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AN ADELIC DESCRIPTION OF MODULAR CURVES

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CONTENTS

INTRODUCTION

The great does not happen through impulse alone, and is a succession of little things that are brought together

Vincent van Gogh

Which integers $x, y \in \mathbb{Z}$ satisfy the equation $x^2 - 3y^2 = 1$ or the equation $x^2 + 19 = y^3$? How many integers satisfy the equation $x^4 + y^4 = z^4$? Questions of this kind, which look very simple and naive at a first glance, have been the leading motivation for the development of *algebraic number theory*. For instance equations of the form $x^2 - dy^2 = 1$ go under the name of *Pell's equation*. The formula $x^2 - dy^2 = (x + \sqrt{d}y)(x - \sqrt{d}y)$ shows that finding all the solutions to Pell's equations involves studying the units of the rings $\mathbb{Z}[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in \mathbb{Z}\}$. It turns out that all the solutions to the Pell equation are generated by a fundamental solution because the units of $\mathbb{Z}[\sqrt{d}]$ can be expressed (up to sign) as powers of a fundamental unit thanks to *Dirichlet's unit theorem*, one of the main results in classical algebraic number theory. Studying equations of the type $x^2 - d = y^3$ leads to the problem of understanding not only the unit group $\mathbb{Z}[\sqrt{d}]^\times$ but also the *class group* $\mathfrak{Cl}(\mathbb{Q}(\sqrt{d}))$ which is a finite group that measures to what extent $\mathbb{Z}[\sqrt{d}]$ is a *principal ideal domain*. Finally we can prove that $x^4 + y^4 = z^4$ has no integral solutions such that $xyz \neq 0$ by studying the unit group $\mathbb{Z}[i]^\times$ and the class group $\mathfrak{Cl}(\mathbb{Q}(i))$ but proving that the general *Fermat equation* $x^n + y^n = z^n$ has no integral solutions when

$n \geq 3$ and $xyz \neq 0$ is one of the most difficult and notable theorems of the history of mathematics, proved by Andrew Wiles in 1995 after 358 years of joint efforts by the most famous mathematicians.

The complicated proof of *Fermat's last theorem* takes place in the wide area of *arithmetic geometry* which studies the deep links between geometry and number theory that have been discovered in the twentieth century. The protagonists of the proof are *elliptic curves* which are curves defined over the rational numbers as the sets of solutions of equations of the form $y^3 = x^2 + ax + b$. The key ingredient of Wiles' work is the proof of the *modularity theorem* which states that for every elliptic curve E over $\mathbb{Q}$ there exists a finite and surjective rational map $X_0(n) \twoheadrightarrow E$, where $X_0(n)$ is the *classical modular curve* of level $n \in \mathbb{N}$. The first chapter of this work reviews the classical, analytic definition of modular curves, which play a fundamental role in number theory.

As we have seen, to study equations over the integers it is usually necessary to study finite extensions $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$, which are called *number fields*. Here $\alpha \in \mathbb{C}$ is any *algebraic* complex number, which is equivalent to say that there exists an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ with such that $f(\alpha) = 0$. To study these extensions it is often crucial to understand the group $\mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha))$ of automorphisms of fields $\mathbb{Q}(\alpha) \xrightarrow{\sim} \mathbb{Q}(\alpha)$ which fix $\mathbb{Q}$, and this group plays a crucial role when $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$ is a *Galois extension*, which means that there exists an irreducible polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(\alpha) = 0$ and if $\beta \in \mathbb{C}$ is such that $f(\beta) = 0$ then $\beta \in \mathbb{Q}(\alpha)$. In particular if we denote by $\overline{\mathbb{Q}}$ the union inside $\mathbb{C}$ of all the number fields $\mathbb{Q}(\alpha)$ then *Galois theory* tells us that if $\mathbb{Q} \subseteq \mathbb{Q}(\alpha)$ is a Galois extension then the group $\mathbf{Gal}\left({}^{\mathbb{Q}(\alpha)}/_{\mathbb{Q}}\right) \overset{\text{def}}{=} \mathrm{Aut}_{\mathbb{Q}}(\mathbb{Q}(\alpha))$ is a quotient of the group $G_{\mathbb{Q}} \overset{\text{def}}{=} \mathrm{Aut}_{\mathbb{Q}}(\overline{\mathbb{Q}})$ which is called the *absolute Galois group* of the rational numbers.

Understanding the topological group $G_{\mathbb{Q}}$ is one of the biggest problems of number theory. A better understanding of $G_{\mathbb{Q}}$ can be achieved for instance by looking at its *continuous representations* over the complex numbers, which are continuous homomorphisms of groups $G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$. Relating these representations to the analytic theory of *automorphic forms* is the subject of the *Langlands program*, a very wide set of conjectures which leads much of the research in number theory today. If $n = 1$ studying characters $G_{\mathbb{Q}} \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^{\times}$ is equivalent to study the abelianization $G_{\mathbb{Q}}^{\mathrm{ab}}$ of $G_{\mathbb{Q}}$ and is what goes under the name of *class field theory*. The main results of this area can be stated as

$$G_K^{\mathrm{ab}} \cong \widehat{C_K} \qquad \text{with} \qquad C_K = \begin{cases} K^\times, \text{ if } K \text{ is local} \\ \\ \mathbb{A}_K^\times/K^\times, \text{ if } K \text{ is global} \end{cases} \tag{1}$$

where $\mathbb{A}_K$ is the *adèle ring* associated to a global field K. The beginning of the second chapter of this report contains the definition and the basic properties of the ring $\mathbb{A}_K$, which was defined to state the main results of class field theory in the very short way outlined in (1).

This thesis deals with the connections between modular curves and adèle rings. First of all we prove in section 2.3 that disjoint unions of some copies of the affine modular curves $\Gamma\backslash\mathfrak{h}$ defined in section 1.1 are homeomorphic to double quotients of the group $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ by the left action of $\mathrm{GL}_2(\mathbb{Q})$ and the right action of the product of a compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ and the group $K_\infty \overset{\mathrm{def}}{=} \mathbb{R}_{>0} \times \mathrm{SO}_2(\mathbb{R})$. The original aim of this thesis was to find a topological space $\mathcal{Z}(\mathbb{A}_\mathbb{Q})$, or maybe a *scheme* of finite type $\mathcal{Z}$ related to the adèle ring $\mathbb{A}_\mathbb{Q}$ such that disjoint unions of some *Baily-Borel compactifications* of modular curves were homeomorphic to the double quotient $\mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{Z}(\mathbb{A}_\mathbb{Q})/K^\infty \times K_\infty$. Trying to pursue this objective we encountered some difficulties in adding the archimedean place to $\mathbb{A}_\mathbb{Q}^\infty$, as we explain in section 3.2. We turned then our attention to the *Borel-Serre compactification* of modular curves, which is another way of compactifying modular curves that is described in section 1.3. Using this compactification we were able to find a topological space $\mathcal{Z}(\mathbb{A}_\mathbb{Q})$ such that disjoint unions of compactified modular curves are homeomorphic to double quotients of the form $\mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{Z}(\mathbb{A}_\mathbb{Q})/K^\infty \times K_\infty$. As we say in the conclusions, the space $\mathcal{Z}(\mathbb{A}_\mathbb{Q})$ is not a scheme of finite type over $\mathbb{Z}$ or over $\mathbb{A}_\mathbb{Q}$ and its definition is not "homogeneous" in the finite and infinite part of $\mathbb{A}_\mathbb{Q}$, which leads to interesting questions concerning its definition.

To sum up, the main objective of this thesis was to give a description of the *projective limit* $\varprojlim_n X(n)$ of compactified modular curves as a *Shimura variety*, i.e. as the projective limit of quotients $G(\mathbb{Q})\backslash\left(X \times G(\mathbb{A}_\mathbb{Q}^\infty)/K^\infty\right)$ where X is a suitable *Hermitian symmetric space*, $G = \mathrm{GL}_2$ or any other reductive algebraic group and $K^\infty \leq G(\mathbb{A}_\mathbb{Q}^\infty)$ runs over all the sufficiently small compact and open subgroups. The most attractive aspect of such a description would be the possibility of defining a suitable theory of *automorphic rep-*

resentations on the compactified modular curves in the spirit of the Langlands program. All starting from three simple, integral equations!

Short acknowledgements

> *Let us be grateful to the people who make us happy: they are the charming gardeners who make our souls blossom.*
>
> Marcel Proust

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The space constraints didn't let me to thank you personally on this page. For this reason you can find a longer version of these acknowledgements here.

CHAPTER 1

MODULAR CURVES AND THEIR COMPACTIFICATIONS

I think I will stop here.

Andrew Wiles, after finishing writing
the proof of Fermat's last theorem

Chapter Abstract

We recall in this chapter the definition of affine modular curves as quotients of the complex upper-half plane. We also introduce two ways of compactifying them which were introduced by Baily and Borel in [1] and [2] and by Borel and Serre in [3]

1.1 The analytic definition of modular curves

In this section we define the modular curves as quotients of the upper half plane by the action of suitable subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$. To do so recall first of all that the group

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}_{2,2}(\mathbb{R}) \mid ad - bc > 0 \right\}$$

of invertible 2×2 matrices with real entries and positive determinant acts on the upper half plane

$$\mathfrak{h} = \{z \in \mathbb{C} \mid \mathfrak{Im}(z) > 0\} \qquad \text{by setting} \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) * z = \frac{az+b}{cz+d}$$

which is a well defined action. Indeed for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}_{2,2}(\mathbb{R})$ and $z \in \mathfrak{h}$ we have $cz + d \neq 0$ and

$$\operatorname{Im}\left(\frac{az+b}{cz+d}\right) = \frac{ad-bc}{|cz+d|^2} \cdot \operatorname{Im}(z)$$

which implies that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) * z \in \mathfrak{h}$ if $z \in \mathfrak{h}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$. Moreover it is straightforward to prove that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) * z = z$ and $A * (B * z) = (A \cdot B) * z$ for every $A, B \in \mathrm{GL}_2^+(\mathbb{R})$ and $z \in \mathbb{R}$, which implies that $\mathrm{GL}_2^+(\mathbb{R}) \circlearrowright \mathfrak{h}$ is a well defined group action. It is important to note that using this action we can describe the full group of bi-holomorphic transformations of $\mathfrak{h}$, as shown in Theorem 1.1.

Theorem 1.1. *Let* $\mathrm{Aut}(\mathfrak{h})$ *be the group of all bi-holomorphic functions* $\mathfrak{h} \to \mathfrak{h}$ *and let*

$$\rho\colon \mathrm{GL}_2^+(\mathbb{R}) \to \mathrm{Aut}(\mathfrak{h}) \qquad \textit{such that} \qquad \rho\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)(z) = \frac{az+b}{cz+d}$$

be the group homomorphism induced by the action $\mathrm{GL}_2^+(\mathbb{R}) \circlearrowright \mathfrak{h}$*. Then* ρ *is surjective and*

$$\ker(\rho) = \{\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) \mid a \in \mathbb{R}^\times\}$$

which implies that

$$\mathrm{Aut}(\mathfrak{h}) \cong \mathrm{PSL}_2(\mathbb{R}) \stackrel{\text{def}}{=} \mathrm{SL}_2(\mathbb{R}) / \left\{\pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\} \quad \textit{where} \quad \mathrm{SL}_2(\mathbb{R}) \stackrel{\text{def}}{=} \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}_{2,2}(\mathbb{R}) \mid ad - bc = 1\right\}.$$

Proof. See **Theorem 2.4** in **Chapter 8** of [16]. □

The previous theorem suggests to consider the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}$ instead of the action of $\mathrm{GL}_2(\mathbb{R})$. Thus for every subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ we will consider the topological space $\Gamma \backslash \mathfrak{h}$ whose elements are the orbits of the action $\Gamma \circlearrowright \mathfrak{h}$ and whose topology is the finest topology such that the quotient map $\mathfrak{h} \twoheadrightarrow \Gamma \backslash \mathfrak{h}$ is continuous. In order to study the topological properties of this quotient we shall review the topological properties of $\mathrm{SL}_2(\mathbb{R})$.

We can define a topology on the space of 2×2 matrices $\mathcal{M}_{2,2}(\mathbb{R})$ using the bijection $\mathcal{M}_{2,2}(\mathbb{R}) \leftrightarrow \mathbb{R}^4$. With this topology we have that $\mathrm{SL}_2(\mathbb{R}) \subseteq \mathcal{M}_{2,2}(\mathbb{R})$ is a closed subset since $\mathrm{SL}_2(\mathbb{R}) = \det^{-1}(1)$ and the map $\det\colon \mathcal{M}_{2,2}(\mathbb{R}) \to \mathbb{R}$ is continuous. Observe moreover that $\mathrm{SL}_2(\mathbb{R})$ endowed with the subspace topology is itself a topological group, which is true over the real numbers but not for every topological ring R.

Recall now that the action $G \circlearrowright X$ of a group G on a topological space X is said to be *properly discontinuous* if for every two compact subsets $K_1, K_2 \subseteq X$ the set

$$\{g \in G : g * K_1 \cap K_2 \neq \emptyset\}$$

is finite. We can characterize the subgroups $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ such that the action $\Gamma \circlearrowright \mathfrak{h}$ is properly discontinuous in the following topological way.

Theorem 1.2. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a subgroup. Then $\Gamma \circlearrowright \mathfrak{h}$ is properly discontinuous if and only if Γ is a discrete subspace of $\mathrm{SL}_2(\mathbb{R})$.*

Proof. See **Proposition 1.6** of [15]. □

We have seen in the previous theorem that discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$ play an important role when looking at the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h}$, and thus they have the special name of *Fuchsian groups*. Some important examples of Fuchsian groups are given by subgroups of the *full modular group*

$$\mathrm{SL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathcal{M}_{2,2}(\mathbb{Z}) \mid ad - bc = 1 \right\}$$

which is clearly a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

Definition 1.3. For every $N \in \mathbb{N}$ we define

$$\begin{aligned} \Gamma_0(N) &\stackrel{\text{def}}{=} \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \bmod N \right\} \\ \Gamma_1(N) &\stackrel{\text{def}}{=} \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \text{ and } a \equiv d \equiv 1 \bmod N \right\} \\ \Gamma(N) &\stackrel{\text{def}}{=} \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \mathrm{SL}_2(\mathbb{Z}) \mid b \equiv c \equiv 0 \text{ and } a \equiv d \equiv 1 \bmod N \right\}. \end{aligned}$$

The group $\Gamma(N)$ is called the *principal congruence subgroup* of level N and the groups $\Gamma_0(N)$ and $\Gamma_1(N)$ are called *modular groups of Hecke type* of level N.

Let us return for now to the general setting in which $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a Fuchsian group. As we said, the fact that the action $\Gamma \circlearrowright \mathfrak{h}$ is properly discontinuous is important to prove that the quotient topological space $\Gamma \backslash \mathfrak{h}$ maintains some of the topological properties of $\mathfrak{h}$, as shown in the following theorem.

Theorem 1.4. *Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group. Then the quotient $\Gamma\backslash\mathfrak{h}$ is a locally compact, connected and Hausdorff topological space. Moreover, it admits a unique Riemann surface structure such that the quotient map $\mathfrak{h} \twoheadrightarrow \Gamma\backslash\mathfrak{h}$ is holomorphic.*

Proof. See **Section 1.5** of [15]. □

Using Theorem 1.4 we define $Y_\Gamma \overset{\text{def}}{=} \Gamma\backslash\mathfrak{h}$ to be the *affine modular curve* associated to the Fuchsian group $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$. In particular we define

$$Y(N) \overset{\text{def}}{=} Y_{\Gamma(N)} \qquad \text{and} \qquad Y_0(N) \overset{\text{def}}{=} Y_{\Gamma_0(N)} \qquad \text{and} \qquad Y_1(N) \overset{\text{def}}{=} Y_{\Gamma_1(N)}$$

for every $N \in \mathbb{N}$. We recall now that many important results from the theory of Riemann surfaces hold only if a Riemann surface is compact. The most important of these results is *Riemann's existence theorem* which implies that every compact Riemann surface S admits an embedding $S \hookrightarrow \mathbb{P}^3(\mathbb{C})$ whose image is an algebraic curve, which allows us to use the tools of algebraic geometry to study compact Riemann surfaces. For a proof of this theorem we refer to **§10** of [13].

Nevertheless not many of the surfaces Y_Γ are compact, as it is shown in the following proposition.

Proposition 1.5. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group such that the Riemann surface Y_Γ is compact. Then for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \{\pm I_2\}$ we have $(a+d)^2 \neq 4$.*

Proof. See **Proposition 1.33** of [15]. □

Corollary 1.6. *For every $N \in \mathbb{N}_{\geq 1}$ the Riemann surfaces $Y(N)$, $Y_0(N)$ and $Y_1(N)$ are not compact.*

Proof. We have $A_N = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma(N)$ and $\Gamma(N) \leq \Gamma_1(N) \leq \Gamma_0(N)$ for every $N \in \mathbb{N}$ which implies that $\Gamma(N)$, $\Gamma_1(N)$ and $\Gamma_0(N)$ contain a matrix A_N such that $\mathrm{tr}(A_N)^2 = 4$. □

Aside: Moduli problems and the algebraic definition of modular curves

We want to add here a small aside about another possible definition of modular curves as moduli spaces for elliptic curves over $\mathbb{Q}$. This is useful to understand why modular curves arise as a natural object to consider in arithmetic geometry, and how the three classical families of congruence subgroups $\Gamma(n)$, $\Gamma_0(n)$ and $\Gamma_1(n)$ have been defined.

As we already said in the introduction an *elliptic curve* over any field $\mathbb{K}$ of characteristic different from 2 and 3 is simply the algebraic curve defined by an equation of the form $y^2 = x^3 + ax + b$ where $a, b \in \mathbb{K}$ and $4a^3 + 27b^2 \neq 0$. It can be proved that an elliptic curve over the complex numbers is always isomorphic (as a Riemann surface) to $\mathbb{C}/_{\mathbb{Z}\oplus\mathbb{Z}\tau}$ where $\tau \in \mathfrak{h}$ is unique up to the action of $\mathrm{SL}_2(\mathbb{Z})$. This shows that the quotient $\Gamma(1)\backslash\mathfrak{h} = \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}$ parametrizes the set of isomorphism classes of elliptic curves over the complex numbers and gives us a first insight into the interpretation of modular curves as moduli spaces of elliptic curves.

Definition 1.7. A moduli space is a geometric object (for example a topological space, a scheme or something more general) whose points parametrize a family of geometric objects (for example, elliptic curves) or the solutions to a geometric problem.

Definition 1.7 is very vague but it can be made precise by looking at the theory of moduli spaces as representatives of contravariant functors from a category of "geometric objects" (for instance schemes) to the category of sets. For the precise definition of moduli problems in this general context and their properties we refer to **Chapter** 4 of [11]. We can nevertheless see how the modular curves $\Gamma(n)\backslash\mathfrak{h}$, $\Gamma_0(n)\backslash\mathfrak{h}$ and $\Gamma_1(n)\backslash\mathfrak{h}$ can be interpreted as spaces classifying suitable isomorphism classes of elliptic curves, as it is stated in Theorem 1.10.

Proposition 1.8. *Let $\mathbb{Q} \subseteq L$ be an extension of fields, and let E be an elliptic curve defined over L. Then for every extension $L \subseteq K$ the set $E(K)$ of K-rational points of E has a natural structure of abelian group. We denote by $E[n](K)$ the set of all points $P \in E(K)$ such that $nP = 0$.*

Definition 1.9. Let $\mathbb{Q} \subseteq L$ be an extension of fields, let $\mathbf{Ell}_L$ be the category of elliptic curves over L and let $n \in \mathbb{N}_{\geq 1}$. We define the categories $\mathcal{E}^L(n)$, $\mathcal{E}_1^L(n)$ and $\mathcal{E}_0^L(n)$ whose objects are given by

$$\mathcal{E}^L(n) = \{(E, \varphi_E) \mid E \in \mathbf{Ell}_L \text{ and } \varphi_E \colon \left(\mathbb{Z}/n\mathbb{Z}\right)^2 \xrightarrow{\sim} E[n](L) \text{ is an isomorphism of groups}\}$$
$$\mathcal{E}_1^L(n) = \{(E, P_E) \mid E \in \mathbf{Ell}_L \text{ and } P_E \in E(L) \text{ is a point of order } N\}$$
$$\mathcal{E}_0^L(n) = \{(E, H_E) \mid E \in \mathbf{Ell}_L \text{ and } H_E \leq E(L) \text{ is a cyclic subgroup of order } n\}$$

and whose morphisms are given by morphisms $f \colon E \to E'$ of elliptic curves such that respectively $f \circ \varphi_E = \varphi_{E'}$, $f(P_E) = P_{E'}$ and $f(H_E) = H_{E'}$.

Theorem 1.10. *For every $n \in \mathbb{N}_{\geq 3}$ there exist three smooth affine curves $Y(n)$, $Y_0(n)$ and $Y_1(n)$ defined over $\mathbb{Q}$ such that:*

- *for every extension $\mathbb{Q} \subseteq L$ we have two bijections*

$$Y(n)(L) \leftrightarrow \mathcal{E}^L(n)/\sim \qquad \textit{and} \qquad Y_1(n)(L) \leftrightarrow \mathcal{E}_1^L(n)/\sim;$$

- *for every extension $\mathbb{Q} \subseteq L$ such that L is algebraically closed we have a bijection*

$$Y_0(n)(L) \leftrightarrow \mathcal{E}_0^L(n)/\sim;$$

- *we have three isomorphisms of Riemann surfaces*

$$Y(n)(\mathbb{C}) \cong \bigsqcup_{j=1}^{\varphi(n)} \Gamma(n)\backslash\mathfrak{h} \qquad Y_1(n)(\mathbb{C}) \cong \Gamma_1(n)\backslash\mathfrak{h} \qquad Y_0(n)(\mathbb{C}) \cong \Gamma_0(n)\backslash\mathfrak{h}.$$

where $\varphi(n) = \#\left(\mathbb{Z}/n\mathbb{Z}\right)^\times$ is the Euler totient function and $\mathcal{E}^L(n)/\sim$, $\mathcal{E}_1^L(n)/\sim$ and $\mathcal{E}_0^L(n)/\sim$ indicate the set of isomorphism classes of objects of $\mathcal{E}^L(n)$, $\mathcal{E}_1^L(n)$ and $\mathcal{E}_0^L(n)$ respectively.

Proof. See **Corollary 2.7.3**, **Theorem 3.7.1** and **Corollary 4.7.1** of [11], where the previous results are proved using the language of schemes. □

Theorem 1.10 shows thus that the curves $Y(n)$, $Y_1(n)$ and $Y_0(n)$ can be defined in an algebraic way and that they are the solutions to three different moduli problems for elliptic curves. Nevertheless they are still not compact, and that is why we will define in the next section one first possible way of compactifying them.

1.2 The Baily-Borel compactification

What we have just proved shows the necessity to find a good candidate for the *compactification* of the Riemann surface $Y_\Gamma = \Gamma \backslash \mathfrak{h}$. In particular we want to find a compact Riemann surface X_Γ together with an embedding $\iota \colon Y_\Gamma \hookrightarrow X_\Gamma$ such that $\iota(Y_\Gamma)$ is an open subset of X_Γ. To do so, we will extend the space $\mathfrak{h}$ on which Γ acts by adding some points. Observe first of all that $\mathrm{SL}_2(\mathbb{R})$ acts also on $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$ by setting

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} * x = (ax_0 + bx_1 : cx_0 + dx_1) = \begin{cases} \infty, \text{ if } c \neq 0 \text{ and } x = -{}^{d}\!/_{c} \text{ or } c = 0 \text{ and } x = \infty \\ {}^{a}\!/_{c}, \text{ if } x = \infty \text{ and } c \neq 0 \\ {}^{ax+b}\!/_{cx+d}, \text{ otherwise} \end{cases}$$

for every $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and every $x = (x_0 : x_1) \in \mathbb{P}^1(\mathbb{R})$. Observe in particular that for every $x \in \mathbb{R}$ we have $\sigma_x * \infty = x$ where $\sigma_x \overset{\text{def}}{=} \left(\begin{smallmatrix} x & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$.

We can easily classify the fixed points of the action $\mathrm{SL}_2(\mathbb{R}) \circlearrowright \mathfrak{h} \cup \mathbb{P}^1(\mathbb{R})$ using the following proposition.

Proposition 1.11. *For every matrix* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}) \setminus \{\pm I_2\}$ *we have:*

- $(a+d)^2 < 4$ *if and only if there exists* $z \in \mathfrak{h}$ *such that* $A * z = z$. *In this case* $A * w \neq w$ *for every* $w \in \mathfrak{h} \cup \mathbb{R} \cup \{\infty\} \setminus \{z\}$;
- $(a+d)^2 = 4$ *if and only if there exists a unique* $x \in \mathbb{R} \cup \{\infty\}$ *such that* $A * x = x$. *In this case* $A * w \neq w$ *for every* $w \in \mathfrak{h} \cup \mathbb{R} \cup \{\infty\} \setminus \{x\}$;
- $(a+d)^2 > 4$ *if and only if there exist* $x, y \in \mathbb{R} \cup \{\infty\}$ *such that* $x \neq y$, $A*x = x$ *and* $A*y = y$. *In this case* $A * w \neq w$ *for every* $w \in \mathfrak{h} \cup \mathbb{R} \cup \{\infty\} \setminus \{x, y\}$.

Proof. See **Proposition 1.12** and **Proposition 1.13** of [15]. □

Let now $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a subgroup. We call a point $x \in \mathbb{R} \cup \{\infty\}$ a *cusp* of Γ if there exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma \setminus \{\pm I_2\}$ such that $(a+d)^2 = 4$ and $A * x = x$ and we call a point $z \in \mathfrak{h}$ an *elliptic point* of Γ if there exists $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ such that $(a+d)^2 < 4$ and $A * z = z$. Let now $\mathfrak{E}_\Gamma$ be the set of all elliptic points of Γ, $\mathfrak{P}_\Gamma$ be the set of all cusps of Γ and let $\mathfrak{h}_\Gamma^* = \mathfrak{h} \cup \mathfrak{P}_\Gamma$.

For every Fuchsian group $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ we can define a topology on the set $\mathfrak{h}_\Gamma^*$ by taking as a fundamental system of neighbourhoods of a cusp $s \in \mathfrak{P}_\Gamma$ the collection

$$\{\sigma_s * (\{z \in \mathfrak{h} \mid \mathfrak{Im}(z) > l\} \cup \{\infty\})\}_{l>0} \qquad \text{where} \qquad \sigma_s = \left(\begin{smallmatrix} s & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

and as a fundamental system of neighbourhoods of a point $z \in \mathfrak{h}$ the usual system of open balls $\{B(z;\varepsilon)\}_{\varepsilon>0}$. Observe now that $\mathfrak{Im}(\sigma_x^{-1} * z) = \mathfrak{Im}(z) \cdot |x - z|^{-2}$ for every $x \in \mathbb{R}$ and every $z \in \mathfrak{h}$, which implies that for every cusp $s \in \mathfrak{P}_\Gamma \setminus \{\infty\}$ we have

$$\sigma_s * \{z \in \mathfrak{h} \mid \mathfrak{Im}(z) > l\} = \left\{z \in \mathfrak{h} \mid |s - z| < \sqrt{\frac{\mathfrak{Im}(z)}{l}}\right\} = \left\{z \in \mathfrak{h} \mid \left|s + \frac{i}{2l} - z\right| < \frac{1}{2l}\right\}$$

and thus in particular that $\sigma_s * U_l$ is an open ball of radius ${}^{1}\!/_{2l}$ whose boundary ∂U_l is a circle tangent to the real line at the point s. It is now easy to prove that $\mathfrak{h}_\Gamma^*$ endowed with this topology is an Hausdorff topological space but is not locally compact unless $\mathcal{P}_\Gamma = \emptyset$. It is also easy to prove that the action $\Gamma \circlearrowright \mathfrak{h}_\Gamma^*$ is still continuous, which allows us to consider the quotient topological space $\Gamma\backslash\mathfrak{h}_\Gamma^*$.

Theorem 1.12. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group. Then $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is a locally compact and Hausdorff topological space with a natural structure of a Riemann surface. Moreover, if $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is compact then the sets $\Gamma\backslash\mathfrak{P}_\Gamma$ and $\Gamma\backslash\mathfrak{E}_\Gamma$ are finite.*

Proof. See **Theorem 1.28**, **Proposition 1.29**, **Proposition 1.32** and **Section 1.5** of [15]. □

There exist Fuchsian subgroups $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ such that the Riemann surface $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is not compact. For instance we can take $\Gamma = \left\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right\}$ or $\Gamma = \left\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right\}$. For further information about this distinction, and for examples of Fuchsian groups of the second kind we refer to [10].

Definition 1.13. For every Fuchsian group $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ such that the quotient $\Gamma\backslash\mathfrak{h}_\Gamma^*$ is compact we define the compact Riemann surface $X_\Gamma \overset{\text{def}}{=} \Gamma\backslash\mathfrak{h}_\Gamma^*$ and we call it the *Baily-Borel compactification* of the modular curve Y_Γ

Theorem 1.16 shows that the classical modular groups are the first examples of Fuchsian groups of the first kind.

Definition 1.14. Let G be a group and let $H, H' \leq G$ be two subgroups. We say that H is *commensurable with* H' if $H \cap H'$ is a subgroup of finite index of H and H'.

Lemma 1.15. *Let $\Gamma, \Gamma' \leq \mathrm{SL}_2(\mathbb{R})$ be two commensurable subgroups. Then Γ is Fuchsian if and only if Γ' is Fuchsian, Γ is of the first kind if and only if Γ' is of the first kind and $\mathcal{P}_\Gamma = \mathcal{P}_{\Gamma'}$.*

Proof. See **Proposition 1.11**, **Proposition 1.30** and **Proposition 1.31** of [15]. □

Theorem 1.16. *The groups $\Gamma(N)$, $\Gamma_0(N)$ and $\Gamma_1(N)$ are Fuchsian groups of the first kind for all $N \in \mathbb{N}_{\geq 1}$.*

Proof. Use Lemma 1.15 and the fact that

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = \#\mathrm{SL}_2\left(\mathbb{Z}/_{N\mathbb{Z}}\right) = N^3 \cdot \prod_{p|N} 1 - \frac{1}{p^2} < \infty$$

as shown in **Lemma 1.38** and the following pages of [15]. □

Let now $\mathfrak{h}^* \overset{\text{def}}{=} \mathfrak{h}^*_{\mathrm{SL}_2(\mathbb{Z})}$ and observe that Lemma 1.15 and Theorem 1.16 imply that for every $n \in \mathbb{N}$ the topological spaces

$$X(N) \overset{\text{def}}{=} \Gamma(N)\backslash\mathfrak{h}^* \qquad X_0(N) \overset{\text{def}}{=} \Gamma_0(N)\backslash\mathfrak{h}^* \qquad \text{and} \qquad X_1(N) \overset{\text{def}}{=} \Gamma_1(N)\backslash\mathfrak{h}^*$$

are compact and connected Riemann surfaces.

One of the problems that affect this notion of compactification is the fact that the topological space $\mathfrak{h}^*_\Gamma$ is not locally compact, and that the Baily Borel compactification of more general *locally symmetric spaces* is singular, as it is shown in **§2.10** of [8]. The attempt of finding a better compactification for general locally symmetric spaces led to the notion of *Borel-Serre compactification*, which is the main subject of the following section.

1.3 The Borel-Serre compactification

The Borel-Serre compactification of the affine modular curve Y_Γ associated to a Fuchsian group $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ is a compact real two dimensional topological manifold with boundary whose boundary is a collection of circles S^1 indexed by the set $\Gamma\backslash\mathcal{P}_\Gamma$ where $\mathcal{P}_\Gamma \subseteq \mathbb{P}^1(\mathbb{R})$ is the set of cusps of Γ. It seems at a first sight that this notion of compactification might be slightly more disadvantageous compared to the Baily-Borel compactification, because in this case we obtain as a result a manifold with boundary and not a Riemann surface. Nevertheless this construction has the advantage of not involving any non locally compact topological space, as we will soon understand. We will also need the Borel-Serre compactification in the third chapter, when we will try to give an adelic description of the compactification of our modular curves. For further details and proofs we refer to [8] and to the original article of Borel and Serre [3].

We will start first of all with the definition of the space $\mathfrak{h}_\Gamma^{**}$ that we need for this compactification. This space is obtained by glueing to $\mathfrak{h}$ a family of lines indexed by $\mathcal{P}_\Gamma$ such that if we endow $\mathfrak{h}$ with the Poincaré metric

$$\rho \colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}_{\geq 0} \qquad \text{defined as} \qquad \rho(z_1, z_2) = 2\tanh^{-1} \frac{|z_1 - z_2|}{|z_1 - \overline{z_2}|}$$

then each line attached to a cusp parametrizes the set of all the possible geodesics of $\mathfrak{h}$ which end at this point. For instance if $\infty \in \mathcal{P}_\Gamma$ then the line at infinity will be a parameter space for all the half lines contained in $\mathfrak{h}$ and parallel to the imaginary axis.

Definition 1.17. Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian subgroup, and let $\mathcal{P}_\Gamma$ be its set of cusps. We define the set $\mathfrak{h}_\Gamma^{**}$ as

$$\mathfrak{h}_\Gamma^{**} = \mathfrak{h} \sqcup \mathcal{L}_\Gamma \qquad \text{where} \qquad \mathcal{L}_\Gamma \overset{\text{def}}{=} \bigsqcup_{x \in \mathcal{P}_\Gamma} L_x \qquad \text{and} \qquad L_x \overset{\text{def}}{=} \mathbb{P}^1(\mathbb{R}) \setminus \{x\}.$$

Observe first of all that we still have an action of Γ on $\mathfrak{h}_\Gamma^{**}$ defined as

$$A * (x, \lambda) = (A * x, A * \lambda) \qquad \text{for every} \qquad (x, \lambda) \in \mathcal{L}_\Gamma = \bigsqcup_{x \in \mathcal{P}_\Gamma} \mathbb{P}^1(\mathbb{R}) \setminus \{x\}$$

and as the usual action on $\mathfrak{h}$. We endow $\mathfrak{h}_\Gamma^{**}$ with the coarsest topology such that the inclusions $\mathfrak{h} \hookrightarrow \mathfrak{h}_\Gamma^{**}$ and $\mathcal{L}_\Gamma \hookrightarrow \mathfrak{h}_\Gamma^{**}$ are topological embeddings and the subsets

$$(\sigma_x * \{z \in \mathfrak{h} \mid \mathfrak{Im}(z) > l\}) \cup L_x \subseteq \mathfrak{h}_\Gamma^{**} \qquad \text{where} \qquad \sigma_x = \left(\begin{smallmatrix} x & -1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$$

are open for every $x \in \mathcal{P}_\Gamma$. Observe that $\mathfrak{h}_\Gamma^{**}$ has a natural structure of *manifold with boundary* in the sense of Definition 1.20. To define what a manifold with boundary is we give the more general definition of *manifold with corners* as stated in [9].

Definition 1.18. Let $n, m \in \mathbb{N}$ and let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ be any two subsets. We say that a map of sets $f : A \to B$ is *smooth* if there exist an open subset $U \subseteq \mathbb{R}^n$ and a smooth map $g : U \to \mathbb{R}^m$ such that $A \subseteq U$ and $g|_A \equiv f$.

Definition 1.19. Let X be a topological space. We define an *atlas with corners* of dimension $n \in \mathbb{N}$ on X to be a collection $\mathcal{A} = \{(U, \varphi)\}$ where $U \subseteq X$ is an open subset and $\varphi : U \to (\mathbb{R}_{\geq 0})^k \times \mathbb{R}^{n-k}$ is a homeomorphism for some $k \in \{0, \dots, n\}$ such that for every other $(V, \psi) \in \mathcal{A}$ the maps $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are smooth homeomorphisms. We define a *manifold with corners* to be a couple $(X, \mathcal{A})$ where X is a Hausdorff, second countable topological space and $\mathcal{A}$ is an atlas with corners which is maximal in the poset of all n-dimensional atlases with corners on X ordered with respect to the inclusion.

We have also a definition of morphisms between manifolds with corners which is given in **Definition 3.1** of [9]. Observe now that on every n-dimensional manifold with corners $(X, \mathcal{A})$ we can define a "stratification" $\{X_k\}_{k=0}^n$. To do so we define first of all a function

$$d_k : \mathbb{R}^k_{\geq 0} \to \{0, \dots, k\} \qquad \text{by setting} \qquad d_k(x_1, \dots, x_k) = \#\{j \in \{1, \dots, k\} \mid x_j = 0\}$$

and then we define the sets X_k as

$$X_k \stackrel{\text{def}}{=} \{x \in X \mid \varphi(x) = (\mathbf{v}, \mathbf{w}) \in (\mathbb{R}^l_{\geq 0} \times \mathbb{R}^{n-l}) \text{ with } d_k(\mathbf{v}) = k\}$$

where $(U, \varphi) \in \mathcal{A}$ is any chart such that $x \in U$. It is not difficult to prove that $d_k(\mathbf{v})$ does not depend on the choice of (U, φ) and thus the sets X_k are well defined.

Definition 1.20. We say that a manifold with corners X is a *smooth manifold* if $X_j = \emptyset \iff j > 0$ and we say that it is a *smooth manifold with boundary* if $X_j = \emptyset \iff j > 1$.

It seems pointless to have introduced the complicated definition of manifold with corners if the only thing that we need is a manifold with boundary, but this definition is widely used in [3] and [8] to define the Borel-Serre compactification of general *modular varieties*, which are the generalization of modular curves to higher dimensions.

We see that almost by definition the action $\Gamma \circlearrowright \mathfrak{h}^{**}_\Gamma$ is continuous with respect to this topology, and so we can consider the topological quotient space $\Gamma \backslash \mathfrak{h}^{**}_\Gamma$. This space is exactly the compactification that we are looking for, as we show in the following theorem.

Theorem 1.21. *Let $\Gamma \leq \mathrm{SL}_2(\mathbb{R})$ be a Fuchsian group of the first kind. Then the topological space $\Gamma\backslash\mathfrak{h}_\Gamma^{**}$ is compact and has a natural structure of manifold with boundary such that the quotient map $\mathfrak{h}_\Gamma^{**} \twoheadrightarrow \Gamma\backslash\mathfrak{h}_\Gamma^{**}$ is a smooth map between manifolds with boundary and the inclusion $\Gamma\backslash\mathfrak{h} \hookrightarrow \Gamma\backslash\mathfrak{h}_\Gamma^{**}$ is an embedding of a smooth manifold inside a manifold with boundary.*

Proof. Let $\mathfrak{f} \subseteq \mathfrak{h}_\Gamma^{**}$ be a connected fundamental domain for the action $\Gamma \circlearrowright \mathfrak{h}_\Gamma^{**}$, i.e. a connected subset $\mathfrak{f} \subseteq \mathfrak{h}_\Gamma^{**}$ such that the quotient map $\mathfrak{h}_\Gamma^{**} \twoheadrightarrow \Gamma\backslash\mathfrak{h}_\Gamma^{**}$ becomes a bijection when restricted to $\mathfrak{f}$. It is not difficult to see that we can cover this fundamental domain with a finite number of closed subsets which become compact in the quotient $\Gamma\backslash\mathfrak{h}_\Gamma^{**}$. Indeed we know from Theorem 1.12 that the set $\Gamma\backslash\mathcal{P}_\Gamma$ is finite, and thus we can choose a finite set of representatives $\mathfrak{S} \subseteq \mathcal{P}_\Gamma$ for the quotient $\Gamma\backslash\mathcal{P}_\Gamma$. Now for every $s \in \mathfrak{S}$ we can choose a sufficiently big neighbourhood of the form $\sigma_s * (\{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq l\} \cup L_\infty)$ such that to exhaust $\mathfrak{f}$ it will be sufficient to take a sufficiently wide compact rectangle $[a,b]\times[c,d] \subseteq \mathfrak{f}$. This proves that the fundamental domain is contained in the union of a finite number of closed subsets, and it is not difficult to see that the projection $\mathfrak{h}_\Gamma^{**} \twoheadrightarrow \Gamma\backslash\mathfrak{h}_\Gamma^{**}$ is a homeomorphism when restricted to the rectangle $[a,b]\times[c,d] \subseteq \mathfrak{f}$ and sends the neighbourhoods $\sigma_s * (\{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq l\} \cup L_\infty)$ to compact subsets of $\Gamma\backslash\mathfrak{h}_\Gamma^{**}$ homeomorphic to the compact annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$. This finally shows that the quotient $\Gamma\backslash\mathfrak{h}_\Gamma^{**}$ is the union of finitely many compact subsets, and thus is compact. $\square$

For a more detailed proof of Theorem 1.21 we refer to §7 and **Theorem 9.3** of [3], which prove the theorem for the more general Borel-Serre compactification of a modular variety associated to an arithmetic group Γ, which is a subgroup of some algebraic group G defined over $\mathbb{Q}$ which is commensurable to the integer points of G. In our case $G = \mathrm{SL}_2$ and thus we will use the following definition of arithmetic group.

Definition 1.22. Any subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Q})$ is an *arithmetic group* if and only if it is commensurable to $\mathrm{SL}_2(\mathbb{Z})$ in the sense of Definition 1.14.

From now on for every arithmetic subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Q})$ we define

$$C_\Gamma^{\mathrm{BS}} \stackrel{\text{def}}{=} \Gamma\backslash\mathcal{L}_\Gamma \qquad \text{and} \qquad X_\Gamma^{\mathrm{BS}} \stackrel{\text{def}}{=} \Gamma\backslash\mathfrak{h}_\Gamma^{**} \tag{1.1}$$

and we say that X_Γ^{BS} is the Borel-Serre compactification of the modular curve Y_Γ. In particular the congruence subgroups $\Gamma(n)$, $\Gamma_0(n)$ and $\Gamma_1(n)$ are arithmetic groups for all $n \in \mathbb{N}$.

CHAPTER 2

ADÈLES AND AFFINE MODULAR CURVES

Nothing in life is to be feared, it is only to be understood. Now is the time to understand more, so that we may fear less.

Marie Curie

Chapter Abstract

In this chapter we define the ring of adèles $\mathbb{A}_K$ associated to a global field K and we show that suitable disjoint unions of affine modular curves are homeomorphic to double quotients of the topological group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$

2.1 The adèle ring

In this section we review the definition of the adèle ring $\mathbb{A}_K$ of a number field K, which will be the protagonist of our new description of the modular curves.

Definition 2.1. Let $\mathbb{K}$ be a field. We say that a function $|\cdot|\colon\mathbb{K}\to\mathbb{R}_{\geq 0}$ is a *norm* on $\mathbb{K}$ if $\left|xy\right| = |x|\left|y\right|$ and $\left|x+y\right| \leq |x| + \left|y\right|$ for every $x,y\in\mathbb{K}$ and $|x| = 0$ if and only if $x = 0$. We say that a norm $|\cdot|$ is *non-Archimedean* if $\left|x+y\right| \leq \max(|x|, \left|y\right|)$ and we say that $|\cdot|$ is *Archimedean* otherwise.

Definition 2.2. Let $\mathbb{K}$ be a field and let $|\cdot|$ be a norm on $\mathbb{K}$. We say that a sequence $\{x_n\}_{n=1}^{+\infty} \subseteq \mathbb{K}$ is a *Cauchy sequence* for $|\cdot|$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for every $n, m \geq n_0$. We say that $|\cdot|$ is *equivalent* to another norm $|\cdot|'$ defined on $\mathbb{K}$ if every sequence $\{x_n\}_{n=1}^{+\infty} \subseteq \mathbb{K}$ is a Cauchy sequence for $|\cdot|$ if and only if it is a Cauchy sequence for $|\cdot|'$.

It can be shown that two norms $|\cdot|_1$ and $|\cdot|_2$ defined on the same field $\mathbb{K}$ are equivalent if and only if there exists $\alpha \in \mathbb{R}_{>0}$ such that $|\cdot|_2 = |\cdot|_1^{\alpha}$. Moreover if $|\cdot|_1$ is equivalent to $|\cdot|_2$ then $|\cdot|_1$ is Archimedean if and only if $|\cdot|_2$ is Archimedean. We will denote by $\Sigma_{\mathbb{K}}$ the set of all the *places* of $\mathbb{K}$, i.e. the set of all the equivalence classes of norms defined on $\mathbb{K}$, and we will denote by $\Sigma_{\mathbb{K}}^{\infty}$ the set of the equivalence classes of the non-Archimedean norms. The following theorem of Ostrowski characterizes the set Σ_K when K is a number field.

Definition 2.3. Let K be a number field and let I_K be the set of all possible embeddings $K \hookrightarrow \mathbb{C}$. Then we define an equivalence relation on I_K by saying that $\sigma \sim \tau$ if and only if $\sigma = \tau$ or $\sigma = \overline{\tau}$, where $\overline{(-)}: \mathbb{C} \to \mathbb{C}$ is the complex conjugation.

Theorem 2.4. *Let K be a number field, and denote by $\|\cdot\|: \mathbb{C} \to \mathbb{R}_{\geq 0}$ the usual absolute value. Then the maps*

$$\begin{aligned} \Sigma_K^{\infty} &\to \operatorname{Spec}(\mathcal{O}_K) \setminus \{0\} \\ [|\cdot|] &\mapsto \{x \in \mathcal{O}_K \mid |x| < 1\} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{I}_K/\!\sim &\to \Sigma_K \setminus \Sigma_K^{\infty} \\ \sigma &\mapsto \|\cdot\| \circ \sigma \end{aligned} \tag{2.1}$$

are bijections.

Proof. See [7]. □

To define the adèle ring $\mathbb{A}_K$ we have to define the completions of a field with respect to a norm defined on it. We say that a normed field $(\mathbb{K}, |\cdot|)$ is *complete* if every Cauchy sequence converges, i.e. if for every Cauchy sequence $\{x_n\} \subseteq \mathbb{K}$ there exists $l \in K$ such that $|x_n - l| \to 0$ when $n \to +\infty$. Let now $(\mathbb{K}, |\cdot|_{\mathbb{K}})$ be a normed field and let $(\mathbb{F}, |\cdot|_{\mathbb{F}})$ be a complete normed field with a fixed inclusion of normed fields $\iota: \mathbb{K} \hookrightarrow \mathbb{F}$ such that $|\iota(x)|_{\mathbb{F}} = |x|_{\mathbb{K}}$ for every $x \in \mathbb{K}$. We say that $\mathbb{F}$ is a completion of $\mathbb{K}$ if for every inclusion of normed fields $f: (\mathbb{K}, |\cdot|_{\mathbb{K}}) \hookrightarrow (\mathbb{L}, |\cdot|_{\mathbb{L}})$ with the property that for every Cauchy sequence $\{x_n\} \subseteq \mathbb{K}$ the sequence $\{f(x_n)\} \subseteq \mathbb{L}$ converges there exists a unique inclusion of normed fields $\hat{f}\colon(\mathbb{F},|\cdot|_{\mathbb{F}}) \hookrightarrow (\mathbb{L},|\cdot|_{\mathbb{L}})$ such that $f = \hat{f} \circ \iota$. It can be shown that for every field $\mathbb{K}$ and every place $v \in \Sigma_{\mathbb{K}}$ there exists a completion $\mathbb{K}_v$ which is unique up to a unique isomorphism of normed fields. For example the completion of $\mathbb{Q}$ with respect to the usual, Euclidean absolute value is the field of the real numbers $\mathbb{R}$ and the completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$ is the field $\mathbb{Q}_p$ of p-adic numbers.

The last notion that we need to define the ring of adèles of a number field K is the notion of *restricted product* of topological spaces. Let $\{X_i\}_{i\in\mathcal{I}}$ be a family of topological spaces and let $\{Y_i\}_{i\in\mathcal{I}}$ be another family of topological spaces such that Y_i is a subspace of X_i for every $i \in \mathcal{I}$. Then we define the restricted product $\prod'_{i\in\mathcal{I}}(X_i\colon Y_i)$ by setting

$$\prod_{i\in\mathcal{I}}{}'(X_i\colon Y_i) \stackrel{\text{def}}{=} \left\{(a_i) \in \prod_{i\in\mathcal{I}} X_i \;\middle|\; \{i \in \mathcal{I} \mid a_i \in X_i \setminus Y_i\} \text{ is finite}\right\}$$

and we endow it with the topology generated by the basis made of sets of the form $\prod_{i\in\mathcal{I}} U_i$ such that $U_i \subseteq X_i$ are open and $U_i = Y_i$ for all but finitely many $i \in \mathcal{I}$.

Let now $(\mathbb{K},|\cdot|)$ be a non-Archimedean normed field. We define the *ring of integers* $\mathcal{O}_{\mathbb{K}}$ and its maximal ideal $\mathfrak{m}_{\mathbb{K}}$ by setting

$$\mathcal{O}_{\mathbb{K}} \stackrel{\text{def}}{=} \{x \in \mathbb{K} \mid |x| \leq 1\} \qquad \text{and} \qquad \mathfrak{m}_{\mathbb{K}} \stackrel{\text{def}}{=} \{x \in \mathbb{K} \mid |x| < 1\}$$

and we observe that $(\mathcal{O}_{\mathbb{K}}, \mathfrak{m}_{\mathbb{K}})$ is a discrete valuation ring if $(\mathbb{K},|\cdot|)$ is a non-Archimedean *local field*, i.e. it is complete and $|\mathbb{K}^\times| \leq \mathbb{R}_{>0}$ is a non trivial discrete subgroup. In particular, if K is a number field and $v \in \Sigma_K^\infty$, then the completion K_v is a non-Archimedean local field with ring of integers $\mathcal{O}_{K_v}$, whereas if $v \in \Sigma_K \setminus \Sigma_K^\infty$ we define $\mathcal{O}_{K_v} \stackrel{\text{def}}{=} K_v$. Using these conventions we can define the ring of adèles $\mathbb{A}_K$ of a number field K as the restricted product

$$\mathbb{A}_K \stackrel{\text{def}}{=} \prod_{v\in\Sigma_K}{}'(K_v\colon\mathcal{O}_{K_v}) \qquad \text{with the operations} \qquad \begin{gathered}(a_v) + (b_v) = (a_v + b_v) \\ (a_v)\cdot(b_v) = (a_v \cdot b_v)\end{gathered}$$

defined for every pair of sequences $(a_v)_{v\in\Sigma_K}, (b_v)_{v\in\Sigma_K} \in \mathbb{A}_K$. With these operations and this topology the ring $\mathbb{A}_K$ becomes a topological ring, i.e. a ring which is also a topological space such that all the operations are continuous.

We can give also another description of the adèle ring $\mathbb{A}_K$ using the topological ring

$\widehat{\mathcal{O}}_K$ defined as

$$\widehat{\mathcal{O}}_K \stackrel{\text{def}}{=} \varprojlim_{I \subseteq \mathcal{O}_K} \frac{\mathcal{O}_K}{I} = \varprojlim_{N \in \mathbb{N}_{\geq 1}} \frac{\mathcal{O}_K}{N\mathcal{O}_K} \cong \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_K \qquad \text{with} \qquad \widehat{\mathbb{Z}} \stackrel{\text{def}}{=} \varprojlim_{N \in \mathbb{N}_{\geq 1}} \frac{\mathbb{Z}}{N\mathbb{Z}}$$

which is called the *profinite completion* of the ring of integers $\mathcal{O}_K$. The topology on $\widehat{\mathcal{O}}_K$ is defined as the coarsest topology such that for every ideal $I \subseteq \mathcal{O}_K$ the map $\widehat{\mathcal{O}}_K \twoheadrightarrow {}^{\mathcal{O}_K}/_I$ is continuous. We want now to prove that there exists a canonical isomorphism

$$\mathbb{A}_K \cong (\widehat{\mathcal{O}}_k \otimes_{\mathcal{O}_K} K) \times (K \otimes_{\mathbb{Q}} \mathbb{R})$$

as topological rings, where the topology on $\widehat{\mathcal{O}}_k \otimes_{\mathcal{O}_K} K$ is the strongest topology such that the map

$$\widehat{\mathcal{O}}_k \times K \to \widehat{\mathcal{O}}_k \otimes_{\mathcal{O}_K} K \qquad (x,y) \mapsto x \otimes 1 + 1 \otimes y$$

is continuous. To prove our claim we define $\mathbb{A}_K^\infty \stackrel{\text{def}}{=} \prod'_{v \in \Sigma_K^\infty} (K_v : \mathcal{O}_{K_v})$ and we observe that $\mathbb{A}_K = \mathbb{A}_K^\infty \times (K \otimes_{\mathbb{Q}} \mathbb{R})$ since $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ where $r_1 = \#\{K \hookrightarrow \mathbb{R}\}$ and $r_1 + 2r_2 = [K:\mathbb{Q}]$. Now we have to prove that $\mathbb{A}_K^\infty \cong \widehat{\mathcal{O}}_k \otimes_{\mathcal{O}_K} K$ and to do so it is sufficient to observe that

$$\widehat{\mathcal{O}}_k \otimes_{\mathcal{O}_K} K \cong \widehat{\mathcal{O}}_k \otimes_{\mathcal{O}_K} \left(\varinjlim_{N \in \mathbb{N}_{\geq 1}} \mathcal{O}_K \left[\frac{1}{N} \right] \right) \cong \varinjlim_{N \in \mathbb{N}_{\geq 1}} \widehat{\mathcal{O}}_k \otimes_{\mathcal{O}_K} \mathcal{O}_K \left[\frac{1}{N} \right] \cong \varinjlim_{N \in \mathbb{N}_{\geq 1}} \left(\prod_{\mathfrak{p} \ni N} K_{\mathfrak{p}} \times \prod_{\mathfrak{p} \not\ni N} \mathcal{O}_{K_{\mathfrak{p}}} \right)$$

where for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_K) \setminus \{0\}$ we define $K_{\mathfrak{p}}$ as the completion of the field K with respect to the place $v_{\mathfrak{p}} \in \Sigma_K^\infty$ which corresponds to $\mathfrak{p}$ through the bijection defined in Equation 2.1.

2.2 Properties of the adèle ring

The aim of this section is to recall some further generalities about adèle rings that we will use in the next section to find a relationship between $\mathbb{A}_{\mathbb{Q}}$ and affine modular curves. Thus let K be a number field and observe that the canonical embeddings $K \hookrightarrow K_v$ for every $v \in \Sigma_K^\infty$ give rise to the diagonal embeddings $K \hookrightarrow \mathbb{A}_K^\infty$ and $K \hookrightarrow \mathbb{A}_K$. Indeed for every ${}^x/_y \in K$ with $x, y \in \mathcal{O}_K$ we have that $x \in \mathfrak{p}_v$ or $y \in \mathfrak{p}_v$ for only finitely many places $v \in \Sigma_K^\infty$, where $\mathfrak{p}_v \in \operatorname{Spec}(\mathcal{O}_K)$ is the prime ideal corresponding to v through the bijection defined in Equation 2.1, and thus $x \in \mathcal{O}_{K_v}$ for all but finitely many places $v \in \Sigma_K^\infty$. It is not difficult to see that the image of this embedding $K \hookrightarrow \mathbb{A}_K$ is discrete, whereas for every finite subset $\emptyset \neq S \subseteq \Sigma_K$ the image of the embedding

$$K \hookrightarrow \mathbb{A}_K^S \qquad \text{where} \qquad \mathbb{A}_K^S \overset{\text{def}}{=} \prod_{\substack{v \in \Sigma_K \\ v \notin S}}{}' (K_v : \mathcal{O}_{K_v})$$

is dense, as it is proved in §II.15 of [6]. We refer to this result by saying that the affine line $\mathbb{A}^1$ satisfies *strong approximation* away from the subset $S \subseteq \Sigma_K$. More generally for every algebraic variety X over a number field K and for every subset $S \subseteq \Sigma_K$ we have the inclusions

$$X(K) \hookrightarrow X(\mathbb{A}_K^S) \hookrightarrow \prod_{v \in \Sigma_K \setminus S} X(K_v) \tag{2.2}$$

and we say that X satisfies *weak approximation* away from S if $X(K)$ is dense in the product $\prod_{v \in \Sigma_K \setminus S} X(K_v)$ whereas we say that X satisfies *strong approximation* away from S if $X(K)$ is dense in $X(\mathbb{A}_K^S)$. Observe that the inclusions mentioned in (2.2) are continuous but the topology on $X(\mathbb{A}_K^S)$ is finer than the subspace topology induced by the inclusion $X(\mathbb{A}_K^S) \hookrightarrow \prod_{v \in \Sigma_K \setminus S} X(K_v)$. For further information and for a precise definition of the topology of $X(\mathbb{A}_{\mathbb{K}}^S)$ we refer to §5.1 and §7.1 of [14].

Lemma 2.5. *The algebraic group* SL_n *over a number field* K *satisfies strong approximation away from any finite set* $\emptyset \neq S \subseteq \Sigma_K$.

Proof. Let G be the closure of $\mathrm{SL}_n(K)$ inside $\mathrm{SL}_n(\mathbb{A}_K^S)$. Since K is dense in $\mathbb{A}_K^S$ we see immediately that G contains all the matrices of the form $I_n + x \cdot E_{i,j}$ for $i \neq j$, where I_n is the $n \times n$ identity matrix, $x \in \mathbb{A}_K^S$ and $E_{i,j} = (e_{k,l})$ with $e_{k,l} = 0$ if $(k,l) \neq (i,j)$ and $e_{i,j} = 1$.

Recall now that for every field $\mathbb{F}$ the group $\mathrm{SL}_n(\mathbb{F})$ is generated by the matrices of the form $I_n + y \cdot E_{i,j}$ where $i \neq j$ and $y \in \mathbb{F}$ as it is proved in **Proposition 17** of **Chapter III.8** of [4]. Let now $M \in \mathrm{SL}_n(\mathbb{A}_K^S)$ and for every $v \in \Sigma_K^S$ let $M_v \in \mathrm{SL}_n(K_v)$ be the image of M via the canonical projection $\mathrm{SL}_n(\mathbb{A}_K^S) \twoheadrightarrow \mathrm{SL}_n(K_v)$. What we have proved shows that for every $v_0 \in \Sigma_K^S$ G contains all the matrices $M \in \mathrm{SL}_n(\mathbb{A}_K^S)$ such that $M_v = I_n$ for every $v \in \Sigma_K^S \setminus \{v_0\}$. Since the closure of a subgroup in a topological group is again a subgroup this implies that G contains all the matrices $M \in \mathrm{SL}_n(\mathbb{A}_K^S)$ such that $M_v = I_n$ for all but finitely many $v \in \Sigma_K^S$.

To conclude it is sufficient to recall that a basis of open subsets for the topological space $\mathrm{SL}_n(\mathbb{A}_K^S)$ is given by the sets of the form $\prod_{v \in \Sigma_K^S} U_v$ where $U_v \subseteq \mathrm{SL}_n(K_v)$ is open for every $v \in \Sigma_K^S$ and $U_v = \mathrm{SL}_n(\mathcal{O}_{K_v})$ for all but finitely many $v \in \Sigma_K^S$. Since each of the elements of this basis contains at least one matrix $M \in \mathrm{SL}_n(\mathbb{A}_K^S)$ such that $M_v = I_n$ for all but finitely many $v \in \Sigma_K^S$ we have that $G = \mathrm{SL}_n(\mathbb{A}_K^S)$ because G is closed and contains the dense subset of all the matrices $M \in \mathrm{SL}_n(\mathbb{A}_K^S)$ such that $M_v = I_n$ for all but finitely many $v \in \Sigma_K^S$. □

Corollary 2.6. *Let K be a number field, and let $S \subseteq \Sigma_K$ be a finite and non-empty subset. Then for every open subgroup $U \leq \mathrm{SL}_2(\mathbb{A}_K^S)$ we have that $\mathrm{SL}_2(\mathbb{A}_K^S) = \mathrm{SL}_2(K) \cdot U$.*

We recall finally that the embedding $K \hookrightarrow \mathbb{A}_K^\infty$ is also used in §II.19 of [6] to prove that

$$\frac{\mathbb{A}_K^{\infty,\times}}{K^\times \cdot \widehat{\mathcal{O}_K}^\times} \cong \mathfrak{Cl}(\mathcal{O}_K) \tag{2.3}$$

where $\mathfrak{Cl}(\mathcal{O}_K)$ is the *ideal class group* of the Dedekind domain $\mathcal{O}_K$, defined as the multiplicative group of all fractional ideals $I \subseteq K = \mathrm{Frac}(\mathcal{O}_K)$ quotiented out by the subgroup of all principal fractional ideals $x\mathcal{O}_K \subseteq K$. In particular this implies that

$$\mathbb{A}_{\mathbb{Q}}^{\infty,\times} = \mathbb{Q}^\times \cdot \widehat{\mathbb{Z}}^\times = \mathbb{Q}_{>0} \cdot \widehat{\mathbb{Z}}^\times \cong \mathbb{Q}_{>0} \times \widehat{\mathbb{Z}}^\times$$

since $\mathbb{Z}$ is a principal ideal domain and $\mathbb{Q}_{>0} \cap \widehat{\mathbb{Z}}^\times = \{1\}$.

Consider now the topological group $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ and take any compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$. The following lemma will be used in the next section to show that a disjoint union of $[\widehat{\mathbb{Z}}^\times : \det(K^\infty)]$ affine modular curves is homeomorphic to a suitable double quotient which involves $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ and K^∞.

Lemma 2.7. *For every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ we have that $\det(K^\infty) \subseteq \widehat{\mathbb{Z}}^\times$, the quotient ${}^{\widehat{\mathbb{Z}}^\times}/_{\det(K^\infty)}$ is finite and discrete and we have a homeomorphism*

$$\mathbb{Q}^\times \backslash \left(\{\pm 1\} \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} / \det(K^\infty) \right) \cong \frac{\widehat{\mathbb{Z}}^\times}{\det(K^\infty)}$$

where the actions $\mathbb{Q}^\times \circlearrowright \{\pm 1\} \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} \circlearrowleft \det(K^\infty)$ are defined as

$$\alpha * (\varepsilon, x) * \beta = (\mathrm{sign}(\alpha) \cdot \varepsilon, \alpha \cdot x \cdot \beta).$$

Proof. Recall first of all that $\widehat{\mathbb{Z}}^\times$ is the unique maximal compact subgroup of $\mathbb{A}_{\mathbb{Q}}^{\infty,\times}$. Since the determinant map is continuous and K^∞ is compact by hypothesis, this implies that $\det(K^\infty) \leq \mathbb{Z}^\times$.

It is now easy to see from the definition of the action of $\mathbb{Q}^\times$ that

$$\mathbb{Q}^\times \backslash \left(\{\pm 1\} \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} / \det(K^\infty) \right) \cong \mathbb{Q}_{>0} \backslash \mathbb{A}_{\mathbb{Q}}^{\infty,\times} / \det(K^\infty) \cong \frac{\widehat{\mathbb{Z}}^\times}{\det(K^\infty)}$$

where the last isomorphism comes from the fact that $\mathbb{A}_{\mathbb{Q}}^{\infty,\times} = \mathbb{Q}_{>0} \cdot \widehat{\mathbb{Z}}^\times$ as we have showed in Equation 2.3.

We finally prove that $\det(K^\infty) \leq \widehat{\mathbb{Z}}$ is open. Observe first of all that $\det(K^\infty)$ is compact since $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\times)$ is compact and the determinant map is continuous, and thus $\det(K^\infty) \leq \widehat{\mathbb{Z}}^\times$ is a closed subgroup because $\widehat{\mathbb{Z}}^\times$ is an Hausdorff topological space. Since the group $\widehat{\mathbb{Z}}^\times$ is a profinite group we have only to prove that $\det(K^\infty) \leq \widehat{\mathbb{Z}}^\times$ is a subgroup of finite index. To show this we observe that the subgroups

$$K(n) \stackrel{\text{def}}{=} \left\{ A \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) \mid A \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \bmod n \right\} = \ker\left(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \twoheadrightarrow \mathrm{GL}_2\left(\mathbb{Z}/_{n\mathbb{Z}} \right) \right)$$

form a basis of open neighbourhoods of the identity in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$. Moreover it is not difficult to show that

$$\det(K(n)) = \{ x \in \widehat{\mathbb{Z}}^\times \mid x \equiv 1 \bmod n \} = \ker\left(\widehat{\mathbb{Z}}^\times \twoheadrightarrow \left(\mathbb{Z}/_{n\mathbb{Z}} \right)^\times \right)$$

which implies finally that $\det(K^\infty) \leq \widehat{\mathbb{Z}}^\times$ is a subgroup of finite index because $\det(K(n))$ is a subgroup of finite index in $\widehat{\mathbb{Z}}^\times$ and $\det(K(n)) \leq \det(K^\infty)$.

Using thus the fact that $\widehat{\mathbb{Z}}^\times$ is compact and that $\det(K^\infty) \leq \widehat{\mathbb{Z}}^\times$ is open we see that the quotient $\widehat{\mathbb{Z}}^\times/_{\det(K^\infty)}$ is compact and discrete, and thus finite, as we wanted to prove. □

2.3 An adelic description of affine modular curves

We are now ready to prove the main theorem of this chapter, which for every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ describes a homeomorphism between the topological disjoint union of $[\widehat{\mathbb{Z}}^\times : \det(K^\infty)]$ affine modular curves and the double quotient $\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / (K^\infty \times \mathbb{R}_{>0} \times \mathrm{SO}_2(\mathbb{R}))$. All the material in this section is not original, and can also be found in [12].

Let now $K^\infty \le \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ be a compact and open subgroup, let $n = [\widehat{\mathbb{Z}}^\times : \det(K^\infty)]$ and let $\{A_j\}_{j=1}^n \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be any set of matrices such that $\{\overline{\det(A_j)}\}_{j=1}^n = {}^{\widehat{\mathbb{Z}}^\times}/_{\det(K^\infty)}$. We define the subgroups $\Gamma_j \le \mathrm{SL}_2(\mathbb{Q})$ as $\Gamma_j \overset{\text{def}}{=} A_j \cdot K^\infty \cdot A_j^{-1} \cap \mathrm{SL}_2(\mathbb{Q})$ and we observe that Γ_j is an arithmetic subgroup for every $j = 1, \ldots, n$. Indeed every two compact and open subgroups $H, H' \subseteq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ are commensurable, since the quotients

$$\frac{H}{H \cap H'} \qquad \text{and} \qquad \frac{H'}{H \cap H'}$$

are compact and discrete, and thus finite. This implies that $A_j \cdot K^\infty \cdot A_j^{-1}$ is commensurable with $\mathrm{SL}_2(\widehat{\mathbb{Z}})$ for every $j \in \{1, \ldots, n\}$ and hence that $\Gamma_j = A_j \cdot K^\infty \cdot A_j^{-1} \cap \mathrm{SL}_2(\mathbb{Q})$ is commensurable with $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{SL}_2(\widehat{\mathbb{Z}}) \cap \mathrm{SL}_2(\mathbb{Q})$ for every $j \in \{1, \ldots, n\}$.

Observe now that every Γ_j contains the principal congruence subgroup $\Gamma(m_j)$ for some $m_j \in \mathbb{N}$. Indeed we have already shown in the proof of Lemma 2.7 that the subgroups $K(m) = \ker\left(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \twoheadrightarrow \mathrm{GL}_2\left({}^{\mathbb{Z}}/_{m\mathbb{Z}}\right)\right)$ form a basis of open neighbourhoods of the identity in $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$. This implies in particular that there exist $\{m_1, \ldots, m_n\} \subseteq \mathbb{N}$ such that $K(m_j) \le A_j \cdot K^\infty \cdot A_j^{-1}$ for every $j \in \{1, \ldots, n\}$. Thus $\Gamma(m_j) \le \Gamma_j$ because $K(m_j) \cap \mathrm{SL}_2(\mathbb{Q}) = \Gamma(m_j)$.

We define now the topological space Y_{K^∞} as

$$Y_{K^\infty} \overset{\text{def}}{=} \bigsqcup_{j=1}^n Y_{\Gamma_j} = \bigsqcup_{j=1}^n \Gamma_j \backslash \mathfrak{h}$$

and we observe that Y_{K^∞} is well defined up to homeomorphisms due to different choices of the matrices $\{A_j\}$. The goal of this section is to provide a homeomorphism

$$Y_{K^\infty} \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / (K^\infty \times \mathbb{R}_{>0} \times \mathrm{SO}_2(\mathbb{R}))$$

and to do so we start with a lemma concerning subgroups of $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ where we don't have problems coming from the determinant.

Lemma 2.8. *Let $U \le \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ be a compact and open subgroup and let $\Gamma = U \cap \mathrm{SL}_2(\mathbb{Q})$. Then Γ is an arithmetic group and the map*

$$\varphi \colon \mathfrak{h} \to \mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^\infty) \qquad \textit{defined as} \qquad \varphi(z) = \left(z, \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)$$

induces a homeomorphism $\overline{\varphi} \colon \Gamma \backslash \mathfrak{h} \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Q}) \backslash \left(\mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^\infty) / U\right)$.

Proof. Let $z, w \in \mathfrak{h}$, and let $\pi\colon \mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \to \mathrm{SL}_2(\mathbb{Q})\backslash(\mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/U)$ be the quotient map. Then for every $z, w \in \mathfrak{h}$ we have $\pi(\varphi(z)) = \pi(\varphi(w))$ if and only if there exist $A \in \mathrm{SL}_2(\mathbb{Q})$ and $B \in U$ such that $w = A * z$ and $A \cdot B = I_2$. This shows that $A = B^{-1} \in U \cap \mathrm{SL}_2(\mathbb{Q}) = \Gamma$ and thus that $\pi(\varphi(z)) = \pi(\varphi(w))$ if and only if there exists $A \in \Gamma$ such that $w = A * z$. This implies that $\overline{\varphi}$ is well defined and injective.

Let now $(z, A) \in \mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and observe that we can write $A = X \cdot Y$ where $X \in \mathrm{SL}_2(\mathbb{Q})$ and $Y \in U$ because $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) = \mathrm{SL}_2(\mathbb{Q}) \cdot U$ as proved in Corollary 2.6. Then we have

$$\pi(X^{-1} * z, I_2) = \pi(z, A) \qquad \text{because} \qquad (z, A) = (z, X \cdot Y) = X * (X^{-1} * z, I_2) * Y$$

and thus we see that $\overline{\varphi}(X^{-1} * z) = \pi(z, A)$, which implies that $\overline{\varphi}$ is surjective.

To conclude we have to prove that $\overline{\varphi}$ is continuous and open. First of all observe that φ is clearly continuous and thus $\overline{\varphi}$ is continuous. Observe moreover that ${}^{\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})}/_{U}$ is a discrete topological space because $U \subseteq \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is open, and thus the map

$$\mathfrak{h} \to \mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/U \qquad z \mapsto \left(z, \left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\right]\right)$$

induced by φ is open, which implies that $\overline{\varphi}$ is a continuous and open bijection, and thus it is a homeomorphism. $\square$

Theorem 2.9. *Let $K^{\infty} \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a compact and open subgroup and let*

$$\mathfrak{h}^{\pm} \overset{\mathrm{def}}{=} \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \neq 0\} \cong \mathfrak{h} \times \{\pm 1\}.$$

Then the maps $\varphi_j\colon \mathfrak{h} \to \mathfrak{h}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ defined as $\varphi_j(x) = (x, A_j)$ induce a homeomorphism $Y_{K^{\infty}} \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q})\backslash\left(\mathfrak{h}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K^{\infty}\right)$.

Proof. Observe now that for every $j \in \{1, \ldots, n\}$ the map

$$\begin{aligned} \Gamma_j\backslash\mathfrak{h} &\xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Q})\backslash\mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/(\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \cap A_j \cdot K^{\infty} \cdot A_j^{-1}) \\ [z] &\mapsto [z, I_2] \end{aligned}$$

is a homeomorphism, as proved in Lemma 2.8. Observe moreover that the map

$$\mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \to \mathfrak{h} \times \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \qquad (z, X) \mapsto (z, X \cdot A_j)$$

is clearly a homeomorphism, and induces a homeomorphism

$$\begin{aligned}\Gamma_j\backslash\mathfrak{h} &\xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Q})\backslash\left(\mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cdot A_j/(\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cap K^{\infty})\right)\\ [z] &\mapsto [z,A_j]\end{aligned}$$

where the right action $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cdot A_j \circlearrowleft K^{\infty}\cap\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is defined as $(X\cdot A_j)*Y = X\cdot Y'\cdot A_j$ where $Y' = A_j\cdot X\cdot A_j^{-1}$. Using these facts we only have to prove that

$$\mathrm{GL}_2(\mathbb{Q})\backslash\left(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K^{\infty}\right)\cong\bigsqcup_{j=1}^{n}\mathrm{SL}_2(\mathbb{Q})\backslash\left(\mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cdot A_j/(\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cap K^{\infty})\right)$$

to conclude.

To do so we observe first of all that the surjective map

$$\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\twoheadrightarrow\{\pm1\}\times\mathbb{A}_{\mathbb{Q}}^{\infty,\times}\qquad (z,A)\mapsto(\mathrm{sign}(\mathfrak{Im}(z)),\det(A))$$

induces a surjective map $\psi\colon\mathrm{GL}_2(\mathbb{Q})\backslash\left(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K^{\infty}\right)\twoheadrightarrow\mathbb{Q}^{\times}\backslash\left(\{\pm1\}\times\mathbb{A}_{\mathbb{Q}}^{\infty,\times}/\det(K^{\infty})\right)$. Let now

$$\begin{aligned}\pi&\colon\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\twoheadrightarrow\mathrm{GL}_2(\mathbb{Q})\backslash\left(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K^{\infty}\right)\\ \pi'&\colon\{\pm1\}\times\mathbb{A}_{\mathbb{Q}}^{\infty,\times}\twoheadrightarrow\mathbb{Q}^{\times}\backslash\{\pm1\}\times\mathbb{A}_{\mathbb{Q}}^{\infty,\times}/\det(K^{\infty}\end{aligned}$$

be the quotient maps and observe that for every $x\in\mathbb{A}_{\mathbb{Q}}^{\infty,\times}$ we have

$$\psi^{-1}(\pi'(+1,x)) = \mathrm{SL}_2(\mathbb{Q})\backslash\left(\mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cdot A/(\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cap K^{\infty})\right)$$

where $A\in\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is any matrix such that $\det(A)=x$.

Indeed let G be any group and let X and Y be two sets with a left action of G. Let moreover $f\colon X\to Y$ be a map such that $f(g*x)=g*f(x)$ for every $x\in X$ and $g\in G$. Then f induces a map

$$\overline{f}\colon G\backslash X\to G\backslash Y\qquad\text{and for every } y\in Y \text{ we have } \left(\overline{f}\right)^{-1}(G\cdot y)=G_y\backslash f^{-1}(y)$$

where $G_y=\{g\in G\mid g*y=y\}\leq G$ is the stabilizer of y in G, and the analogous statement is true if G acts from the right on X and Y. We can first of all apply this statement to the map of left $\mathrm{GL}_2(\mathbb{Q})$-sets

$$\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\to\{\pm1\}\times\mathbb{A}_{\mathbb{Q}}^{\infty,\times}\qquad (z,A)\mapsto(\mathrm{sign}(\mathfrak{Im}(z)),\det(A))$$

to conclude that for every $x \in \mathbb{A}_{\mathbb{Q}}^{\infty,\times}$ the fiber of the induced map

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathfrak{h}^{\pm} \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \to \mathbb{Q}^{\times}\backslash\{\pm 1\} \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} \tag{2.4}$$

over $\mathbb{Q}^{\times}\cdot(+1,x)$ is given by $\mathrm{SL}_2(\mathbb{Q})\backslash\mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cdot A$, where $A \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is any matrix such that $\det(A) = x$. Now if we view the map (2.4) as a map between two right K^{∞}-sets we see immediately that for every $x \in \mathbb{A}_{\mathbb{Q}}^{\infty,\times}$ the stabilizer of $\mathbb{Q}^{\times}\cdot(+1,x)$ in K^{∞} is given by $K^{\infty} \cap \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and thus the fiber over $\pi'(+1,x)$ of ψ is indeed given by

$$\mathrm{SL}_2(\mathbb{Q})\backslash\Big(\mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cdot A/(\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cap K^{\infty})\Big)$$

where $A \in \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is any matrix such that $\det(A) = x$.

To conclude it is now sufficient to recall that

$$\mathbb{Q}^{\times}\backslash\Big(\{\pm 1\}\times\mathbb{A}_{\mathbb{Q}}^{\infty,\times}/\det(K^{\infty})\Big) \cong \frac{\widehat{\mathbb{Z}}^{\times}}{\det(K^{\infty})}$$

as we have proved in Lemma 2.7, which implies that the fibers of ψ are both open and closed, because the topological space $\mathbb{Q}^{\times}\backslash\Big(\{\pm 1\}\times\mathbb{A}_{\mathbb{Q}}^{\infty,\times}/\det(K^{\infty})\Big)$ is discrete. This implies in turn that the fibers of ψ are the connected components of $\mathrm{GL}_2(\mathbb{Q})\backslash\Big(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K^{\infty}\Big)$ because we have proved that they are homeomorphic to $\Gamma_j\backslash\mathfrak{h}$ and thus they are connected. This shows indeed that

$$\bigsqcup_{j=1}^{n}\Gamma_j\backslash\mathfrak{h} \cong \bigsqcup_{j=1}^{n}\mathrm{SL}_2(\mathbb{Q})\backslash\Big(\mathfrak{h}\times\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cdot A_j/(\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cap K^{\infty})\Big) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\Big(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K^{\infty}\Big)$$

as we wanted to prove. □

We can now apply Theorem 2.9 to the group $K^{\infty} = K(n) = \ker\Big(\mathrm{GL}_2(\widehat{\mathbb{Z}}) \twoheadrightarrow \mathrm{GL}_2\big(\mathbb{Z}/n\mathbb{Z}\big)\Big)$ to get a homeomorphism

$$\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^{\times}\times Y(n) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\Big(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K(n)\Big).$$

In the same way we can define the subgroups

$$K_1(n) \overset{\text{def}}{=} \widehat{\pi}_n^{-1}\left(\left\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\right\}\right) \qquad \text{and} \qquad K_0(n) \overset{\text{def}}{=} \widehat{\pi}_n^{-1}\left(\left\{\left(\begin{smallmatrix}* & *\\ 0 & *\end{smallmatrix}\right)\right\}\right)$$

where $\widehat{\pi}_n\colon \mathrm{SL}_2(\widehat{\mathbb{Z}}) \twoheadrightarrow \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})$ is the reduction map. If we do so we obtain two homeomorphisms

$$\left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^{\times}\times Y_1(n) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\Big(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K_1(n)\Big) \qquad Y_0(n) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\Big(\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/K(n)\Big).$$

Observe now that the adelic quotients $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K$ are defined over the rational numbers, whereas the curves $\Gamma\backslash\mathfrak{h}$ are usually defined over extensions of the rational numbers. In particular if $\pi_n\colon\mathrm{SL}_2(\mathbb{Z})\twoheadrightarrow\mathrm{SL}_2\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)$ is the reduction map, $H\leq\mathrm{GL}_2\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)$ is any subgroup and $\Gamma_H\stackrel{\text{def}}{=}\pi_n^{-1}\left(H\cap\mathrm{SL}_2\left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)\right)$ then the Riemann surface $\Gamma_H\backslash\mathfrak{h}$ can be viewed as an affine curve naturally defined over $\mathbb{Q}(\zeta_n)^{\det(H)}$. Thus the curves $Y(n)$ and $Y_1(n)$ are naturally defined over $\mathbb{Q}(\zeta_n)$ whereas $Y_0(n)$ is naturally defined over $\mathbb{Q}$.

Observe finally that the product $\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ can also be described in a completely adelic way. To show this we need a general lemma which concerns the action $G\circlearrowright X$ of a topological group G on a topological space X.

Lemma 2.10. *Suppose that X is a locally compact and Hausdorff topological space, and that G is a locally compact and second countable topological group. Suppose moreover that G acts continuously and transitively on X. Then for every $x\in X$ the map*

$$G/G_x\to X\qquad [g]\mapsto g*x\qquad \textit{where}\qquad G_x\stackrel{\text{def}}{=}\{g\in G\mid g*x=x\}$$

is a homeomorphism.

Proof. See **Theorem 1.1** of [15]. □

Consider now the action $\mathrm{GL}_2(\mathbb{R})\circlearrowright\mathfrak{h}^{\pm}$ and observe first of all that the map

$$\mathbb{R}_{>0}\times\mathrm{SO}_2(\mathbb{R})\to\mathrm{GL}_2(\mathbb{R})_i=\{A\in\mathrm{GL}_2(\mathbb{R})\mid A*i=i\}\qquad(\alpha,M)\mapsto\alpha\cdot M$$

is an isomorphism, and so we obtain that

$$\mathfrak{h}^{\pm}\cong{}^{\mathrm{GL}_2(\mathbb{R})}/_{K_\infty}\qquad\text{where}\qquad K_\infty\stackrel{\text{def}}{=}\mathbb{R}_{>0}\times\mathrm{SO}_2(\mathbb{R})$$

by applying the previous Lemma 2.10. Observe finally that $\mathbb{A}_{\mathbb{Q}}=\mathbb{R}\times\mathbb{A}_{\mathbb{Q}}^{\infty}$ and so we have

$$\mathfrak{h}^{\pm}\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cong(\mathrm{GL}_2(\mathbb{R})/K_\infty)\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})\cong\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_\infty\tag{2.5}$$

since $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})\cong\mathrm{GL}_2(\mathbb{R})\times\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Thus for every compact and open subgroup $K^{\infty}\leq\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ we have

$$Y_{K^\infty}\cong\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K$$

where $K=K^{\infty}\times K_{\infty}$.

To conclude we observe that we have a homeomorphism $Y \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_\infty$, where

$$Y \overset{\text{def}}{=} \varprojlim_{K^\infty} Y_{K^\infty} \cong \varprojlim_{n} Y_{K(n)} \cong \varprojlim_{n} \left(\frac{\mathbb{Z}}{n\mathbb{Z}}\right)^{\times} \times Y(n) \cong \widehat{\mathbb{Z}}^{\times} \times \varprojlim_{n} Y(n)$$

is the projective limit of all the affine modular curves. To prove this we will use the following general lemma.

Lemma 2.11. *Let G be a topological group which acts continuously from the right on a T1 topological space X, and let $\{G_i\}_{i\in\mathcal{I}}$ be an inverse system of compact subgroups of G. Then the natural map*

$$\frac{X}{\bigcap G_i} \to \varprojlim_{\mathcal{I}} \frac{X}{G_i} \qquad [x] \mapsto (x)_{i\in\mathcal{I}} \tag{2.6}$$

is bijective.

Proof. For every $x, y \in X$ we have $(x)_{i\in\mathcal{I}} = (y)_{i\in\mathcal{I}}$ in $\varprojlim X/G_i$ if and only if for every $i \in \mathcal{I}$ there exists $g_i \in G_i$ such that $x*g_i = y$. This implies that the sets $\{g \in G_i \mid x*g = y\}$ form an inverse system of non-empty subsets of G which are also compact because the subgroups G_i are compact and the points of X are closed. Thus using *Cantor's intersection theorem* we obtain that $(x)_{i\in\mathcal{I}} = (y)_{i\in\mathcal{I}}$ if and only if there exists $g \in \bigcap G_i$ such that $x*g = y$, which proves that the map (2.6) is injective.

Let now $([x_i]) \in \varprojlim X/G_i$. Since the action $X \circlearrowleft G$ is continuous and all the subgroups $G_i \leq G$ are compact the orbits $\{x_i * G_i\}$ form an inverse system of non-empty compact subsets of X. Thus using again *Cantor's intersection theorem* we can take $x \in \bigcap x_i * G_i$ and observe that the map (2.6) sends $[x] \in X/\bigcap G_i$ to $([x_i]) \in \varprojlim X/G_i$. □

Now it is sufficient to apply Lemma 2.11 to $X = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_\infty$ to obtain that

$$\varprojlim \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_\infty \times K^\infty \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})/K_\infty.$$

Indeed $\bigcap K^\infty = \left\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right\}$ because $\bigcap n\widehat{\mathbb{Z}} = \{0\}$ and thus $\bigcap K(n) = \left\{\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)\right\}$.

Moreover, for every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ we have that $Y_{K^\infty} \cong Y/K^\infty$. These results are precisely the results that we would like to generalize to compact modular curves, and the next chapter is entirely devoted to this aim.

CHAPTER 3

ADELIC DESCRIPTIONS OF COMPACTIFIED MODULAR CURVES

One day I will find the right words, and they will be simple.

Jack Kerouac

Chapter Abstract

In this chapter we give a completely adelic description of the classical set of cusps $\Gamma\backslash\mathcal{P}_\Gamma$ and of the topological spaces C_Γ^{BS} and X_Γ^{BS} associated to the Borel-Serre compactification of the affine modular curve Y_Γ.

3.1 Equivalence classes of cusps and spaces of finite adèles

The first section of this chapter is devoted to giving an adelic description of the set $\Gamma\backslash\mathcal{P}_\Gamma$, where $\Gamma \leq \mathrm{SL}_2(\mathbb{Q})$ is a *congruence subgroup* in the sense of the following definition.

Definition 3.1. A subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Q})$ is a *congruence subgroup* if there exists a compact and open subgroup $U \leq \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $\Gamma = U \cap \mathrm{SL}_2(\mathbb{Q})$.

It is important to recall that every congruence subgroup of $\mathrm{SL}_2(\mathbb{Q})$ is an arithmetic subgroup (i.e. it is commensurable with $\mathrm{SL}_2(\mathbb{Z})$), but the converse is not true. Moreover every congruence subgroup contains a principal subgroup $\Gamma(n)$ for some $n \in \mathbb{N}$, as we proved in Theorem 2.9. This implies that

$$\mathrm{cusps}(\Gamma) = \mathrm{cusps}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{Q} \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q})$$

as it is proved in **Proposition 1.30** and on **page 14** of [15].

Let now $K^\infty \le \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ be a compact and open subgroup and define

$$C_{K^\infty} \stackrel{\text{def}}{=} \bigsqcup_{j=1}^{n} C_{\Gamma_j} = \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) \qquad \text{where} \qquad \Gamma_j \stackrel{\text{def}}{=} A_j \cdot K^\infty \cdot A_j^{-1} \cap \mathrm{SL}_2(\mathbb{Q})$$

and $\{A_j\}_{j=1}^n \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is any set of matrices such that $\{\det(A_j)\}_{j=1}^n \subseteq \widehat{\mathbb{Z}}^\times$ is a set of representatives for the quotient ${}^{\widehat{\mathbb{Z}}^\times}/_{\det(K^\infty)}$. The aim of this section is to define a topological space $\mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty)$ with two actions $\mathrm{GL}_2(\mathbb{Q}) \circlearrowright \mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty) \circlearrowleft \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ such that $C_{K^\infty} \cong C/K^\infty$ where $C \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty)$. Observe that we could take $\mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty) = (\mathbb{P}^1(\mathbb{Q}) \times \{\pm 1\}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ as follows easily from Lemma 3.11, where on $\mathbb{P}^1(\mathbb{Q})$ we take the discrete topology. The problem with this definition is that the quotient $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ is not locally compact. Thus to give a better definition of $\mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty)$ we will introduce some new notation in the following pages.

Definition 3.2. Let R be a commutative ring. We denote by R^2_{prim} the set of all $\binom{a}{b} \in R^2$ such that $Ra + Rb = R$.

Observe that we have a left action $\mathrm{GL}_2(R) \circlearrowright R^2_{\mathrm{prim}}$ induced by the natural action $\mathrm{GL}_2(R) \circlearrowright R^2$. Indeed if $\binom{x}{y} \in R^2_{\mathrm{prim}}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$ then $ad - bc \in R^\times$ and there exist $r, s \in R$ such that $rx + sy = 1$, which implies that

$$\left(\frac{rd - sc}{ad - bc}\right) \cdot (ax + by) + \left(\frac{sa - rb}{ad - bc}\right) \cdot (cx + dy) = 1$$

and thus that $\binom{ax+by}{cx+dy} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \binom{x}{y} \in R^2_{\mathrm{prim}}$. Observe finally that we clearly have a right action $R^2_{\mathrm{prim}} \circlearrowleft R^\times$ defined as $\binom{x}{y} * \alpha = \binom{\alpha \cdot x}{\alpha \cdot y}$.

It is a problem in general to define an appropriate topology on R^2_{prim} if R is a topological ring. Nevertheless if K is a global field and $S \subseteq \Sigma_K$ is finite and non-empty we see that

$$(\mathbb{A}_K^S)^2_{\text{prim}} = \prod_{v \in \Sigma_K^S}{}' ((K_v)^2_{\text{prim}} : (\mathcal{O}_{K_v})^2_{\text{prim}})$$

as sets. For every $v \in \Sigma_K$ we define the topology on $(K_v)^2_{\text{prim}}$ to be the subspace topology from K_v^2 and then we define the topology on $(\mathbb{A}_K^S)^2_{\text{prim}}$ as the restricted product topology.

Definition 3.3. For every ring R and every $n, m \in \mathbb{N}_{\geq 1}$ we denote with $\mathcal{M}_{n,m}(R)$ the set of $n \times m$ matrices with coefficients in R. If R is a topological ring we endow $\mathcal{M}_{n,m}(R)$ with the topology induced by the bijection $\mathcal{M}_{n,m}(R) \leftrightarrow R^{n \cdot m}$.

Definition 3.4. Let R be a commutative $\mathbb{Q}$-algebra, i.e. a commutative ring with unity with a morphism of rings $\mathbb{Q} \to R$. We define the set $W(R)$ as

$$W(R) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}_{2,2}(R) \colon Ra + Rb + Rc + Rd = R \text{ and there exists } (\alpha\colon\beta) \in \mathbb{P}^1(\mathbb{Q}) \\ \text{such that } \alpha \cdot (a,b) + \beta \cdot (c,d) = (0,0)\}.$$

Observe that we have two actions $\mathrm{GL}_2(\mathbb{Q}) \circlearrowright W(R) \circlearrowleft \mathrm{GL}_2(R)$. Indeed let $\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \in W(R)$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ and $\left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$, and suppose that

$$\alpha \cdot (a,b) + \beta \cdot (c,d) = (0,0)$$

for some $(\alpha \cdot \beta) \in \mathbb{P}^1(\mathbb{Q})$. Then

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \left(\begin{smallmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{smallmatrix}\right) \qquad \text{and} \qquad \left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} e & f \\ g & h \end{smallmatrix}\right) = \left(\begin{smallmatrix} ex+gy & fx+hy \\ ez+gw & fz+hw \end{smallmatrix}\right)$$

and clearly $\alpha \cdot (ex + gy, fx + hy) + \beta \cdot (ez + gw, fz + hw) = (0,0)$, which implies that $\mathrm{GL}_2(R)$ acts from the right on $W(R)$. Moreover if $\left(\begin{smallmatrix} \gamma \\ \delta \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1} \cdot \left(\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right) \in \mathbb{Q}^2$ then it is easy to prove that $\gamma \cdot (ax + bz, ay + bw) + \delta \cdot (cx + dz, cy + dw) = (0,0)$, which implies that $\mathrm{GL}_2(\mathbb{Q})$ acts from the left on $W(R)$.

The problem is now to give the correct topology to the set $W(R)$. Observe first of all that for every matrix $A \in W(R)$ there exists a unique $(\alpha\colon\beta) \in \mathbb{P}^1(\mathbb{Q})$ such that $(\alpha, \beta) \cdot A = (0,0)$. This implies that

$$W(R) = \bigsqcup_{(\alpha:\beta) \in \mathbb{P}^1(\mathbb{Q})} \left\{ \left(\begin{smallmatrix} \alpha x & \alpha y \\ \beta x & \beta y \end{smallmatrix}\right) \colon \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in R^2_{\text{prim}} \right\}$$

and thus that we have a bijection $W(R) \leftrightarrow \bigsqcup_{\mathbb{P}^1(\mathbb{Q})} R^2_{\text{prim}}$. If we have defined a suitable topology on R^2_{prim} (as we did before when $R = \mathbb{A}^S_K$) we define the topology on $W(R)$ as the unique topology such that this bijection is a homeomorphism.

We are now almost ready to introduce the fundamental result of this section. We would like to start from any compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}^\infty_\mathbb{Q})$ but in the current context we needed to restrict ourselves to integral subgroups, i.e. to subgroups $K^\infty \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$. This is not a problem as we prove in the following lemma.

Lemma 3.5. *Let $K^\infty \leq \mathrm{GL}_2(\mathbb{A}^\infty_\mathbb{Q})$ be a compact and open subgroup. Then there exists a matrix $A \in \mathrm{GL}_2(\mathbb{Q})$ such that $A \cdot K^\infty \cdot A^{-1} \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$. Moreover if we consider any matrix $B \in \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and we define*

$$\Gamma \overset{\text{def}}{=} B \cdot K^\infty \cdot B^{-1} \cap \mathrm{SL}_2(\mathbb{Q}) \qquad \textit{and} \qquad \Gamma' \overset{\text{def}}{=} B \cdot (A \cdot K^\infty \cdot A^{-1}) \cdot B^{-1} \cap \mathrm{SL}_2(\mathbb{Q})$$

then the maps

$$\begin{matrix} \mathfrak{h} \to \mathfrak{h} \\ \mathbb{P}^1(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q}) \end{matrix} \qquad \textit{and} \qquad \begin{matrix} \mathfrak{h}^* \to \mathfrak{h}^* \\ \mathfrak{h}^{**} \to \mathfrak{h}^{**} \end{matrix} \qquad \textit{defined as} \qquad x \mapsto A * x$$

induce isomorphisms $Y_\Gamma \xrightarrow{\sim} Y_{\Gamma'}$, $\Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \xrightarrow{\sim} \Gamma' \backslash \mathbb{P}^1(\mathbb{Q})$, $X_\Gamma \xrightarrow{\sim} X_{\Gamma'}$ and $X^{BS}_\Gamma \xrightarrow{\sim} X^{BS}_{\Gamma'}$.

Proof. We can write $K^\infty = \prod_p K^\infty_p$ where $K^\infty_p \leq \mathrm{GL}_2(\mathbb{Q}_p)$ for every prime number $p \in \mathbb{N}$ and since K^∞ is compact and open we have that $K^\infty_p = \mathrm{GL}_2(\mathbb{Z}_p)$ for all but finitely many prime numbers $\{p_1, \ldots, p_n\} \subseteq \mathbb{Z}$. It is now not difficult to prove that for every $j \in \{1, \ldots, n\}$ there exists a matrix $A_j \in \mathrm{GL}_2(\mathbb{Q})$ such that $A_j \in \mathrm{GL}_2(\mathbb{Z}_q)$ for every prime $q \neq p_j$ and $A_j \cdot K^\infty_{p_j} \cdot A_j^{-1} \leq \mathrm{GL}_2(\mathbb{Z}_{p_j})$. This implies that we can take $A = A_1 \cdots A_n$ to have $A \cdot K^\infty \cdot A^{-1} \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$. The rest of the proof is straightforward. $\square$

Using the previous lemma we can assume that $K^\infty \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ without loss of generality, and we can prove the following results which describe the set C_{K^∞} as a double quotient of the adelic space $W(\mathbb{A}^\infty_\mathbb{Q}) \times \mathbb{A}^{\infty,\times}_\mathbb{Q}$. Observe that all the homeomorphisms in the proofs of Lemma 3.6 and Theorem 3.7 are between discrete topological spaces: it is thus sufficient to prove that these maps are bijective to prove that they are homeomorhpisms.

Lemma 3.6. *Let $U \le \mathrm{SL}_2(\widehat{\mathbb{Z}})$ be a compact and open subgroup and let $\Gamma \overset{\text{def}}{=} U \cap \mathrm{SL}_2(\mathbb{Q})$. Then the inclusion map $\mathbb{Q} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{\infty}$ induces homeomorphisms*

$$\Gamma\backslash\mathbb{P}^1(\mathbb{Q}) \xrightarrow{\sim} U\backslash(\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{prim}/\mathbb{Q}^{\times} \cong U\backslash\widehat{\mathbb{Z}}^2_{prim}/\{\pm 1\}.$$

Proof. Observe first of all that $\Gamma \le \mathrm{SL}_2(\mathbb{Z})$ because $\mathrm{SL}_2(\mathbb{Q}) \cap \mathrm{SL}_2(\widehat{\mathbb{Z}}) = \mathrm{SL}_2(\mathbb{Z})$. Moreover since the action $\mathrm{SL}_2(\mathbb{Q}) \circlearrowright \mathbb{P}^1(\mathbb{Q})$ is transitive we can apply Lemma 2.10 to show that the map

$$\mathrm{SL}_2(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q}) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} * \infty = (a\colon c)$$

induces a homeomorphism $\mathrm{SL}_2(\mathbb{Q})/\mathrm{SL}_2(\mathbb{Q})_\infty \xrightarrow{\sim} \mathbb{P}^1(\mathbb{Q})$ where

$$\mathrm{SL}_2(\mathbb{Q})_\infty \overset{\text{def}}{=} \{A \in \mathrm{SL}_2(\mathbb{Q}) \mid A * \infty = \infty\} = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Q}) \mid c = 0\right\} = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \mid a \in \mathbb{Q}^{\times},\ b \in \mathbb{Q}\right\}$$

is the stabilizer of $\{\infty\}$ under the action $\mathrm{SL}_2(\mathbb{Q}) \circlearrowright \mathbb{P}^1(\mathbb{Q})$ and the topologies on $\mathrm{SL}_2(\mathbb{Q})$ and $\mathbb{P}^1(\mathbb{Q})$ are discrete. Observe now that the inclusion map $\mathrm{SL}_2(\mathbb{Q}) \hookrightarrow \mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ induces a homeomorphism

$$\Gamma\backslash\mathrm{SL}_2(\mathbb{Q})/\mathrm{SL}_2(\mathbb{Q})_\infty \xrightarrow{\sim} U\backslash\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/\mathrm{SL}_2(\mathbb{Q})_\infty$$

because $\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) = U \cdot \mathrm{SL}_2(\mathbb{Q})$ as we proved in Lemma 2.5. Observe moreover that the map

$$\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \to (\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{\text{prim}} \qquad \text{defined as} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}$$

induces a homeomorphism $U\backslash\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})/\mathrm{SL}_2(\mathbb{Q})_\infty \xrightarrow{\sim} U\backslash(\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{\text{prim}}/\mathbb{Q}^{\times}$. Observe finally that the inclusion map $\widehat{\mathbb{Z}} \hookrightarrow \mathbb{A}_{\mathbb{Q}}^{\infty}$ induces a homeomorphism $\widehat{\mathbb{Z}}^2_{\text{prim}}/\{\pm 1\} \cong (\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{\text{prim}}/\mathbb{Q}^{\times}$. Using all the homeomorphisms that we defined in this proof we finally obtain two homeomorphisms

$$\Gamma\backslash\mathbb{P}^1(\mathbb{Q}) \xrightarrow{\sim} U\backslash(\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{\text{prim}}/\{\pm 1\} \xrightarrow{\sim} U\backslash\widehat{\mathbb{Z}}^2_{\text{prim}}/\{\pm 1\}$$

such that the first one is induced by the inclusion $\mathbb{Q}^2_{\text{prim}} \hookrightarrow (\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{\text{prim}}$. □

Theorem 3.7. *Let $K^\infty \le \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be a compact and open subgroup and consider any minimal set of matrices $\{A_j\}_{j=1}^n \subseteq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that $\widehat{\mathbb{Z}}^{\times}/_{\det(K^\infty)} = \{\overline{\det(A_j)}\}_{j=1}^n$. Let now*

$$K_j^\infty \overset{\text{def}}{=} A_j \cdot K^\infty \cdot A_j^{-1} \cap \mathrm{SL}_2(\widehat{\mathbb{Z}}) \qquad \textit{and} \qquad \Gamma_j \overset{\text{def}}{=} \mathrm{SL}_2(\mathbb{Q}) \cap K_j^\infty$$

and define the map

$$(-)^{(j)} : (\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{prim} \to (\mathbb{A}_{\mathbb{Q}}^{\infty})^2_{prim} \qquad \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \begin{pmatrix} x^{(j)} \\ y^{(j)} \end{pmatrix} \stackrel{\text{def}}{=} A_j^{-1} \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right).$$

Then the map

$$\bigsqcup_{j=1}^{n} \mathbb{Q}^2_{prim} \to W(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} \qquad \textit{defined as} \qquad \left(j, \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right) \mapsto \left(\det(A_j)^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -y^{(j)} & x^{(j)} \end{smallmatrix}\right), \det(A_j)\right)$$

induces a homeomorphism $C_{K^\infty} \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q}) \backslash \left(W(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} \right) / K^\infty$*, where the actions*

$$\mathrm{GL}_2(\mathbb{Q}) \circlearrowright W(\mathbb{A}_{\mathbb{Q}}^{\infty}) \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} \circlearrowleft K^\infty$$

are defined as $A * (X, \alpha) * B = (A \cdot X \cdot B, \det(A) \cdot \alpha \cdot \det(B))$.

Proof. Using the previous Lemma 3.6 we obtain immediately a homeomorphism

$$\bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) = \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathbb{Z}^2_{\mathrm{prim}} / \{\pm 1\} \xrightarrow{\sim} \bigsqcup_{j=1}^{n} K_j^\infty \backslash \widehat{\mathbb{Z}}^2_{\mathrm{prim}} / \{\pm 1\}$$

induced by the inclusion $\mathbb{Z}^2_{\mathrm{prim}} \hookrightarrow \widehat{\mathbb{Z}}^2_{\mathrm{prim}}$.

Observe now that the map

$$\bigsqcup_{j=1}^{n} \widehat{\mathbb{Z}}^2_{\mathrm{prim}} \to \widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times \qquad (j, \mathbf{v}) \mapsto \left(A_j^{-1} \cdot \mathbf{v}, \det(A_j)\right) \tag{3.1}$$

induces a homeomorphism $\bigsqcup_{j=1}^{n} K_j^\infty \backslash \widehat{\mathbb{Z}}^2_{\mathrm{prim}} / \{\pm 1\} \xrightarrow{\sim} K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}} / \{\pm 1\} \times \widehat{\mathbb{Z}}^\times \right)$, where the action $K^\infty \circlearrowright \widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times$ is defined as $A * (\mathbf{v}, \alpha) = (A \cdot \mathbf{v}, \det(A) \cdot \alpha)$. Indeed let $(i, \mathbf{v}), (j, \mathbf{w}) \in \bigsqcup_{j=1}^{n} \widehat{\mathbb{Z}}^2_{\mathrm{prim}}$ and suppose that

$$\varepsilon \cdot M \cdot A_i^{-1} \cdot \mathbf{v} = A_j^{-1} \cdot \mathbf{w} \qquad \text{and} \qquad \det(A_i) = \det(M) \cdot \det(A_j)$$

for some $M \in K^\infty$ and $\varepsilon \in \{\pm 1\}$. The second equation implies that $i = j$ and $\det(M) = 1$ because by hypothesis $\{\overline{\det(A_j)}\}_{j=1}^{n}$ is a set of representatives for the quotient $\widehat{\mathbb{Z}}^\times / _{\det(K^\infty)}$. The first equation can thus be written as $\mathbf{v} = (A_j \cdot M \cdot A_j^{-1}) \cdot \mathbf{w} \cdot \varepsilon$ and we have $A_j \cdot M \cdot A_j^{-1} \in K_j^\infty$ because $\det(A_j \cdot M \cdot A_j^{-1}) = \det(M) = 1$. This implies that the map (3.1) induces a well defined and injective map $\bigsqcup_{j=1}^{n} K_j^\infty \backslash \widehat{\mathbb{Z}}^2_{\mathrm{prim}} / \{\pm 1\} \hookrightarrow K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}} / \{\pm 1\} \times \widehat{\mathbb{Z}}^\times \right)$. Let now $(\mathbf{v}, \alpha) \in \widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times$. Since again by definition we have that $\widehat{\mathbb{Z}}^\times / _{\det(K^\infty)} = \{\overline{\det(A_j)}\}_{j=1}^{n}$ there exists a matrix $M \in K^\infty$ such that $\det(M)\cdot\alpha = \det(A_j)$ for a unique $j \in \{1,\dots,n\}$. Thus the map

$$\bigsqcup_{j=1}^{n} \widehat{\mathbb{Z}}^2_{\mathrm{prim}} \to K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right)$$

given by the composition of (3.1) with the quotient map

$$\widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times \twoheadrightarrow K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right)$$

sends $(j, A_j \cdot M \cdot \mathbf{v})$ to the equivalence class of $(\mathbf{v}, \alpha)$ in the quotient $K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right)$. This implies that the injective map $\bigsqcup_{j=1}^{n} K_j^\infty \backslash \widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \hookrightarrow K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right)$ induced by (3.1) is also surjective, and thus it is a homeomorphism because it is clearly continuous and open.

To conclude it is sufficient to prove that the map

$$\widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times \to W(\mathbb{A}_{\mathbb{Q}}^\infty) \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times} \qquad \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \alpha\right) \mapsto \left(\alpha^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -y & x \end{smallmatrix}\right), \alpha\right) \tag{3.2}$$

induces a homeomorphism $K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q}) \backslash W(\mathbb{A}_{\mathbb{Q}}^\infty) \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times}/K^\infty$. Indeed let $\left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \alpha\right), \left(\left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right), \beta\right) \in \widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times$ and suppose that

$$A \cdot \alpha^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -y & x \end{smallmatrix}\right) \cdot B = \beta^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -w & z \end{smallmatrix}\right) \qquad \text{and} \qquad \det(A) \cdot \alpha \cdot \det(B) = \beta$$

for some $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ and $B = \left(\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}\right) \in K^\infty$. The first equation implies that $a_2 = 0$ and the second equation implies that $\det(A) = a_1 \cdot a_4 \in \mathbb{Q}^\times \cap \widehat{\mathbb{Z}}^\times = \mathbb{Z}^\times = \{\pm 1\}$. Moreover the first equation can be written as

$$\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = a_4 \cdot \alpha \cdot \beta^{-1} \cdot \left(\begin{smallmatrix} b_4 & -b_2 \\ -b_3 & b_1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) = a_1^{-1} \cdot B^{-1} \cdot \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \qquad \text{with} \qquad \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), B^{-1} \cdot \left(\begin{smallmatrix} z \\ w \end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2_{\mathrm{prim}}$$

which implies that $a_1 \in \mathbb{Q}^\times \cap \widehat{\mathbb{Z}}^\times = \{\pm 1\}$ and thus that the map (3.2) induces a well defined and injective map $K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right) \hookrightarrow \mathrm{GL}_2(\mathbb{Q}) \backslash \left(W(\mathbb{A}_{\mathbb{Q}}^\infty) \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times}\right)/K^\infty$. Let now $(M, \alpha) \in W(\mathbb{A}_{\mathbb{Q}}^\infty) \times \mathbb{A}_{\mathbb{Q}}^{\infty,\times}$. By the definition of $W(\mathbb{A}_{\mathbb{Q}}^\infty)$ there exists $(a, b) \in \mathbb{P}^1(\mathbb{Q})$ such that $(a, b) \cdot M = (0, 0)$. We also know that there exists $c \in \mathbb{Q}^\times$ such that $c \cdot \alpha \in \widehat{\mathbb{Z}}^\times$. Thus if we define $A = \left(\begin{smallmatrix} a & b \\ 0 & ca^{-1} \end{smallmatrix}\right)$ if $a \neq 0$ and $A = \left(\begin{smallmatrix} 0 & -c \\ 1 & 0 \end{smallmatrix}\right)$ if $a = 0$ then $A \in \mathrm{GL}_2(\mathbb{Q})$ and

$$A * (M, \alpha) = \left(\left(\begin{smallmatrix} 0 & 0 \\ x & y \end{smallmatrix}\right), \beta\right)$$

with $\beta \in \widehat{\mathbb{Z}}^\times$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in (\mathbb{A}_{\mathbb{Q}}^\infty)^2_{\mathrm{prim}}$ by the definition of $W(\mathbb{A}_{\mathbb{Q}}^\infty)$. If we now take $d \in \mathbb{Q}^\times$ such that $d \cdot \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2_{\mathrm{prim}}$ and we define $A' = \left(\begin{smallmatrix} d^{-1} & 0 \\ 0 & d \end{smallmatrix}\right) \cdot A \in \mathrm{GL}_2(\mathbb{Q})$ then

$$A' * (M, \alpha) = \left(\left(\begin{smallmatrix} 0 & 0 \\ z & w \end{smallmatrix}\right), \beta\right)$$

with $\binom{z}{w} = d \cdot \binom{x}{y} \in \widehat{\mathbb{Z}}^2_{\text{prim}}$. This shows that the map

$$\widehat{\mathbb{Z}}^2_{\text{prim}} \times \widehat{\mathbb{Z}}^\times \to \mathrm{GL}_2(\mathbb{Q})\backslash\Big(W(\mathbb{A}^\infty_\mathbb{Q}) \times \mathbb{A}^{\infty,\times}_\mathbb{Q}\Big)/K^\infty$$

given by the composition of (3.2) and the quotient map

$$W(\mathbb{A}^\infty_\mathbb{Q}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash\Big(W(\mathbb{A}^\infty_\mathbb{Q}) \times \mathbb{A}^{\infty,\times}_\mathbb{Q}\Big)/K^\infty$$

sends $(\beta \cdot \binom{w}{-z}, \beta)$ to the class of (M, α) in $\mathrm{GL}_2(\mathbb{Q})\backslash\Big(W(\mathbb{A}^\infty_\mathbb{Q}) \times \mathbb{A}^{\infty,\times}_\mathbb{Q}\Big)/K^\infty$. This implies that the map $K^\infty\backslash\Big(\widehat{\mathbb{Z}}^2_{\text{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\Big) \hookrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash\Big(W(\mathbb{A}^\infty_\mathbb{Q}) \times \mathbb{A}^{\infty,\times}_\mathbb{Q}\Big)/K^\infty$ induced by (3.2) is a homeomorphism, because it is bijective and it is clearly continuous and open. □

What we have proved shows that if we define $\mathcal{W}(R) \stackrel{\text{def}}{=} W(R) \times R^\times$ for every topological commutative $\mathbb{Q}$-algebra R then we have that

$$C_{K^\infty} \cong C/K^\infty \qquad \text{where} \qquad C \stackrel{\text{def}}{=} \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{W}(\mathbb{A}^\infty_\mathbb{Q})$$

for every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}^\infty_\mathbb{Q})$.

3.2 A real problem

Our aim for this section is now to replace the ring of the finite adéles $\mathbb{A}^\infty_\mathbb{Q}$ with $\mathbb{A}_\mathbb{Q}$ in what we have proved in section 3.1. In particular we will find a relationship between the set C_{K^∞} and the double quotient $\mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{W}(\mathbb{A}_\mathbb{Q})/K$ where $K = K^\infty \times K_\infty \subseteq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ with $K_\infty \stackrel{\text{def}}{=} \mathbb{R}_{>0} \cdot \mathrm{SO}_2(\mathbb{R}) \leq \mathrm{GL}_2(\mathbb{R})$. We did so in the aim to use the adelic description of Y_{K^∞} given in section 2.3 to obtain an adelic description of the Baily-Borel compactification $X_{K^\infty} \stackrel{\text{def}}{=} \bigsqcup_{j=1}^n \Gamma_j\backslash\mathfrak{h}^*$. This is not possible using the space $\mathcal{W}(\mathbb{A}_\mathbb{Q})$ because Theorem 3.8 tells us that the double quotient $\mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{W}(\mathbb{A}_\mathbb{Q})/K$ is not compact.

Theorem 3.8. *Let $K^\infty \leq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ be a compact and open subgroup and consider any minimal set of matrices $\{A_j\}_{j=1}^n \subseteq \mathrm{GL}_2(\mathbb{A}^\infty_\mathbb{Q})$ such that $\widehat{\mathbb{Z}}^\times/_{\det(K^\infty)} = \{\overline{\det(A_j)}\}_{j=1}^n$. Define now the groups $K_j^\infty \stackrel{\text{def}}{=} A_j \cdot K^\infty \cdot A_j^{-1} \cap \mathrm{SL}_2(\widehat{\mathbb{Z}})$ and $\Gamma_j \stackrel{\text{def}}{=} \mathrm{SL}_2(\mathbb{Q}) \cap K_j^\infty$ and the map*

$$(-)^{(j)} \colon (\mathbb{A}^\infty_\mathbb{Q})^2_{prim} \to (\mathbb{A}^\infty_\mathbb{Q})^2_{prim} \qquad \binom{x}{y} \mapsto \binom{x^{(j)}}{y^{(j)}} \stackrel{\text{def}}{=} A_j^{-1} \cdot \binom{x}{y}.$$

Then the map $\bigsqcup_{j=1}^n \mathbb{Q}^2_{prim} \times \mathbb{R}^2_{prim} \times \mathbb{R}_{>0} \to W(\mathbb{A}_\mathbb{Q}) \times \mathbb{A}_\mathbb{Q}^\times$ *defined as*

$$(j,(x,y),(z,w),t) \mapsto \left(\left(\det(A_j)^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -y^{(j)} & x^{(j)} \end{smallmatrix}\right), t^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -w & z \end{smallmatrix}\right)\right), \det(A_j) \cdot t\right)$$

induces a homeomorphism $\bigsqcup_{j=1}^n \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) \times \mathbb{R}_{>0} \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q})/K$.

Proof. Using Lemma 3.6 and the homeomorphism induced by (3.1) we obtain immediately a homeomorphism

$$\bigsqcup_{j=1}^n \Gamma_j \backslash \mathbb{P}^1(\mathbb{Q}) \times \mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0} \times K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right)$$

which is the identity on $\mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0}$. Observe moreover that we have a natural left action $K_\infty \circlearrowleft \mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0}$ defined as $A * (\mathbf{v}, t) = (A \cdot \mathbf{v}, \det(A) \cdot t)$ and that the inclusion

$$\mathbb{R}_{>0} \hookrightarrow \mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0} \qquad t \mapsto \left(\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), t\right)$$

induces a homeomorphism $\mathbb{R}_{>0} \xrightarrow{\sim} K_\infty \backslash \left(\mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0}\right)$.

Consider now the map

$$\begin{gathered} \widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \mathbb{R}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \to W(\mathbb{A}_\mathbb{Q}) \times \mathbb{A}_\mathbb{Q}^\times \\ \left(\left(\begin{smallmatrix} x^\infty \\ y^\infty \end{smallmatrix}\right), \left(\begin{smallmatrix} x_\infty \\ y_\infty \end{smallmatrix}\right), t^\infty, t_\infty\right) \mapsto \left(t^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -y & x \end{smallmatrix}\right), t\right) \end{gathered} \tag{3.3}$$

where $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\left(\begin{smallmatrix} x^\infty \\ y^\infty \end{smallmatrix}\right), \left(\begin{smallmatrix} x_\infty \\ y_\infty \end{smallmatrix}\right)\right) \in (\mathbb{A}_\mathbb{Q})^2_{\mathrm{prim}} = (\mathbb{A}_\mathbb{Q}^\infty)^2_{\mathrm{prim}} \times \mathbb{R}^2_{\mathrm{prim}}$ and $t = (t^\infty, t_\infty) \in \mathbb{A}_\mathbb{Q}^\times = \mathbb{A}_\mathbb{Q}^{\infty,\times} \times \mathbb{R}^\times$. To conclude we have to prove that (3.3) induces a homeomorphism

$$K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right) \times K_\infty \backslash \left(\mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0}\right) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q})/K.$$

Let $\left(\left(\begin{smallmatrix} x_\infty \\ y_\infty \end{smallmatrix}\right), t_\infty\right), \left(\left(\begin{smallmatrix} z_\infty \\ w_\infty \end{smallmatrix}\right), s_\infty\right) \in \mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0}$ and suppose that

$$A \cdot t_\infty^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -y_\infty & x_\infty \end{smallmatrix}\right) \cdot B = s_\infty^{-1} \cdot \left(\begin{smallmatrix} 0 & 0 \\ -w_\infty & x_\infty \end{smallmatrix}\right) \qquad \text{and} \qquad \det(A) \cdot t_\infty \cdot \det(B) = s_\infty$$

for some $A = \left(\begin{smallmatrix} a_1 & a_2 \\ a_3 & a_4 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ and $B \in K_\infty$. The first equation implies that $a_2 = 0$ and it is equivalent to the fact that

$$\left(\begin{smallmatrix} x_\infty \\ y_\infty \end{smallmatrix}\right) = a_1^{-1} \cdot B^{-1} \cdot \left(\begin{smallmatrix} z_\infty \\ w_\infty \end{smallmatrix}\right)$$

with $a_1^{-1} \cdot B^{-1} \in K_\infty$. This fact together with what we proved in Theorem 3.7 implies that the map (3.3) induces a well defined and injective map

$$K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right) \times K_\infty \backslash \left(\mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0}\right) \hookrightarrow \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q})/K.$$

Let now $(M,t) \in W(\mathbb{A}_\mathbb{Q}) \times \mathbb{A}_\mathbb{Q}^\times$ and suppose that $t = (t^\infty, t_\infty)$ with $t^\infty \in \mathbb{A}_\mathbb{Q}^{\infty,\times}$ and $t_\infty \in \mathbb{R}^\times$. We know by the definition of $W(\mathbb{A}_\mathbb{Q})$ that there exists $(a,b) \in \mathbb{P}^1(\mathbb{Q})$ such that $(a,b)\cdot M = (0,0)$. We also know that there exists $c \in \mathbb{Q}^\times$ such that $c \cdot t^\infty \in \widehat{\mathbb{Z}}^\times$ and $c \cdot t_\infty \in \mathbb{R}_{>0}$. Thus if we define $A = \left(\begin{smallmatrix} a & b \\ 0 & ca^{-1}\end{smallmatrix}\right)$ if $a \neq 0$ and $A = \left(\begin{smallmatrix} 0 & -c \\ 1 & 0\end{smallmatrix}\right)$ if $a = 0$ then $A \in \mathrm{GL}_2(\mathbb{Q})$ and

$$A * (M,t) = \left(\left(\begin{smallmatrix} 0 & 0 \\ x & y\end{smallmatrix}\right), s\right) \qquad \text{where} \qquad s = (s^\infty, s_\infty) \in \mathbb{A}_\mathbb{Q}^{\infty,\times} \times \mathbb{R}^\times$$

with $s^\infty \in \widehat{\mathbb{Z}}^\times$, $s_\infty \in \mathbb{R}_{>0}$ and $\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right) \in (\mathbb{A}_\mathbb{Q})^2_{\mathrm{prim}}$ by the definition of $W(\mathbb{A}_\mathbb{Q})$. Suppose now that $\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right) = \left(\left(\begin{smallmatrix} x^\infty \\ y^\infty\end{smallmatrix}\right), \left(\begin{smallmatrix} x_\infty \\ y_\infty\end{smallmatrix}\right)\right) \in (\mathbb{A}_\mathbb{Q}^\infty)^2_{\mathrm{prim}} \times \mathbb{R}^2_{\mathrm{prim}}$ and take $d \in \mathbb{Q}^\times$ such that $d \cdot \left(\begin{smallmatrix} x^\infty \\ y^\infty\end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2_{\mathrm{prim}}$. Then if we define $A' = \left(\begin{smallmatrix} d^{-1} & 0 \\ 0 & d\end{smallmatrix}\right) \cdot A \in \mathrm{GL}_2(\mathbb{Q})$ we have that

$$A' * (M, \alpha) = \left(\left(\begin{smallmatrix} 0 & 0 \\ z & w\end{smallmatrix}\right), s\right)$$

with $\left(\begin{smallmatrix} z \\ w\end{smallmatrix}\right) = d \cdot \left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right) \in \widehat{\mathbb{Z}}^2_{\mathrm{prim}}$. This shows that the map

$$\widehat{\mathbb{Z}}^2_{\mathrm{prim}} \times \mathbb{R}^2_{\mathrm{prim}} \times \widehat{\mathbb{Z}}^\times \times \mathbb{R}_{>0} \to \mathrm{GL}_2(\mathbb{Q})\backslash \mathcal{W}(\mathbb{A}_\mathbb{Q})/K$$

given by the composition of (3.3) with the quotient map $\mathcal{W}(\mathbb{A}_\mathbb{Q}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{Q})\backslash \mathcal{W}(\mathbb{A}_\mathbb{Q})/K$ sends $\left(s^\infty \cdot \left(\begin{smallmatrix} x^\infty \\ y^\infty\end{smallmatrix}\right), s_\infty \cdot \left(\begin{smallmatrix} x_\infty \\ y_\infty\end{smallmatrix}\right), s^\infty, s_\infty\right)$ to the class of (M,α) in $\mathrm{GL}_2(\mathbb{Q})\backslash \mathcal{W}(\mathbb{A}_\mathbb{Q})/K$. Observe finally that the maps

$$\begin{array}{cc} (\mathbb{A}_\mathbb{Q})^2_{\mathrm{prim}} \times \mathbb{A}_\mathbb{Q}^\times \to (\mathbb{A}_\mathbb{Q})^2_{\mathrm{prim}} & (\mathbb{A}_\mathbb{Q})^2_{\mathrm{prim}} \to W(\mathbb{A}_\mathbb{Q}) \\ \left(\left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right), t\right) \mapsto t^{-1} \cdot \left(\begin{smallmatrix} -y \\ x\end{smallmatrix}\right) & \left(\begin{smallmatrix} x \\ y\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} 0 & 0 \\ x & y\end{smallmatrix}\right) \end{array} \tag{3.4}$$

are continuous and open. This implies that the bijective map

$$K^\infty \backslash \left(\widehat{\mathbb{Z}}^2_{\mathrm{prim}}/\{\pm 1\} \times \widehat{\mathbb{Z}}^\times\right) \times K_\infty \backslash \left(\mathbb{R}^2_{\mathrm{prim}} \times \mathbb{R}_{>0}\right) \to \mathrm{GL}_2(\mathbb{Q})\backslash \mathcal{W}(\mathbb{A}_\mathbb{Q})/K.$$

is continuous and open, and thus it is a homeomorphism. □

Theorem 3.8 shows indeed that the space $W \times \mathbb{G}_m$ is not the right space to use if we want to describe X_{K^∞} using the ring of adèles $\mathbb{A}_\mathbb{Q}$. Actually it seems that maybe the Baily-Borel compactification was not the right compactification to look at. In the following sections we will look at the Borel-Serre compactification $X^{\mathrm{BS}}_{K^\infty}$ and we will obtain a completely adelic description of them.

3.3 Adèles and the Borel-Serre cusps

This section is devoted to finding a new space related to the full adèle ring $\mathbb{A}_{\mathbb{Q}}$ and using it to describe the disjoint union $\bigsqcup_{j=1}^{n} C_{\Gamma_j}^{\mathrm{BS}}$ where $\Gamma_j \leq \mathrm{SL}_2(\mathbb{Q})$ are the arithmetic groups related to a compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ that we defined in section 2.3 and $C_{\Gamma_j}^{\mathrm{BS}}$ are the "Borel-Serre cusps" that we defined in (1.1). To do so we will first of all find a topological space $\mathcal{C}(\mathbb{A}_{\mathbb{Q}})$ with a left action of $\mathrm{GL}_2(\mathbb{Q})$ and a right action of $K_\infty = \mathbb{R}_{>0} \times \mathrm{SO}_2(\mathbb{R})$ such that for every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^\infty)$ we will have a homeomorphism $C_{K^\infty}^{\mathrm{BS}} \cong C^{\mathrm{BS}}/K^\infty$ where $C^{\mathrm{BS}} = \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{C}(\mathbb{A}_{\mathbb{Q}})/K_\infty$ and

$$C_{K^\infty}^{\mathrm{BS}} \stackrel{\text{def}}{=} \bigsqcup_{j=1}^{n} \Gamma_j\backslash\mathcal{L} \qquad \text{with} \qquad \begin{aligned} n &= [\widehat{\mathbb{Z}}^\times : \det(K^\infty)] \\ \mathcal{L} &= \bigsqcup_{x \in \mathbb{P}^1(\mathbb{Q})} \mathbb{P}^1(\mathbb{R}) \setminus \{x\} \end{aligned}$$

is the set that needs to be "glued" to the affine modular curve $Y_{K^\infty} = \bigsqcup_{j=1}^{n} \Gamma_j\backslash\mathfrak{h}$ to obtain its Borel-Serre compactification, as we described in section 1.3.

We see from this description that, similarly to what we have done for affine modular curves, we should have a homeomorphism $C^{\mathrm{BS}} \cong \widehat{\mathbb{Z}}^\times \times \varprojlim_n \Gamma(n)\backslash\mathcal{L}$ because

$$\begin{aligned} C^{\mathrm{BS}} &\cong \varprojlim C^{\mathrm{BS}}/K^\infty \cong \varprojlim_n C^{\mathrm{BS}}/K(n) \cong \varprojlim_n C_{K(n)}^{\mathrm{BS}} \cong \\ &\cong \varprojlim_n \left({}^{\mathbb{Z}}/_{n\mathbb{Z}}\right)^\times \times \Gamma(n)\backslash\mathcal{L} \cong \widehat{\mathbb{Z}}^\times \times \varprojlim_n \Gamma(n)\backslash\mathcal{L}. \end{aligned}$$

To study the topological space $\varprojlim_n \Gamma(n)\backslash\mathcal{L}$ we prove a general lemma concerning the action of a group on a topological space and its possible profinite completions.

Lemma 3.9. *Let G be a topological group and let X be a topological space endowed with a continuous action $G \circlearrowright X$. Let moreover $\{H_j\}_{j\in\mathcal{J}}$ be an inverse system of closed normal subgroups of G. Then the map*

$$\left(\varprojlim_{j\in\mathcal{J}} H_j\backslash G\right) \times X \to \varprojlim_{j\in\mathcal{J}} H_j\backslash X \qquad \left((g_j)_{j\in\mathcal{J}}, x\right) \mapsto (g_j^{-1} * x)_{j\in\mathcal{J}} \tag{3.5}$$

induces a homeomorphism $G\backslash\left(\varprojlim_{j\in\mathcal{J}} H_j\backslash G\right) \times X \xrightarrow{\sim} \varprojlim_{j\in\mathcal{J}} H_j\backslash X$ where the left action $G \circlearrowright \left(\varprojlim_{j\in\mathcal{J}} H_j\backslash G\right) \times X$ is defined as $h * \left((g_j), x\right) = \left((h \cdot g_j), h * x\right)$.

Proof. It is clear that the map (3.5) is continuous and open, so we have only to show that it induces a bijective map $G\backslash\left(\varprojlim_{j\in\mathcal{J}} H_j\backslash G\right)\times X \xrightarrow{\sim} \varprojlim_{j\in\mathcal{J}} H_j\backslash X$. To do so let

$$\big((g_j),x\big),\big((h_j),y\big)\in\left(\varprojlim_{j\in\mathcal{J}} H_j\backslash G\right)\times X$$

and suppose that $g_j^{-1}*x = h_j^{-1}*y\in H_j\backslash X$, $\forall j\in\mathcal{J}$ for some $k\in G$. We see immediately that this happens if and only if there exists $k\in G$ such that $x=k*y$ and $g_j\cdot h_j^{-1}=k$ for every $j\in J$, which is true if and only if $\big((g_j),x\big)$ and $\big((h_j),y\big)$ are equivalent under the action $G\circlearrowleft\left(\varprojlim_{j\in\mathcal{J}} H_j\backslash G\right)\times X$.

Observe now that for every $j\in\mathcal{J}$ we have a surjective map $H_j\backslash X\twoheadrightarrow G\backslash X$. These maps induce clearly a well defined and surjective map $\pi\colon \varprojlim_{j\in\mathcal{J}} H_j\backslash X\twoheadrightarrow G\backslash X$. Let now $\xi=(H_j*x_j)_{j\in\mathcal{J}}\in\varprojlim_{j\in\mathcal{J}} H_j\backslash X$ and let $x\in X$ be any element such that $G*x=\pi(\xi)$. This implies that $G*x_j=G*x$ for every $j\in J$ and thus that for every $j\in J$ there exists $g_j\in G$ such that $g_j*x=x_j$. Observe that for every $i,j\in\mathcal{J}$ such that $H_i\le H_j$ we have that $H_j*x_i=H_j*x_j$ and this implies that we can choose the elements $\{g_j\}_{j\in\mathcal{J}}\subseteq G$ such that $(g_j)_{j\in\mathcal{J}}\in\varprojlim_{j\in\mathcal{J}} H_j\backslash G$. Thus the map (3.5) is surjective, because for every $\xi\in\varprojlim_{j\in\mathcal{J}} H_j\backslash X$ the couple $\big((g_j)_{j\in\mathcal{J}},x\big)$ that we have just defined is mapped to ξ by construction. □

We can now apply the lemma to the inverse system $\{\Gamma(n)\}_{n\in\mathbb{N}_{\ge1}}$ of subgroups of $\mathrm{SL}_2(\mathbb{Z})$ acting on the topological space $\mathcal{L}$ to obtain a homeomorphism

$$\varprojlim_{n\in\mathbb{N}}\Gamma(n)\backslash\mathcal{L}\xrightarrow{\sim}\mathrm{SL}_2(\mathbb{Z})\backslash\big(\mathcal{L}\times\mathrm{SL}_2(\widehat{\mathbb{Z}})\big).$$

It is now easy to check that the map

$$\widehat{\mathbb{Z}}^\times\times\mathcal{L}\times\mathrm{SL}_2(\widehat{\mathbb{Z}})\to\mathcal{L}\times\mathrm{GL}_2(\widehat{\mathbb{Z}})\qquad(\alpha,\lambda,A)\mapsto\left(\lambda,A\cdot\left(\begin{smallmatrix}1&0\\0&\alpha\end{smallmatrix}\right)\right)$$

induces a homeomorphism $\widehat{\mathbb{Z}}^\times\times\mathrm{SL}_2(\mathbb{Z})\backslash\big(\mathcal{L}\times\mathrm{SL}_2(\widehat{\mathbb{Z}})\big)\xrightarrow{\sim}\mathrm{SL}_2(\mathbb{Z})\backslash\big(\mathcal{L}\times\mathrm{GL}_2(\widehat{\mathbb{Z}})\big)$ and that the map

$$\begin{aligned}\mathcal{L}\times\mathrm{GL}_2(\widehat{\mathbb{Z}})&\to\mathcal{L}^\pm\times\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)\\(\lambda,A)&\mapsto((\lambda,1),A)\end{aligned}\qquad\text{where}\qquad\mathcal{L}^\pm\overset{\text{def}}{=}\mathcal{L}\times\{\pm1\}$$

induces a homeomorphism $\mathrm{SL}_2(\mathbb{Z})\backslash\big(\mathcal{L}\times\mathrm{GL}_2(\widehat{\mathbb{Z}})\big)\xrightarrow{\sim}\mathrm{GL}_2(\mathbb{Q})\backslash\big(\mathcal{L}^\pm\times\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)\big)$. Putting together all the homeomorphisms we see that

$$\widehat{\mathbb{Z}}^\times\times\varprojlim_{n\in\mathbb{N}}\Gamma(n)\backslash\mathcal{L}\cong\mathrm{GL}_2(\mathbb{Q})\backslash\big(\mathcal{L}^\pm\times\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)\big)\tag{3.6}$$

which is already a partially adelic description of the topological space $\mathcal{C}(\mathbb{A}_{\mathbb{Q}})$ which should be used to describe the topological spaces

$$\mathcal{C}_{K^\infty} \overset{\text{def}}{=} \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathcal{L} \cong \bigsqcup_{j=1}^{n} (\Gamma_j \backslash \mathbb{P}^1(\mathbb{Q})) \times S^1 \qquad \text{where} \qquad n = [\widehat{\mathbb{Z}}^\times : \det(K^\infty)]$$

as quotients $\mathcal{C}_{K^\infty} \cong \mathcal{C}(\mathbb{A}_{\mathbb{Q}})/K^\infty$.

We look now for a description of the topological space $\mathcal{L}^\pm$ as a quotient $\mathcal{B}(\mathbb{R})^\pm/K_\infty$ where $\mathcal{B}(\mathbb{R})$ is a topological space related to the real numbers $\mathbb{R}$ which will be the archimedean component of $\mathcal{C}(\mathbb{A}_{\mathbb{Q}})$. We define the topological space $\mathcal{B}(\mathbb{R})$ as

$$\mathcal{B}(\mathbb{R}) \overset{\text{def}}{=} \bigsqcup_{x \in \mathbb{P}^1(\mathbb{Q})} B_x \qquad \text{where} \qquad B_{(x_0:x_1)} \overset{\text{def}}{=} \{M \in \mathcal{M}_{2,2}(\mathbb{R}) \mid \det(M) = 0, (-x_1, x_0) \cdot M \neq (0,0)\}$$

where each B_x has the subspace topology induced by the inclusion $B_x \hookrightarrow \mathcal{M}_{2,2}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R})$ has the disjoint union topology. Observe that for every $x \in \mathbb{Q}$ we have $B_x = \sigma_x \cdot B_\infty$ where $\sigma_x \in \mathrm{SL}_2(\mathbb{Q})$ is any matrix such that $\sigma_x * \infty = x$. Observe moreover that for every matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in B_\infty$ we have that $(0,1) \cdot M = (c,d) \neq (0,0)$. This allows us to define a family of maps $\varphi_x : B_x \to \mathbb{P}^1(\mathbb{R}) \setminus \{x\}$ by defining

$$\varphi_\infty : B_\infty \to L_\infty \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \frac{ac+bd}{c^2+d^2} = \mathfrak{Re}\left(\frac{ai+b}{ci+d}\right) \tag{3.7}$$

and then by setting $\varphi_x(M) \overset{\text{def}}{=} \sigma_x * \varphi_\infty(\sigma_x^{-1} \cdot M)$.

Lemma 3.10. *The maps $\{\varphi_x\}$ allow us to define a map*

$$\mathcal{B}(\mathbb{R})^\pm \to \mathcal{L}^\pm \qquad (x, \varepsilon, M) \mapsto (x, \varepsilon, \varphi_x(M)) \qquad \textit{where} \qquad \mathcal{B}(\mathbb{R})^\pm \overset{\text{def}}{=} \mathcal{B}(\mathbb{R}) \times \{\pm 1\} \tag{3.8}$$

which induces a homeomorphism $\mathcal{B}(\mathbb{R})^\pm/K_\infty \xrightarrow{\sim} \mathcal{L}^\pm$.

Proof. Observe first of all that the map

$$\mathbb{R} \times \mathbb{R}^2_{\text{prim}} \to B_\infty \qquad (\lambda, \mathbf{v}) \mapsto \begin{pmatrix} \lambda \cdot \mathbf{v} \\ \mathbf{v} \end{pmatrix} \tag{3.9}$$

is a homeomorphism and that the map $\mathbb{R} \times \mathbb{R}^2_{\text{prim}} \to L_\infty = \mathbb{R}$ given by the composition of φ_∞ and (3.9) is simply the projection onto the first factor. This shows that φ_∞ is continuous and open and that it induces a homeomorphism $B_\infty/K_\infty \xrightarrow{\sim} L_\infty$ because the map (3.9) is invariant under the right action $\mathbb{R}^2_{\text{prim}} \circlearrowleft K_\infty$ which is transitive.

It is now sufficient to observe that for every $x \in \mathbb{P}^1(\mathbb{Q})$ the homeomorphism

$$B_\infty \longrightarrow B_x \qquad M \mapsto \sigma_x \cdot M$$

is equivariant with respect to the right action of K_∞ and that the square

$$\begin{tikzcd} B_\infty \arrow[r, "\varphi_\infty"] \arrow[d, "\sigma_x\cdot(-)"'] & L_\infty \arrow[d, "\sigma_x*(-)"] \\ B_x \arrow[r, "\varphi_x"] & L_x \end{tikzcd}$$

commutes to conclude that φ_x induces a homeomorphism $B_x/K_\infty \xrightarrow{\sim} L_x$ for every $x \in \mathbb{P}^1(\mathbb{Q})$ and thus that (3.8) induces a homeomorphism $\mathcal{B}(\mathbb{R})^\pm/K_\infty \xrightarrow{\sim} \mathcal{L}^\pm$ as we wanted to prove. $\square$

Using now Lemma 3.10 and (3.6) we see that

$$\widehat{\mathbb{Z}}^\times \times \varprojlim_{n\in\mathbb{N}} \Gamma(n)\backslash\mathcal{L} \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{B}(\mathbb{R})^\pm \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)/K_\infty$$

which implies that we can define $\mathcal{C}(\mathbb{A}_\mathbb{Q}) \stackrel{\text{def}}{=} \mathcal{B}(\mathbb{R})^\pm \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ to have a homeomorphism

$$C_{K^\infty}^{\mathrm{BS}} \cong C^{\mathrm{BS}}/K^\infty \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{C}(\mathbb{A}_\mathbb{Q})/K \qquad \text{where} \qquad K = K^\infty \times K_\infty$$

for every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$. This definition is still not completely satisfactory because it separates the archimedean part and the non-archimedean part in the definition of $\mathcal{C}(\mathbb{A}_\mathbb{Q})$ but it seems to us difficult if not impossible to avoid this, as we will explain in the conclusions.

3.4 The full Borel-Serre compactification

We will use now what we have proved in the previous section to give an adelic description of the full Borel-Serre compactified modular curve $X_{K^\infty}^{\mathrm{BS}}$ associated to a compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ which is defined as the disjoint union

$$X_{K^\infty}^{\mathrm{BS}} \stackrel{\text{def}}{=} \bigsqcup_{j=1}^{[\widehat{\mathbb{Z}}^\times:\det(K^\infty)]} X_{\Gamma_j}^{\mathrm{BS}}$$

where $X^{\mathrm{BS}}_{\Gamma_j} = \Gamma_j\backslash\mathfrak{h}^{**}$ is the Borel-Serre compactification of $Y_{\Gamma_j} = \Gamma_j\backslash\mathfrak{h}$ defined in section 1.3. As we did in the previous section we will find a topological space $\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ with two actions $\mathrm{GL}_2(\mathbb{Q}) \circlearrowright \mathcal{Z}(\mathbb{A}_{\mathbb{Q}}) \circlearrowleft K_\infty$ such that if $X^{\mathrm{BS}} \overset{\text{def}}{=} \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})/K_\infty$ then $X^{\mathrm{BS}}_{K^\infty} \cong X^{\mathrm{BS}}/K^\infty$ for every compact and open subgroup $K^\infty \le \mathrm{GL}_2(\mathbb{A}^\infty_{\mathbb{Q}})$.

To do so we prove first of all the following lemma, which is related to what we have proved in section 2.3.

Lemma 3.11. *Let T be any connected topological space with a left action $\mathrm{SL}_2(\mathbb{Q}) \circlearrowright T$ and let $K^\infty \le \mathrm{GL}_2(\mathbb{A}^\infty_{\mathbb{Q}})$ be a compact and open subgroup. Let moreover $\{A_j\}_{j=1}^n \subseteq \mathrm{GL}_2(\mathbb{A}^\infty_{\mathbb{Q}})$ be any minimal set of matrices such that $\widehat{\mathbb{Z}}^\times/_{\det(K^\infty)} = \{\overline{\det(A_1)}, \dots, \overline{\det(A_n)}\}$. If we define*

$$K^\infty_j \overset{\text{def}}{=} A_j \cdot K^\infty \cdot A_j^{-1} \cap \mathrm{SL}_2(\mathbb{A}^\infty_{\mathbb{Q}}) \qquad \Gamma_j \overset{\text{def}}{=} \mathrm{SL}_2(\mathbb{Q}) \cap K^\infty_j \qquad \text{and} \qquad T^\pm \overset{\text{def}}{=} T \times \{\pm 1\}$$

then the maps

$$\varphi_j \colon T \to T^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty_{\mathbb{Q}}) \qquad \text{defined as} \qquad \varphi_j(t) = ((t,1), A_j)$$

induce a homeomorphism $\bigsqcup_{j=1}^n \Gamma_j\backslash T \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q})\backslash T^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty_{\mathbb{Q}})/K^\infty$.

Proof. The proof is exactly the same as the one that we did in the case $T = \mathfrak{h}$ which is contained in Lemma 2.8 and Theorem 2.9. $\square$

Observe now that we could have applied Lemma 3.11 to the case $T = \mathcal{L}$ to obtain immediately the homeomorphism (3.6) but we preferred to show that this homeomorphism can be obtained also in a more "intrinsic" way by looking at the inverse limit $\varprojlim_n \Gamma(n)\backslash\mathcal{L}$. Nevertheless we will now apply Lemma 3.11 to the case $T = \mathfrak{h}^{**}$ to obtain a homeomorphism

$$X^{\mathrm{BS}} \cong \varprojlim_n X^{\mathrm{BS}}_{K(n)} \cong \varprojlim_n \mathrm{GL}_2(\mathbb{Q})\backslash\Big((\mathfrak{h}^{**})^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty_{\mathbb{Q}})/K(n)\Big) \cong \mathrm{GL}_2(\mathbb{Q})\backslash(\mathfrak{h}^{**})^\pm \times \mathrm{GL}_2(\mathbb{A}^\infty_{\mathbb{Q}}).$$

Now, as we did in the previous section, we will find a topological space $\mathcal{G}(\mathbb{R})$ with a right action of K_∞ such that $(\mathfrak{h}^{**})^\pm \cong \mathcal{G}(\mathbb{R})^\pm/K_\infty$. To do so we define

$$U_{(x_0:x_1)} \overset{\text{def}}{=} \{M \in \mathcal{M}_{2,2}(\mathbb{R}) \mid \det(M) \ge 0,\ (-x_1, x_0)\cdot M \ne (0,0)\} \qquad \text{for every } x \in \mathbb{P}^1(\mathbb{Q})$$

endowed with the subspace topology induced by the inclusion $U_x \hookrightarrow \mathcal{M}_{2,2}(\mathbb{R})$. Now we set $\mathcal{G}(\mathbb{R}) \overset{\text{def}}{=} \left(\bigsqcup_{x\in\mathbb{P}^1(\mathbb{Q})} U_x\right)/\sim$ where $(x,A) \sim (y,B)$ if and only if $A, B \in \mathrm{GL}_2^+(\mathbb{R})$ and $A = B$.

Observe that for every $x \in \mathbb{P}^1(\mathbb{Q})$ we have $U_x = \mathrm{GL}_2^+(\mathbb{R}) \sqcup B_x$ as sets but not as topological spaces. This implies that we have a bijection $\mathcal{G}(\mathbb{R}) \leftrightarrow \mathrm{GL}_2^+(\mathbb{R}) \sqcup \mathcal{B}(\mathbb{R})$ such that the inclusions

$$\mathrm{GL}_2^+(\mathbb{R}) \hookrightarrow \mathcal{G}(\mathbb{R}) \qquad \text{and} \qquad \mathcal{B}(\mathbb{R}) \hookrightarrow \mathcal{G}(\mathbb{R})$$

are continuous. This bijection is not a homeomorphism, which is the key fact to prove the next fundamental result of this paper.

Theorem 3.12. *Using the functions $\{\varphi_x\}$ defined in (3.7) we can define two maps*

$$\begin{aligned} \mathrm{GL}_2^+(\mathbb{R}) &\to \mathfrak{h} \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\mapsto \frac{ac+bd}{c^2+d^2} + \frac{c^2+d^2}{ad-bc} \cdot i \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \mathcal{B}(\mathbb{R}) &\to \mathcal{L} \\ (x, M) &\mapsto (x, \varphi_x(M)) \end{aligned} \tag{3.10}$$

which induce a homeomorphism $\mathcal{G}(\mathbb{R})/K_\infty \xrightarrow{\sim} \mathfrak{h}^{**}$.

Proof. Let now $x \in \mathbb{P}^1(\mathbb{Q})$ and let $\sigma_x \in \mathrm{SL}_2(\mathbb{Q})$ be any matrix such that $\sigma_x * \infty = x$. We define a map

$$\Psi_x \colon U_x \to \mathfrak{h} \sqcup L_x \qquad M \mapsto \begin{cases} M * i, \text{ if } M \in \mathrm{GL}_2^+(\mathbb{R}) \\ \varphi_x(M), \text{ if } M \in B_x \end{cases}$$

using the fact that $U_x = \mathrm{GL}_2^+(\mathbb{R}) \sqcup M_x$ as sets. Observe now that for every $x \in \mathbb{P}^1(\mathbb{Q})$ the maps

$$\mathfrak{m}_x \colon U_\infty \to U_x \qquad M \mapsto \sigma_x \cdot M$$

are homeomorphisms such that the square

$$\begin{tikzcd} U_\infty \arrow[r, "\Psi_\infty"] \arrow[d, "\mathfrak{m}_x"'] & L_\infty \sqcup \mathfrak{h} \arrow[d, "\sigma_x * (-)"] \\ U_x \arrow[r, "\Psi_x"] & L_x \sqcup \mathfrak{h} \end{tikzcd}$$

is always commutative and $\mathfrak{m}_x(\mathrm{GL}_2^+(\mathbb{R})) = \mathrm{GL}_2^+(\mathbb{R})$. Thus to conclude it is sufficient to prove that Ψ_∞ induces a homeomorphism $U_\infty/K_\infty \xrightarrow{\sim} L_\infty \sqcup \mathfrak{h}$.

To do so we observe that the map

$$\rho \colon L_\infty \sqcup \mathfrak{h} \to \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq 0\} \qquad x \mapsto \begin{cases} \mathfrak{Re}(x) + \mathfrak{Im}(x)^{-1} \cdot i, \text{ if } x \in \mathfrak{h} \\ x + 0 \cdot i, \text{ if } x \in L_\infty = \mathbb{R} \end{cases}$$

is a homeomorphism, as it is immediate to prove if we recall the definition of the topology on $L_\infty \sqcup \mathfrak{h}$ given in section 1.3. Moreover we have that

$$\rho \circ \Psi_\infty \colon U_\infty \to \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq 0\} \qquad \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \frac{ac+bd}{c^2+d^2} + \frac{ad-bc}{c^2+d^2} \cdot i = \frac{ai+b}{ci+d}$$

is clearly continuous and open. Moreover it induces a bijective map $U_\infty/K_\infty \to \{z \in \mathbb{C} \mid \mathfrak{Im}(z) \geq 0\}$ because we already know that the maps

$$\begin{aligned} \mathrm{GL}_2^+(\mathbb{R})/K_\infty &\to \mathfrak{h} \\ \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] &\mapsto \frac{ai+b}{ci+d} \end{aligned} \qquad \text{and} \qquad \begin{aligned} B_\infty/K_\infty &\to \mathbb{R} \\ \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right] &\mapsto \frac{ac+bd}{c^2+d^2} \end{aligned}$$

are well defined and bijective and that $U_\infty = \mathrm{GL}_2^+(\mathbb{R}) \sqcup B_\infty$ as sets. □

What we have proved shows that we can define $\mathcal{Z}(\mathbb{A}_\mathbb{Q}) \overset{\text{def}}{=} \mathcal{G}(\mathbb{R})^\pm \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$ to have a homeomorphism

$$X_{K^\infty}^{\mathrm{BS}} = \bigsqcup_{j=1}^{n} \Gamma_j \backslash \mathfrak{h}^{**} \cong \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{Z}(\mathbb{A}_\mathbb{Q})/K \qquad \text{where} \qquad n = [\widehat{\mathbb{Z}}^\times \colon \det(K^\infty)] \text{ and } K = K^\infty \times K_\infty$$

for every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$. In particular we have three continuous maps

$$\begin{aligned} Y &\hookrightarrow X^{\mathrm{BS}} \\ C^{\mathrm{BS}} &\twoheadrightarrow C \\ C^{\mathrm{BS}} &\hookrightarrow X^{\mathrm{BS}} \end{aligned} \qquad \text{where} \qquad \begin{aligned} Y &= \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_\infty \\ C^{\mathrm{BS}} &= \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{C}(\mathbb{A}_\mathbb{Q})/K_\infty \\ X^{\mathrm{BS}} &= \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{Z}(\mathbb{A}_\mathbb{Q})/K_\infty \end{aligned} \qquad \text{and} \qquad C = \mathrm{GL}_2(\mathbb{Q}) \backslash \mathcal{W}(\mathbb{A}_\mathbb{Q}^\infty)$$

such that

$$\begin{aligned} Y &\hookrightarrow X^{\mathrm{BS}} && \text{is an open embedding} \\ C^{\mathrm{BS}} &\twoheadrightarrow C && \text{is a continuous surjection} \\ C^{\mathrm{BS}} &\hookrightarrow X^{\mathrm{BS}} && \text{is a closed embedding} \end{aligned}$$

and the squares

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, two heads] & X^{\mathrm{BS}} \arrow[d, two heads] \\ Y_{K^\infty} \arrow[r, hook] & X_{K^\infty}^{\mathrm{BS}} \end{tikzcd} \qquad \begin{tikzcd} C^{\mathrm{BS}} \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\ C_{K^\infty}^{\mathrm{BS}} \arrow[r, two heads] & C_{K^\infty} \end{tikzcd} \qquad \begin{tikzcd} C^{\mathrm{BS}} \arrow[r, hook] \arrow[d, two heads] & X^{\mathrm{BS}} \arrow[d, two heads] \\ C_{K^\infty}^{\mathrm{BS}} \arrow[r, hook] & X_{K^\infty}^{\mathrm{BS}} \end{tikzcd}$$

commute for every compact and open subgroup $K^\infty \leq \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty)$.

CONCLUSIONS

I may not have gone where I intended to go, but I think I have ended up where I needed to be.

Douglas Adams

The aim of this thesis that we outlined in the introduction was to find a description of the projective limit of compactified modular curves as a quotient of the form

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{X}(\mathbb{A}_{\mathbb{Q}})/K_\infty$$

for some topological space $\mathcal{X}(\mathbb{A}_{\mathbb{Q}})$. Unfortunately we have not been able to do so for the projective limit $\varprojlim_n X(n)$ of classical compact modular curves because there is no clear way to extend the homeomorphism $\varprojlim_n \Gamma(n)\backslash\mathbb{P}^1(\mathbb{Q}) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{W}(\mathbb{A}_{\mathbb{Q}}^\infty)$ to the entire projective limit $\varprojlim_n \Gamma(n)\backslash X(n)$ as we explained in section 3.2. We turned then our attention to the "less classical" Borel-Serre compactifications $X^{\mathrm{BS}}(n)$ and we could find a topological space $\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ such that $\varprojlim_n X^{\mathrm{BS}}(n) \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})/K_\infty$ as we outlined in section 3.4.

These results leave us with some unanswered questions. It would be interesting for instance to know more about the definition of the space $\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ and to try to make it more symmetric in the archimedean and non archimedean parts of the ring of adèles. Another interesting fact is that the archimedean part $\mathcal{B}(\mathbb{R})^\pm$ of $\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ is defined as a disjoint union over $\mathbb{P}^1(\mathbb{Q})$, which seems to be a manifestation of the "global nature" of the compactification of a modular curve, since $\mathbb{P}^1(\mathbb{Q})$ parametrizes all the parabolic subgroups of $\mathrm{GL}_2(\mathbb{Q})$. It would finally be of great interest to understand adelic automorphic forms on the space $X^{\mathrm{BS}} = \mathrm{GL}_2(\mathbb{Q})\backslash\mathcal{Z}(\mathbb{A}_{\mathbb{Q}})$ as a generalisation of the notion of adelic automorphic forms for the quotient $Y = \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ which is outlined in **Chapter 7** of [5].

BIBLIOGRAPHY

[1] W. L. Baily, Jr. and A. Borel, *On the compactification of arithmetically defined quotients of bounded symmetric domains*, Bull. Amer. Math. Soc. **70** (1964), 588–593. MR 0168802 6

[2] ———, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. of Math. (2) **84** (1966), 442–528. MR 0216035 6

[3] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault. MR 0387495 6, 15, 16, 17

[4] Nicolas Bourbaki, *Elements of mathematics. Algebra, Part I: Chapters 1-3*, Hermann, Paris; Addison-Wesley Publishing Co., Reading Mass., 1974, Translated from the French. MR 0354207 23

[5] D. Bump, J. W. Cogdell, E. de Shalit, D. Gaitsgory, E. Kowalski, and S. S. Kudla, *An introduction to the Langlands program*, Birkhäuser Boston, 2003, Edited by Joseph Bernstein and Stephen Gelbart. MR 1990371 50

[6] John William Scott Cassels and Albrecht Fröhlich, *Algebraic number theory*, Academic Press, London; Thompson Book Co., Inc., Washington, D.C., 1967. MR 0215665 23, 24

[7] Keith Conrad, *Ostrowski for number fields*, http://www.math.uconn.edu/~kconrad/math321/ostrowski.pdf. 20

[8] Mark Goresky, *Compactifications and cohomology of modular varieties*, Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 551–582. MR 2192016 14, 15, 16

[9] Dominic Joyce, *On manifolds with corners*, Advances in geometric analysis, Adv. Lect. Math. (ALM), vol. 21, Int. Press, Somerville, MA, 2012, pp. 225–258. MR 3077259 15, 16

[10] Svetlana Katok, *Fuchsian groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1992. MR 1177168 13

[11] Nicholas Michael Katz and Barry Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, vol. 108, Princeton University Press, Princeton, NJ, 1985. MR 772569 10, 11

[12] James S. Milne, *Canonical models of Shimura curves*, http://www.jmilne.org/math/articles/2003a.pdf. 25

[13] Raghavan Narasimhan, *Compact Riemann surfaces*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1992. MR 1172116 9

[14] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR 1278263 23

[15] Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Princeton University Press, Princeton, N.J., 1971, Kanô Memorial Lectures, No. 1. MR 0314766 8, 9, 12, 13, 14, 30, 33

[16] Elias M. Stein and Rami Shakarchi, *Complex analysis*, Princeton Lectures in Analysis, vol. 2, Princeton University Press, Princeton, NJ, 2003. MR 1976398 7