\} = \emptyset$. Then, by the chinese remainder theorem, the canonical morphism

$$\mathcal{O}_X(U) \longrightarrow \bigoplus_{s \in S} \frac{\mathcal{O}_X(U)}{\mathfrak{m}_s^2} \tag{2.1}$$

is surjective as well and moreover it induces another surjective map

$$I_i \longrightarrow \bigoplus_{s \in S} \frac{I_i}{I_i \mathfrak{m}_s^2}, \qquad \text{for } i = 1, ..., m \tag{2.2}$$

Then for any $i = 1, ..., m$, consider the element $f_{s,i} \in I_i - \mathfrak{m}_s^2$. In particular[8] it is in $I_i - I_i \mathfrak{m}_s^2$, so we can consider $(f_{s,i})_{s \in S} \in \bigoplus_{s \in S} \frac{I_i}{I_i \mathfrak{m}_s^2}$ and we can find $f_i \in I_i$ which is mapped in it[9] by (2.2). In the same way, by (2.1), for $i = m+1, ..., n$ we can find $f_i \in \mathcal{O}_X(U)$ mapped in $(f_{s,i})_{s \in S} \in \bigoplus_{s \in S} \frac{\mathcal{O}_X}{\mathfrak{m}_s^2}$. Thus we found $f_1, ..., f_n \in \mathcal{O}_X(U)$ vanishing on D_i for $i = 1, ..., m$, such that their images in the cotangent space of each $s \in S$ form a base.
Now since $U = X - D = X_\sigma$, by Lemma 1.2.2 applied to the coherent sheaf $\mathcal{O}_X$, there exists an integer $r > 0$ such that $f_i \otimes (\sigma|_{X_U})^{\otimes r}$ lifts to a global section $\beta_i \in \mathcal{L}^{\otimes r}(X)$ (N.B. $\mathcal{O}_X \otimes \mathcal{L}^{\otimes r} \cong \mathcal{L}^{\otimes r}$).
Now by Lemma 2.2.4 we can choose global sections $\sigma_1, ..., \sigma_n \in \mathcal{L}^{\otimes l}$ for some integer $m > 0$, such that σ_i vanishes along D_i for $i = 1, ..., m$, for any $i = 1, ..., n$ σ_i vanishes on S and they have no common zeroes on D.
Now we define for $i = 1, ..., n$

$$\delta_i = \beta_i \otimes \sigma^{\otimes(2l-r)} + \sigma_i^{\otimes 2} \in \mathcal{L}^{\otimes(2l)}.$$

We have that on $D = \mathcal{V}(\sigma)$ the first addendum vanishes and what remains ($\sigma_i^{\otimes 2}$) has no common zeroes on D, so the same holds for δ_i. Furthermore the map g induced by $\sigma^{\otimes 2l}, \delta_1, ..., \delta_n$, on $U = X_\sigma$ behaves like

$$\begin{array}{ccc} U & \longrightarrow & \operatorname{Spec} k[T_1, ..., T_n] \\ \dfrac{\delta_i}{\sigma^{\otimes 2l}} & \longleftarrow\!\!\shortmid & T_i \end{array}$$

[8] $I_i \mathfrak{m}_s^2 \subseteq \mathfrak{m}_s^2$
[9] Supposing $f_i \in I_i$ such that $f_i = f_{s,i} + I_i \mathfrak{m}_s^2$; if $f_{s,i} \notin \mathfrak{m}_s^2$, then $f_i \notin \mathfrak{m}_s^2$

and on $s \in S$ (we avoid the notation for the stalk, but keep in mind that on these points $\beta_i = f_i \otimes \sigma^{\otimes r}$)[10]

$$\begin{aligned}\frac{\delta_i}{\sigma^{\otimes 2l}} &= \frac{\beta_i}{\sigma^{\otimes r}} + \left(\frac{\sigma_i}{\sigma^{\otimes l}}\right)^{\otimes 2}\\ &= f_i + \left(\frac{\sigma_i}{\sigma^{\otimes l}}\right)^2.\end{aligned}$$

The class in the cotangent space corresponds to $f_{s,i}$, since the second addendum is a square. So the associated cotangent map is an isomorphism for any $s \in S$, since we choosed $f_{s,1}, ..., f_{s,n}$ to be a base for the cotangent space. Moreover δ_i vanishes on S, this means that for all $s \in S$, $g(s) = (\sigma^{\otimes(2l)}(s) : \delta_1(s) : ... : \delta_n(s))$, which corresponds to the point[11] $(1 : 0 : ... : 0)$, whose residue field is k. This last remark, together with remark 2.2.10, implies that the morphism g is unramified. □

Now we can prove the main result of this chapter, we highlight that, as far as these previous results are concerned, we didn't suppose any hypothesis on the characteristic of the field k, but this condition is going to be foundamental for what follows. We point out a little remark before stating the theorem.

Remark 2.2.11. Recall that a morphism of $k-$schemes $X \xrightarrow{f} Y$ is unramified on a point $x \in X$ if and only if $\Omega^1_{X/Y,x} = 0$ (see [L] Corollary 6.2.3). If we consider the canonical exact sequence[12]

$$f^*\Omega^1_{Y/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0\ ,$$

which becomes on the stalk on x (recall that $(f^*\Omega^1_{Y/k})_x = \Omega^1_{Y/k,f(x)}$)

$$\Omega^1_{Y/k,f(x)} \xrightarrow{df} \Omega^1_{X/k,x} \longrightarrow \underbrace{\Omega^1_{X/Y,x}}_{=0} \longrightarrow 0\ ,$$

we can deduce that f is unramified in x if and only if the canonical morphism $df : \Omega^1_{Y/k,f(x)} \longrightarrow \Omega^1_{X/k,x}$ is surjective. Moreover if X (resp. Y) is smooth over k at x (resp. $f(x)$), the morphism df is an isomorphism if and only if it is étale at x (see [L] Proposition 6.2.10).

[10] For the last equality: $\frac{\beta_i}{\sigma^{\otimes r}} \in \mathcal{O}_{X,s}$, hence the tensor product becomes the standard product

[11] Recall that $\sigma^{\otimes(2l)}(s) \neq 0$.

[12] For the part relative to the sheaf of differentials we refer to the theory developped in [L] Chapter 6 Section 2.

Theorem 2.2.12. *Let X be a geometrically reduced projective variety of (pure) dimension n over a field k of characteristic $p > 0$ and let $\mathcal{L}$ be an invertible sheaf on it. Suppose there are the closed subschemes $D \subseteq X$ of dimension less than n and $S \subseteq X_{\text{smooth}}$ (subscheme of smooth points of X) zero-dimensional, whose points have residue fields separable over k. Assume that D and S have empty intersection. Then there exists a finite $k-$morphism $f : X \longrightarrow \mathbb{P}^n_k$ such that:*

(a) f is étale away from the hyperplane at infinity $H = \mathcal{V}(T_0) \subset \mathbb{P}^n_k$;

(b) $f(D) \subseteq H$;

(c) $f(S)$ does not meet H.

We notice that the hypothesis of being geometrically reduced is needed for having each irreducible component of X generically smooth. This last condition is given for any scheme admitting a generically étale map to a smooth scheme[13].

Proof. Without loss of generality we may enlarge S so that it meets each irreducible component of X. By Lemma 2.2.3, we can replace $\mathcal{L}$ with a suitable tensor power, such that there exists a global sections $\sigma \in \mathcal{L}(X)$ vanishing along D but with no zeroes on S. Proposition 2.2.8 assures that, for some integer $m > 0$, we can find global sections $\sigma_1, ..., \sigma_n \in \mathcal{L}^{\otimes m}(X)$ such that they have no common zeroes on $\mathcal{V}(\sigma)$ and the finite morphism $g : X \longrightarrow \mathbb{P}^n_k$, induced by $\sigma^{\otimes m}, \sigma_1, ..., \sigma_n$ (Lemma 2.2.2), is unramified at each point of S. Consider the subset of $X - \mathcal{V}(\sigma)$ where g is unramified: it is open and its intersection with each irreducible component of X is nonempty, since g in unramified on S meeting each of them; let E be the complement of such open set in X. By applying Lemma 2.2.3 to the closed subset E and S, we have a global section $\tau \in \mathcal{L}^{\otimes rm}(X)$, for some $r > 0$, vanishing on E but nowhere on S. By Lemma 2.2.4 applied to the closed subset $\mathcal{V}(\tau)$ (no zero-dimensional subscheme needed), we can find global sections $\tau_1, ..., \tau_n \in \mathcal{L}^{\otimes lrm}(X)$, for some integer l, which can be assumed greather than 2, such that they have no common zeroes on $\mathcal{V}(\tau)$. Now we leave temporarily the notation of tensor product and we adopt the more light notation of simple product. We put

$$\begin{cases} \gamma_0 = \tau^{pl} \\ \gamma_i = \sigma_i \sigma^{m(pr-1)} \tau^{p(l-1)} + \tau_i^p \qquad i = 1, ..., n. \end{cases}$$

[13]Given $f : X \longrightarrow Y$ étale for all points outside of a closed subscheme, with Y smooth and $\dim X = \dim Y = d$. Then by remark 2.2.11, for any $x \in X$ such that f is étale at x, $d = \dim Y = \operatorname{rank} \Omega^1_{Y/k, f(x)} = \operatorname{rank} \Omega^1_{X/k,x}$. Thus X is smooth at x

Then $\mathcal{V}(\gamma_0) = \mathcal{V}(\tau)$, while for $i = 1, ..., n$ we have $\mathcal{V}(\gamma_i|_{\mathcal{V}(\tau)}) = \mathcal{V}(\tau_i^p|_{\mathcal{V}(\tau)})$, since the first addendum vanishes, but the τ_i have no common zeroes on $\mathcal{V}(\tau)$, hence $\bigcap_{i\neq 0} \mathcal{V}(\gamma_i|_{\mathcal{V}(\tau)})$ is empty. So $\gamma_0, \gamma_1, ..., \gamma_n$ have no common zeroes, thus they define a finite morphism $f : X \longrightarrow \mathbb{P}_k^n$ by Lemma 2.2.2.
Now we prove that f satisfies (a),(b),(c).
First of all, since $\mathcal{V}(\gamma_0) = \mathcal{V}(\tau) \supseteq E \supseteq D$, we have (b) and since $\mathcal{V}(\tau) \cap S = \emptyset$, we have (c). Let us now consider $Z := \mathcal{V}(\sigma) \cup \mathcal{V}(\tau)$. For a point $y \in X - Z$, the map g in unramified at y; c'est-à-dire if we consider the linear morphism

$$\begin{array}{ccc} \Omega^1_{\mathbb{P}^n_k/k,f(y)} & \xrightarrow{df} & \Omega^1_{X/k,y} \\ d\left(\frac{T_i}{T_0}\right)_{f(y)} & \longmapsto & d\left(\frac{\sigma_i}{\sigma^m}\right)_y \end{array},$$

by remark 2.2.11, it is surjective, thus $d = \operatorname{rank} \Omega^1_{\mathbb{P}^n_k/k,f(y)} \leq \operatorname{rank} \Omega^1_{X/k,y}$. By Nakayama Lemma, $\operatorname{rank} \Omega^1_{X/k,y} = \dim(\Omega^1_{X/k,y} \otimes k(y)) \geq \dim_x X = d$, thus $\Omega^1_{X/k,y}$ is of rank d and df is an isomorphism. Therefore we can state that the differentials at y of the regular functions $\sigma_1/\sigma^m, ..., \sigma_n/\sigma^m \in \mathcal{O}_X(X - Z)$ are a basis. On the other hand

$$\begin{aligned} d\frac{\gamma_i}{\gamma_0} &= d\frac{\sigma_i \sigma^{m(pr-1)} \tau^{p(l-1)}}{\tau^{pl}} + d\frac{\tau_i^p}{\tau^{pl}} \\ &= d\left[\frac{\sigma_i}{\sigma^m}\left(\frac{\sigma^{mr}}{\tau}\right)^p\right] + d\left[\left(\frac{\tau_i}{\tau^l}\right)^p\right] \\ &= \left(\frac{\sigma^{mr}}{\tau}\right)^p d\frac{\sigma_i}{\sigma^m}, \end{aligned}$$

where the last equality holds by applying Leibnitz rule and keeping in mind that we are working on a field of characteristic p. Now $(\sigma^{mr}/\tau)^p$ is invertible on $X - Z$, therefore the differentials at y of the regular functions γ_i/γ_0 are a basis, i.e. f is étale at y. □

As we stated at the beginning of this section, in positive characteristic we have a morphism which is étale away from an hyperplane of $\mathbb{P}_k^d$, hence it is étale over the affine space $\mathbb{A}_k^d$. This result can be improved with the following Theorem.

Theorem 2.2.13. *Given a separated scheme of finite type over a field k of positive characteristic. Let n be its (pure) dimension and $x \in X$ a smooth point. Suppose $D_1, ..., D_m$ be irreducible subscheme of codimension 1 intersecting transversely at x. Then there exist a finite étale morphism $f : U \longrightarrow \mathbb{A}_k^n$, for an open dense subset $U \subseteq X$ containing x, such that $D_1, ..., D_m$ are mapped to coordinate hyperplanes.*

Proof. We can replace X by a projective compactification of an affine open neighbourhood of x, so that we can consider S as irreducible and projective. Furthermore, by blowing up in a suitable way (avoiding x) we can suppose that for any intersection of the D_i, we obtain an irreducible subscheme.
Let us consider an ample invertible sheaf $\mathcal{L}$ on X (which is projective). Then we can repeat the same argument as before, with $D = \emptyset$ and $S = \{x\}$. Further since, for any $T \subseteq \{1, ..., m\}$, we arranged the intersection $\cap_{t \in T} D_t$ to be irreducible and it does not lie in $\mathcal{V}(\sigma)$ or $\mathcal{V}(\tau)$, we can impose the additional restriction that σ_i and τ_i both vanish along D_i provided by Proposition 2.2.8. □

Chapter 3

Schemes over a ring

Now we want to consider the case of a scheme $X \longrightarrow S$, where $S = \operatorname{Spec} R$, for some commutative ring R with unity. There are few results in this case, concerning particular classes of rings.

3.1 Pictorsion Rings

Definition 3.1.1. The ring R is said to be *pictorsion* if for any finite morphism $Z \longrightarrow S$, $\operatorname{Pic}(Z)$ is a torsion group.

It is not easy to face interesting examples of such class of rings with the tools developed in the previous chapters, thus we provide some results without going deeply in the details, which can be found in [GLL] Section 8 and [CMPT] Sections 1,2,3.

We recall the fact that, since S is affine, any finite morphism of schemes $Z \longrightarrow S$ corresponds to a finite R-algebra extension $R \longrightarrow R'$.
A zero dimensional ring is pictorsion; a finitely generated algebra R over a field k is pictorsion if and only if k is algebraic over a finite field and $\dim(R) \leq 1$ (e.g. $\mathbb{F}_p[x]$).
We have also the following result (whose proof can be found in [GLL]).

Lemma 3.1.2. *Let R be a commutative ring and R^{red} the quotient of R by its nilradical. Then R is pictorsion if and only if R^{red} is pictorsion.*

A *local-global* ring R is a commutative ring with the following property: whenever $f \in R[x_1, ..., x_n]$ is such that the ideal generated by $\{f(r) \in R : r \in R^n\}$ corresponds to R, then there exists $r \in R^n$ such that $f(r) \in R^*$. It can be proved the following statement (see [GLL]).

Proposition 3.1.3. *Let R be a local-global ring. Then every finite R-algebra R' has trivial Picard group. In particular R is pictorsion.*

Hereafter we are going to work on affine schemes $S = \operatorname{Spec} R$, where R is a noetherian ring, and morphisms of finite type $X \longrightarrow S$. Actually a large part of the following results can be proved for any commutative ring and any morphism of schemes $X \longrightarrow S$ finitely presented (see [GLL]). Our choice of a stronger hypothesis is due to the fact that all the examples we are considering of pictorsion rings are taken among the Noetherian rings, for which the condition of a finitely presented morphism as before is equivalent to a morphism of finite type.

3.2 Hypersurfaces

Let us consider an invertible sheaf $\mathcal{L}$ over a scheme X, given a global section $\sigma \in \mathcal{L}(X)$ we define the closed subset[1]

$$H_\sigma := \{x \in X : \sigma_x \mathcal{O}_{X,x} \neq \mathcal{L}_x\}.$$

We notice that $\sigma\mathcal{O}_X \subseteq \mathcal{L}$, so $\mathcal{I} := \sigma\mathcal{O}_X \otimes \mathcal{L}^{-1} \subseteq \mathcal{L} \otimes \mathcal{L}^{-1} = \mathcal{O}_X$ ($\mathcal{L}^{-1}$, locally free, is flat, so it preserves inclusions), so $\mathcal{I}$ is a sheaf of ideals on X, which defines[2] H_σ, so that the latter can be view as a closed subscheme of X.

Definition 3.2.1. Given a morphism of schemes $X \longrightarrow S$, let $\mathcal{L}$ be an invertible sheaf on X and $\sigma \in \mathcal{L}(X)$ a global section, the closed subscheme H_σ is called *hypersurface* (relative to $X \longrightarrow S$), if, for all $s \in S$, H_σ does not contains any irreducible components of positive dimension of the fiber X_s.

The condition on the fibers poses on a central role the morphism $X \longrightarrow S$. In particular we want H_σ to be an hypersurface in the usual sense on every fibers, as the following result states.

Lemma 3.2.2. *Given $S = \operatorname{Spec} R$, with R noetherian ring, let $X \longrightarrow S$ be a morphism of finite type and $H := H_\sigma$ a hypersurface relative to it, as in the previous lines. If $\dim X_s \geq 1$, then $\dim H_s \leq \dim X_s - 1$.*
If, moreover, $X \longrightarrow S$ is projective, $\mathcal{L}$ is ample and H nonempty, then H_s meets every irreducible component of positive dimension of X_s and, in particular, $\dim H_s = \dim X_s - 1$.

[1] Notice that it is the complementary of X_σ.

[2] Recall that $\mathcal{V}(I) = \{x \in X : I_x \neq \mathcal{O}_{X,x}\}$ and $\sigma_x\mathcal{O}_{X,x} = \mathcal{L}_x$ if and only if $\sigma_x\mathcal{O}_{X,x} \otimes (\mathcal{L}^{-1})_x = \mathcal{O}_{X,x}$.

Proof. Without loss of generality we can consider $H_s \neq \emptyset$, otherwise $\dim H_s < 0 \leq \dim X_s - 1$.
As in Lemma 1.2.2, locally, H_s is defined by one equation and it does not contain any irreducible components of X_s of positive dimension. In particular this means that $\dim H_s \leq \dim X_s - 1$.
We avoid the proof of the second part (which can be found in [GLL] Lemma 3.2), since it is not useful for our pourposes. $\square$

For the next part we need the following result, which can be proved in a more general setting as Theorem 5.1 in [GLL]. It is another version of the avoindance lemma presented in Chapter 2.

Theorem 3.2.3. *Let S be an affine scheme and $X \longrightarrow S$ be a projective morphism of finite type. Let $\mathcal{O}_X(1)$ be a very ample invertible sheaf relative to such morphism and let $F \subseteq X$ be a subscheme of finite type over S. Then there exists $n_0 > 0$, such that for all $n \geq n_0$ there exists a global section $f \in \mathcal{O}_X(n)(X)$, such that, for all $s \in S$, H_f does not contain any irreducible component of positive dimension of F_s.*

Remark 3.2.4. Suppose $X \longrightarrow \operatorname{Spec} R$ be a projective morphism, i.e.

$$X = \operatorname{Proj} \frac{R[T_0, ..., T_N]}{I} \text{ for some homogeneous ideal } I \subseteq R[T_0, ..., T_N].$$

Then, by Lemma 1.1.12, we can identify global sections $f \in \mathcal{O}_X(n)(X)$ with homogeneous elements $f' \in \dfrac{R[T_0, ..., T_N]}{I}$ of degree n and, in particular, $H_f = \mathcal{V}_+(f')$. Hereafter we are going to use the same notation for global sections and homogeneous elements.

3.3 Finite morphisms to projective spaces

Finally in this section we want to guarantee the existence of a finite S-morphism $X \longrightarrow \mathbb{P}^d_S$, when $X \longrightarrow \operatorname{Spec} R$ is projective with R pictorsion and $d := \max\{\dim X_s\}$. We are going to provide a converse of this statement too.

3.3.1 Finite morphisms as necessary condition

Theorem 3.3.1. *Let R be a noetherian pictorsion ring and $S = \operatorname{Spec} R$. Given a projective scheme X over S, consider $d := \max\{\dim X_s,\ s \in S\}$. Then there exists a finite morphism*

$$r : X \longrightarrow \mathbb{P}^d_S.$$

In particular if for every $s \in S$ $\dim X_s = d$, then r is surjective.

Proof. Since X is projective over S, we can consider it as a closed subscheme of $\mathbb{P}^N_S$. In particular $X \longrightarrow S$ is of finite type ([L] Corollary 3.3.10).
We have that $\mathbb{P}^N_S \longrightarrow S$ is projective and of finite type and $\mathcal{O}_{\mathbb{P}^N_S}(1)$ is a very ample sheaf relative to it. Then considering $F = X$ (trivially a subscheme of finite type over S), by Theorem 3.2.3, we can find $n_0 > 0$ and $f_0 \in \mathcal{O}_{\mathbb{P}^N_S}(n_0)(\mathbb{P}^N_S)$ such that H_{f_0} does not contain any irreducible components of positive dimension of $F_s = X_s$. Hence, by definition, $X \cap H_{f_0}$ is an hypersurface relative to $X \longrightarrow S$ (it is $\mathcal{V}_+(f|_X)$ using the notation in remark 3.2.4) and, by Lemma 3.2.2,

$$\dim(X \cap H_{f_0})_s \leq \dim X_s - 1 \leq d - 1, \qquad \forall s \in S. \tag{3.1}$$

Applying again Theorem 3.2.3 to $\mathbb{P}^N_S \longrightarrow S$, $\mathcal{O}_{\mathbb{P}^N_S}(n_0)$ and $F = X \cap H_{f_0}$ (it is of finite type since it is again projective, as closed in a projective sheme), we obtain $n_1 > 0$ and $f_1 \in \left(\mathcal{O}_{\mathbb{P}^N_S}(n_0)\right)(n_1)(\mathbb{P}^N_S) = \mathcal{O}_{\mathbb{P}^N_S}(n_0 n_1)(\mathbb{P}^N_S)$ such that H_{f_1} does not contain any irreducible components of positive dimension of $F_s = (X \cap H_{f_0})_s$ and, again by Lemma 3.2.2,

$$\dim(X \cap H_{f_0} \cap H_{f_1})_s \leq d - 2, \qquad \forall s \in S. \tag{3.2}$$

By continuing the process other $d-2$ times, we obtain a family of homogeneous polynomials $f_0, ..., f_{d-1}$, such that $Y := X \cap H_{f_0} \cap ... \cap H_{f_{d-1}}$ is a closed subscheme with fibers of dimension at most 0, as a consequence of continuing the results (3.1),(3.2). Hence $Y \longrightarrow S$ is quasi-finite (N.B. it is of finite type) and projective, as closed subscheme of a projective scheme. Therefore Y is finite over S (as in Lemma 2.2.2).
Since $H_{f_i} = H_{f_i^{\otimes m}}$ (remark 1.2.13) for any $m > 0$, we can suppose, without loss of generality, that the sections are of the same degree, i.e. $f_0, ..., f_{d-1} \in \mathcal{O}_{\mathbb{P}^N_S}(n)(\mathbb{P}^N_S)$ for some $n > 0$.
Now S is pictorsion by assumpion and $Y \longrightarrow S$ is finite, thus $\operatorname{Pic} Y$ is a torsion group. In particular we have that, for any $j > 0$, $\mathcal{O}_{\mathbb{P}^N_S}(nj)|_Y \in \operatorname{Pic} Y$, so for some power $j > 0$, $\mathcal{O}_{\mathbb{P}^N_S}(nj)|_Y \cong \mathcal{O}_Y$. Let $e \in \mathcal{O}_{\mathbb{P}^N_S}(nj)|_Y(Y)$ be a basis (i.e. $\forall y \in Y$ $(\mathcal{O}_{\mathbb{P}^N_S}(nj)|_Y)_y \cong e_y \mathcal{O}_{Y,y}$). We can prove that the canonical morphism

$$\mathcal{O}_{\mathbb{P}^N_S}(njk)(\mathbb{P}^N_S) \longrightarrow \mathcal{O}_{\mathbb{P}^N_S}(njk)|_Y(Y)$$

is surjective for k sufficiently large[3]. Therefore for such $k > 0$, $e^{\otimes k}$ lifts to a global section $f_d \in \mathcal{O}_{\mathbb{P}^N_S}(njk)(\mathbb{P}^N_S)$. Since $e_y \mathcal{O}_{\mathbb{P}^N_S, y} = (\mathcal{O}_{\mathbb{P}^N_S}(nj)|_Y)_y$ for any

[3] It is a consequence of Serre's Vanishing Theorem applied to the very ample sheaf $\mathcal{O}_{\mathbb{P}^N_S}$

$y \in Y$, then $H_e \cap Y = \emptyset$ and in particular $H_{f_d} \cap Y = \emptyset$.
As a result we have $f_0^{\otimes jk}, ..., f_{d-1}^{\otimes jk}, f_d \in \mathcal{O}_{\mathbb{P}^N_S}(njk)(\mathbb{P}^N_S)$ which generate $\mathcal{O}_{\mathbb{P}^N_S}(njk)|_X$, since they have no common zeroes on X. Thus their restriction on X induces a morphism

$$r : X \longrightarrow \mathbb{P}^d_S.$$

Since $X \longrightarrow S$ is of finite type by assumpion and $\mathbb{P}^d_S \longrightarrow S$ is separated and of finite type ([L] Corollary 3.3.10), the morphism $r : X \longrightarrow \mathbb{P}^d_S$ is of finite type as well (see [EGA],IV.1.6.2(v)). By using the same argument as Lemma 2.2.2 (where the finite type condition was easy to check), we can conclude that $r : X \longrightarrow \mathbb{P}^d_S$ is finite. This means in particular that $X_s \longrightarrow \mathbb{P}^d_{k(s)}$ is finite for all $s \in S$; if in addition $\dim X_s = d$ for all $s \in S$, then it is surjective[4]. □

We first point out a class of projective morphisms $X \longrightarrow \operatorname{Spec} R$, where R need not to be pictorsion, but they satisfy the thesis of the previous Theorem.

Proposition 3.3.2. *Given R connected Noetherian ring of dimension 1. Let $S = \operatorname{Spec} R$ and let $\mathcal{E}$ be an $\mathcal{O}_S$-module locally free of rank $r \geq 2$. Then there exists a finite S-morphism*

$$\mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}^{r-1}_S.$$

Proof. Given a locally free sheaf of rank r (on a scheme S) of the form $\mathcal{L}_1 \oplus ... \oplus \mathcal{L}_r$, with $\mathcal{L}_i$ invertible for $i = 1, ..., r$, there exists a finite S-morphism

$$\mathbb{P}(\mathcal{L}_1 \oplus ... \oplus \mathcal{L}_r) \longrightarrow \mathbb{P}(\mathcal{L}_1^{\otimes d} \oplus ... \oplus \mathcal{L}_r^{\otimes d})$$

for any integer d, defined on local trivialization (recall that $\mathbb{P}(\mathcal{E})|_{\pi^{-1}(U)} \cong \mathbb{P}^n_U$ on some open sets) by raising the coordinates to the dth tensor power.
In our case S in connected Noetherian of dimension 1, so, by [S] Proposition

on $\mathbb{P}^N_S$ (see remark 2.2.1). We put $\mathcal{J} \subset \mathcal{O}_{\mathbb{P}^N_S}$ to be the sheaf of ideals defining Y, so we have the canonical short exact sequence

$$0 \longrightarrow \mathcal{J}(njk) \longrightarrow \mathcal{O}_{\mathbb{P}^N_S}(njk) \longrightarrow \mathcal{O}_{\mathbb{P}^N_S}(njk)|_Y \cong e^{\otimes k}\mathcal{O}_Y \longrightarrow 0 ,$$

whose associated long exact sequence in cohomology produce the following

$$\mathcal{O}_{\mathbb{P}^N_S}(njk)(\mathbb{P}^N_S) \longrightarrow \mathcal{O}_{\mathbb{P}^N_S}(njk)|_Y(Y) \longrightarrow H^1(\mathbb{P}^N_S, \mathcal{J}(njk)) ,$$

and, for k sufficiently large, $H^1(\mathbb{P}^N_S, \mathcal{J}(njk)) = 0$.

[4]The image of r is of dimension d, because it is a finite morphism. Since such image is contained in $\mathbb{P}^d_S$, which is irreducible and of the same dimension, we have that they are equal

7, any locally free $\mathcal{O}_S$-module like $\mathcal{E}$ can be seen (modulo an isomorphism) as $\mathcal{O}_S^{r-1} \oplus \mathcal{L}$, for some invertible sheaf $\mathcal{L}$. In particular the previous morphism, with $d = r$, becomes

$$\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{O}_S^{\otimes r-1} \oplus \mathcal{L}) \longrightarrow \mathbb{P}((\mathcal{O}_S^{\otimes r})^{r-1} \oplus \mathcal{L}^{\otimes r}) = \mathbb{P}(\mathcal{O}_S^{r-1} \oplus \mathcal{L}^{\otimes r}).$$

Now applying again the result above $\mathcal{L}^r \cong \mathcal{O}_X^{r-1} \oplus \mathcal{L}'$ for some invertible sheaf $\mathcal{L}'$. By using the theory of the determinant[5], we have that

$$\begin{aligned} \mathcal{L}^{\otimes r} &\cong \det(\mathcal{L}^r) \\ &\cong \det(\mathcal{O}_X^{r-1} \oplus \mathcal{L}') \\ &\cong \det(\mathcal{O}_X^{r-1}) \otimes \det \mathcal{L}' \\ &\cong \det \mathcal{L}' \\ &\cong \mathcal{L}'. \end{aligned}$$

Hence $\mathcal{L}^r \cong \mathcal{O}_X^{r-1} \oplus \mathcal{L}' \cong \mathcal{O}_X^{r-1} \oplus \mathcal{L}^{\otimes r}$. Now $\mathbb{P}(\mathcal{L}^r)$ is S-isomorphic to $\mathbb{P}(\mathcal{O}_S^r) = \mathbb{P}_S^{r-1}$ (see [EGA]II.4.1.4), thus the finite morphism above becomes

$$\mathbb{P}(\mathcal{E}) \longrightarrow \mathbb{P}(\mathcal{O}_S^{r-1} \oplus \mathcal{L}^{\otimes r}) = \mathbb{P}_S^{r-1}$$

and we are done. □

3.3.2 Finite morphisms as sufficient condition

To prove a converse of Theorem 3.3.1 we need the following result.

Proposition 3.3.3. *Given S Noetherian connected scheme, $\mathcal{E}$ locally free sheaf on it of rank $n+1$ and $\pi : \mathbb{P}(\mathcal{E}) \longrightarrow S$ the canonical projection morphism (see Definition 1.4.1).*

(a) Any invertible sheaf on $\mathbb{P}(\mathcal{E})$ is isomorphic to a sheaf of the form $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(m) \otimes \pi^*(\mathcal{L})$; $m \in \mathbb{Z}$ and $\mathcal{L}$ invertible sheaf on S.

(b) If $S = \operatorname{Spec} R$ is affine, $\mathcal{E} = \mathcal{O}_X^{n+1}$ and it is given a finite morphism $f : \mathbb{P}_S^n \longrightarrow \mathbb{P}_S^n$, then $f^*(\mathcal{O}_{\mathbb{P}_S^n}(1))$ is isomorphic to a sheaf of the form $\mathcal{O}_{\mathbb{P}_S^n}(m) \otimes \pi^*\mathcal{L}$; $m > 0$ and $\mathcal{L}$ invertible sheaf on S of finite order in $\operatorname{Pic} S$.

[5] In particular the determinant changes direct sums in tensor products and the determinant of an invertible sheaf is the sheaf itself (see [L] Subsection 6.4.1 for more details)

Proof. For the point (a) see [EGA]II.4.2.7 and in particular [GLL] Proposition 8.4(a) for a complete proof.
Se we have that $f^*(\mathcal{O}_{\mathbb{P}^n_S}(1)) \cong \mathcal{O}_{\mathbb{P}^n_S}(m) \otimes \pi^*(\mathcal{L})$, for some invertible sheaf $\mathcal{L}$ and $m \in \mathbb{Z}$. In particular m is positive because over each point $s \in S$ $f^*(\mathcal{O}_{\mathbb{P}^n_S}(1))_s$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n_S}(m)_s$ (locally $\pi^*(\mathcal{L})$ is trivial) and it is ample, since it is the pullback of the ample sheaf $\mathcal{O}_{\mathbb{P}^n_S}(1)_s$ via a finite morphism (see [L] Remark 5.3.9 based on Exercise 5.2.16).
It remains to prove that $\mathcal{L}$ is of finite order. Let $M := \mathcal{L}(S)$, then, by Lemma 1.1.12 and Example 1.1.3, $(\pi^*\mathcal{L})(m)(\mathbb{P}^n_S) = M \otimes_R R[x_0, ..., x_n]_m$ ($R[x_0, ..., x_n]_m$ are the homogeneous polynomials of degree m) and the global sections $x_i \in R[x_0, ..., x_n]_1 = \mathcal{O}_{\mathbb{P}^n_S}(1)(\mathbb{P}^n_S)$ are sent to some elements $F_i \in M \otimes_R R[x_0, ..., x_n]_m$. Now M is locally free of rank 1, thus there exist an affine covering $S = \bigcup_{j=1}^t D(s_j)$, $s_j \in R$, such that[6] $M_{s_j} = M \otimes_R R[s_j^{-1}]$ has a basis t_j, $j = 1, ..., t$. Hence in $D(s_j)$ we can write $F_i = t_j \otimes G_{ij}$, with $G_{ij} \in R[s_j^{-1}][x_0, ..., x_n]_m$.
Let $\operatorname{Res}(G_{0j}, ..., G_{nj})$ be the resultant of $G_{0j}, ..., G_{nj}$ (see [J] 2.3). The morphism $f_{D(s_j)} := f|_{D(s_j)} : \mathbb{P}^n_{D(s_j)} \longrightarrow \mathbb{P}^n_{D(s_j)}$ is given by the global sections of $\mathcal{O}_{\mathbb{P}^n_S}(1)|_{D(s_j)}$ corresponding to $G_{0j}, ..., G_{nj} \in R[s_j^{-1}][x_0, ..., x_n]_m$, hence they generate $\mathcal{O}_{\mathbb{P}^n_S}(1)|_{D(s_j)}$ and then they have no common zeroes. This means that $\operatorname{Res}(G_{0j}, ..., G_{nj}) \in R[s_j^{-1}]^*$.
Now let us define $r_j := \operatorname{Res}(G_{0j}, ..., G_{nj}) t_j^{\otimes(n+1)m^n} \in R[s_j^{-1}] \otimes_R M^{\otimes(n+1)m^n}$; since $\operatorname{Res}(G_{0j}, ..., G_{nj})$ is invertible, r_j is a basis. Now it can be proved[7] that $r_j|_{D(s_j)\cap D(s_k)} = r_k|_{D(s_j)\cap D(s_k)}$, so they can be glued to an element $r \in M^{\otimes(n+1)m^n}$, which results to be a basis. So we can conclude that $M^{\otimes(n+1)m^n}$ is free of rank 1, i.e. $\mathcal{L}^{\otimes(n+1)m^n} \cong \mathcal{O}_S$, so $\mathcal{L}$ is of finite order in $\operatorname{Pic} S$. □

The following example will provide us an useful construction of a projective morphism $X \longrightarrow S$ not admitting a finite S-morphism to $\mathbb{P}^d_S$.

Example 3.3.4. Let S be a connected Noetherian affine scheme. Suppose there exists $\mathcal{L} \in \operatorname{Pic} S$ of infinite order (i.e. S is not pictorsion). Let us assume that $\mathcal{L}$ is generated by $d + 1$ sections, for some $d \geq 0$. Then there exists a projective morphism $X_{\mathcal{L}} \longrightarrow S$, with fibers of dimension d, such that there exists no finite S-morphisms $X_{\mathcal{L}} \longrightarrow \mathbb{P}^d_S$.

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[6] $D(s_j) = \operatorname{Spec} R\left[\dfrac{1}{s_j}\right]$.

[7] There exists $a \in R[s_j^{-1}, s_k^{-1}]$ such that $at_j = t_k$. Then, since $t_j \otimes G_{ij} = F_i = t_k \otimes G_{ik}$, we have that $G_{ij} = aG_{ik}$. Thus $\operatorname{Res}(G_{0j}, ..., G_{nj}) = a^{(n+1)m^n} \operatorname{Res}(G_{0k}, ..., G_{nk})$ (see [J] 5.11.2) so that r_j equals r_k in the considered restriction.

Since $\mathcal{L}$ is generated by $d+1$ sections, we have a closed S-immersion[8] $i_1 : S \longrightarrow \mathbb{P}^d_S$ such that $i_1^*\mathcal{O}_{\mathbb{P}^d_S}(1) = \mathcal{L}$ (by Proposition 1.2.7). Further, consider the closed S-immersion[9]

$$\begin{array}{rccc} i_0 : & S & \longrightarrow & U_0 \subseteq \mathbb{P}^d_S \\ & 0 & \longleftarrow\!\!\!\!\dashv & T_i/T_0 \end{array},$$

so that[10] $i_0^*\mathcal{O}_{\mathbb{P}^d_S}(1) = \mathcal{O}_S$. We define $X_{\mathcal{L}}$ to be the scheme defined by glueing two copies of $\mathbb{P}^d_S$ over the closed subschemes $\Im(i_0)$, $\Im(i_1)$, i.e. it is the push-out

$$\begin{array}{ccc} S & \xrightarrow{i_1} & \mathbb{P}^d_S \\ i_0 \downarrow & & \downarrow \varphi_0 \\ \mathbb{P}^d_S & \xrightarrow{\varphi_1} & X_{\mathcal{L}} \end{array},$$

where in particular φ_0, φ_1 are closed immersions. Under our hypothesis we have a separated morphism of finite type $\pi : X_{\mathcal{L}} \longrightarrow S$ (see [A]1.1.5). Further, the morphism $(\varphi_0, \varphi_1) : \mathbb{P}^d_S \sqcup \mathbb{P}^d_S \longrightarrow X_{\mathcal{L}}$ is clearly finite and surjective, this means that the same holds on each fiber, thus $\dim(X_{\mathcal{L}})_s = \dim(\mathbb{P}^d_S \sqcup \mathbb{P}^d_S)_s = d$ for all $s \in S$. Finally, since $\mathbb{P}^d_S \longrightarrow S$ is proper (see [L] Theorem 3.3.30), $X_{\mathcal{L}} \longrightarrow S$ is proper as well. It can also be proved (see [GLL] Example 8.5) that $\pi : X_{\mathcal{L}} \longrightarrow S$ is a projective morphism.
Now suppose there exists a finite S-morphism $f : X_{\mathcal{L}} \longrightarrow \mathbb{P}^d_S$, then $f \circ \varphi_i$ is finite as well (recall that a closed immersion is finite). By Proposition 3.3.3(b) $(f \circ \varphi_i)^*\mathcal{O}_{\mathbb{P}^d_S}(1) \cong \mathcal{O}_{\mathbb{P}^d_S}(m_i) \otimes \pi_i\mathcal{M}_i$, $m_i > 0$ and $\mathcal{M}_i$ of finite order. Then[11]

$$\begin{aligned} (f \circ \varphi_0 \circ i_0)^*\mathcal{O}_{\mathbb{P}^d_S}(1) &= i_0^*(f \circ \varphi_0)^*\mathcal{O}_{\mathbb{P}^d_S}(1) \\ &= i_0^*\mathcal{O}_{\mathbb{P}^d_S}(m_0) \otimes (\pi_0 \circ i_0)^*\mathcal{M}_0 \\ &= \mathcal{O}_S^{\otimes m_0} \otimes \mathcal{M}_0 \\ &= \mathcal{M}_0 \end{aligned}$$

[8] It is a section, since it is a morphism over S, of the separated morphism $\mathbb{P}^d_S \longrightarrow S$ (The identity is proper, hence i_1 is proper by [L] Proposition 3.3.16; further it is a mono, since it is a section; hence by [EGAIV]18.12.6 it is a closed immersion).

[9] It corresponds to the morphism $s \longmapsto (1:0:...:0) \in \mathbb{P}^d_{k(s)}$.

[10] We have that $i_0(S)$ is contained in $D_+(T_0)$, hence $i_0^*(\mathcal{O}_{\mathbb{P}^d_S}(1)) = i_0^*(\mathcal{O}_{\mathbb{P}^d_S}(1)|_{D_+(T_0)})$. Now, since $\mathcal{O}_{\mathbb{P}^d_S}(1)|_{D_+(T_0)})$ is free, the same holds for $i_0^*(\mathcal{O}_{\mathbb{P}^d_S}(1))$.

[11] Notice that $\pi_i \circ i_i = \mathrm{id}$.

and

$$\begin{aligned}(f \circ \varphi_1 \circ i_1)^* \mathcal{O}_{\mathbb{P}^d_S}(1) &= i_1^*(f \circ \varphi_1)^* \mathcal{O}_{\mathbb{P}^d_S}(1) \\ &= i_1^* \mathcal{O}_{\mathbb{P}^d_S}(m_1) \otimes (\pi_1 \circ i_1)^* \mathcal{M}_1 \\ &= \mathcal{L}^{\otimes m_1} \otimes \mathcal{M}_1.\end{aligned}$$

But $f \circ \varphi_0 \circ i_0 = f \circ \varphi_1 \circ i_1$, so $\mathcal{M}_0$ and $\mathcal{L}^{\otimes m_1} \otimes \mathcal{M}_1$ must be isomorphic, but $\mathcal{L}$ has no torsion, therefore such finite morphism f cannot exist.

Proposition 3.3.5. *Let R be a Noetherian ring and $S = \operatorname{Spec} R$. Suppose that for any $d \geq 0$ and any projective morphism $X \longrightarrow S$, such that $\dim X_s = d$ for all $s \in S$, there exists a finite surjective S-morphism $X \longrightarrow \mathbb{P}^d_S$. Then R is pictorsion.*
If R is of finite Krull dimension, we can weaken our hypothesis. Assume that for all projective morphism $X \longrightarrow S$, such that $\dim X_s \leq \dim R$ for all $s \in S$, there exists a finite S-morphism $X \longrightarrow \mathbb{P}^{\dim R}_S$. Then R is pictorsion.

Proof. Consider an invertible sheaf $\mathcal{L}$. For each connected component S_i of S, let d_i be such that $\mathcal{L}|_{S_i}$ can be generated by $d_i + 1$ global sections. Put $d = \max\{d_i\}$ (note that if R is of finite dimension $d \leq \dim R$). Suppose the order of $\mathcal{L}$ in $\operatorname{Pic} S$ is not finite. Then it must be of infinite order on some connected components of S.
For such connected components we can apply the construction of Example 3.3.4 using d global sections generating $\mathcal{L}|_{S_i}$. Hence we have a projective scheme $X_i := X_{\mathcal{L}|_{S_i}} :\longrightarrow S_i$, with fibers of dimension d and not admitting any finite morphism $X_i \longrightarrow \mathbb{P}^d_{S_i}$. For connected components of S where $\mathcal{L}$ has finite order we consider the morphism $X_j := \mathbb{P}^d_{S_j} \longrightarrow S_j$. Now consider the induced morphism $X := \sqcup X_i \longrightarrow \sqcup S_i = S$. It has fibers of dimension d, but it does not admit a finite morphism $X \longrightarrow \mathbb{P}^d_S$, which is a contradiction. $\square$

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