PROPER BASE CHANGE OVER HENSELIAN PAIRS

ALGANT MASTER'S THESIS

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Introduction

In his famous paper [Art69], M. Artin was able to provide a new proof of the following theorem:

Let (A, m) be an henselian local ring and let X be a finitely presented, proper scheme over $S = Spec(A)$. Then the functor $- \times_S S_0$, where $S_0 = Spec(A/m)$, induces an equivalence of categories between the category of finite étale schemes over X and the category of finite étale schemes over $X_0 = X \times_S S_0$

This was first shown by Artin himself, A. Grothendieck and J. L. Verdier in [SGAIV]. The importance of this theorem comes from the fact that it is the main ingredient for the proof of the *Proper Base Change* theorem in étale cohomology. Our ultimate goal is to face the nonlocal case. In fact, since we have a definition which generalizes that of henselian ring to the nonlocal case, it is reasonable to ask ourselves what happens if we consider a ring that is not local. This question appears, for example, in [EGA IV.4, Remarks 18.5.16 (i)]. In the case where (A, m) is an henselian pair and $X = S$ the theorem still holds. This is a consequence of the work R. Elkik in [Elk] and of O. Gabber in [Gab].

The key ingredient in Artin's proof is a theorem which, roughly speaking, tells us that henselizations at a prime ideal of algebras of finite type over a field or over an excellent discrete valuation ring have a certain *approximation property*. This means that, under some hypothesis, for any *structure* over the completion $\hat{A}$ of the given ring A, we can find a *structure* over A which approximates the given one. This idea was made precise by Artin in [Art69].

By means of D. Popescu's characterization of regular homomorphisms between noetherian rings, it is possible to generalize Artin's theorem to the nonlocal case. This allows us to adapt Artin's proof of the theorem stated above to the case where (A, m) is an henselian pair.

This theorem has some implications on the level of étale cohomology. In particular, it tells us that a statement which was conjectured in [SGAIV, Exposé XII, Remarks 6.13] holds under certain assumptions.

Outline

This thesis is divided into six chapters. In the first one we treat henselian local rings, which had been introduced by G. Azumaya and play a central role in Algebraic Geometry.

We follow the exposition given by M. Raynaud in [Ray] and we investigate the main properties of this class of local rings. We also give a number of characterizations of henselian rings. Then we introduce a particularly important algebra associated to any finite free A-algebra B, where A is any ring. It is a representative of the functor which associates to any A-algebra C the set of idempotent elements of $B \otimes_A C$. At the end of the chapter we describe an universal way to associate an henselian local ring to any local ring.

In Chapter 2 we introduce the class of henselian pairs. Also in this part we mainly follow Raynaud's exposition [Ray]. It will be immediately clear that the definition of henselian local ring we give does not generalize to the nonlocal case. Anyway, considering the local case as a guideline and using the characterizations we prove in Chapter 1, it is possible to give a satisfactory definition of henselian pair. We will see that in this context it is convenient to work with idempotents. For this reason, we prove an important theorem which deals with liftings of idempotents. This result is useful to give several characterizations of henselian pairs. In the last part of this chapter, similarly to what we do for local rings, we give an universal way to associate an henselian pair to any pair.

Chapter 3 is devoted to the investigation of projective limits in the category of schemes. In particular, if $X = \varprojlim X_j$, we focus on results of the kind

X has the property $\mathscr{P}$ if and only if there exists some j such that X_j has the same property

We investigate modules over X which come from a family of modules over the X_j. We also study morphisms between such modules.
These results are of particular interest to us as they play a crucial role both in Artin's proof of the theorem cited above and in its generalization to the nonlocal setting. In this chapter we follow the exposition given in [EGA IV.3, §8].

Chapter 4 starts with a brief account on the class of excellent rings, which play a central role in Artin's paper [Art69] and in Algebraic Geometry in general. We do not enter into the details of the theory, but we try to underline the main reasons which led to its introduction instead. We give the precise definition of what an excellent ring is and we state some of the most important properties of this class of noetherian rings.

Then we expose Artin's approximation in some detail. The main idea is to consider a pair (A, I) and its I-adic completion $\hat{A}$. Then, given some *structure* $\hat{X}$ over $\hat{A}$, we wonder if there exists a *structure* over A that approximates $\hat{X}$ in some sense. With some assumptions, it is possible to reduce this abstract problem to a very concrete one:

Given a finite number of polynomial equations with coefficients in A, a solution $\hat{y}$ in $\hat{A}^n$, and an integer N, can we find a solution y in A^n such that

$$y \equiv \hat{y} \ mod\ I^N?$$

Artin in [Art69] was able to answer this question in the local setting, putting some restrictions on A. He also conjectured that the answer is always positive if A is an excellent henselian local ring. Moreover, some years later and precisely in [Art82], he conjectured an even stronger result. We discuss this second conjecture and Popescu's theorem, which provides an answer.

We also show how this theorem allows us to generalize Artin's answer to the question above, in the setting we are interested in.

This chapter's references include [EGA IV.2], [Mats], [Rot], [Eis] for the section on excellent rings. Moreover, we refer to [Art69], [Art82], [Rot] for the second section an to [Pop85], [Pop86], [Pop90], [Rot] and [Swan] for the last one.

In the fifth chapter we give the proof of the theorem stated above in the case where (A, m) is an henselian pair. This is done in two steps:

1. we use the results and techniques of Chapter 3 to reduce ourself to the case where (A, m) is the henselization of a finitely generated $\mathbb{Z}$-algebra.

2. we prove essential surjectivity and fully faithfulness of the functor $- \times_{Spec(A)} Spec(A/m)$ using Artin's approximation and [EGA IV.3, Theorem 18.3.4], which is a consequence of Grothendieck's existence theorem.

In the last chapter we introduce the more general notion of *henselian couple*, which coincides with that of henselian pair in the affine case. In particular, we show that couples (X, X_0) which arise as in the theorem stated above are henselian. Finally, we use this fact to give an answer to a question which appears in [SGAIV, Exposé XII, Remarks 6.13] in a particular case, namely when the given henselian couple (X, X_0) is such that X is proper over a noetherian ring A and $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. The affine case is a consequence of the work of R. Elkik and of O. Gabber.

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This script is dedicated to the all of you.
Questo scritto è dedicato a tutti voi.

Regensburg, 14/06/2017
M.P.

Contents

Chapter 1

Henselian rings

In this chapter we introduce the class of henselian local rings. We give a number of characterizations of these rings and we study some of their fundamental properties. Finally, we prove that there is a universal way to associate an henselian ring to any local ring.

1.1 Henselian Local Rings

Definition 1.1.1. A local ring A is said to be *henselian* if every finite A-algebra is decomposed, i.e. it is a product of local rings.

Remark 1.1.2. It is immediate to observe that if A is henselian, every finite local A-algebra is henselian.

Proposition 1.1.3. *Let (A, m) denote a local ring and let k be its residue field. Let B be a finite A-algebra and set $\bar{B} = B/mB \cong B \otimes_A k$. Then B is a semilocal ring whose maximal ideals are the prime ideals which lie over m.*

Proof. Let $n \subseteq B$ be a maximal ideal. Set $p = \phi^{-1}(n)$, where $\phi : A \to B$ denotes the structure homomorphism. Then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ A/p \arrow[r, "\bar{\phi}"] & B/n \end{tikzcd}$$

Moreover, since B is finite over A, also B/n is finite over A/p. By the Going up/Going down theorems we get that $Spec(B/n) \longrightarrow Spec(A/p)$ is surjective. This means that A/p is a field, that is to say, p is the maximal ideal m of A. This proves that every maximal ideal of B lies over A.

Conversely, let q be a prime ideal of B lying over m. As above, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & B \arrow[d] \\ k \arrow[r, "\bar{\phi}"] & B/q \end{tikzcd}$$

and since B is finite over A, the integral domain B/q is finite over k. Then B/q is a field and q is a maximal ideal.
Moreover, let us notice that B/mB is a finite dimensional k-vector space. Consider $n_1, n_2, \dots$ maximal ideals of B. Then we have a chain of k-vector subspaces of B/mB

$$n_1/mB \supset (n_1 \cap n_2)/mB \supset (n_1 \cap n_2 \cap n_3)/mB \supset \dots$$

Then there exists an integer $s \geq 1$ such that

$$(n_1 \cap \dots \cap n_s)/mB = (n_1 \cap \dots n_{s+1})/mB$$

Thus, we get $n_1 \cap \dots \cap n_s = n_1 \cap \dots \cap n_{s+1}$. Then $n_{s+1} \subseteq n_1 \cup \dots \cup n_s$ and by [AM, Proposition 1.11] we have $n_{s+1} \subseteq n_j$ for some $j \in \{1, \dots, s\}$. Since n_{s+1} is maximal, we necessarily have $n_{s+1} = n_j$. Therefore, B has only finitely many maximal ideals. □

In what follows, we will label $I = \{n_1, \dots, n_s\}$ the set of maximal ideals of the finite A-algebra B.

Proposition 1.1.4. *The canonical map*

$$\bar{B} \to \Pi_{i \in I} \bar{B}_{\bar{n}_i}$$

is an isomorphism.

Proof. It suffices to observe that $\bar{B}$ is a finite k-algebra. Hence, it is an artinian ring. In particular, by the structure theorem, $\bar{B}$ is isomorphic to the product of the localizations in its maximal ideals, which are

$$\bar{n}_i = n_i/mB$$

□

Proposition 1.1.5. *Let B be a finite algebra over the local ring (A, m). The following statements are equivalent:*

1. *B is decomposed*
2. *the canonical morphism*

   $$B \longrightarrow \Pi_{i \in I} B_{n_i}$$

   is an isomorphism

3. the decomposition of $\bar{B}$ lifts to a decomposition of B

Proof. 1.$\Rightarrow$ 2. We assume that
$$B \cong \Pi_{j\in J} B_j$$
where each B_j is a local ring. Let m_j be the maximal ideal of B_j for each j. Then,
$$n_j = m_j \times \Pi_{k\neq j} B_k$$
is a maximal ideal of B. Conversely, every maximal ideal of B has such shape. Therefore, J is a finite set and there exists a bijection $f : I \to J$ such that $n_i \cong n_{f(i)}$ for every $i \in I$. Finally, observe that there are isomorphisms
$$B_{n_j} \cong B_j$$

2. $\Rightarrow$ 3. Consider the given isomorphism
$$B \cong \Pi_{i\in I} B_{n_i}$$
Applying the functor $- \otimes_A A/m$ we obtain isomorphisms
$$\bar{B} \cong B \otimes_A A/m \cong (\Pi_{i\in I} B_{n_i}) \otimes_A A/m \cong \Pi_{i\in I}(B_{n_i} \otimes_A A/m) \cong \Pi_{i\in I} \bar{B}_{\bar{n}_i}$$

3. $\Rightarrow$ 1. There is nothing to show. □

Notation: For any ring R, we shall label $Id(R)$ the set of idempotent elements of R.
The following proposition gives us a criterion to recognize when a finite A algebra is decomposed.

Proposition 1.1.6. *Let (A, m) be a local ring and let B be a finite A-algebra. Set $\bar{B} = B/mB$. The function induced by the natural projection*
$$Id(B) \longrightarrow Id(\bar{B})$$
is injective. Moreover, it is a bijection if and only if B is decomposed

Proof. Let $e \in Id(B)$. It is obvious that $\bar{e}$ lies in $Id(\bar{B})$. Therefore, our function is well defined. Let e and f be two idempotent elements of B such that their images in $\bar{B}$ coincide. Let $x = e - f \in mB$. Then,
$$x^3 = (e - f)^3 = e^3 - f^3 - 3e^2 f + 3ef^2 = e - f = x$$
and
$$x(1 - x^2) = 0$$
Moreover, as mB is contained in the Jacobson ideal of B, $1 - x^2$ is a unit in B, i.e. $x = 0$.
Assume now that B is decomposed,
$$B \cong \Pi_{i\in I} B_{n_i}$$
Fact: A local ring has no nontrivial idempotents: let g be an idempotent element in a local ring. If it lies in the maximal ideal, $1 - g$ is a unit; since $g(1 - g) = 0$, we obtain $g = 0$. If g is

a unit instead , we obtain $g = 1$.

Let $e \in Id(B)$. Its image via the canonical map $\pi_i : B \to B_{n_i}$ is an idempotent element of B_{n_i}. Hence, it is either 0 or 1. In particular we can describe explicitly the idempotent element e:

$$e = (e_1, \dots, e_s) \qquad \text{with } e_i \in \{0, 1\}$$

Similarly, any idempotent element of $\bar{B}$ has the same shape. Therefore, we have that the function $Id(B) \to Id(\bar{B})$ is surjective.
Conversely, assume that the above function is a bijection. We have

$$\bar{B} \cong \Pi_{i \in I} \bar{B}_{\bar{n}_i}$$

Let $\bar{e}_i = (0, \dots, 1, \dots, 0)$ be the idempotent element with all but the i-th coordinate equal to zero. Let $e_i \in Id(B)$ be an idempotent of B which maps to $\bar{e}_i$. Then $B_{n_i} \cong B_{e_i}$ is a direct factor of B as

$$B \cong B_{e_i} \times B_{1-e_i}$$

□

Remark 1.1.7. In order to show that $Id(B) \longrightarrow Id(\bar{B})$ is injective we only used the fact that (A, m) is a pair (see **Definition** 2.1.1) with m contained in the Jacobson ideal of A.

Our next aim is to give several characterizations of henselian rings. First we need a lemma.

Lemma 1.1.8. *Let C be a finite A-algebra, where A is a local ring. Assume that $\bar{C} = C/mC \cong k[X]/(\bar{Q})$ for some monic polynomial $\bar{Q}$ of degree n. Let $\bar{X}$ be the image of X in $\bar{C}$ and let $x \in C$ such that $\bar{X} = x + mC$. Then x generates C and is a root of a monic polynomial Q of degree n such that $Q + m[X] = \bar{Q}$.*

Proof. Label I the ideal generated by $1, x, x^2, \dots, x^{n-1}$ in C. Then

$$I + mC = C$$

and therefore, by Nakayama's lemma, $I = C$. Write

$$x^n = a_0 + a_1 x + \dots + a_{n-1} x^{n-1}$$

and consider the polynomial

$$Q = -a_0 - a_1 X - \dots - a_{n-1} X^{n-1} + X^n \in C[X]$$

Then x is a root of Q and $Q + m[X]$ is a multiple of $\bar{Q}$. Since $deg(Q) = deg(\bar{Q}) = n$ and both the polynomials are monic, we have $\bar{Q} = Q + m[X]$. □

Remark 1.1.9. Notice that we only need that $m \subset A$ is an ideal contained in the Jacobson radical. The local hypothesis is not needed.

We are now able to give a first result on some characterizations of henselian local rings.

Theorem 1.1.10. *Let (A, m) be a local ring. The following statements are equivalent:*

1. *A is henselian*

2. *Every finite free A-algebra is decomposed*

3. *For every monic polynomial $P \in A[X]$, $A[X]/(P)$ is decomposed*

4. *Every monic polynomial $P \in A[X]$ such that $\bar{P}(X) \in k[X]$ has a decomposition $\bar{P} = \bar{Q}\bar{R}$, where $\bar{Q}$ and $\bar{R}$ are monic polynomials in $k[X]$ and $(\bar{Q}, \bar{R}) = k[X]$ admits a decomposition $P = QR$, where Q and R are monic polynomials in $A[X]$ and their reductions modulo $mA[X]$ are $\bar{Q}$ and $\bar{R}$ respectively.*

Proof. The implications *1.* $\Rightarrow$ *2.* $\Rightarrow$ *3.* are trivial.
3. $\Rightarrow$ *4.* Let P be a polynomial in $A[X]$ and let $\bar{P} = \bar{Q}\bar{R}$ be a decomposition of its reduction modulo $mA[X]$ into the product of two monic polynomials which generate the unit ideal. Let x be the image of X in $A[X]/(P)$. Notice that

$$(\bar{Q}\bar{R}) = (\bar{Q}) \cap (\bar{R})$$

In fact, $(\bar{Q}) \cap (\bar{R}) = (\bar{Q}\bar{R}/\bar{S})$, where $\bar{S}$ is the greatest common divisor of the two polynomials. But, as they generate the unit ideal, $\bar{S}$ is a unit and therefore $(\bar{Q}) \cap (\bar{R}) = (\bar{Q}\bar{R})$. By the Chinese Remainder Theorem, we thus get

$$k[X]/(\bar{P}) \cong k[X]/(\bar{Q}) \times k[X]/(\bar{R})$$

By **Proposition** 1.1.5, the decomposition of $k[X]/(\bar{P})$ lifts to a decomposition of $A[X]/(P)$

$$A[X]/(P) \cong B_1 \times B_2$$

Each B_i, $i = 1, 2$, is a finite A-algebra such that the hypothesis of the previous lemma are verified. Then B_1 is generated by the an element x_1 over $X + \bar{Q}$, which is a root of some monic polynomial $Q \in B_1[X]$, with $deg(Q) = deg(\bar{Q})$ and $Q + m[X] = \bar{Q}$. Similarly, B_2 is generated by an element x_2 over $X + \bar{R}$, which is a root of a monic polynomial $R \in B_2[X]$, with $deg(R) = deg(\bar{R})$ and $R + m[X] = \bar{R}$. We can also choose x_1 and x_2 such that $x = (x_1, x_2)$. Therefore, x is a root of QR. In particular, we get that QR is a multiple of P. Since $deg(P) = deg(QR)$ and both polynomials are monic, we actually have an equality $P = QR$.
4. $\Rightarrow$ *3.* Let P be a monic polynomial in $A[X]$. Let $\bar{P} = \Pi_{i \in I} \bar{P}_i \in k[X]$ be the decomposition of $\bar{P}$ in powers of irreducible monic polynomials pairwise coprime. Then

$$P = \Pi_{i \in I} P_i$$

where each P_i is monic. Consider the morphism

$$u : A[X]/(P) \to \Pi_{i \in I} A[X]/(P_i)$$

The induced morphism $\bar{u}$ is surjective. Therefore, by Nakayama's lemma, we get that u is surjective as well. Moreover, since $deg(P) = \sum_{i \in I} deg(P_i)$, it follows that $A[X]/(P)$ and

$\Pi_{i\in I}A[X]/(P_i)$ are two free A-modules of the same rank. Hence, u is also injective.
Notice that each $A[X]/(P_i)$ is a local ring: in fact, it is a finite A-algebra and by **Proposition** 1.1.3 its maximal ideals are exactly those above m. Therefore, the set of maximal ideals of $A[X]/(P_i)$ is in bijection with the set of prime ideals of $A[X]/(P_i)\otimes_A A/m \cong k[X]/(\bar{P}_i)$, which is an artinian local ring.
3. $\Rightarrow$ *1.* Let B be a finite A-algebra. By **Proposition** 1.1.6, B is decomposed if and only if

$$Id(B) \to Id(\bar{B})$$

is a bijection. Let $\bar{e}_i$ be the idempotent element of $\bar{B}$ corresponding to the maximal ideal n_i of B. Let $b \in B$ be such that $b + mB = \bar{e}_i$ and let $P \in A[X]$ be a monic polynomial such that $P(b) = 0$. Consider the A-morphism

$$\phi : A[X]/(P) \to B$$

which maps $X + (P)$ onto b. Set $p = \phi^{-1}(n_i)$. By the choice of $\bar{e}_i$, n_i is the only prime ideal of B lying over p. By hypothesis, $A[X]/(P)$ is decomposed. Let e be the elementary idempotent element such that $e+p = 1$. Then $\phi(e) \in B$ is an idempotent element and $\phi(e)+mB = \bar{e}_i$. □

Remark 1.1.11. Let (A, m) be a local ring and let Q and R be two polynomials in $A[X]$, with Q monic, such that their reduction modulo m generate the unit ideal. Then also Q and R generate the unit ideal. In fact, let J be the ideal in $A[X]$ generated by these two polynomials and set $M = A[X]/J$. Then M is a finitely generated A-module, as Q is monic. By the assumptions, it follows that $J + mA[X] = A[X]$. This implies that $mM = M$. Then, by Nakayama's lemma, $M = 0$.
We can apply this fact to see that the factorization $P = QR$ in **Theorem** 1.1.10 *4.* is unique. In fact, if $P = UV$ is another factorization with the same properties, by what we have just shown, Q and V generate the unit ideal $A[X]$. Then there exist two polynomials $F, G \in A[X]$ such that $FQ + GV = 1$. Multiplying by U:

$$U = UFQ + UGV = UFQ + GQR$$

Thus, Q divides U. As both Q and U are monic and they have the same degree, they coincide.

Corollary 1.1.12. *Let A be an henselian ring and let J be an ideal. Then A/J is an henselian ring.*

Proof. It is immediate to observe that the ring A/J is a local ring which satisfies property *4.* in **Theorem** 1.1.10. □

The most trivial example of an henselian ring is that of a field.
The following result provides other examples:

Lemma 1.1.13. *If I is a locally nilpotent ideal of R, the function*

$$Id(R) \to Id(R/I)$$

is bijective.

Proof. *injectivity:* Let $e_1, e_2 \in Id(R)$ such that $e_1 + I = e_2 + I$. Then, $e_1 - e_2 = x \in I$. Therefore, there exists an odd integer $n \geq 1$ such that $x^n = 0$, i.e.

$$0 = (e_1 - e_2)^n = \sum_{i=0}^{n} \binom{n}{i} e_1^i (-e_2)^{n-i} = e_1^n - n e_1^{n-1} e_2 + \cdots + n e_1 e_2^{n-1} - e_2^n = e_1 - e_2$$

surjectivity: let $\bar{e} = e + I \in Id(R/I)$. Define $f = 1 - e$. Then $ef \in I$, as $\bar{e}\bar{f} = 0 \in R/I$. Let $k \in \mathbb{N}$ such that $e^k f^k = 0$. Consider the element $x = 1 - e^k - f^k \in I$. As x lies in I, it is a nilpotent element. Hence, $1 - x$ is a unit in R. Let $u = (1-x)^{-1}$. Then

$$ue^k + uf^k = u(1-x) = 1$$

Multiplying by ue^k we see that

$$(ue^k)^2 = ue^k \in Id(R)$$

Moreover, we have that

$$\bar{u}\bar{e}^k = \bar{1}\bar{e}^k = \bar{e}$$

□

As an immediate consequence, we obtain the following corollary:

Corollary 1.1.14. *Let A be a local ring. Then A is henselian if and only if A_{red} is henselian.*

In particular, every artinian local ring is henselian.
This allows us to show that complete separate local rings are henselian.

Proposition 1.1.15. *If (A, m) is complete and separate with respect to the m-adic topology. Then A is henselian.*

Proof. Let B be a finite A-algebra, which is free as an A-module. Then also B is complete and separate with respect to the m-adic topology:

$$B \cong \varprojlim B/m^n B$$

For every $n \in \mathbb{N}$, $B/m^n B$ is a finite A/m^n-algebra. Note that A/m^n is an artinian local ring and therefore it is henselian. Thus $B/m^n B$ is decomposed:

$$B/m^n B = \Pi_{i \in I} (B/m^n B)_{\bar{n}_i} = \Pi_{i \in I} B_{n_i}/m^n B_{n_i}$$

Applying the functor $\varprojlim$, we get an isomorphism

$$B = \varprojlim B/m^n B = \varprojlim \Pi_{i \in I} B_{n_i}/m^n B_{n_i} = \Pi_{i \in I} \varprojlim (B/m^n B)_{\bar{n}_i}$$

□

1.2 Idempotents of a finite algebra over a local ring

As we have seen in **Proposition** 1.1.6, in order to study henselian rings (A, m) it is very important to understand the behavior of idempotent elements of finite A-algebras. Moreover, thanks to **Theorem** 1.1.10, it is sufficient to focus to the case where B is a finite free A-algebra. The aim of this section is to study the obstruction of an idempotent element of B/mB to be lifted to an idempotent element of B. With this in mind, let B be a finite A-algebra and let $\{e_1, \dots, e_r\}$ be an A-basis of B. Let

$$e_i e_j = \sum_{k=1}^{r} \mu(i,j,k) e_k$$

be the multiplicative table of B. Pick an element $b = \sum_{i=1}^{r} a_i e_i \in B$. Then

$$b^2 = \sum_{i,j=1}^{r} a_i a_j e_i e_j = \sum_{i,j=1}^{r} a_i a_j \sum_{k=1}^{r} \mu(i,j,k) e_k = \sum_{k=1}^{r} \big(\sum_{i,j=1}^{r} a_i a_j \mu(i,j,k) \big) e_k$$

Then b is an idempotent element if and only if for every $k = 1, \dots, r$

$$\sum_{i,j=1}^{r} a_i a_j \mu(i,j,k) = a_k$$

Define, for $k = 1, \dots, r$, the polynomials $P_k(T_1, \dots, T_r) \in A[T_1, \dots, T_r]$ as

$$P_k(T_1, \dots, T_r) = \sum_{i,j=1}^{r} \mu(i,j,k) T_i T_j - T_k$$

Then the element $b = \sum_{i=1}^{r} a_i e_i \in B$ is idempotent if and only if the r-tuple $(a_1, \dots, a_r)$ satisfies the system of equations

$$\begin{gathered} P_1(a_1, \dots, a_r) = 0 \\ \dots \\ P_r(a_1, \dots, a_r) = 0 \end{gathered}$$

Set

$$E(B) = E = \frac{A[T_1, \dots, T_r]}{(P_1, \dots, P_r)} = A[t_1, \dots, t_r]$$

Then, we have just shown that there exists a bijection between the sets

$$Hom_A(E, A) \longleftrightarrow Id(B)$$

More generally, let C be an A-algebra. Then, with the same notation we used above, we have that $\{e_1 \otimes 1, \dots, e_r \otimes 1\}$ is a C-basis of the C-algebra $B \otimes_A C$. Let $\beta = \sum_{i=1}^{r} e_i \otimes c_i \in B \otimes_A C$. Then

$$\beta^2 = \sum_{i,j=1}^{r} e_i e_j \otimes c_i c_j = \sum_{i,j=1}^{r} \big(\sum_{k=1}^{r} \mu(i,j,k) e_k \big) \otimes c_i c_j =$$

$$\sum_{k=1}^{r} e_k \otimes \big(\sum_{i,j=1}^{r} \mu(i,j,k)c_i c_j\big)$$

Therefore, $\beta^2 = \beta$ if and only if the r-tuple $(c_1, \dots, c_r)$ satisfies the system of equations

$$\begin{gathered} P_1(c_1, \dots, c_r) = 0 \\ \dots \\ P_r(c_1, \dots, c_r) = 0 \end{gathered}$$

Then, the bijection we considered above holds for every A-algebra C:

$$Hom_A(E, C) \longleftrightarrow Id(B \otimes_A C)$$

$$\phi \mapsto \sum_{i=1}^{r} e_i \otimes \phi(t_i)$$

Then, if we define the functor

$$F : A - algebras \longrightarrow Sets$$

$$C \mapsto Id(B \otimes_A C)$$

what we have proved so far is that F is represented by the object E (it is easy to see that the bijection we described above is natural).
In this new setting, the problem of whether or not an idempotent element of B/mB can be lifted to an idempotent element of B corresponds to the following question: given an A-homomorphism $\bar{u} : E \to k = A/m$, can we lift it to a A-morphism $u : E \to A$? In other words, can we find an arrow such that the following diagram commutes?

$$\begin{tikzcd} E \arrow[rr, "u"] \arrow[dr, "\bar{u}"'] & & A \arrow[dl] \\ & k & \end{tikzcd}$$

Remark 1.2.1. Notice that at this point we just assumed that (A, m) is a pair (see **Definition** 2.1.1).

Assume that (A, m) is local. Set $q = Ker(\bar{u})$, which is a maximal ideal of E. Then, $\bar{u}$ factors uniquely through a morphism $\bar{v} : E_q \to k$

$$\begin{tikzcd} E \arrow[rr, "\bar{u}"] \arrow[dr] & & k \\ & E_q \arrow[ur, "\bar{v}"'] & \end{tikzcd}$$

Therefore, we can reformulate the problem as follows: given an A-homomorphism $\bar{v} : E_q \to k$, can we lift it to an A-homomorphism $v : E_q \to A$? In general, the answer is no. The main advantage of the last reformulation is that we can restrict our attention to local homomorphisms between local rings. Consider the collection of local homomorphisms

$$\phi_i : (A, m) \to (A_i, m_i)$$

such that the residue field extension is trivial and such that, if $B_i = B \otimes_A A_i$, every idempotent element of B_i/m_iB_i can be lifted to an idempotent element of B_i. Notice that, since $A_i/m_i = A/m$, $B_i/m_iB_i \cong B/mB$. Therefore, the collection of local A-algebras we are considering is the one consisting of A-algebras such that, for every $\bar{u}_{\bar{e}} : E_q \to k$ corresponding to some idempotent element $\bar{e}$ of B/mB, there exist a lifting arrow $v_{i,\bar{e}} : E_q \to A_i$

$$\begin{tikzcd} E_q \arrow[r, "v_{i,\bar{e}}"] \arrow[dr, "\bar{u}_{\bar{e}}"'] & A_i \arrow[d] \\ & k \end{tikzcd}$$

We will say that $j \leq i$ if there exists an A-homomorphism $\chi_{j,i} : A_i \to A_j$ such that, for every $\bar{e} \in Id(B/mB)$, the following diagram commutes

$$\begin{tikzcd} E_q \arrow[rr, "v_{i,\bar{e}}"] \arrow[dr, "v_{j,\bar{e}}"] \arrow[ddr, "\bar{u}_{\bar{e}}"'] & & A_j \arrow[ddl] \\ & A_i \arrow[ur, "\chi_{j,i}"'] \arrow[d] & \\ & k & \end{tikzcd}$$

It is clear that $i \leq i$ and that $i \leq j$, $j \leq k$ imply $i \leq k$. Also assume that

$$\chi_{i,i} = id_{A_i}$$

$$\chi_{i,k} = \chi_{i,j} \circ \chi_{j,k} \quad \forall\, i \leq j \leq k$$

Therefore, we have just defined a diagram

$$D : I^{op} \to A - algebras$$

Remark 1.2.2. Let $\phi_i : A \to A_i$ as above. Each idempotent element $\bar{e}$ of B/mB corresponds uniquely to a r-tuple of elements of $(a_{1,i,\bar{e}}, \ldots, a_{r,i,\bar{e}}) \in A_i$. Consider the A-subalgebra of A_i generated by those (finitely many, as $Id(B/mB)$ is a finite set) elements. After localizing such algebra in the maximal ideal corresponding to m_i, we obtain a local A-algebra, whose structure homomorphism is local. Therefore, we can restrict our attention to the algebras A_i which are localizations of A-algebras of finite type.

Assume that the limit associated to the diagram we constructed above exists. Then the following result holds:

Lemma 1.2.3. *With the same notation we have used above, B is decomposed if and only if $A \cong \varprojlim A_i$*

Proof. ($\Rightarrow$) If B is decomposed, then A is one of the A-algebras A_i, say $A = A_{i_0}$. Notice that in this case χ_{i,i_0} has to be ϕ_i for every i. Hence, if $\{\psi_i : C \to A_i\}$ is a collection of A-morphisms such that for $i \leq j$ $\psi_i = \chi_{i,j} \circ \psi_j$, there exists a unique morphisms ψ such that the following diagram commutes

$$\begin{tikzcd}
 & C \arrow[d, "\psi"] \arrow[ddl, "\psi_j"'] \arrow[ddr, "\psi_i"] & \\
 & A \arrow[dl, "\phi_j"] \arrow[dr, "\phi_i"'] & \\
A_j \arrow[rr, "\chi_{i,j}"] & & A_i
\end{tikzcd}$$

Namely, $\psi = \psi_{i_0}$. Therefore, (A, ϕ_i) satisfies the universal property, whence $A \cong \varprojlim A_i$
($\Leftarrow$) Let $\bar{e}$ be an idempotent element of B/mB and let $\bar{v} : E_q \to k$ be the corresponding A-morphism. For each i, let $v_i : E_q \to A_i$ be the lifting of $\bar{v}$ associated to A_i. By definition, for every $i \leq j$, the following diagram commutes

$$\begin{tikzcd}
 & E_q \arrow[ddl, "v_j"'] \arrow[ddr, "v_i"] & \\
 & \varprojlim A_i \arrow[dl] \arrow[dr] & \\
A_j \arrow[rr, "\chi_{i,j}"] & & A_i
\end{tikzcd}$$

and therefore there exists a unique A-morphism $v : E_q \to \varprojlim A_i \cong A$. It is immediate to verify that v is a lifting of $\bar{v}$. As this holds for every idempotent element of B/mB, we have $Id(B/mB) = Id(B)$. Hence, B is decomposed. □

With this new approach we also find a new proof of **Proposition** 1.1.15:

Proof. Let $A = \varprojlim A/m^n$ and let B be a finite free A-algebra. For every A-homomorphism $\bar{v} : E_q \to k = A/m$ and for every $n \geq 1$, there exists a lifting homomorphism $v_n : E_q \to A/m^n$. In fact, by **Lemma** 1.1.13, we have that $Id(B/mB) = Id(B/m^nB)$. Moreover, since each of those arrow is unique, we have that the canonical morphism $A/m^{n+1}A \to A/m^nA$ make the following diagram commutative

$$\begin{tikzcd} E_q \arrow[r, "v_{n+1}"] \arrow[dr, "v_n"'] & A/m^{n+1} \arrow[d] \\ & A/m^n \end{tikzcd}$$

In particular, we get an A-homomorphism $v : E_q \to \varprojlim A/m^n = A$ that lifts $\bar{v}$. As this is possible for every idempotent element of B/mB, we find out that B is decomposed. This means that every finite free A-algebra is decomposed. Applying **Theorem** 1.1.10, we find that A is henselian. $\square$

Remark 1.2.4. Notice that the A-algebra E defined above has the following important property: for every A-algebra C and for every ideal J in C such that $J^2 = 0$, there is a bijection

$$Id(C \otimes_A B) = Hom_A(E, C) \longleftrightarrow Hom_A(E, C/J) = Id(C/J \otimes_A B)$$

Therefore, $Spec(E) \longrightarrow Spec(A)$ is a formally étale morphism. Since, by construction, E is also a finitely presented A-algebra, we find that $Spec(E) \longrightarrow Spec(A)$ is actually an étale morphism, that is to say, E is an étale A-algebra.

1.3 New characterizations of henselian rings

We can extend **Theorem** 1.1.10 in the following way.

Theorem 1.3.1. *Let (A, m) be a local ring. The following statements are equivalent:*

1. *A is henselian.*
2. *If $P \in A[T]$ is a monic polynomial and its reduction modulo m has a simple root $\bar{a}$, there exists a unique simple root a of P such that $a + m = \bar{a}$.*
3. *If B is an étale A-algebra and $n \in Spec(B)$ is an ideal lying over m such that $k(n) = B_n/nB_n \cong k = A/m$, then $A \longrightarrow B_n$ is an isomorphism.*
4. *If $P_1, \dots, P_n \in A[T_1, \dots, T_n]$ and $\bar{a} = (\bar{a}_1, \dots, \bar{a}_n) \in k^n$ is an element such that $\bar{P}_i(\bar{a}) = 0$ for every $i = 1, \dots, n$ and $det\big(\frac{\partial \bar{P}_i}{\partial T_j}\big) \neq 0$, there exists an element $a \in A^n$ which lifts $\bar{a}$ and such that $P_i(a) = 0$, for every $i = 1, \dots, n$.*
5. *Let $P \in A[T]$. If $\bar{P}$ factorizes as $\bar{P} = \bar{Q}\bar{R}$ in $k[T]$, with $\bar{Q}$ monic and $\bar{Q}, \bar{R}$ coprime in $k[T]$, then P factors as $P = QR$, where Q and R are liftings of $\bar{Q}$ and $\bar{R}$ respectively.*

Proof. 1. ⇒ 2. Assume that $\bar{a}$ is a simple root of the polynomial $\bar{P} = P + m[T] \in k[T]$. Then $\bar{P} = (T - \bar{a})\bar{Q}$ for some monic polynomial $\bar{Q} \in k[T]$. Moreover, since $\bar{a}$ is a simple root, $T - \bar{a}$ and $\bar{Q}$ generate the unit ideal $k[T]$. Therefore, as A is henselian by hypothesis, there exist

liftings $T-a, Q \in A[T]$ of $T-\bar{a}$ and $\bar{Q}$ respectively such that $P=(T-a)Q$. Moreover, a is necessarily a simple root of the polynomial P, as otherwise $\bar{a}$ would be a multiple root of $\bar{P}$. Finally, the element a is unique as the factorization $P=(T-a)Q$ as above is unique (see *Remark* 1.1.11).
2. $\Rightarrow$ *3.* We can assume without loss of generality that $B=\big(\frac{A[T]}{(F)}\big)_{\bar{G}}$ is standard-étale (F monic polynomial, $F' \in B^{\times}$). As $k(n)=k$, n corresponds to a root $\bar{a}$ of $\bar{F}$. Moreover, since F' is invertible in B, $\bar{a}$ has to be simple. By hypothesis, we can factor the polynomial F in a unique way as $F=(T-a)Q$. Consider the natural morphism

$$\phi : \frac{A[T]}{F} \longrightarrow \frac{A[T]}{(T-a)} \times \frac{A[T]}{(Q)}$$

Notice that

$$\phi\Big(\frac{A[T]}{F}\Big) + m\Big(\frac{A[T]}{(T-a)} \times \frac{A[T]}{(Q)}\Big) = \frac{A[T]}{(T-a)} \times \frac{A[T]}{(Q)}$$

By Nakayama's Lemma, therefore, we obtain that ϕ is surjective. As both the A-algebra on the left and the A-algebra on the right are free A-modules of rank $deg(F)$, ϕ is also injective. Whence, n corresponds to a maximal ideal of $\frac{A[T]}{(T-a)} \cong A$. Therefore, $B_n \cong A$.
3. $\Rightarrow$ *1.* Let B be a finite free A-algebra and let E be its associated étale A-algebra. By what we proved in the previous section, B is decomposed if and only if

$$Hom_A(E, A) \longrightarrow Hom_A(E, k)$$

is a bijection.
Let $\bar{u} \in Hom_A(E,k)$ and set $n = Ker(\bar{u})$. Then $n \in Spec(E)$ is a maximal ideal of E which lies over m and such that $E_n/nE_n = k$. Applying the hypothesis, we thus get $E_n \cong A$. Therefore, we can lift the A-morphism $\bar{v}$ associated to $\bar{u}$ to an A-morphism $v : E_n \longrightarrow A$:

$$Hom_A(E_n, k) \cong Hom_A(A, k) \cong Hom_A(A, A) \cong Hom_A(E_n, A)$$

Composing v with the canonical morphism $E \longrightarrow E_n$, we obtain a lifting $u : E \longrightarrow A$ of $\bar{u}$. Therefore, B is decomposed.
3. $\Rightarrow$ *4.* Consider the A-algebra

$$B = \frac{A[T_1, \dots, T_n]}{(P_1, \dots, P_n)}$$

Consider the ideal n of B generated by m and $T_i - b_i$, where $\bar{b}_i = \bar{a}_i$ for every $i = 1, \dots, n$. Notice that $B/n \cong k$. Then the fact that $det\big(\frac{\partial \bar{P}_i}{\partial T_j}\big)_{|(\bar{b}_1, \dots, \bar{b}_n)} \neq 0$ implies that $det\big(\frac{\partial P_i}{\partial T_j}\big) \notin n$. Therefore, we can find an element b in $B - n$ such that $det\big(\frac{\partial P_i}{\partial T_j}\big)$ is invertible in B_b. This means that B_b is an étale A-algebra. Moreover, the ideal nB_b lies over m and $B_b/nB_b = k$. Using *3.*, we can find a section $\psi : B_b \longrightarrow A$ of the structure morphism of B_b. Therefore, setting $a_i = \psi(T_i)$, we find the desired element.
4. $\Rightarrow$ *5.* Let $P = a_n T^n + \cdots + a_1 T + a_0$. Let $r = deg(\bar{Q})$, $s = n - r$ and consider the set of

equations

$$\begin{gathered} X_0Y_0 = a_0 \\ X_0Y_1 + X_1Y_0 = a_1 \\ X_0Y_2 + X_1Y_1 + X_2Y_0 = a_0 \\ \vdots \\ X_{r-1}Y_s + Y_{s-1} = a_{n-1} \\ Y_s = a_n \end{gathered}$$

$(b_0, \dots, b_{r-1}, c_0, \dots, c_s)$ is a solution of this system of equations if and only if

$$P = (T^n + b_{r-1}T^{r-1} + \cdots + b_0)(c_sT^s + \cdots + c_1T + c_0)$$

The Jacobian matrix associated to this set of equations is

$$J = \begin{bmatrix} Y_0 & 0 & \dots & X_0 & 0 & \dots & \\ Y_1 & Y_0 & \dots & X_1 & X_0 & \dots & \\ Y_2 & Y_1 & \dots & X_2 & X_1 & \dots & \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ Y_s & Y_{s-1} & \dots & \dots & \dots & \dots & \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \\ 0 & \dots & \dots & 0 & \dots & 0 & 1 \end{bmatrix}$$

whose determinant is the resultant $res(Q, R)$ of the two polynomials Q and R defined as

$$Q = T^r + X_{r-1}T^{r-1} + \cdots + X_0$$

$$R = Y_sT^s + \cdots + Y_1T + Y_0$$

Moreover, notice that $res(\bar{Q}, \bar{R}) \neq 0$ as $\bar{Q}$ is monic and $\bar{Q}, \bar{R}$ are coprime. Therefore we can apply *4.* and we find a solution $(b_0, \dots, b_{r-1}, c_0, \dots, c_s)$ of the system, which yields the desired factorization of P.
5. $\Rightarrow$ *1.* Using **Theorem** 1.1.10, this implication is trivial. □

The last theorem gives some algebraic characterizations of henselian rings. However, we can also characterize henselian rings in geometric terms. This is what we will do in the next result.

Theorem 1.3.2. *Let (A, m) be a local ring and let k be its residue field. Let $S = Spec(A)$ and let s be the closed point of S. The following statements are equivalent:*

1. *A is henselian.*

2. *If $f : X \longrightarrow S$ is a quasi-finite and separated morphism, then $X = X_0 \amalg X_1 \amalg \cdots \amalg X_n$, where $s \notin f(X_0)$ and $X_i = Spec(B_i)$ is finite over S for every $i = 1, \dots n$, B_i being a local ring.*

3. *If $f : X \longrightarrow S$ is étale and $x \in X$ is a point such that $f(x) = s$ and $k(x) = k(s) = k$. Then f has a section $g : S \longrightarrow X$.*

Proof. 1. $\Rightarrow$ *2.* By Zariski's Main Theorem, f factorizes as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & S \\ & Y \arrow[ur, "f'"'] & \end{tikzcd}$$

where j is an open immersion and f' is a finite morphism. In particular, f' is an affine morphism and therefore $Y = (f')^{-1}(S)$ is an affine scheme, say $Y = Spec(B)$. Since A is henselian, B can be written as

$$B = \Pi_{i=1}^m B_{\mu_i}$$

where μ_i are its maximal ideals. Then $Y = \coprod_{i=1}^m Spec(B_{\mu_i})$. Let $\mu_1, \ldots, \mu_n$ be the closed points of Y which lie also in X and set $X_i = Spec(B_{\mu_i})$. $\coprod_{i=1}^n X_i$ is open and closed in X, as it is open and closed in Y. Therefore, if we set $X_0 = X - \coprod_{i=1}^n X_i$,

$$X = X_0 \amalg X_1 \amalg \cdots \amalg X_n$$

and it is immediate to observe that $s \notin f(X_0)$ (**Proposition** 1.1.3). Finally, as each B_{μ_i} is a quotient of a finite A-algebra, each X_i is finite over S for $i = 1, \ldots, n$.
2. $\Rightarrow$ *1.* Let B be a finite A-algebra. Then $f : X = Spec(B) \longrightarrow S$ is a finite morphism. In particular, it is quasi-finite and separated. Then we can write X as

$$X = X_0 \amalg X_1 \amalg \cdots \amalg X_n$$

with the properties listed above. If $X_0 \neq \emptyset$, then it would contain a closed point, as it is closed in an affine scheme. This is impossible since closed points of B are mapped onto s and $s \notin f(X_0)$. Therefore,

$$X = X_1 \amalg \cdots \amalg X_n = Spec(B_1) \amalg \cdots \amalg Spec(B_n)$$

that is to say, $B \cong \Pi_{i=1}^n B_i$ is decomposed.
1. $\Rightarrow$ *3.* Let $f : X \longrightarrow S$ be an étale morphism, $x \in X$ a point such that $f(x) = s$ and $k(x) = k$. Then there exists an affine open neighborhood $U = Spec(B)$ of x such that $A \longrightarrow B$ is standard étale. Let n be the prime ideal of B corresponding to x. Then, proceeding as in the proof of the previous theorem, we see that B_n is isomorphic to A and that it is a direct factor of B. This yields a section

$$S = Spec(A) \cong Spec(B_n) \longrightarrow Spec(B) \longrightarrow X$$

3. $\Rightarrow$ *1.* Let B be an étale A-algebra and let n be an ideal of B lying over m and such that $B_n/nB_n = k$. Then the morphism $Spec(B) \longrightarrow Spec(A)$ has a section, that is to say, there exists a morphism $B \longrightarrow A$ such that, when composed with the structural morphism of B, gives the identity on A. Moreover, we can find $B \longrightarrow A$ in such a way that n is the inverse image of m (see [StacksProj, Tag 04GH]). Consider the induced morphism

$$B_n \longrightarrow A$$

and the two compositions

$$B_n \longrightarrow A \longrightarrow B_n$$

$$A \longrightarrow B_n \longrightarrow A$$

which induce the identity on the residue fields. Applying **Proposition** 1.4.7 below, we immediately see that both compositions have to be the identity. □

1.4 Henselization

In this section we will show how to associate in an universal way an henselian ring to a local ring.

Definition 1.4.1. Let A be a local ring. An *henselization* of A is a couple (A^h, i), where A^h is an henselian ring and $i : A \longrightarrow A^h$ is a local homomorphism satisfying the following universal property: for every henselian ring B and for every local homomorphism $\phi : A \longrightarrow B$, there exists a unique homomorphism $\phi^h : A^h \longrightarrow B$ such that $\phi = \phi^h \circ i$

$$\begin{tikzcd} A \arrow[rr, "\phi"] \arrow[dr, "i"'] & & B \\ & A^h \arrow[ur, "\phi^h"'] & \end{tikzcd}$$

Notice that, as a consequence of the universal property, the henselization of A is unique up to (a unique) isomorphism.

Definition 1.4.2. Let (A, m) be a local ring. We say that an A-algebra is *local-étale* if it is of the shape B_n, where B is an étale A-algebra and n is a prime ideal of B which lies over m.

Remark 1.4.3. In general, a local-étale A-algebra is not étale. In fact, it has not to be finitely presented. Anyway, it is formally étale. Let B be an étale A-algebra and let n be a prime ideal of B over m. Let C be an A-algebra and $J \subseteq C$ an ideal such that $J^2 = 0$. Consider a commutative square

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "j"] \arrow[dd, "i"'] & & C \arrow[dd, "p"] \\ & B \arrow[dl, "\psi"] & \\ B_n \arrow[rr, "g"'] & & C/J \end{tikzcd}$$

Since B is an étale A-algebra, there exists a unique A-morphism $g' : B \longrightarrow C$ such that $g' \circ j = f$ and $p \circ g' = g \circ \psi$. Pick an element $x \in B - n$. We have that $p \circ g'(x) = g \circ \psi(x)$ is a unit in C/J. Let $y \in C$ such that $g'(x)y + J = 1 + J$. Then $g'(x)y = 1 + z$ for some $z \in J$, which is a nilpotent element. Therefore, $g'(x)y$ is a unit in C, whence the same is true for $g'(x)$. By the universal property of localization, g' factorizes through B_n. This means that we can lift the morphism g to a morphism $B_n \longrightarrow C$. The uniqueness of this map is clear. Thus, B_n is a formally étale A-algebra.

Definition 1.4.4. Let (A, m) be a local ring. A *local ind-étale* A-algebra is an inductive limit of local-étale A-algebras, where the transition maps are local homomorphisms.

Lemma 1.4.5. *Let (A, m) be a local ring, B' a local-étale A-algebra and C' a local-étale B'-algebra. Then C' is a local-étale A-algebra.*

Proof. By definition, $B' = B_n$, with B étale over A and $n \in Spec(B)$ an ideal over m. Moreover, $C' = C_p$, where C is an étale B_n-algebra and $p \in Spec(B)$ is an ideal which lies over nB_n. By the local structure of étale morphisms, there exists an element $f \in C - p$ such that C_f is a standard étale B_n-algebra, say

$$C_f \cong \left(\frac{B_n[T]}{(F)}\right)_{\bar{G}}$$

where F is a monic polynomial and F' is a unit in C_f. As $C_p \cong (C_f)_{pC_f}$, we can assume without loss of generality that C is a standard étale B_n-algebra. Let $h \in B - n$ such that F, G lie in $B_h[T]$, with F monic and F' a unit in $D = \left(\frac{B_h[T]}{(F)}\right)_{\bar{G}}$. Since D is a (standard) étale B_h-algebra and B_h is an étale A-algebra, D is an étale A-algebra. Now, we have two morphisms

$$\phi : D \longrightarrow C \qquad \psi : C \longrightarrow C_p$$

Let q be the prime ideal of D which corresponds to p. Consider a morphism $\chi : D \longrightarrow E$ such that for every $x \notin q$, $\chi(x) \in E^{\times}$. If $b \in B - n$, then $b \notin q$ (as q lies over n). Hence, there exists an homomorphism $\chi' : C \longrightarrow E$ such that $\chi' \circ \phi = \chi$. Every element of $C - p$ maps to a unit via χ', providing a unique homomorphism $C_p \longrightarrow E$. Hence, $C_p \cong D_q$ and C' is a local-étale A-algebra. □

In order to prove the next proposition, we will need a lemma.

Lemma 1.4.6. *Let B be an unramified A-algebra and C any A-algebra. Let $p \in Spec(C)$ and $\pi : C \longrightarrow C/p = k(p)$ the canonical projection. Let $u, v \in Hom_A(B, C)$ such that $\pi \circ u = \pi \circ v$. Then there exists an element $f \in C - p$ such that the compositions of u and v with the canonical morphism $C \longrightarrow C_f$ coincide.*

Proof. Let $S = Spec(A)$, $X = Spec(B)$, $Y = Spec(C)$ and let $u^*, v^* : Y \longrightarrow X$ be the morphisms of S-schemes which corresponds to u and v respectively. Label $y \in Y$ the point which corresponds to p. Let $\Delta : X \longrightarrow X \times_S X$ denote the diagonal morphism. It is an open immersion as X is unramified over S. Let W be the fiber product of X and Y over $X \times_S X$

$$\begin{tikzcd} W \arrow[r] \arrow[d, "i"] & X \arrow[d, "\Delta"] \\ Y \arrow[r, "u^* \times v^*"] & X \times_S X \end{tikzcd}$$

As Δ is an open immersion, i is an open immersion as well. W is the biggest subscheme of Y on which u^* and v^* coincide. Moreover, $y \in W$. Therefore, there exists a principal open subset $y \in D(f) \subseteq W \subseteq Y$. This concludes the proof of the lemma. □

Proposition 1.4.7. *Let (A, m) be a local ring, (B, n) a local ind-étale A-algebra and (C, p) a local A-algebra whose structure morphism is local. Set $k(A) = A/m$, $k(B) = B/n$, $k(C) = C/p$ and $Hom_A^{loc}(B, C) = \{\phi \in Hom_A(B, C) : \phi$ is local$\}$.*

1. *$Hom_A^{loc}(B, C) = Hom_A(B, C)$*
2. *$\Phi : Hom_A(B, C) \longrightarrow Hom_{k(A)}(k(B), k(C))$ is injective. It is also surjective if C is henselian.*

Proof. *1.* Let $f \in Hom_A(B, C)$. We need to show that $f^{-1}(p) = n$. Since

$$Hom_A(B, C) = Hom_A(\varinjlim B_i, A) = \varprojlim Hom_A(B_i, C)$$

we can assume without loss of generality that B is a local-étale A-algebra. Therefore, Let $B = (B')_{n'}$, where B' is an étale A-algebra and n' lies over m. Clearly, we have

$$Hom_A(B, C) = \{\phi \in Hom_A(B', C) : \phi^{-1}(p) = n\}$$

Therefore, it is sufficient to show that if q is another ideal of B' which lie over m, there are no inclusion relations between n and q. This can be seen as follows : since B' is étale over A, the fiber of m is a finite product of finite separable extensions of $k(A)$. Therefore, neither $q \subseteq n$ or $n \subseteq q$. In particular,

$$Hom_A^{loc}(B, C) = Hom_A(B, C)$$

2. Let $B = \varinjlim B_i$. Then $k(B) = \varinjlim k(B_i)$. We have that

$$Hom_A(B, C) = \varprojlim Hom_A(B_i, C)$$

and, similarly,

$$Hom_{k(A)}(k(B), k(C)) = \varprojlim Hom_{k(A)}(k(B_i), k(C))$$

This means that we can assume without loss of generality that B is a local-étale A-algebra. Let B' be an étale A-algebra and let $B = (B')_{n'}$. As B' is étale over A, $A/m \longrightarrow B'/mB'$ is an étale morphism. In particular, there are only finitely many prime ideals of B', $n' = n'_0, n'_1, \ldots, n'_s$ which lie over m. There exists an element $f \notin n'$ which lies in n'_i for every $i = 1, \ldots, s$. Interchanging B' with B'_f, we can therefore assume that n' is the only point in the fiber of m.

Let $u : B' \longrightarrow C$ be an A-homomorphism. As $u^{-1}(p)$ lies over m, we have that it coincides with n'. Therefore,

$$Hom_A(B', C) = Hom_A(B'_{n'}, C)$$

Moreover, it is also true that

$$Hom_{k(A)}(k(B), k(C)) = Hom_{k(A)}(B'/mB', k(C))$$

Hence, we can replace B with B'. So far, we reduced to the case where B is an étale A-algebra. Let $u, v \in Hom_A(B, C)$ such that the induced morphisms $\bar{u}, \bar{v} : B \longrightarrow k(C)$ coincide. By the previous lemma, there exists an element $f \in C - p$ such that the compositions of u and v with the canonical morphism $C \longrightarrow C_f$ coincide. As C is a local ring with maximal ideal p, $C \cong C_f$ and therefore $u = v$.
Assume that C is henselian. Set $D = B \otimes_A C$. Then $Hom_A(B, C) = Hom_C(D, C)$. In a similar way, $Hom_{k(A)}(k(B), k(C)) = Hom_A(B, k(C)) = Hom_C(D, k(C))$. The set $Hom_C(D, k(C))$ can be identified with the prime ideals q of D which lie over p and such that $k(q) = k(C)$. By the characterization theorem of henselian rings we proved above (**Theorem** 1.3.1), any such ideal is such that $D_q \cong C$. Therefore, any homomorphism $\bar{u} : D \longrightarrow k(C)$ with $q := Ker(\bar{u})$ can be lifted to an homomorphism $u : D \longrightarrow C$. □

The last proposition is the key result which is used to prove the next important result.

Corollary 1.4.8. *Let (A, m) be an henselian ring. Consider the functors*

$$\mathscr{F} : \{\textit{finite étale local } A - \textit{algebras}\} \longrightarrow \{\textit{finite separable extensions of } k(A)\}$$

$$\mathscr{G} : \{\textit{finite étale } A - \textit{algebras}\} \longrightarrow \{\textit{finite étale } k(A) - \textit{algebras}\}$$

both defined by $B \mapsto B \otimes_A k(A)$. Then F and G are equivalences of categories.

Proof. Let us start with the local case: let B and C be two local finite étale A-algebras. In particular, we have that B is local ind-étale algebra and C is henselian. Therefore, it is an immediate consequence of the previous proposition that

$$Hom_A(B, C) = Hom_{k(A)}(k(B), k(C))$$

which means that F is a fully faithful functor. To show essential surjectivity, consider a finite separable extension L of $k(A)$. By the primitive element theorem, there exists a monic separable polynomial $\bar{F}(T) \in k(A)[T]$ such that $L \cong k(A)[T]/(\bar{F}(T))$. Let $F(T) \in A[T]$ be any polynomial such that its reduction modulo m coincides with $\bar{F}(T)$ and set $B = A[T]/(F(T))$. Since $\bar{F}(T)$ is separable, we have

$$F'(T)B + mB = B$$

Therefore, Nakayama's lemma implies that $F'(T)B = B$, i.e. $F'(T)$ is a unit in B. This implies that B is a finite étale algebra. Moreover, we know that the maximals ideals of B are those which lie over m. As $B \otimes_A k(A) = L$, B is local. This shows that F is an equivalence of categories.

Let us now show that also $\mathscr{G}$ is an equivalence of categories. Let B and C be two finite étale A-algebras. Then they decompose in the product of finite local A-algebras (A is henselian)

$$B = \prod_{i=1}^{m} B_i \quad C = \prod_{j=1}^{n} C_j$$

By the universal property of the direct product we get

$$Hom_A(B, C) = \prod_{j=1}^{n} Hom_A(B, C_j)$$

Furthermore, we have that

$$\bar{C} = \prod_{j=1}^{n} \bar{C}_j = \prod_{j=1}^{n} k(C_j)$$

and

$$Hom_{k(A)}(\bar{B}, \bar{C}) = \prod_{j=1}^{n} Hom_{k(A)}(\bar{B}, k(C_j))$$

Hence, we can assume without loss of generality that C is local. In particular, C is henselian. Moreover, we have bijections

$$Hom_A(B, C) \longleftrightarrow Hom_{Spec(A)}(Spec(C), Spec(B)) =$$

$$Hom_{Spec(A)}(Spec(C), \coprod_{i=1}^{m} Spec(B_i))$$

As $Spec(C)$ is connected, we have that

$$Hom_{Spec(A)}(Spec(C), \coprod_{i=1}^{m} Spec(B_i)) = \coprod_{i=1}^{m} Hom_{Spec(A)}(Spec(C), Spec(B_i))$$

Similarly,

$$Hom_{Spec(k(A))}(Spec(k(C)), \coprod_{i=1}^{m} Spec(k(B_i))) = \coprod_{i=1}^{m} Hom_{Spec(k(A))}(Spec(k(C)), Spec(k(B_i)))$$

Therefore, using the previous result, we get

$$Hom_A(B, C) = \coprod_{i=1}^{m} Hom_A(B_i, C) =$$

$$\coprod_{i=1}^{m} Hom_{k(A)}(k(B_i), k(C)) = Hom_{k(A)}(B/mB, k(C))$$

and $\mathscr{G}$ is a fully faithful functor. Let L be a finite étale $k(A)$-algebra. Thus, it is isomorphic to a finite product of finite separable extensions of $k(A)$: $L = L_1 \times \cdots \times L_s$. Let B_i be a finite, local étale A-algebra such that $F(B_i) = L_i$. Set $B = B_1 \times \cdots \times B_s$. It follows that $\mathscr{G}(B) = L$. This shows that also G is an equivalence of categories. $\square$

In the last part of this section, we will show that an henselization of a local ring always exists.

Proposition 1.4.9. *Let (A, m) be a local ring.*

1. *There exist a set Λ and a family $\{A_\lambda\}$ of local-étale A-algebras such that every local-étale A-algebra is isomorphic to A_λ for some λ. For every $\lambda \in \Lambda$, let us denote with m_λ the maximal ideal of A_λ.*

2. *Let $I = \{\lambda \in \Lambda : A_\lambda/m_\lambda = A/m\}$. For $i, j \in I$ we will write $i \leq j$ if there exists a local homomorphism $\phi_{ji} : A_i \longrightarrow A_j$. Then $\leq$ is a order relation and I is a directed set with respect to such relation.*

Proof. 1. Put

$$\Lambda_0 = \{(P, q) \in A[T] \times Spec(A[T]) : P \text{ is monic}, q \text{ lies over } m, P' \notin q\}$$

For $\lambda_0 = (P, q) \in \Lambda_0$, set $B_{\lambda_0} = A[T]/(P(T))$ and $A_{\lambda_0} = (B_{\lambda_0})_q$. Then the local structure theorem of étale morphisms implies that every local-étale A-algebra is isomorphic to A_{λ_0} for some $\lambda_0 \in \Lambda_0$. Define the equivalence relation $\sim$ on Λ_0: $\lambda_0 \sim \mu_0$ if and only if $A_{\lambda_0} \cong A_{\mu_0}$. set $\Lambda = \Lambda_0/\sim$. This set verifies the properties listed in *1.*
2. It is clear that the relation $\leq$ is reflexive and transitive. Let $i \leq j$ and $j \leq i$. The composition $\phi_{ij} \circ \phi_{ji}$ maps to $id_{k(A_i)}$ when we consider the induced map on the residue fields. Since the map Φ of **Proposition** 1.4.7 is injective, we see that $\phi_{ij} \circ \phi_{ji} = id_{A_i}$. In a similar way we can show that $\phi_{ji} \circ \phi_{ij} = id_{A_j}$. Therefore $A_i \cong A_j$, that is to say, $i = j$. It remains to be shown that I is directed. Let $i, j \in I$ and let A_i, A_j be the corresponding local-étale A-algebras. By the definition of local-étale A-algebras

$$A_i \cong (B_i)_{n_i} \qquad A_j \cong (B_j)_{n_j}$$

where B_i and B_j are étale A-algebras and n_i, n_j are ideals over m. Moreover, by definition of the set I, the residue fields of A_i and A_j are equal to $k(A)$. As a consequence

$$(A_i \otimes_A A_j) \otimes_A k(A) \cong (A_i \otimes_A k(A)) \otimes_{k(A)} (A_j \otimes_A k(A)) \cong k(A) \otimes_{k(A)} k(A) \cong k(A)$$

i.e. there exists a unique prime ideal of $A_i \otimes_A A_j$ which lies over m. Let A' be the localization of $A_i \otimes_A A_j$ in such ideal. It is clear that we have two local homomorphisms $A_i \longrightarrow A'$ and $A_j \longrightarrow A'$. Finally, since A_i and A_j are localizations of the étale A-algebras B_i and B_j respectively, A' is a localization of the étale A-algebra $B_i \otimes_A B_j$. Finally $A' \cong A_k$ for some $k \in I$ and $i, j \leq k$. $\square$

We are finally able to prove the existence of an henselization of a local ring. We will use the same notation as above.

Theorem 1.4.10. *Set $A^h = \varinjlim A_i$ and let $i : A \longrightarrow A^h$ be the canonical local homomorphism. Then (A^h, i) is an henselization of the local ring A.*

Proof. First of all, we need to show that A^h is an henselian ring. We will show that if B is an étale A-algebra and n is a point of $Spec(B)$ which lies in the special fiber and determine a trivial extensions of the residue fields, then $B_n \cong A$. Without loss of generality, we can assume that B is a standard étale A-algebra, say

$$B = \Big(A^h[T]/(F(T))\Big)_{\bar{G}}$$

For a sufficiently large i, the coefficients of $F(T), G(T)$ and of the inverse of F' come from A_i. Set

$$B_i = \Big(A_i[T]/(F(T))\Big)_{\bar{G}}$$

Then we have a cocartesian diagram

$$\begin{tikzcd} B_i \arrow[r] & B \\ A_i \arrow[u] \arrow[r] & A^h \arrow[u] \end{tikzcd}$$

Label n_i the prime ideal of B_i corresponding to n. Clearly, we have that the residue field of $(B_i)_{n_i}$ is $k(A)$ and $(B_i)_{n_i}$ is a local-étale A_i-algebra. By **lemma** 1.4.5, it is also a local-étale A-algebra. Therefore, there exists an index $j \geq i$ such that $(B_i)_{n_i} \cong A_j$. Let B_j be the A_j-algebra $B_i \otimes_{A_i} A_j$ and label n_j the ideal of B_j corresponding to n. Define $S = Spec(A_i)$ and $X = Spec(B_i)$. Since B_i is unramified over A_i, the diagonal morphism $\Delta : X \longrightarrow X \times_S X$ is both an open and a closed immersion. If we set $T = Spec(A_j)$, then we have an S-morphism $s : T \longrightarrow X$ corresponding to $B_i \longrightarrow (B_i)_{n_i} \longrightarrow A_j$. Consider the cartesian square

$$\begin{tikzcd} X \times_S X & X \times_S T \arrow[l] \\ X \arrow[u, "\Delta"] & T \arrow[l, "s"] \arrow[u, "f"'] \end{tikzcd}$$

Then f is an open and closed immersion and $(B_j)_{n_j}$ is a direct factor of B_j and is isomorphic to A_j. By base change $A_j \longrightarrow A^h$, we find that $B_n \cong A^h$

To show that (A^h, i) owns the desired universal property, notice that A^h is local ind-étale and that $k(A^h) = k(A)$. Thus, applying **Proposition** 1.4.7, we have that for any henselian ring B,

$$Hom_A(A^h, B) = Hom_{k(A)}(k(A^h), k(B))$$

consists of only one morphism. ☐

Chapter 2

Henselian pairs

In this chapter we generalize the notion of henselian rings. In particular, we remove the hypothesis which foresees A to be local.

As we have seen, the main feature of an henselian ring (A, m) is the possibility to decompose finite algebras over A. This decomposition corresponds to a decomposition of the corresponding finite A/m-algebra.
If A is not local, we have first of all to choose an ideal $I \subseteq A$. Moreover, notice that it does not make sense to consider the same property used to define henselian rings. We define henselian pairs in terms of their behavior with respect to idempotents instead.

2.1 Lifting of idempotent elements

We take the previous chapter as a guideline.

Definition 2.1.1.

1. a *pair* (A, I) is a ring A together with an ideal $I \subseteq A$. We denote with $\bar{A}$ the quotient ring A/I
2. a *morphism of pairs* $\phi : (A, I) \longrightarrow (B, J)$ is a ring homomorphism $\phi : A \longrightarrow B$ such that $\phi(I) \subseteq J$

   The induced morphism between quotients is denoted with $\bar{\phi} : \bar{A} \longrightarrow \bar{B}$
3. a morphism of pairs $\phi : (A, I) \longrightarrow (B, J)$ is *strict* if $\phi(I)B = J$

Definition 2.1.2. Let (A, I) be a pair. An *étale neighborhood* of (A, I) is a pair (B, J) together with a strict morphism $\phi : (A, I) \longrightarrow (B, J)$ such that $\phi : A \longrightarrow B$ is an étale ring map and $\bar{\phi} : \bar{A} \longrightarrow \bar{B}$ is an isomorphism.

Similarly, if S is a scheme and $\bar{S}$ is a closed subscheme, an *étale neighborhood* of $\bar{S}$ in S is an étale morphism $T \longrightarrow S$ that induces an isomorphism $T \times_S \bar{S} \cong \bar{S}$.

In the local case, following Raynaud's exposition, we defined a ring to be henselian if every finite algebra was decomposed. It is clear that it doesn't make sense to extend this definition to the nonlocal case. Anyway, considering what we proved for henselian rings, it seems reasonable to work with idempotents in an analogous way. In fact, this is exactly what we will do. With this in mind, it should be self evident that **Theorem** 2.1.6 below is a key result. In order to be able to prove it, we need some preliminary results.

The following lemma characterizes the structure of finite, monogenic, torsion free algebras over normal rings. It was first showed by Hamet Seydi.

Lemma 2.1.3. *Let A be an integral normal ring with fraction field K. Let B be a finite, monogenic, torsion free A-algebra and let $x \in B$ be a generator. If $P(X) \in K[X]$ is the characteristic polynomial of x, then $P(X) \in A[X]$ and $B \cong A[X]/(P(X))$.*

Proof. The coefficients of $P(X)$ lie in A since the latter is normal (see [Bou, Chapter V §1, Corollary 1 to Proposition 17]). Applying Hamilton Cayley Theorem, we get that $P(x) \in B \otimes_A K$ is zero. Since B is torsion free, this means that $P(x) \in B$ is zero. Therefore, the homomorphism

$$A[X]/(P(X)) \longrightarrow B$$

is surjective. Consider the following short exact sequence:

$$0 \longrightarrow N \to \tfrac{A[X]}{(P(X))} \to B \longrightarrow 0$$

Then we get the following exact sequence

$$0 = Tor_1^A(B, K) \to N \otimes_A K \to \frac{A[X]}{(P(X))} \otimes_A K \to B \otimes_A K \to 0$$

Since $\frac{A[X]}{(P(X))} \otimes_A K \to B \otimes_A K$ is an isomorphism (it is a surjective morphism between two K-vector spaces of the same dimension), we have $N \otimes_A K = 0$. But $N \subseteq \frac{A[X]}{(P(X))}$ is torsion free, whence $N = 0$ and $B \cong \frac{A[X]}{(P(X))}$. □

Lemma 2.1.4. *Let A be a noetherian, reduced ring of finite type over $\mathbb{Z}$. Let $I \subseteq A$ be an ideal and B a finite, monogenic A-algebra. Assume that*

1. *B is flat outside $V(I)$ and A is normal outside $V(I)$*

2. *The ideal of B whose elements are killed by some power of I is zero*

Let $\bar{e} \in Id(B/IB)$. Then there exist an element $t \in B$ which lifts $\bar{e}$, an integer $m \geq 0$ and a monic polynomial $P(X) \in A[X]$ such that

$$P(t) = 0$$

and

$$P(X) = (X^2 - X)^m \ mod \ I$$

Proof. Let $p_1, \dots, p_r$ be the minimal prime ideals of A. Set $A_i = A/p_i$ and let $\tilde{A}_i$ be the normalization of A_i. Moreover, let $\tilde{A} = \prod_{i=1}^r \tilde{A}_i$, $\tilde{I} = I\tilde{A}$, $\tilde{I}_i = I\tilde{A}_i$, $\tilde{B}_i = B \otimes_A \tilde{A}_i$, $\tilde{B} = \prod_{i=1}^r \tilde{B}_i$. Let T_i be the ideal of $\tilde{B}_i$ formed by those elements of $\tilde{B}_i$ that are killed by some power of $\tilde{I}_i$ and let $C_i = \tilde{B}_i/T_i$. Let x be a generator of B and let $\tilde{x}, \tilde{x}_i, c_i$ be the images of x in $\tilde{B}, \tilde{B}_i$ and C_i respectively. As A is excellent (see **Chapter 4**), $\tilde{A}$ of finite type over A. Moreover, since A is normal outside $V(I)$, the morphism $Spec(\tilde{A}) \longrightarrow Spec(A)$ is an isomorphism outside $V(I)$. Let $\mathcal{C} = \{a \in A : a\tilde{A} \subseteq A\}$. Then $\mathcal{C}$ contains a power of I. Thanks to **Lemma** 1.1.13, we can substitute I with one of its powers, whence we can assume that $\mathcal{C}$ contains I. Then we can replace I with $\tilde{I}$ and we can assume that $I = \tilde{I}$.
Apply **Lemma** 2.1.3 to each A_i-algebra C_i to deduce that they are finite and free. Label $m_i = rank_{A_i} B_i$. Let $t \in B$ an element over $\bar{e}$ and set $y = t^2 - t \in IB$. Let $z = (z_1, \dots, z_r) \in C = \prod_{i=1}^r C_i$ be its image. In particular, we have that $z_i \in IC_i$. Therefore, the characteristic polynomial P_i of z_i in the A_i free algebra B_i is a monic polynomial in $A_i[X]$ of degree m_i such that $P_i = X^{m_i} \ mod \ IA_i$ (because of Hamilton Cayley theorem). Let $m = max\{m_i : i = 1, \dots, r\}$ and $Q_i = X^{m-m_i} P_i$ for every i. Then the polynomials Q_i define a monic polynomial $Q \in \tilde{A}[X]$ of degree m such that $Q = X^m \ mod \ \tilde{I}$ and such that $Q(z) = 0$. As $\tilde{I} \subseteq \mathcal{C}$, $Q \in A[X]$ and $Q(z)$ is killed by some power of I. Then $Q(z) = 0$ because of the hypothesis. Finally, since $I = \tilde{I}$, we have that

$$Q = X^m \ mod \ I$$

To conclude, it suffices to choose $P(X) = Q(X^2 - X)$. □

Finally, we will need the following simple lemma in scheme theory.

Lemma 2.1.5. *Let S be a scheme and let X, Y S-schemes. Then*

$$(X \times_S Y)_{red} = (X_{red} \times_{S_{red}} Y_{red})_{red}$$

Proof. Let T be a reduced scheme and assume we are given a morphism $T \longrightarrow X \times_S Y$. Consider the following diagram

$$\begin{tikzcd}
T \arrow[rrr] \arrow[dr] \arrow[ddd] & & & Y_{red} \arrow[d] \arrow[ddl] \\
& X \times_S Y \arrow[rr] \arrow[dd] & & Y \arrow[dd] \\
& & S_{red} \arrow[dr] & \\
X_{red} \arrow[r] \arrow[urr] & X \arrow[rr] & & S
\end{tikzcd}$$

Then there exists a unique morphism $T \longrightarrow X_{red} \times_{S_{red}} Y_{red}$ that fits into the diagram. As T is reduced, such morphism factors uniquely through $(X_{red} \times_{S_{red}} Y_{red})_{red}$. Therefore, we can conclude by the universal property of $(X \times_S Y)_{red}$. □

Theorem 2.1.6. *Let (A, I) be a pair, B a finite A algebra and $\bar{e} \in Id(\bar{B})$ an idempotent element. Then there exists an étale neighborhood (A', I') of (A, I) such that, if $B' = B \otimes_A A'$ and $\bar{e}'$ is the idempotent of $\bar{B}' = B'/I'B'$ that corresponds to $\bar{e}$, then $\bar{e}'$ lifts to an idempotent element e' of B'.*

Proof. Step 1. reduction to the case where B is monogenic.
Let x be any element of B lying over $\bar{e}$. Label C the sub A-algebra of B generated by x. In particular, $C \subseteq B$ is an integral ring extension. By the Going up/Going down theorems we deduce that $Spec(B) \longrightarrow Spec(C)$ is surjective. Moreover, notice that the following diagram is cartesian

$$\begin{tikzcd} Spec(\bar{B}) \arrow[r] \arrow[d] & Spec(\bar{C}) \arrow[d] \\ SpecB \arrow[r] & Spec(C) \end{tikzcd}$$

In fact, we have

$$\bar{B} = B \otimes_A A/I$$

$$B \otimes_C \bar{C} = B \otimes_C C/IC = B \otimes_C C \otimes_A A/I = B \otimes_A A/I$$

Since surjective morphisms are stable under base change (see [StacksProj, Tag 01S1]), also $\bar{\phi}^* : Spec(\bar{B}) \longrightarrow Spec(\bar{C})$ is surjective. As we have a decomposition $Spec(\bar{B}) = V(\bar{e}) \amalg V(1-\bar{e})$, we obtain the following decomposition of $Spec(\bar{C})$:

$$Spec(\bar{C}) : \bar{\phi}^*(V(\bar{e})) \amalg \bar{\phi}^*(V(1-\bar{e}))$$

Claim: $\bar{\phi}^*(V(\bar{e})) = V(\bar{x})$

Proof. (**Claim**) First of all, notice that $\bar{\phi}^*(V(\bar{e})) = V(\phi^{-1}(\bar{e}))$, where we denoted ϕ the morphism $\bar{C} \longrightarrow \bar{B}$ By construction, we have that $\bar{x} \mapsto \bar{e}$. Therefore, $(\bar{x}) \subseteq (\phi^{-1}(\bar{e}))$ and $V(\phi^{-1}(\bar{e})) \subseteq V(\bar{x})$. Conversely, if $p = \bar{\phi}^*(q) \in V(\bar{x})$, then $\bar{x} \in p$. Hence, $\bar{e} \in q$ and $\phi^{-1}(\bar{e}) \subseteq p = \bar{\phi}^*(q)$, giving us the other inclusion. □

In a similar way, we obtain that $\bar{\phi}^*(V(1-\bar{e})) = V(1-\bar{x})$. This means that $Spec(\bar{C}) = V(\bar{x}) \amalg V(1-\bar{x})$. Then there exists an idempotent element $\bar{f}$ in $\bar{C}$ such that $\phi(\bar{f}) = \bar{e}$. It is clear that it suffices to show the theorem for C and $\bar{f}$.
Step 2. reduction to the case where A is of finite type over $\mathbb{Z}$.
Step 2.1. reduction to the case where B is finitely presented over A.
Let $P(X)$ be a monic polynomial in $A[X]$ such that $P(x) = 0$, where x is a generator of B. Set $C = A[X]/(P(X))$. Thus $B \cong C/J$ for some ideal J of C. Write $J = \varinjlim J_i$, where $J_i \subseteq J$ are finitely generated subideals. Label $C_i = C/J_i$ for each i. Then

$$B = \varinjlim C_i$$

$$\bar{B} = B \otimes_A A/I = \varinjlim C_i \otimes_A A/I = \varinjlim (C_i \otimes_A A/I) = \varinjlim \bar{C}_i$$

Then $\bar{e}$ is the image of an idempotent element $\bar{e_i} \in \bar{C_i}$ for some sufficiently large index i. Up to a substitution of B with C_i, we can assume B to be finitely presented over A.
Step 2.2. reduction to the case where I is finitely generated.
Write I as the direct limit of its finitely generated subideals I_i. Then

$$\bar{B} = \varinjlim B/I_iB$$

and the element $\bar{e}$ is the image of an idempotent $\bar{e_i} \in B/I_iB$ for a sufficiently large index i. Suppose we solved our problem for the pair (A, I_i), i.e. we found an étale neighborhood (A', I'_i) of (A, I_i) such that the thesis is satisfied. Define $I' = u(I)A'$, where u is the étale homomorphisms $A \longrightarrow A'$. Notice that $I'_i \subseteq I'$. Then $A/I \longrightarrow A'/I'$ is injective by definition of I and is surjective since $A \longrightarrow A'/I'_i$ is surjective. Therefore, we get an isomorphism

$$A/I \cong A'/I'$$

that is to say, (A', I') is an étale neighborhood of (A, I) that verifies the thesis. We reduced to the case when I is finitely generated.
Step 2.3. reduction to the case where A is a $\mathbb{Z}$-algebra of finite type.
Let $a_1, \ldots, a_r$ be a set of generators for I. Write $A = \varinjlim A_i$, where each A_i is a subalgebra of A of finite type over $\mathbb{Z}$ containing the set $\{a_1, \ldots, a_r\}$. Let I_i be the ideal in A_i generated by such set. Then

$$(A, I) = \varinjlim (A_i, I_i)$$

Since B is finitely presented over A, we can find a sufficiently large index i such that there exists a monogenic, finite A_i-algebra B_i verifying $B \cong B_i \otimes_{A_i} A$. Up to substituting i with a bigger index, we can also assume that $\bar{e}$ comes from an idempotent element $\bar{e_i} \in \bar{B_i} = B_i/I_iB_i$. Suppose that we have an étale neighborhood (A'_i, I'_i) of (A_i, I_i) such that there exists an idempotent element e'_i of $B_i \otimes_{A_i} A'_i$ mapping to $\bar{e_i}$. Then it is easy to see that $(A'_i \otimes_{A_i} A, I'_i(A'_i \otimes_{A_i} A))$ is an étale neighborhood of (A, I) and the image e' of e'_i in $A'_i \otimes_{A_i} A$ maps to $\bar{e}'$. In this way we reduced to the case where A is of finite type over $\mathbb{Z}$.

Set $S = Spec(A)$, $\bar{S} = Spec(\bar{A})$, $X = Spec(B)$, $\bar{X} = Spec(\bar{B})$. Let J be the kernel of the morphism $A \longrightarrow B$ and let Y be the closed subscheme $V(J)$.
Step 3. reduction to the reduced case.
Let us assume that we are able to find an étale neighborhood (A'_0, I'_0) of (A_{red}, IA_{red}) that solves the problem for B_{red}. Since we have an equivalence

$$\{\text{étale } A\text{-algebras}\} \longrightarrow \{\text{étale } A_{red}\text{-algebras}\}$$

(see [SGAI, Exposé I, Théorème 8.3]) we can lift A'_0 to an étale A-algebra A'.
Claim: (A', IA') is an étale neighborhood of (A, I).

Proof. It is clear that $(A, I) \longrightarrow (A', IA')$ is a strict homomorphism. The morphism $A \longrightarrow A'$ is étale by construction and

$$A'_0/I'_0 \cong A_{red}/IA_{red}$$

implies that

$$(A'/IA')_{red} \cong (A'_{red} \otimes_{A_{red}} (A/I)_{red})_{red} \cong (A'_0/I'_0)_{red} \cong A/\sqrt{I} \cong (A/I)_{red}$$

Therefore, applying the equivalence cited before with A/I instead of A, we get an isomorphism

$$A'/IA' \cong A/I$$

□

Moreover, applying **Lemma** 2.1.5 once again we obtain that

$$Id(B \otimes_A A') = Id((B \otimes_A A')_{red}) = Id((B_{red} \otimes_{A_{red}} A'_{red})_{red})$$

$$= Id(B_{red} \otimes_{A_{red}} A'_{red}) = Id(B_{red} \otimes_{A_{red}} A'_0)$$

In fact, since A'_0 is étale over A_{red} it is reduced (see [SGAI, Exposé I, Proposition 9.2]) and we have that $A'_{red} = A'_0$ by construction. In this way, we can assume that S, X and Y are reduced.
Step 4. reduction to the case when Y is reduced and normal outside $V(I)$ and X is flat over Y outside $V(I)$.
We make an induction on $dim_{Krull}(Y - \bar{S})$, i.e. on the dimension of $im(X \longrightarrow S) - \bar{S}$.
Step 4.1. base of the induction.
If $dim_{Krull}(Y - \bar{S}) < 0$, then $Y - \bar{S} = \emptyset$. Hence, $Y \subseteq \bar{S}$ and therefore $\sqrt{I} \subseteq \sqrt{J}$. In this case, it is immediate to observe that $IB \subseteq Nil(B)$. Therefore, $B = B_{red} = \bar{B}_{red}$ and

$$Id(B) = Id(\bar{B}_{red}) = Id(\bar{B})$$

Step 4.2. $dim_{Krull}(Y - \bar{S}) \geq 0$
We can assume Y reduced because of *Step 3.* and because $V(J) = V(\sqrt{J})$. Moreover, because of *Step 2*, Y is excellent (A is of finite type over $\mathbb{Z}$). Consider the open subscheme $U = Y - \bar{S} \subseteq Y$. Then there exists an open dense subset of U which is normal (see [EGA IV.2, Scholie 7.8.3 (iv)]). The fact that Y is reduced also implies that X is flat over the generic points of the irreducible components of U. Then there exists an open dense subset $V \subseteq U$ which is normal and such that X is flat over V. Let K be the ideal of A defining the following closed subset of S

$$\bar{S} \cup (Y - V) = \bar{S} \cup (U - V)$$

As $\bar{S} \subseteq \bar{S} \cup (Y - V)$, we have that $K \subseteq \sqrt{I}$. Since A is noetherian, eventually replacing K with some power, we can assume that $K \subseteq I$. Let Z be the affine scheme $Spec(B/KB)$. Then, from the commutativity of the involved cartesian square, it is immediate to observe that the image of Z in S is $Y - V$. Therefore,

$$(Y - V) - \bar{S} = U - V$$

By the construction of V, it follows that

$$dim_{Krull}(U - V) < dim_{Krull}(U) = dim_{Krull}(Y - \bar{S})$$

Thus, we can apply the induction hypothesis: there exists an étale neighborhood (A', I') of (A, I) such that $\bar{e}$ lifts to an idempotent of $B'/K'B' = (B/KB) \otimes_A A'$, where $B' = B \otimes_A A'$

and $K' = KA'$. Let $S' = Spec(A')$ and $X' = Spec(B')$. Since $X \longrightarrow S$ is a finite morphism, we get that the image Y' of X' inside S' is closed. Moreover, we also have that $Y' = Y \times_S S'$. By definition, Y is normal outside $V(K)$ and therefore Y' is normal outside $V(K') = V(K) \times_S S'$. By the same argument, we also get that X' is flat outside $V(K')$.
To conclude this reduction step, notice that an étale neighborhood of (A', K') provides us an étale neighborhood of (A', I') as $K' \subseteq I'$, and an étale neighborhood of (A', I') is an étale neighborhood of (A, I). Finally, eventually replacing A with A' and I with K', we can assume that Y is reduced, normal outside $V(I)$ and X is flat over Y outside $V(I)$.
Step 5. reduction to the case where the ideal T of elements of B killed by some power of I is zero.
Set $C = B/T$ and let $\bar{f}$ be the image of $\bar{e}$ inside $\bar{C} = C/IC$. Suppose we are able to lift $\bar{f}$ to an idempotent element f of C. We are assuming to work with noetherian rings, therefore we can apply Artin-Rees Lemma: there exists an integer $k > 0$ such that

$$I^n B \cap T = I^{n-k}(I^k B \cap T)$$

Therefore, choosing a sufficiently large $l \in \mathbb{N}$, we have

$$I^l B \cap T = 0$$

Eventually replacing I with one of its powers, we can assume $IB \cap T = 0$ (remember that $Id(B/IB) = Id(B/I^r B)$ for every r). With this assumption, we have the following commutative diagram, where the lines are exact:

$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "="] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T \arrow[r] & \bar{B} \arrow[r] & \bar{C} \arrow[r] & 0
\end{tikzcd}$$

Let $e \in B$ such that $e + T = f$:

$$e^2 - e = t \in T \cap IB = 0$$

Then $\bar{e}$ lifts to an idempotent element of B. It is then sufficient to show the problem in the case when $T = 0$.
Step 6. reduction to the case where B is a finite free A-algebra.
Let R be the ring A/J, i.e. the image of A inside B. Then we can apply **Lemma** 2.1.4 to R, IR and B: there exists $x \in B$ which lifts $\bar{e}$ and an integer m such that x is a root of some monic polynomial $P(X) \in R[X]$ which verifies

$$P(X) = (X^2 - X)^m \ mod \ IR$$

Let $Q(X) \in A[X]$ be a monic polynomial over $P(X)$ such that

$$Q(X) = (X^2 - X)^m \ mod \ I$$

Let $D = A[X]/(Q(X))$ and label ψ the natural morphism $D \longrightarrow B$. Notice that $\bar{D} = D/ID = \bar{A}[X]/(X^2 - X)^m$. Denote the unique idempotent of $\bar{D} = D/ID$ which is zero in $V(X)$ and 1 in $V(1-X)$ with $\bar{g}$. Then it is clear that $\bar{\psi}(\bar{g}) = \bar{e}$ and it is sufficient to consider the case when B is a finite free A-algebra.
Step 7. end of the proof.
Let E be the étale A-algebra which represents the idempotent elements of B that we introduced in **Chapter 1 §2**. Then $\bar{e}$ corresponds to an $\bar{A}$-homomorphism

$$\bar{u} : E/IE = \bar{E} \longrightarrow \bar{A}$$

In particular, $\bar{u}$ is surjective. Moreover, as $id_{\bar{S}} : \bar{S} \longrightarrow \bar{S}$ is étale and $Spec(\bar{E}) \longrightarrow \bar{S}$ is unramified, $\bar{S} \longrightarrow Spec(\bar{E})$ is étale. Therefore, the image of $\bar{S}$ in $Spec(\bar{E})$ is both closed and open. Then, it corresponds to an idempotent element of $\bar{E}$ which takes value 0 over points in the image of $\bar{S}$ and 1 elsewhere. Let $h \in E$ be any element over $\bar{h}$. Then (E_h, IE_h) is an étale neighborhood of (A, I). By the characterizing property of E, the canonical morphism $E \longrightarrow E_h$ corresponds to an idempotent element of $B \otimes_A E_h$ which lifts $\bar{e}$. This concludes the proof of the theorem. □

Lemma 2.1.7. *Let A be a ring and let B be an A-algebra of finite type. Let $\phi : Spec(B) \longrightarrow Spec(A)$ be the structure morphism. Let*

$$U = \{p \in Spec(B) : p \text{ is isolated in } \phi^{-1}(\phi(p))\}$$

Then U is an open subset of $Spec(B)$.

Proof. Let $p \in U$ and let A' be the integral closure of A in B. By [Ray, Chapitre IV, Théorème 1] there exists an element $f \in A'$, $f \notin p$ such that $A'_f \cong B_f$. Write

$$A' = \varinjlim A_i \quad A_i \text{ subalgebra of} A' \text{ finite over} A$$

Since tensor products commute with direct limits, we have that

$$B_f \cong A'_f \cong \varinjlim (A_i)_f$$

As B_f is of finite type over A, $B_f \cong (A_i)_f$ for a sufficiently large index i. Then B_f is a localization of a finite A-algebra, whence $Spec(B_f) \longrightarrow Spec(A)$ is a quasi-finite morphism. □

Corollary 2.1.8. *Let S be an affine scheme, $\bar{S} \subseteq S$ a closed subscheme and X an affine scheme of finite type over S such that $\bar{X} = X \times_S \bar{S}$ is finite over S. Then there exists a commutative diagram*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

with the following properties

1. *S' is an étale affine neighborhood of $\bar{S}$ in S*

2. *Z is an étale neighborhood of $\bar{X}$ in X*

3. *Z is finite over S'*

Proof. Since $\bar{X}$ is finite over S, X is quasi-finite over S in every point of $\bar{X}$. Applying the previous lemma, we can find an element $f \in \mathcal{O}_X(X)$ invertible over $\bar{X}$ such that $X_f = Spec(\mathcal{O}_X(X)_f)$ is quasi-finite over S. Eventually replacing X with X_f, we can thus assume that X is quasi-finite over S.
By Zariski's Main Theorem, we can find a scheme Y which is finite over S and such that X is an open subscheme of Y. Let $\bar{Y}$ be the scheme $Y \times_S \bar{S}$. Then $\bar{X}$ is an open subscheme of $\bar{Y}$. Moreover, since $\bar{X}$ is finite over $\bar{S}$ (as $\bar{X}$ is finite over S and $\bar{S} \longrightarrow S$ is separated), we get that $\bar{X}$ is a finite $\bar{Y}$-scheme ($\bar{Y} \longrightarrow \bar{S}$ is finite, whence separated). Then $\bar{X}$ is both an open and a closed subset of $\bar{Y}$. Let $\bar{e}$ be the idempotent element of $\mathcal{O}_{\bar{Y}}(\bar{Y})$ that is 1 over $\bar{X}$ and zero elsewhere. By **Theorem** 2.1.6, there exists an étale affine neighborhood S' of S such that $\bar{e}$ lifts to an idempotent element e' in $\mathcal{O}_{Y \times_S S'}(Y \times_S S')$. Let $Y' = Y'_1 \amalg Y'_2$ be the corresponding decomposition, where $Y' = Y \times_S S'$. Then $X' = X \times_S S'$ is an open subset of Y' and $\bar{X}' = \bar{X} \times_S S' = Y'_1 \times_S \bar{S}$ because of the choice of e'. Consider the closed subset $Y'_1 - X' \subseteq Y'_1$. Then its image in S' is a closed subset (since Y'_1 is finite over S') that does not meet $\bar{S}' = \bar{S} \times_S S'$. Eventually replacing S' with S'_f, where f is a suitable element in $\mathcal{O}_{S'}(S')$ invertible over $\bar{S}'$, we can assume that $Y'_1 \subseteq X'$. Then we can take $Z = Y'_1$, which is an étale neighborhood of $\bar{X}$ in X, and the corollary is proved. □

2.2 Henselian pairs

Let (A, I) be a pair and let S be the multiplicative system in A formed by elements of the form $1 + x$, $x \in I$. Then it is clear that the ideal $IS^{-1}A$ is contained in the Jacobson radical.
Notation: We will denote the pair $(S^{-1}A, IS^{-1}A)$ with (A_S, I_S).
Inspired by the local case, we give the following definition:

Definition 2.2.1. A pair (A, I) is *henselian* if I is contained in the Jacobson radical and if for every finite A-algebra B, given an idempotent element $\bar{e} \in B \otimes_A A/I$, there exists an idempotent element $e \in B$ which maps onto $\bar{e}$.

In order to give some characterizations of henselian pairs, we will need the following preliminary result.

Lemma 2.2.2. *Let (A', I') be an étale neighborhood of (A, I). There exist an integer $m \geq 0$ and a monic polynomial $P(X) \in A[X]$ such that*

1. *$P(X) = (X^2 - X)^m$ mod I*

2. *If E is the étale A-algebra which represents the idempotent elements of $A[X]/(P(X))$, then there exists some $h \in E$ such that (E_h, IE_h) is an étale neighborhood of (A, I) and the structure morphism $A \longrightarrow E_h$ factors through A'.*

Proof. Proceeding as in the proof of **Theorem** 2.1.6, we can reduce to the case when A is a finitely generated $\mathbb{Z}$-algebra.

Therefore, A is a quotient of $\mathbb{Z}[T_1, \ldots, T_n]$ for a suitable n. Let J denote the inverse image of I in $\mathbb{Z}[T_1, \ldots, T_n]$. Let $S = Spec(\mathbb{Z}[T_1, \ldots, T_n])$, $\bar{S} = Spec(\mathbb{Z}[T_1, \ldots, T_n]/J) = Spec(A/I)$, $X = Spec(A')$. Notice that X is quasi finite over S (as $A \longrightarrow A'$ is an étale morphism) and it is finite over $\bar{S}$ (as (A', I') is an étale neighborhood of (A, I)). We can apply **Corollary** 2.1.8: there exists a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

where

1. S' is an étale affine neighborhood of $\bar{S}$ in S
2. Z is an étale neighborhood of $\bar{X}$ in X
3. Z is finite over S'

Moreover, looking at the proof of such result, we also see that we can assume Z to be an open and closed subscheme of $X' = X \times_S S'$. Let $T = Spec(A)$. Notice that T is a closed subscheme of S which contains $\bar{S}$. Let T' and $\bar{S}'$ denote the inverse images in S' of T and S' respectively. With our notation, X is an étale neighborhood of $\bar{S}$ in T. As Z is open in X', it is an étale neighborhood of $\bar{S}'$ in T'. Being finite over S', Z is finite over T'. Then $Z \longrightarrow T'$ is an isomorphism in a neighborhood of $\bar{S}'$. Eventually replacing S' with a convenient affine neighborhood of $\bar{S}'$, we can assume that $Z \longrightarrow T'$ is an isomorphism. Then T' dominates X and to prove the lemma we can replace A' with $\mathcal{O}_{T'}(T')$. In this way, we reduced to the case where A' lifts to an étale neighborhood B' of $B = \mathbb{Z}[T_1, \ldots, T_n]$ where B is normal. Therefore, it suffices to show the lemma for the étale neighborhood B' of B.
Let us assume that A is a normal integral domain with fraction field K. Let $K' = K \otimes_A A'$ and let B' be the normalization of A in K'. Label S, X, Y the spectra of A, A' and B' respectively. Zariski's Main Theorem implies that $B' \longrightarrow A'$ corresponds to an open immersion $X \longrightarrow Y$ (since A' is normal). Let $\bar{S} = Spec(A/I), \bar{X} = X \times_S \bar{S}$ and $\bar{Y} = Y \times_S \bar{S}$. Then $\bar{X}$ is an open subscheme of $\bar{Y}$. It is also closed since $\bar{X}$ is finite over $\bar{S}$. Let $\bar{e}$ be the idempotent element of $\bar{B}' = B'/IB'$ that is 1 over $\bar{X}$ and 0 elsewhere. Let t be any element of B' over $\bar{e}$ and let $y = t^2 - t$. Notice that $y = 0 \bmod IB'$. Label $q(X)$ the characteristic polynomial of y in $A[X]$. By Hamilton-Cayley, we know that $q(X) = X^m \; mod \; I$ (where m is the rank of B' over A). Then $p(X) = q(X^2 - X)$ is a monic polynomial of degree m, killed by t, such that $p(X) = (X^2 - X)^m \; mod \; I$. Then we have an A-homomorphism

$$C = A[X]/(p(X)) \longrightarrow B'$$

$$X \mapsto t$$

If $\bar{f}$ is the idempotent element of $\bar{C} = C/IC$ that has the same image of X in $\bar{C}_{red}$, it is clear that $\bar{f} \mapsto \bar{e} \in \bar{B}'$. Let E be the étale A-algebra which represents idempotent elements of C and label $\bar{u} : \bar{E} = E/IE \longrightarrow \bar{A}$ the $\bar{A}$-homomorphism corresponding to $\bar{f}$. Then $Spec(\bar{u}) : Spec(\bar{A}) \longrightarrow Spec(\bar{E})$ is an immersion both closed and open. After we replace E with E_h, for a convenient element $h \in E$, we can assume that $\bar{u}$ is an isomorphism. In this case, (E, IE) is an étale neighborhood of (A, I). Furthermore, it comes from the definition of E that the idempotent $\bar{f} \in \bar{C}$ lifts to an idempotent in $C \otimes_A E$. Then $\bar{e}$ lifts to an idempotent element of $B' \otimes_A E$. From this, we deduce that E dominates A'. □

We are now able to provide some characterizations of henselian pairs.

Theorem 2.2.3. *Let (A, I) be a pair such that I is contained in the Jacobson radical. The following statements are equivalent.*

1. *(A, I) is henselian.*
2. *For every finite free A-algebra B, every idempotent of $\bar{B} = B/IB$ lifts to an idempotent of B.*
3. *For every A-algebra $B = A[X]/(p(X))$, where $p(X)$ is a monic polynomial such that $p(X) = (X^2 - X)^m \bmod I$, every idempotent element in $\bar{B} = B/IB$ lifts to an idempotent element of B.*
4. *If $p(X) \in A[X]$ is a monic polynomial such that its reduction modulo I $\bar{p}(X)$ can be factored as the product of two monic polynomials $\bar{q}(X)$, $\bar{r}(X)$ that generate the unit ideal in $\bar{A}[X]$, there exist two monic polynomials $q(X)$, $r(X) \in A[X]$ over $\bar{q}(X)$ and $\bar{r}(X)$ respectively such that $p(X) = q(X)r(X)$.*
5. *If (A', I') is an étale neighborhood of (A, I), there exists an A-homomorphism $A' \longrightarrow A$.*

Proof. *1.* $\Rightarrow$ *2.*: obvious.
2. $\Rightarrow$ *3.*: obvious.
3. $\Rightarrow$ *4.*: Let $p(X) \in A[X]$ as in *4.* and set $B = A[X]/(p(X))$. Then $\bar{B} = \bar{A}[X]/(\bar{q}(X)) \oplus \bar{A}[X]/(\bar{r}(X))$. Applying *3.*, we get an idempotent e of B corresponding to $(1, 0)$. Let $B = B_1 \oplus B_2$ be the corresponding decomposition. Applying **Lemma** 1.1.8 and *Remark* 1.1.9, we get that

$$B_1 = A[X]/(q(X)) \quad \text{where } q(X) \text{ is a monic polynomial over } \bar{q}(X)$$
$$\text{with } deg(q(X)) = deg(\bar{q}(X))$$
$$B_2 = A[X]/(r(X)) \quad \text{where } r(X) \text{ is a monic polynomial over } \bar{r}(X)$$
$$\text{with } deg(r(X)) = deg(\bar{r}(X))$$

It is easy to see that $X + (p(X))$ is a root of $q(X)r(X)$. Therefore, $q(X)r(X)$ is a multiple of $p(X)$. Since they are both monic polynomials of the same degree, they coincide.
4. $\Rightarrow$ *3.*: it suffices to notice that $(X^m, (1 - X)^m) = (1)$.
3. $\Rightarrow$ *5.*: it follows immediately from the previous lemma.
5. $\Rightarrow$ *1.*: let B be a finite A-algebra and let $\bar{e}$ be an idempotent element of $\bar{B} = B/IB$.

Theorem 2.1.6 provides us an étale neighborhood (A', I') of (A, I) such that $\bar{e}$ lifts to an idempotent $e' \in B \otimes_A A'$. Then by the hypothesis we have an A-homomorphism $A' \longrightarrow A$. Then the image of e' in $B = B \otimes_A A$ is an idempotent element over $\bar{e}$. □

Proposition 2.2.4. *1. Let (A, I) be an henselian pair and let B be an integral A-algebra. Then (B, IB) is an henselian pair.*

2. If $\{(A_i, I_i)\}$ is an inductive system of henselian pairs, then $(\varinjlim A_i, \varinjlim I_i)$ is an henselian pair.

Proof. *2.* Label $A = \varinjlim A_i$ and $I = \varinjlim I_i$. Let $B = A[X]/(p(X))$, where $p(X) \in A[X]$ is a monic polynomial such that $p(X) = (X^2 - X)^m \ mod\ I$. Let $\bar{e}$ be an idempotent element of $\bar{B} = B/IB$. There exists a sufficiently large index i such that the coefficients of $p(X)$ lie in A_i and $p(X) = (X^2 - X)^m \ mod\ I_i$. Thus, we can consider the A_i-algebra $B_i = A_i[X]/(p(X))$. Moreover, we can also assume that $\bar{e}$ comes from an idempotent element $\bar{e_i} \in \bar{B_i} = B_i/I_iB_i$. Let e_i be the idempotent element in B_i over $\bar{e_i}$. Its image $e \in B = A \otimes_{A_i} B_i$ is an idempotent over $\bar{e}$. Therefore, (A, I) is an henselian pair.
1. It is clear from the definition that, if B is a finite A-algebra, then (B, IB) is an henselian pair.
Let B be an integral A-algebra and write it as an inductive limit $B = \varinjlim B_i$, where B_i are subalgebras of B finite over A. By the previous remark, $\{(B_i, IB_i)\}$ form an inductive system of henselian pairs. Therefore,

$$(B, IB) = (\varinjlim B_i, \varinjlim I_iB_i)$$

is an henselian pair by *2.* □

2.3 Henselization

As in the case of local henselian ring, it would be desirable to have a universal way to associate to each pair (A, I) an henselian pair (A^h, I^h). We will take the local case as a guideline once again.

Definition 2.3.1. Let (A, I) be a pair. An *henselization* of (A, I) is a pair (A^h, I^h) endowed with a morphism $\phi : (A, I) \longrightarrow (A^h, I^h)$ having the following universal property: for any henselian pair (B, J) together with a morphism $\psi : (A, I) \longrightarrow (B, J)$, there exists a unique morphism $\tilde{\psi} : (A^h, I^h) \longrightarrow (B, J)$ such that $\psi = \tilde{\psi} \circ \phi$

$$\begin{tikzcd}
(A, I) \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & (B, J) \\
& (A^h, I^h) \arrow[ur, dashed, "\tilde{\psi}"'] &
\end{tikzcd}$$

Remark 2.3.2. It is a formal consequence of the universal property that, if it exists, an henselization is unique up to (a unique) isomorphism.

Remark 2.3.3. Since we want $I^h \subseteq Jac(A^h)$, it is clear that a morphism $(A,I) \longrightarrow (A^h, I^h)$ factors through $(S^{-1}A, IS^{-1}A)$, where S is the multiplicative system $\{1+x : x \in I\}$. Therefore, we can assume that $I \subseteq Jac(A)$.

Definition 2.3.4. Let (A,I) be a pair with $I \subseteq Jac(A)$. A *local-étale neighborhood* of (A,I) is a pair (B,J) isomorphic to the localization in $1+I'$ of an étale neighborhood (A',I') of (A,I).

The next theorem guarantees the existence of henselizations. Its proof is similar to the local case (**Theorem** 1.4.10).

Theorem 2.3.5. *Let (A,I) be a pair with $I \subseteq Jac(A)$.*

1. *There exists a set $\{(A_j, I_j)\}_{j \in J}$ of local-étale neighborhoods of (A,I) such that every local-étale neighborhood (A',I') of (A,I) is isomorphic to a unique (A_j, I_j).*
2. *J is filtrant with respect to the order relation*

$$j \leq k \text{ } iff \text{ } A_k \text{ dominates } A_j$$

3. *$(A^h, I^h) = (\varinjlim A_j, \varinjlim I_j)$ endowed with the canonical morphism $(A,I) \longrightarrow (A^h, I^h)$ is an henselization of (A,I).*

The previous theorem can be improved as follows:

Theorem 2.3.6. *The henselization process is left adjoint to the inclusion functor*

$$\textit{Henselian Pairs} \longrightarrow \textit{Pairs}$$

Proof. See [StacksProj, Tag 0A02]. □

Remark 2.3.7. The same result is valid in the local case, see [StacksProj, Tag 0A03].

Remark 2.3.8. If (A,I) is a pair and (A,I) is its henselization, then $A \longrightarrow A^h$ is flat, $I^h = IA^h$ and $A/I^n \longrightarrow A^h/I^nA^h$ is an isomorphism for all n. For a proof, see [StacksProj, Tag 0AGU]

Chapter 3

Projective limits of schemes

The aim of this chapter is to introduce the concept of projective limits of schemes. We also study some of their fundamental properties. These will be crucial in the following part, when we will use the results and the strategies of this chapter to make some fundamental reduction steps. We will follow the exposure given in [EGA IV.3, §8].

3.1 Introduction

Assume we are given an inductive system of rings (A_i, ϕ_{ji}), indexed by the set I. Let $A = \varinjlim A_i$ be the inductive limit in *Rings*. Moreover, assume we are also given an A_i-scheme X_i, for some index $i \in I$. Then we can form a projective system in *Schemes*$/A_i$ as follows:

$$X_j = X_i \times_{Spec(A_i)} Spec(A_j) \text{ for every } j \geq i$$

$$\text{define } g_{i,j} : X_j \longrightarrow X_i \text{ as}$$

$$\begin{tikzcd} X_j \arrow[r] \arrow[d, "g_{ij}"] & Spec(A_j) \arrow[d] \\ X_i \arrow[r] & Spec(A_i) \end{tikzcd}$$

Moreover, let

$$X = X_i \times_{Spec(A_i)} Spec(A)$$

Notice that $X = X_j \times_{Spec(A_j)} Spec(A)$ for every $j \geq i$. Then we can define the morphisms $g_j : X \longrightarrow X_j$ as

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g_j"] & Spec(A) \arrow[d] \\ X_j \arrow[r] & Spec(A_j) \end{tikzcd}$$

The first important result that we will prove is the following

$$X = \varprojlim X_j \text{ in } \textit{Schemes} \text{ and in } \textit{Schemes}/A_i$$

This agrees with the specific case where X_i is the spectrum of an A_i-algebra, which is a consequence of the equivalence

$$\textit{Rings}^{op} \longrightarrow \textit{Affine} \quad \textit{Schemes}$$

Given such isomorphism, it is natural to ask if there exist hypothesis on the A_j and X_j that guarantee that X has some property $\mathscr{P}$ if and only if, for some sufficiently large index j, X_j has the same property $\mathscr{P}$.
Furthermore, we will see that, for every finitely presented A-scheme X, there exists some index $j >> 0$ and a finitely presented A_i-scheme such that

$$X = X_j \times_{Spec(A_j)} Spec(A)$$

This kind of results have a broad field of application. We will be mainly interested in the following one: assume we want to study some property of a finitely presented Y-scheme, local on Y. Then we can suppose that $Y = Spec(A)$ is affine. Write $A = \varinjlim A_i$ as the direct limit of its subalgebras A_i that are finitely generated over $\mathbb{Z}$. The results of this chapter allow us to reduce to the case when A is a finitely generated A_i-algebra. As we will see, this will be extremely important for our purpose.

3.2 Projective limits of schemes

3.2.1 Existence of inductive limits in the category of

Let S_0 be a ringed space and let $(\mathscr{A}, \phi_{ji})_{i\in I}$ be a direct system of $\mathscr{O}_{S_0}$-algebras. Then we can consider them as $\mathscr{O}_{S_0}$-modules. We know that a limit $\mathscr{A} = \varinjlim \mathscr{A}_i$ exists in $\mathscr{O}_{S_0}$*-modules.* Label $\phi_i : \mathscr{A} \longrightarrow \mathscr{A}_i$ the morphisms of $\mathscr{O}_{S_0} - modules$ that make $\mathscr{A}$ a colimit. For every $i \in I$, let $m_i : \mathscr{A}_i \otimes_{\mathscr{O}_{S_0}} \mathscr{A}_i \longrightarrow \mathscr{A}_i$ be the homomorphism of $\mathscr{O}_{S_0} - modules$ which defines the multiplication on $\mathscr{A}_i$. Then $\{m_i\}_{i\in I}$ is a direct system of morphisms of $\mathscr{O}_{S_0} - modules$. Since $\otimes_{\mathscr{O}_{S_0}}$ commutes with colimits, $m = \varinjlim m_i$ is an homomorphism of $\mathscr{O}_{S_0} - modules$ too. Taking limits of diagrams which express associativity of m_j, existence of a unit in $\mathscr{A}_i$ and commutativity, we get that m endows $\mathscr{A}$ with the structure of an $\mathscr{O}_{S_0} - algebra$. Moreover, it turns out that $\phi_i : \mathscr{A}_i \longrightarrow \mathscr{A}$ are morphisms of $\mathscr{O}_{S_0} - algebras$, for every index i. Furthermore, $(\mathscr{A}, \phi_i) = \varinjlim \mathscr{A}_i$ in $\mathscr{O}_{S_0} - algebras$, i.e. for every $\mathscr{O}_{S_0}$-algebra $\mathscr{B}$, we have a bijection

$$Hom_{\mathscr{O}_{S_0}-alg}(\mathscr{A}, \mathscr{B}) \longrightarrow \varprojlim Hom_{\mathscr{O}_{S_0}-alg}(\mathscr{A}_i, \mathscr{B})$$

$$f \mapsto (f \circ \phi_i)$$

In fact, it we know that

$$Hom_{\mathscr{O}_{S_0}-alg}(\mathscr{A}, \mathscr{B}) \subseteq Hom_{\mathscr{O}_{S_0}-mod}(\mathscr{A}, \mathscr{B}) = Hom_{\mathscr{O}_{S_0}-mod}(\mathscr{A}_i, \mathscr{B})$$

and therefore the above map is injective.
If $\{f_i\}_{i\in I} \in \varprojlim Hom_{\mathscr{O}_{S_0}-alg}(\mathscr{A}_i, \mathscr{B})$, the following diagrams commute

$$\begin{tikzcd}
\mathscr{A}_i \otimes \mathscr{A}_i \arrow[r, "m_i"] \arrow[d, "f_i \otimes f_i"] & \mathscr{A}_i \arrow[d, "f_i"] \\
\mathscr{B} \otimes \mathscr{B} \arrow[r] & \mathscr{B}
\end{tikzcd}$$

Applying the $\varinjlim$ functor, we get the commutative square

$$\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A} \arrow[r, "m"] \arrow[d, "f \otimes f"] & \mathscr{A} \arrow[d, "f"] \\
\mathscr{B} \otimes \mathscr{B} \arrow[r] & \mathscr{B}
\end{tikzcd}$$

This means that $f = \varinjlim f_i$ is a morphism of $\mathscr{O}_{S_0}$-algebras, and the above map is also surjective.

3.2.2 Projective limits of schemes

Let us consider the following situation:

$(\dagger)$ S_0 a scheme and $\mathscr{A}_i$ a quasi-coherent $\mathscr{O}_{S_0}$-algebras. Then also $\mathscr{A} = \varinjlim \mathscr{A}_i$ is quasi-coherent. Put $S_i = Spec(\mathscr{A}_i)$, $S = Spec(\mathscr{A})$,

$$u_{i,j} : S_j \longrightarrow S_i \quad \text{corresponding to } \phi_{i,j} : \mathscr{A}_i \longrightarrow \mathscr{A} \text{ for } i \leq j$$

$$u_i : Spec(\mathscr{A}) \longrightarrow Spec(\mathscr{A}_i) \quad \text{corresponding to } \phi_i : \mathscr{A}_i \longrightarrow \mathscr{A}$$

Remark 3.2.3.
1. $(S_i, u_{ji})_{i\in I}$ form a projective system in *Schemes*$/S_0$
2. u_{ji} and u_i are affine morphism, whence quasi-compact and separated (see [EGAII, Proposition 1.6.2])

Having in mind the affine case, it is natural to ask whether

$$\mathscr{A} = \varinjlim \mathscr{A}_i = \varprojlim S_i$$

This follows immediately from the next general lemma.

Lemma 3.2.4. *Let $\mathbb{C}$ be a category and let T be an object. Label $\mathbb{C}/T$ the slice category over T. Let $(S_i, u_{ji})_{i\in I}$ be a projective system in $\mathbb{C}/T$. Then every limit of such system in $\mathbb{C}/T$ is also a limit in $\mathbb{C}$ and reciprocally.*

Proof. Assume (S, u_i) is a limit of $(S_i, u_{ji})_{i\in I}$ in $\mathbb{C}/T$. Let $f_i : S_i \longrightarrow T$ and $f : S \longrightarrow T$ be the structure morphisms of S_i and T respectively. Then the following diagram commutes:

$$\begin{tikzcd}
S_j \arrow[rr, "u_{ij}"] \arrow[ddr, "f_j"'] & & S_i \arrow[ddl, "f_i"] \\
& S \arrow[ul, "u_j"] \arrow[ur, "u_i"'] \arrow[d, "f"] & \\
& T &
\end{tikzcd}$$

Let $(g_i : R \longrightarrow S_i)_{i\in I}$ be a collection of morphisms such that the following condition holds

$$(\star) \qquad g_i = u_{ij} \circ g_j \quad \forall\, i \leq j$$

$$\begin{tikzcd}
& R \arrow[ddl, "g_j"'] \arrow[ddr, "g_i"] \arrow[d, dashed, "g"] \arrow[ddd, bend right=90] \arrow[ddd, bend left=90] & \\
& S \arrow[dl, "u_j"'] \arrow[dr, "u_i"] & \\
S_j \arrow[rr, "u_{ij}"] \arrow[dr, "f_j"'] & & S_i \arrow[dl, "f_i"] \\
& T &
\end{tikzcd}$$

From the equalities $f_j = f_i \circ u_{ij}$ and $(\star)$ we see can R can be endowed in a natural way with the structure of a T-scheme

$$f_i \circ g_i : R \longrightarrow T$$

By the hypothesis we get an unique T-morphism $g : R \longrightarrow S$. In particular, $u_i \circ g = g_i$ for every index i. If $g' : R \longrightarrow S$ also has this property, then we have that it is a T-morphisms which fits into the upper diagram. By the universal property of S in $\mathbb{C}/T$, we obtain $g = g'$. Conversely, if S is a projective limit of the given system in $\mathbb{C}$, it is straightforward to verify that it is also a projective limit in $\mathbb{C}/T$. $\square$

In what follows we will keep the notation we fixed in (†).

Proposition 3.2.5. *(S, u_i) is a projective limit of (S_i, u_{ji}) in Schemes$/S_0$. Moreover, if a morphism $h : S_0 \longrightarrow T$ is given, we can endow every S_0-morphism with the structure of a T-scheme. Then (S, u_i) is a projective limit of (S_i, u_{ji}) in the category of T-schemes.*

Proof. Let $q : X \longrightarrow S_0$ be an S_0-scheme. We have that (see[Bosch, p. 288-289])

$$Hom_{S_0}(X, S) = Hom_{S_0}(X, Spec(\mathscr{A})) = Hom_{S_0}(\mathscr{A}, q_*\mathscr{O}_X) =$$

$$Hom_{\mathscr{O}_{S_0}-alg}(\varinjlim \mathscr{A}_i, q_*\mathscr{O}_X) = \varprojlim Hom_{\mathscr{O}_{S_0}-alg}(\mathscr{A}_i, q_*\mathscr{O}_X) = \varprojlim Hom_{S_0}(X, S_i)$$

The second statement follows from the previous lemma. $\square$

The next step is to understand whether the nice result we have just found for spectra of quasi-coherent algebras can be generalized. Namely, assume that we are given an S_i-scheme X_i. Put $X_j = X_i \times_{S_i} S_j$ for every $i \leq j$, $X = X_i \times_{S_i} S = X_j \times_{S_j} S$, $v_{jk} : X_k \longrightarrow X_j = id_{X_i} \times u_{jk}$ and $u_j : id_{X_i} \times u_j$. It is clear that $(X_j, u_{jk})_{j \geq i}$ form a projective system in the category of X_i-schemes. Then it would be desirable to have that

$$X \cong \varprojlim X_j$$

This is a consequence of the following general lemma.

Lemma 3.2.6. *Let $\mathbb{C}$ be a category with pullbacks, $q : T' \longrightarrow T$ a morphism in $\mathbb{C}$ and $\mathbb{C}/T$, $\mathbb{C}/T'$ the slice categories over T and T' respectively. Let $(S_i, u_{ij})_{i \in I}$ be a projective system in $\mathbb{C}/T$. For every index i, label $S'_i = S_i \times_T T'$ and $u'_{ij} = u_{ij} \times id_{T'}$. Then $(S'_i, u'_{ij})_{i \in I}$ is a projective system in $\mathbb{C}/T'$. Assume a limit (S, u_i) of (S_i, u_{ij}) exists in $\mathbb{C}/T$. Then, if $S' = S \times_T T'$ and $u'_i = u_i \times id_{T'}$, (S', u'_i) is a limit of (S'_i, u'_{ij}) in $\mathbb{C}/T'$.*

Proof. Let $(g'_i : R \longrightarrow S'_i)_{i \in I}$ be a collection of T'-morphisms such that, for every $i \leq j$, the following diagram commutes

$$\begin{tikzcd}
 & & R \arrow[dl, "g'_j"'] \arrow[dr, "g'_i"] \arrow[ddll, bend right=60, "g_j"'] \arrow[ddrr, bend left=60, "g_i"] & & \\
 & S'_j \arrow[rr, "u'_{ij}"] \arrow[dl] \arrow[dr] & & S'_i \arrow[dl] \arrow[dr] & \\
S_j \arrow[rrrr, "u_{ij}"] \arrow[dr] & & T' \arrow[dl, "q"'] \arrow[dr, "q"] & & S_i \arrow[dl] \\
 & T \arrow[rr, "id_T"] & & T &
\end{tikzcd}$$

where g_i is the composition of g'_i with $S'_i \longrightarrow S_i$ for every i. Then $(g_i)_{i \in I}$ form a projective system of morphisms in $\mathbb{C}/T$ and therefore there exists a unique T-morphism $g : R \longrightarrow S$ such that $g_i = u_i \circ g$, for all $i \in I$. It is clear from the previous diagram and from the universal property of S' that there exists a unique morphism $g' : R \longrightarrow S'$ such that $g'_i = u'_i \circ g'$ for all $i \in I$. $\square$

In particular, this applies to the situation we are interested in.

Proposition 3.2.7. $(X, v_j)_{j\geq i}$ *is a projective limit of* $(X_j, v_{jk})_{j\geq i}$ *in Schemes/*X_i.

Remark 3.2.8. It follows from **Lemma** 3.2.4 that $X = \varprojlim X_j$ in the category of schemes too.

Remark 3.2.9. Let S be any ringed space. Inductive limits with respect to any preordered set always exist in $\mathscr{O}_S - algebras$. In fact, we showed that inductive limits exist when we consider filtrant ordered sets. If $(\mathscr{A}_i)$ is any collection of $\mathscr{O}_S$-algebras indexed by a preordered set, we can always add to such collection the algebras $\mathscr{A}_i \otimes \mathscr{A}_j$. It is clear that in this way we obtain a collection of algebras indexed by a directed set.
In particular, if S is a scheme, inductive limits always exist in the category of quasi-coherent $\mathscr{O}_S$-algebras, due to the fact that if $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are quasi-coherent, then also $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{C}$ is also quasi-coherent.

3.3 Finitely presented modules over projective limits

3.3.1 Morphisms between finitely presented modules over projective limits

Let us keep notation (†) we introduced in §**3.2.2**

Remark 3.3.2. We can assume without loss of generality that S_0 is one of the schemes S_i.

Let $(\mathscr{F})_{i\in I}$ be a collection of objects where each $\mathscr{F}_i$ is a $\mathscr{O}_{S_i}$-module, such that

$$\mathscr{F}_j = u_{ij}^* \mathscr{F}_i \quad \forall\, i \leq j$$

Let $\mathscr{F} = u_i^* \mathscr{F}_i$. Notice that $\mathscr{F}$ does not depend on the choice of $i \in I$, because of the congruence conditions imposed to the $\mathscr{F}_i$: if $i, j \in I$, there exists some $k \in I$ with $i \leq k$, $j \leq k$. Then, as $u_i = u_{ik} \circ u_k$ and $u_j = u_{jk} \circ u_k$,

$$u_i^* \mathscr{F}_i = (u_{ik} \circ u_k)^* \mathscr{F}_i = u_k^*(u_{ik}^* \mathscr{F}_i) = u_k^* \mathscr{F}_k = \cdots = u_j^* \mathscr{F}_j$$

Assumption: in this section, it will be implicitly assumed that all collections $(\mathscr{F}_i)_{i\in I}$ of $\mathscr{O}_{S_i}$-modules verify the consistency property considered above.

Let $(\mathscr{F}_i)_{i\in I}$ and $(\mathscr{G}_i)_{i\in I}$ be two families of $\mathscr{O}_{S_i}$-modules. For every couple of indexes $i \leq j$, $u_{ij} : S_j \longrightarrow S_i$ induces an homomorphism in $\mathbb{Z} - modules$.

$$u_{ij}^* : Hom_{\mathscr{O}_{S_i}-mod}(\mathscr{F}_i, \mathscr{G}_i) \longrightarrow Hom_{\mathscr{O}_{S_i}-mod}(\mathscr{F}_j, \mathscr{G}_j)$$

Then $(Hom_{\mathscr{O}_{S_i}-mod}(\mathscr{F}_i, \mathscr{G}_i), u_{ij^*})_{i\in I}$ form a projective system in $\mathbb{Z} - modules$. Furthermore, if we put $\mathscr{F} = \varinjlim \mathscr{F}_i$ and $\mathscr{G} = \varinjlim \mathscr{G}_i$, the morphisms

$$u_i^* : Hom_{\mathscr{O}_{S_i}-mod}(\mathscr{F}_i, \mathscr{G}_i) \longrightarrow Hom_{\mathscr{O}_S-mod}(\mathscr{F}, \mathscr{G})$$

verify the equalities

$$u_i^* = u_j^* \circ u_{ij}^* \quad \forall\, i \leq j$$

which induce a canonical morphism

$$u_{\mathscr{F},\mathscr{G}} : \varinjlim Hom_{\mathscr{O}_{S_i}-mod}(\mathscr{F}_i, \mathscr{G}_i) \longrightarrow Hom_{\mathscr{O}_S-mod}(\mathscr{F}, \mathscr{G})$$

Remark 3.3.3. If $\mathscr{F}_i = \mathscr{O}_{S_i}$ for all $i \in I$, then the coherence condition above is verified and

$$u_{\mathscr{F},\mathscr{G}} = u_{\mathscr{G}} : \varinjlim Hom_{\mathscr{O}_{S_i}-mod}(\mathscr{F}_i, \mathscr{G}_i) = \varinjlim \mathscr{G}_i(S_i) \longrightarrow Hom_{\mathscr{O}_S-mod}(\mathscr{F}, \mathscr{G}) = \mathscr{G}(S)$$

Our aim is now to prove that under some hypothesis on S_0, $\mathscr{F}_i$ and $\mathscr{G}_i$, the homomorphism $u_{\mathscr{F},\mathscr{G}}$ is an isomorphism. In a first place, we will investigate the affine case.

Lemma 3.3.4. *Let A_0 be a ring and let $(A_i)_{i\in I}$ be an inductive system of A_0-algebras. Let A be the inductive limit $\varinjlim A_i$. Consider two A_0-modules M_0 and N_0. For all indexes i let $M_i = M_0 \otimes_{A_0} A_i$, $N_i = N_0 \otimes_{A_0} A_i$, $M = \varinjlim M_i$ and $N = \varinjlim N_i$. If M_0 is an A_0-module of finite type (resp. of finite presentation), then*

$$u_{M,N} : \varinjlim Hom_{A_i}(M_i, N_i) \longrightarrow Hom_A(M, N)$$

is injective (resp. bijective).

Proof. First assume that $M_0 = A_0$. Then $M_i = A_i$ for every i and we have canonical isomorphisms

$$Hom_{A_i}(A_i, N_i) \cong N_i$$

$$Hom_A(A, N) \cong N$$

Moreover, the functors $Hom_{A_i}(-, N_i)$, $Hom_A(-, N)$ and $\varinjlim$ commute with finite sums. Therefore, the lemma is valid if $M_0 \cong A_0^r$ for any $r \in \mathbb{N}$.
Assume that M_0 is a A_0-module of finite type. Let $(f_i) \in \varinjlim Hom_{A_i}(M_i, N_i)$ be an element such that $u_{M,N}((f_i)) = 0$. Consider a surjection

$$A_0^r \longrightarrow M_0 \longrightarrow 0$$

Then also

$$A_i^r \longrightarrow M_i \longrightarrow 0$$

$$A^r \longrightarrow M \longrightarrow 0$$

are exact and we find the following commutative diagram,

$$\begin{tikzcd} 0 \arrow[r] & \varinjlim Hom_{A_i}(M_i, N_i) \arrow[r] \arrow[d, "u_{M,N}"] & \varinjlim Hom_{A_i}(A_i^r, N_i) \arrow[d, "u_{A^r,N}"] \\ 0 \arrow[r] & Hom_A(M, N) \arrow[r] & Hom_A(A^r, N) \end{tikzcd}$$

where rows are exact. Then, as $u_{A^r,N}$ is an isomorphism, it is immediate to notice that $u_{M,N}$ is injective.
Let M_0 be a finitely presented A_0-module. Then there exists a finite presentation

$$A_0^s \longrightarrow A_0^r \longrightarrow M_0 \longrightarrow 0$$

Since the functors $-\otimes_{A_0} A_i$ and $\otimes_{A_0} A$ are right exact, we obtain finite presentations of M_i and M

$$A_i^s \longrightarrow A_i^r \longrightarrow M_i \longrightarrow 0$$

$$A^s \longrightarrow A^r \longrightarrow M \longrightarrow 0$$

We thus obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim Hom_{A_i}(M_i, N_i) \arrow[r] \arrow[d, "u_{M,N}"] & \varinjlim Hom_{A_i}(A_i^r, N_i) \arrow[r] \arrow[d, "u_{A^r,N}"] & \varinjlim Hom_{A_i}(A_i^s, N_i) \arrow[d, "u_{A^s,N}"] \\
0 \arrow[r] & Hom_A(M, N) \arrow[r] & Hom_A(A^r, N) \arrow[r] & Hom_A(A^s, M)
\end{tikzcd}$$

with exact rows. Both $u_{A^r,N}$ and $u_{A^r,N}$ are isomorphisms and we can conclude that $u_{M,N}$ is also an isomorphism with the 5-lemma. □

In order to generalize this result to the non-affine case, our strategy will be to cover the scheme with open affine subsets and try to extrapolate global data from a collection of local ones. For this kind of approach, it seems unlikely to avoid some finiteness assumptions. In fact, in order to guarantee good properties of the map $u_{\mathscr{F},\mathscr{G}}$, we will need to assume that the scheme has properties such as quasi-compactness and quasi-separability. On the other hand, for the $\mathscr{O}_{S_0}$-modules involved, it will be sufficient to consider the analogue concepts of the finite type and finite presentation conditions.

Theorem 3.3.5. *Let S_0 be quasi-compact (resp. quasi compact and quasi-separated). Assume that there exists an index i such that*

1. *$\mathscr{F}_i$ is a quasi-coherent $\mathscr{O}_{S_0}$-module of finite type (resp. of finite presentation).*

2. *$\mathscr{G}_i$ is a quasi-coherent $\mathscr{O}_{S_0}$-module.*

Then $u_{\mathscr{F},\mathscr{G}}$ is injective (resp. bijective).

Proof. Notice that we can assume without loss of generality that S_0 is one of the S_i. In fact, the morphisms $u_{0,i} : S_i \longrightarrow S_0$ are affine, whence quasi-compact and separated (see [StacksProj, Tag 01S7]). Therefore, if S_0 is quasi-compact (resp. quasi-compact and quasi-separated), then the same is true for S_i.
Step 1. Assume that $S_0 = Spec(A_0)$ is an affine scheme. Then the assertion follows from **Lemma** 3.3.4.
Step 2.1. Assume that S_0 is quasi-compact and that i is an index such that $\mathscr{F}_i$ is a quasi-coherent $\mathscr{O}_{S_0}$-algebra of finite type and $\mathscr{G}_i$ is a quasi-coherent $\mathscr{O}_{S_0}$-algebra. Consider an open affine cover (U_λ) of S_0. Since u_{0i} is affine, $U_{i,\lambda} = u_{0i}^{-1}(U_\lambda)$ form an open affine cover of S_i. Similarly, $V_\lambda = u_i^{-1}(U_\lambda)$ form an open affine cover of S. Let $f_i : \mathscr{F}_i \longrightarrow \mathscr{G}_i$ be a representative of an element in the kernel of $u_{\mathscr{F},\mathscr{G}}$, i.e. such that $f = u_i^*(f_i) = 0$. We need to show that there exists an index $j \geq i$ such that $f_j = u_{ij}^*(f_i) = 0$. By the affine case, for every λ there exists

an index i_λ such that $f_{j|U_\lambda} = 0$ for all $j \geq i_\lambda$. As there are finitely many λ and as the set I is directed, we can certainly choose j so large such that $f_{j|U_\lambda} = 0$ for all λ, i.e. $f_j = 0$.
Step 2.2. Let us add to the hypothesis of *Step 2.1.* the assumption that S_0 is quasi-separated and that $\mathscr{F}_i$ is finitely presented. Let $f \in Hom_{\mathscr{O}_S}(\mathscr{F}, \mathscr{G})$. Let us keep he notation of *Step 2.1.* The affine case implies that, for every index λ, we can find a morphism

$$f_{i_\lambda}^{(\lambda)} : \mathscr{F}_{\lambda|\mathscr{U}_\lambda} \longrightarrow \mathscr{G}_{\lambda|\mathscr{U}_\lambda}$$

such that

$$u_{i_\lambda}^*(f_{i_\lambda}) = f_{|U_\lambda}$$

As before, we can assume that the index $i_\lambda = i$ does not depend on λ. Then S_i is quasi-compact and quasi-separated and $\mathscr{F}_i$ is quasi-coherent and finitely presented. As $S_i \longrightarrow S_0$ is quasi-separated, for every couple of indexes λ, μ $U_{i,\lambda\mu} = U_{i,\lambda} \cap U_{i,\mu}$ is quasi-compact. We also have that

$$u_i^*(f_{i|U_{i,\lambda\mu}}^{(\lambda)}) = u_i^*(f_{i|U_{i,\lambda\mu}}^{(\mu)}) = f_{|V_\lambda \cap V_\mu}$$

For each couple (λ, μ) we can find an index $i_{\lambda\mu}$ such that

$$u_{ij}^*(f_{i|U_{i,\lambda\mu}}^{(\lambda})) = u_{ij}^*(f_{i|U_{i,\lambda\mu}}^{(\mu})) \quad \text{for all } j \geq i_{\lambda\mu}$$

With the same argument used above, we can assume that $i = i_{\lambda\mu}$ does not depend on (λ, μ). Then, for $j \geq i$, we find a morphisms $f_j^{(\lambda)} : \mathscr{F}_{j|U_{j,\lambda\mu}} \longrightarrow \mathscr{G}_{|\mathscr{U},\lambda\mu}$ such that

$$u_{ij}^*(f_{j|U_{j,\lambda\mu}}^{(\lambda)}) = u_{ij}^*(f_{j|U_{j,\lambda\mu}}^{(\mu)})$$

Then the collection of maps $(f_j^{(\lambda)})$ define a homomorphism $f_j : \mathscr{F}_j \longrightarrow \mathscr{G}_j$ and it is clear that $u_j^*(f_j) = f$. □

Corollary 3.3.6. *Let S_0 be a quasi-compact scheme. Let $\mathscr{F}_i$ be a quasi-coherent $\mathscr{O}_{S_0}$-module of finite type and let $\mathscr{G}_i$ be a quasi-coherent finitely presented $\mathscr{O}_{S_0}$-module. Consider a morphism $f_i \in Hom_{\mathscr{O}_{S_i}}(\mathscr{F}_i, \mathscr{G}_i)$. Then $f = u_i^*(f_i)$ is an isomorphism if and only if $f_j = u_{ij}^*(f_i)$ is an isomorphism for some $j \geq i$.*

Proof. We can assume without loss of generality that $S_0 = S_i$. Moreover, the property of being an isomorphism is a local one. As S_0 is quasi-compact and I is filtrant, we can therefore assume that S_0 is affine. In particular, S_0 is quasi-compact and quasi-separated.
($\Leftarrow$) obvious.
($\Rightarrow$) as $\mathscr{G}$ is finitely presented, there exists some $j \geq i$ and some $g_j \in Hom_{\mathscr{O}_{S_i}-mod}(\mathscr{G}_i, \mathscr{F}_i)$ such that $u_j^*(g_j) = g = f^{-1}$. As both $u_{\mathscr{F},\mathscr{G}}$ and $u_{\mathscr{G},\mathscr{F}}$ are injective, we can conclude that

$$g_j \circ f_j = if_{\mathscr{F}_j} \qquad f_j \circ g_j = id_{\mathscr{G}_j}$$

□

Corollary 3.3.7. *Let S_0 be a quasi-compact and quasi-separated scheme. Consider two finitely presented $\mathscr{O}_{S_i}$-modules $\mathscr{F}_i$ and G_i. Then $\mathscr{F} \cong \mathscr{G}$ if and only if there exists some index $j \geq i$ such that $\mathscr{F}_j \cong \mathscr{G}_j$. Moreover, given any isomorphism $f : \mathscr{F} \longrightarrow \mathscr{G}$, there exists a sufficiently large index $j \geq k$ and an isomorphism $f_k : \mathscr{F}_k \longrightarrow \mathscr{G}_k$ such that $u_k^*(f_k) = f$.*

Proof. $(\Leftarrow)$: obvious.
$(\Rightarrow)$: it follows from **Theorem** 3.3.5 and from **Corollary** 3.3.6. □

3.3.8 Representation of finitely presented modules over a projective limit of schemes

As we have seen, a set $(\mathscr{F}_i)_{i\in I}$ of compatible (in the sense of §**3.3.1**) $\mathscr{O}_{S_i}$-modules produces an $\mathscr{O}_S$-module $\mathscr{F}$. In this subsection we face the inverse problem. Namely, given an $\mathscr{O}_S$-module $\mathscr{F}$, can we find and index i and an $\mathscr{O}_{S_i}$-module $\mathscr{F}_i$ such that $\mathscr{F} = u_i^*\mathscr{F}_i$? Or, equivalently, can we find a collection of compatible $\mathscr{O}_{S_i}$-modules that produce $\mathscr{F}$?
As usual, we start with the study of the affine case.

Lemma 3.3.9. *Let A_0 be a ring and let $(A_i)_{i\in I}$ be an inductive system of A_0-algebras. Consider a finitely presented A-module, where $A = \varinjlim A_i$. There exists an index i and a finitely presented A_i-module M_i such that $M = M_i \otimes_{A_i} A$.*

Proof. As M is finitely presented over A, there exists an exact sequence

$$A^m \xrightarrow{\ \pi\ } A^n \longrightarrow M \longrightarrow 0$$

The homomorphism π is determined by $\pi(e_j) = (a_{j,1}, \dots, a_{j,n}) \in A^n$, where $e_1, \dots, e_m$ is the canonical basis of A^m and j goes from 1 to m. We can pick i so large that every element $a_{j,s}$ comes from and element $a_{j,s}^{(i)} \in A_i$. Then we can define $\pi_i : A_i^m \longrightarrow A_i^n$ by $\pi_i(e_j) = (a_{j,1}^{(i)}, \dots, a_{j,n}^{i})$. It is clear that $M_i = coker(\pi_i)$ is a finitely presented A_i-module and, as $\pi_i \otimes_{A_i} A = \pi$, we have that $M = M_i \otimes_{A_i} A$. □

Theorem 3.3.10. *Let S_0 be a quasi-compact and quasi-separated scheme. For every quasi-coherent finitely presented $\mathscr{O}_S$-module $\mathscr{F}$, there exists an index $i \in I$ and a quasi-coherent finitely presented $\mathscr{O}_{S_i}$-module $\mathscr{F}_i$ such that $\mathscr{F} = u_i^*(\mathscr{F}_i)$.*

Proof. Step 1. assume that S_0 is affine. Then the assertion of the theorem is equivalent to **Lemma** 3.3.9.
Step 2. let us make no further assumptions on S_0 other than the ones in the statement of the theorem. Consider an open affine finite cover (U_λ) of S_0 and let $U_{j,\lambda} = u_j^{-1}(U_\lambda)$ and $V_\lambda = u_0^{-1}(U_\lambda)$. For each $j \in I$, $(U_{j,\lambda})$ is a finite open affine cover of S_j. We deduce from the local case that for every λ there exist an index i_λ and a quasi-coherent finitely presented $\mathscr{O}_{S_{i_\lambda}}$-module $\mathscr{F}_{i_\lambda}^{(\lambda)}$ such that $u_{i_\lambda}^*(\mathscr{F}_{i_\lambda}^{(\lambda)}) = \mathscr{F}_{|U_\lambda}$. With the same argument we used several times before, we can assume without loss of generality that $i = i_\lambda$ does not depend on λ. For all couples of indexes λ, μ, let $U_{j,\lambda\mu} = U_{j,\lambda} \cap U_{j,\mu}$. Applying **Corollary** 3.3.7, we see that for all couples λ, μ there exists an index $i_{\lambda\mu}$ and an isomorphism

$$\Theta_{\lambda\mu} : u_{i,i_{\lambda\mu}}^*(\mathscr{F}_{i|U_{i,\lambda\mu}}^{(\lambda)}) \longrightarrow u_{i,i_{\lambda\mu}}^*(\mathscr{F}_{i|U_{i,\lambda\mu}}^{(\mu)})$$

with

$$u^*_{i_{\lambda\mu}}(\Theta_{\lambda\mu}) = id_{\mathscr{F}_{|V_\lambda\cap V_\mu}}$$

as the pullbacks of $\mathscr{F}^{(\lambda)}_{i|U_{i,\lambda\mu}}$ and $\mathscr{F}^{(\mu)}_{i|U_{i,\lambda\mu}}$ in $\mathscr{O}_S - modules$ (via $u^*_{i_{\lambda\mu}}$) both coincide with $\mathscr{F}_{V_\lambda\cap V_\mu}$. Once again, we can assume that $j = i_{\lambda\mu}$ does not depend on λ, μ.
For every triplet of indexes λ, μ, η and for every $k \in I$ label $U_{k,\lambda\mu\eta}$ the intersection $U_{k,\lambda\mu} \cap U_{k,\lambda\eta} \cap U_{k,\mu\eta}$ and label $V_{\lambda\mu\eta}$ the intersection $V_{\lambda\mu} \cap V_{\lambda\eta} \cap V_{\mu\eta}$. Resorting to **Corollary** 3.3.7 once again, let

$$\Psi_{\lambda\mu} : u^*_{i,j}(\mathscr{F}^{(\lambda)}_{i|U_{i,\lambda\mu\eta}}) \longrightarrow u^*_{i,j}(\mathscr{F}^{(\mu)}_{i|U_{i,\lambda\mu\eta}})$$

$$\Psi_{\lambda\eta} : u^*_{i,j}(\mathscr{F}^{(\lambda)}_{i|U_{i,\lambda\mu\eta}}) \longrightarrow u^*_{i,j}(\mathscr{F}^{(\eta)}_{i|U_{i,\lambda\mu\eta}})$$

$$\Psi_{\mu\eta} : u^*_{i,j}(\mathscr{F}^{(\mu)}_{i|U_{i,\lambda\mu\eta}}) \longrightarrow u^*_{i,j}(\mathscr{F}^{(\eta)}_{i|U_{i,\lambda\mu\eta}})$$

be the isomorphisms corresponding to $\Theta_{\lambda\mu|U_{i,\lambda\mu\eta}}$, $\Theta_{\lambda\eta|U_{i,\lambda\mu\eta}}$ and $\Theta_{\mu\eta|U_{i,\lambda\mu\eta}}$ respectively. There exists $l \geq j$ such that

$$u^*_{jl}(\Psi_{\lambda\mu} \circ \Psi_{\mu\eta}) = u^*_{jl}(\Psi_{\lambda\eta})$$

Therefore, the isomorphisms

$$u^*_{jl}(\Psi_{\lambda\mu}) : u^*_{jl}(\mathscr{F}^{\lambda}_{j|U_{j,\lambda\mu}}) \longrightarrow u^*_{jl}(\mathscr{F}^{\mu}_{j|U_{j,\lambda\mu}})$$

define a finitely presented quasi-coherent $\mathscr{O}_{S_j}$-module $\mathscr{F}_l$ such that $u^*_l(\mathscr{F}_l) = \mathscr{F}$. □

An immediate consequence of **Theorem** 3.3.5 is the following result (see 3.3.3).

Corollary 3.3.11. *Let S_0 be a quasi-compact and quasi-separated scheme. If $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_S$-module, then the map $u_{\mathscr{G}}$ is a bijection.*

Proposition 3.3.12. *Let S_0 be a quasi-compact scheme. Let $\mathscr{F}_i$ be a finitely presented quasi-coherent $\mathscr{O}_{S_i}$-module. Then $\mathscr{F}$ is locally free (resp. locally free of rank n) in and only if there exists some $j \geq i$ such that $\mathscr{F}_j$ is locally free (resp. locally free of rank n).*

Proof. ($\Leftarrow$) : obvious.
($\Rightarrow$) : Let (V_λ) be a finite open affine cover of S such that $\mathscr{F}_{|V_\lambda} \cong \mathscr{O}^{n_i}_{S|V_\lambda}$ (resp. $\mathscr{O}^{n}_{S|V_\lambda}$) for all λ. By [EGA IV.3, Corollaire 8.2.11]), there exists some $j \geq i$ and a quasi-compact open subset $U_{j,\lambda}$ for each λ such that the equality $V_\lambda = u_j^{-1}(U_{j,\lambda})$ holds for every λ. As S_j is quasi-compact, it can be covered by finitely many open affine subsets. Then we are reduced to the case where S_0 is affine and $\mathscr{F} \cong \mathscr{O}^{n_i}_S$ (resp. $\mathscr{O}^{n_i}_S$). Then we conclude by **Corollary** 3.3.7. □

3.4 Finitely presented schemes over projective limits

Let us keep notation (†) we introduced in §**1.2.**
Assume that for some index i we are given two S_i schemes X_i and Y_i. Let $X_j = X_i \times_{S_i} S_j$, $Y_j = Y_i \times_{S_i} S_j$, $X = X_i \times_{S_i} S$ and $Y = Y_i \times_{S_i} S$. Moreover, let $v_{jk} = id_{X_i} \times u_{jk}$, $w_{jk} = id_{Y_i} \times u_{jk}$, $v_j = id_{X_i} \times u_j$ and $w_j = id_{Y_i} \times u_j$. Notice that for every $i \leq j \leq k$ we have a map

$$e_{jk} : Hom_{S_j}(X_j, Y_j) \longrightarrow Hom_{S_k}(X_k, Y_k)$$

$$f_j \mapsto f_k = f_j \times id_{S_k}$$

Remark 3.4.1. It is immediate to observe that $(Hom_{S_j}(X_j, Y_j), e_{jk})$ form an inductive system in the category of sets. Notice also that the morphisms

$$e_j : Hom_{S_j}(X_j, Y_j) \longrightarrow Hom_S(X, Y)$$

are compatible with the e_{jk}. Therefore, they induce a canonical map

$$e : \varinjlim Hom_{S_j}(X_j, Y_j) \longrightarrow Hom_S(X, Y)$$

that is functorial in S_j, X_j and Y_j.

It is natural to ask whether there exist conditions on S_i, X_i and Y_i which guarantee that e is a bijection. Let us consider the affine case first.

Lemma 3.4.2. *Let A_0 be a ring and let $(A_j)_{j \in J}$ be an inductive system in $A_0 - algebras$. Label A the corresponding colimit. Let B_i and C_i be two A_i-algebras and assume that C_i is of finite type (resp. of finite presentation). Then the canonical homomorphism*

$$\varinjlim Hom_{A_j}(C_i \otimes_{A_i} A_j, B_i \otimes_{A_i} A_j) \longrightarrow Hom_A(C_i \otimes_{A_i} A, B_i \otimes_{A_i} A)$$

is injective (resp. bijective).

Proof. We have canonical isomorphisms

- $Hom_{A_j}(C_i \otimes_{A_i} A_j, B_i \otimes_{A_i} A_j) \cong Hom_{A_i}(C_i, B_i \otimes_{A_i} A_j)$
- $Hom_A(C_i \otimes_{A_i} A, B_i \otimes_{A_i} A) \cong Hom_{A_i}(C_i, B_i \otimes_{A_i} A)$

Hence, it suffices to show that

$$\varinjlim Hom_{A_i}(C_i, B_i \otimes_{A_i} A_j) \longrightarrow Hom_{A_i}(C_i, B_i \otimes_{A_i} A)$$

is injective (resp. bijective).
Assume that C_i is of finite type over A_i and let $c_1, \ldots, c_n$ be a set of generators. Let (f_i) and (g_i) be two compatible systems of morphisms in $\varinjlim Hom_{A_i}(C_i, B_i \otimes_{A_i} A_j)$ such that $f = \varinjlim f_i = \varinjlim g_i = g$. It is immediate to deduce from the following commutative diagram

$$\begin{tikzcd}
& C_i \arrow[d, "f=g"] \arrow[ddl, bend right, "f_j"'] \arrow[ddl, "g_j"] \arrow[ddr, bend left, "f_k"] \arrow[ddr, "g_k"'] & \\
& B_i \otimes_{A_i} A \arrow[dl, "\theta_j"] \arrow[dr, "\theta_k"] & \\
B_i \otimes_{A_i} A_j \arrow[rr, "\theta_{kj}"] & & B_i \otimes_{A_i} A_k
\end{tikzcd}$$

that there exists an index $j >> 0$ such that $f_j(c_s) = g_j(c_s)$ for all $s = 1, \dots, n$, whence $(f_i) = (g_i)$.
Assume that C_i is a finitely presented A_i-algebra, say

$$C_i = A_i[T_1, \dots, T_n]/(F_1, \dots, F_m)$$

Let $f \in Hom_{A_i}(C_i, B_i \otimes_{A_i} A)$. It is uniquely determined by n elements $x_1, \dots, x_n$ such that

$$F_t(x_1, \dots, x_n) = 0 \quad \text{for all } t = 1, \dots, m$$

Choosing j sufficiently large, we can assume that every x_s is the image of some $x_{j,s} \in A_j$. Furthermore, we can also assume that

$$F_t(x_{j,1}, \dots, x_{j,n}) = 0 \quad \text{for all } t = 1, \dots, m$$

Then $x_{j,1}, \dots, x_{j,n}$ determine an element in $Hom_{A_i}(C_i, B_i \otimes_{A_i} A_j)$ that maps onto f. □

Theorem 3.4.3. *Let X_i be a quasi-compact scheme (resp. quasi-compact and quasi-separated). Assume that Y_i is locally of finite type (resp. locally of finite presentation) over S_i. Then*

$$e : \varinjlim Hom_{S_j}(X_j, Y_j) \longrightarrow Hom_S(X, Y)$$

is injective (resp. bijective).

Proof. Step 1. reduction to the case where $X_i = S_i$.
Let $Z_i = X_i \times_{S_i} Y_i$ and $Z_j = Z_i \times_{S_i} S_j = X_j \times_{S_j} Y_j$ for all $j \geq i$. Let $Z = Z_i \times_{S_i} S = X \times_S Y$. Notice that Z_i is of finite type (resp. locally of finite presentation) over X_i. For every $i \leq j \leq k$ we have a commutative diagram

$$\begin{array}{ccccc} Hom_{S_j}(X_j, Y_j) & \to & Hom_{S_k}(X_k, Y_k) & \longrightarrow & Hom_S(X, Y) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ Hom_{X_j}(X_j, Z_j) & \to & Hom_{X_k}(X_k, Z_k) & \longrightarrow & Hom_X(X, Z) \end{array}$$

where vertical arrows are bijections. Hence we can assume that $X_i = S_i$.
Step 2. reduction to the case where X_i is affine.
Since X_i is quasi-compact, we can cover it by finitely many open affine subsets. As I is filtrant, we can assume that X_i is affine.
Step 3. treatment of the case where Y_i is of finite type over S_i.
Let (f_i) and (g_i) be two compatible systems of morphisms $X_i \longrightarrow Y_i$ such that $f = \varinjlim f_i = \varinjlim g_i = g \in Hom_S(X, Y)$. As X_i is quasi-compact, $f_i(X_i) \cup g_i(X_i) \subseteq Y_i$ is quasi-compact. Moreover, Y_i is of finite type over X_i, $f_i(X_i) \cup g_i(X_i)$ can be covered by finitely many affine open subsets $U_{i,\lambda} \subseteq Y_i$ that are of finite type over X_i. Let

$$V_{i,\lambda} = f_i^{-1}(U_{i,\lambda}) \quad W_{i,\lambda} = g_i^{-1}(U_{i,\lambda}) \quad O_{i,\lambda} = V_{i,\lambda} \cap W_{i,\lambda} \quad O_i = \cup_\lambda O_{i,\lambda}$$

Considering the following commutative diagram

$$\begin{tikzcd} X_i \arrow[r, shift left, "f_i"] \arrow[r, shift right, "g_i"'] & Y_i \\ X \arrow[u, "v_i"] \arrow[r, "f=g"] & Y \arrow[u, "w_i"'] \end{tikzcd}$$

it is immediate that

$$v_i^{-1}(V_{i,\lambda}) = v_i^{-1}(f_i^{-1}(U_{i,\lambda})) = v_i^{-1}(g_i^{-1}(U_{i,\lambda})) = v_i^{-1}(W_{i,\lambda}) = g^{-1}(w_i^{-1}(U_{i,\lambda}))$$

As $(V_{i,\lambda})$ is a cover of $f_i(X_i) \cup g_i(X_i)$, we have that $v_i^{-1}(O_i) = f^{-1}(Y) = X$. Moreover, since X_i is quasi-compact and every open subscheme of X_i is ind-constructible, by [EGA IV.3, Corollaire 8.3.4] we get an index $j \geq i$ such that the $O_{j,\lambda} = v_{ij}^{-1}(O_{i,\lambda})$ form a cover of X_j. Hence, we can assume without loss of generality that $(O_{i,\lambda})$ is a cover of X_i.
For every $x \in X_i$, there exists an open affine neighborhood $N_x \subseteq U_{i,\lambda}$ for some λ, i.e. both $f_{|N_x}$ and $g_{|N_x}$ map N_x inside $V_{i,\lambda}$. As X_i is quasi compact, we can cover it with finitely many affine open subsets of type N_x, say $N_{x_1}, \ldots, N_{x_h}$. Since I is filtrant, we can take a j so large that

$$f_{j|v_{ij}^{-1}(N_{x_s})} = g_{j|v_{ij}^{-1}(N_{x_s})} \quad \text{for all } s = 1, \ldots, h$$

i.e. $f_j = g_j$.
Step 4. Assume that X_i is quasi-compact and quasi-separated and that Y_i is locally of finite presentation over S_i. Let $f \in Hom_S(X, Y)$.
Remark: the reduction we made in *Step 1.* is still valid.
As X is quasi-compact (X_i being quasi-compact), $f(X) \subseteq Y$ is quasi-compact. Therefore, there exists an open quasi-compact subset $Y' \subseteq Y$ which contains $f(X)$. By [EGA IV.3, Corollaire 8.2.11]), there exists some $j \geq i$ and an open quasi-compact subset $Y'_j \subseteq Y_j$ such that $Y' = w_j^{-1}(Y'_j)$. Therefore, we can assume without loss of generality that Y_i is quasi-compact. Consider a finite open affine cover V_λ. Then $X = \bigcup f^{-1}(V_\lambda)$. Now, every point of X has an open quasi-compact neighborhood contained in one of the $f^{-1}(V_\lambda)$. As X is quasi-compact, we can thus assume that, for every λ, $V_\lambda = w_{i,\lambda}^{-1}(V_{i,\lambda})$, where $(V_{i,\lambda})$ is an open affine cover of Y_i. Therefore, $X = \bigcup f^{-1}(V_\lambda)$. Every point in X has an open quasi-compact neighborhood contained in one of the $f^{-1}(V_\lambda)$. We can cover X with finitely many such neighborhoods; eventually repeating some of the V_λ, we can also assume that the number of such neighborhoods equals the number of the V_λ. Using [EGA IV.3, Corollaire 8.2.11] once again, we can assume without loss of generality that $U_\lambda = v_i^{-1}(U_{i,\lambda})$, where $U_{i,\lambda} \subseteq X_i$ is an open quasi-compact subset. Moreover, using ([EGA IV.3, Corollaire 8.3.4]), we can also assume without loss of generality that $(U_{i,\lambda})$ is a cover of X_i. Then it suffices to show that there exist some $j \geq i$ and a morphism $f_{j,\lambda} : U_{i,\lambda} \longrightarrow V_{i,\lambda}$ such that $\varinjlim f_{j,\lambda} = f_{|U_\lambda}$ (here $U_{j,\lambda} = v_{ij}^{-1}(U_{i,\lambda})$ and $V_{j,\lambda} = w_{ij}^{-1}(V_{i,\lambda})$).
As X_j is quasi-separated, $U_{j,\lambda} \cap U_{j,\mu}$ are quasi-compact (see [Bosch, Remark 6.9.8 (i)]). By

what we showed above, it suffices to show that there exists some $k \geq j$ such that all $f_{k,\lambda}$ and $f_{k,\mu}$ coincide on the intersections of their domains. In this way, we can assume that Y_i is affine. Furthermore, as we can assume that the image of each $V_{i,\lambda}$ in S_i is contained in an open affine subset, we can also assume that S_i is affine. Let $S_i = Spec(A_i)$ and $Y_i = Spec(C_i)$, where C_i is an A_i-algebra of finite presentation. set $A = \varinjlim A_i$, $C = \varinjlim C_j = \varinjlim C_i \otimes_{A_i} A$, $S = Spec(A)$ and $Y = Spec(C)$. Then

$$Hom_S(X,Y) = Hom_A(C, \mathscr{O}_X(X)) = Hom_{A_i}(C_i, \mathscr{O}_X(X))$$

$$Hom_{S_j}(X_j, Y_j) = Hom_{A_j}(C_j, \mathscr{O}_{X_j}(X_j)) = Hom_{A_i}(C_i, \mathscr{O}_{X_j}(X_j))$$

Since X_i is quasi-compact and quasi-separated, by [EGA IV.3, Corollaire 8.5.4] it follows that $\mathscr{O}_X(X) = \varinjlim \mathscr{O}_{X_j}(X_j)$ and we can apply **Lemma** 3.4.2. □

The following results follow from the theorem.

Corollary 3.4.4. *Let S_0 be a quasi-compact scheme, X_i be a finitely presented S_i-scheme and let Y_i be a quasi-separated S_i-scheme of finite type. Let $f_i : X_i \longrightarrow Y_i$ be an S_i-morphism, $f = \varinjlim f_j : X \longrightarrow Y$ is an isomorphism if and only there exists $j \geq i$ such that $f_j : X_j \longrightarrow Y_j$ is an isomorphism.*

Proof. $(\Leftarrow)$: obvious.
$(\Rightarrow)$: let $g : Y \longrightarrow X$ be the inverse of f. We have that

$$e : \varinjlim Hom_{S_j}(Y_j, X_j) \longrightarrow Hom_S(Y, X)$$

is a bijection. Then there exists $(g_k) \in \varinjlim Hom_{S_j}(X_j, Y_j)$ with $g = \varinjlim g_k$. In particular, as both X_i and Y_i are of finite type, for some $k >> 0$,

$$f_k \circ g_k = id_{Y_k} \qquad g_k \circ f_k = id_{X_k}$$

□

Corollary 3.4.5. *Let S_0 be quasi-compact and quasi-separated, X_i and Y_i of finite presentation over S_i. Then $X \cong Y$ if and only if there exists some $j \geq i$ with $X_j \cong Y_j$. Moreover, for every S-isomorphism $f : X \longrightarrow Y$ there exist some $k \geq j \geq i$ and an isomorphism $f_k : X_k \longrightarrow Y_k$ with $f = f_k \times id_S$.*

Proof. $(\Leftarrow)$ obvious.
$(\Rightarrow)$ by the theorem, we get that $f = f_j \times id_S$ for some $j \geq i$ and some $f_j : X_j \longrightarrow Y_j$. Then f_j is an isomorphism by the previous corollary. □

Theorem 3.4.6. *Let S_0 be quasi-compact and quasi-separated, X a finitely presented S-scheme. Then there exist some index $j \in I$, a finitely presented S_j-scheme X_j and an S-scheme isomorphism*

$$X \longrightarrow X_j \times_{S_j} S$$

Proof. Step 1. assume that $S_0 = Spec(A_0)$ and $X = Spec(B)$, with

$$B = A[T_1, \dots, T_n]/(F_1, \dots, F_m) \quad F_1, \dots, F_m \in A[T_1, \dots, T_n]$$

We can choose $j \geq i$ so large that all coefficients of all F_s come from A_j. Then

$$B_j = A_j[T_1, \dots, T_n]/(F_1, \dots, F_m)$$

is such that $B_j \otimes_{A_j} A = B$.
*Step 2.*let S_0 and X be as in the statement of the theorem. Notice that S is quasi-compact and quasi-separated. Since $X \longrightarrow S$ is a finitely presented morphism and $u_0 : S \longrightarrow S_0$ is an affine morphism, there exists a finite open affine cover (X_r) such that, for all r, the image of X_r in S is contained in $W_{\lambda_r} = u_0^{-1}(U_{\lambda_r})$.

Remark 3.4.7. $\mathscr{O}_X(X_r)$ is a finitely presented $\mathscr{O}_S(W_{\lambda_r})$-algebra, for all r

By *Step 1.* and the fact that I is filtrant, there exist a $j \geq i$ and an affine scheme $Z_{j,k}$ such that $g_r : Z_{j,r} \times_{S_j} S = X_r$ is an isomorphism, with $Z_{j,r}$ finitely presented over $W_{j,\lambda_r} = u_{0j}^{-1}(U_{\lambda_r})$. Let $Z_{rs} = g_r^{-1}(X_r \cap X_s)$; Z_{rs} is quasi-compact as X is quasi-separated (see [Bosch, Remark 6.9.8 (i)]). Let $g'_{rs} = g_{rs|Z_{rs}} : Z_{rs} \cong X_r \cap X_s$. **Corollary** 3.4.5 implies that there exists an index $k \geq j$ and, for every couple (r, s), an opne quasi-compact subscheme $Z_{k,rs} \subseteq Z_{k,r} = v_{jk}^{-1}(Z_{j,r})$ such that Z_{rs} is the inverse image of $Z_{k,rs}$. As S_k is quasi-separated and $W_{k,\lambda_r} \subseteq S_k$ is open and quasi-compact, every $Z_{k,rs}$ is finitely presented over S_k. For every pair (r, s) consider the isomorphism

$$h_{rs} = g_{sr}^{-1} \circ g_{rs} : Z_{rs} \longrightarrow Z_{sr}$$

By **Corollary** 3.4.4 there exists $l \geq k$ and, for every pair (r, s), an isomorphism $h_{l,sr} : Z_{l,rs} \longrightarrow Z_{l,sr}$ such that $h_{sr} = h_{l,sr} \times id_S$. For every triplet (r, s, t), let $h'_{sr} = h_{sr|Z_{rs} \cap Z_{rt}}$. Then

$$h'_{sr} : Z_{rs} \cap Z_{rt} \cong Z_{sr} \cap Z_{st}$$

and $h'_{ts} \cong h'_{sr} = h'_{tr}$. **Theorem** 3.4.2 guarantees the existence of an index $m \geq l$ such that , for every (r, s, t),

$$h'_{m,ts} \circ h'_{m,sr} = h'_{m,tr}$$

Then we can glue the schemes (X_r) and we obtain an S_m-scheme X_m such that $X = X_m \times_{S_m} S$. Moreover, the $Z_{m,r}$ are finitely presented over S_m. If we identify them with open subsets of X_m, $Z_{m,r} \cap Z_{m,s} \cong Z_{m,rs}$ are quasi-compact. Then X_m is finitely presented over S_m (see [EGAI, §6.3]). □

As we mentioned at the beginning of this chapter, at this point we can ask whether good properties of morphisms are preserved under $e : \varinjlim Hom_{S_j}(X_j, Y_j) \longrightarrow Hom_S(X, Y)$. One direction is given by permanence of good properties under base change. Anyway, for reduction purposes, it would be useful to know the converse, i.e. if, given an S-morphism $f : X \longrightarrow Y$ with some property $\mathcal{P}$, there exists a sufficiently large index j such that $f = f_j \times id_S$ and $f_j : X_j \longrightarrow Y_j$ has the same property $\mathcal{P}$. It comes out that e behaves extremely well. For the convenience of the reader, we will state the theorem we will use later on. For a proof, consult [EGA IV.3, §8.10].

Theorem 3.4.8. *Let S_0 be quasi compact and X_i, Y_i two finitely presented S_i-schemes. Let $f_i : X_i \longrightarrow Y_i$ be an S_i-morphism. Let $\mathcal{P}$ be one of the following properties:*

1. *being an isomorphism*
2. *being a monomorphism*
3. *being an immersion*
4. *being an open immersion*
5. *being a closed immersion*
6. *being separated*
7. *being surjective*
8. *being purely inseparable*
9. *being affine*
10. *being quasi-affine*
11. *being finite*
12. *being quasi-finite*
13. *proper*

Then f has $\mathcal{P}$ if and only if there exists an index $j \geq i$ such that f_j has the property $\mathcal{P}$.
Moreover, if in addition S_0 is also quasi-separated, the same is true also for the properties of being projective and quasi-projective.

Chapter 4

Artin's approximation

In the first part of this chapter we briefly define excellent rings, trying to explain why they were introduced. In the second part we discuss Artin's approximation. Finally, we state Popescu's theorem and we prove that it implies that henselian pairs (A, I), with A a G-ring, satisfy Artin's approximation property.

4.1 Excellent rings

4.1.1 Introduction

There are some properties of noetherian rings which are not preserved under some fundemental constructions in Commutative Algebra and Algebraic Geometry. For example, the normalization of a noetherian ring is not always noetherian. Another example was found by M. Nagata, who showed a normal noetherian local domain whose completion is not reduced (see [Nag62]). This motivated the attempt to define a new class of noetherian rings with additional properties. The first question one should ask is the following: which are the good properties we want our class of rings to have? As in Algebraic Geometry we work with algebras of finite type over a field, it seems reasonable to take them as a prototype. Then the following question arises: which are the main properties that characterize affine rings? In the case we are working over an algebraically closed field, it is known that the subset of singular points in an affine variety is closed (see [Liu, Proposition 4.2.24]). Nagata, in his paper *On the closedness of the singular loci* (see [Nag59]) studied the conditions that a noetherian ring has to satisfy in order to guarantee that the singular locus of any finitely generated extension is closed. He proved that this class of noetherian rings is well behaved under certain fundamental operations such as formation of finitely generated extensions. Moreover, he provided an example of a noetherian ring that does not belong to this class. It is remarkable to notice that his counterexample satisfies a number of hypothesis. Anyway, Nagata's rings fail in other desirable properties. This led A. Grothendieck to define a new class of notherian rings. As it is stated in [EGA IV.2, §7.8], one might want that good properties of a noetherian (local) ring are inherited by its completion. Grothendieck studied the formal fibers of the canonical morpihism

$$Spec(\hat{A}) \longrightarrow Spec(A)$$

and found that good properties of the fibers correspond to results of the form

$$A \text{ has the property } \mathscr{P} \text{ if and only if } \hat{A} \text{ has the same property}$$

This is the reason why he introduced the theory of G-rings.
Another desirable property of a ring is that the locus of points of $Spec(A)$ for which A_p has a nice property is open. As we said above, Nagata showed that if the property involved is regularity, this is not always the case. Finally, if A is a noetherian integral domain, one may ask if, given a finite extension L of $Frac(A)$, the integral closure of A in L is a finitely generated A-algebra. In general, the answer is no.
The noetherian rings which have all the desirable properties listed above were called *excellent* by Grothendieck. In the remaining part of this paragraph, we briefly give the definition of excellent ring and we state the main properties of this class of noetherian rings. For more detailed discussions on excellent rings, see [EGA IV.2, §7.8], [Mats] and [StacksProj]. A very nice and readable exposition on excellent rings can be found in [Rot].

4.1.2 The class of excellent rings

First of all, let us recall that a chain of prime ideals of a ring A

$$p_1 \subseteq p_2 \subseteq \dots p_r$$

is *saturated* if there are non prime ideals between two terms of the chain. Equivalently, one could say that p_{i+1}/p_i is a minimal prime ideal in A/p_i for every i.

Definition 4.1.3. A ring A is *catenary* if for any prime ideals $p, q \subseteq A$ with $p \subseteq q$, there exists a saturated chain starting from p and finishing in q. Moreover, any two such chains have the same (finite) length.

Definition 4.1.4. A ring A is *universally catenary* if it is noetherian and every finitely generated A-algebra is catenary.

Excellent rings will be required to be universally catenary as this reflects good properties in terms of dimension.
If (A, m) is a local noetherian ring and $\hat{A}$ is its completion, for any $p \in Spec(A)$ the *formal fibre* of A in p is $\hat{A} \otimes_A k(p)$

Definition 4.1.5. A noetherian ring A that contains a field k is *geometrically regular* if for any finite field extension $k \subseteq l$, $A \otimes_k l$ is a regular ring.

Definition 4.1.6. An homomorphism $\phi : A \longrightarrow B$ between two noetherian rings is *regular* if it is flat and if, for every $p \in Spec(A)$, $B \otimes_A k(p)$ is a geometrically regular ring over $k(p)$.

At this point we can introduce the first class of rings introduced by Grothendieck.

Definition 4.1.7. A noetherian ring A is a *G-ring (Grothendieck ring)* if for any $p \in Spec(A)$ the canonical morphism

$$A_p \longrightarrow \hat{A}_p$$

is regular.

Finally, we are able to define excellent rings.

Definition 4.1.8. A noetherian ring A is *excellent* if

1. A is universally catenary
2. A is a G-ring
3. for any finitely generated A-algebra B, the locus of regular points $Reg(B) \subseteq Spec(B)$ is open

Some very important properties of excellent rings are collected in the next lemma.

Lemma 4.1.9. *Excellent rings are closed under localizations, quotients and finitely generated extensions.*

Finally, the next proposition tells us that most of the rings we encounter in algebraic geometry and in number theory are excellent.

Proposition 4.1.10. *The following rings are excellent:*

1. *complete noetherian local rings.*
2. *Dedekind domains with fraction field of characteristic zero.*
3. *extensions of finite type of the rings above.*

4.2 Artin's approximation

4.2.1 Introduction to the main idea

In his celebrated paper [Art69], Artin treats a method which leads to approximations of structures over $\hat{A}$ with structures over A. We will see that, after Popescu's theorem, we can generalize this method to pairs.
In order to be precise, assume that the kind of structure we are interested in is classified by a functor:

$$\mathscr{F} : A\text{-}algebras \longrightarrow Sets$$

$$B \mapsto \mathscr{F}(B) = \text{set of isomorphism classes of structures over } B$$

If A is given together with an ideal, i.e. if we are given a pair (**Definition** 2.1.1) (A, I), then for every $n \in \mathbb{N}$ we can consider the following arrows, where $\hat{A}$ is the I-adic complition of A:

$$\begin{tikzcd}
F(A) \arrow[dr] & & F(\hat{A}) \arrow[dl] \\
& F(A/I^n) = F(\hat{A}/\hat{I}^n) &
\end{tikzcd}$$

Definition 4.2.2. We say that $\xi \in \mathscr{F}(A)$ and $\hat{\xi} \in \mathscr{F}(\hat{A})$ are *congruent modulo* I^n if they have the same image in $\mathscr{F}(A/I^n)$. In this case we write

$$\xi \equiv \hat{\xi} \ mod \ I^N$$

The first question that comes to the mind is the following one:

Question 4.2.3. Given $\mathscr{F}$ as above, a positive integer $n \in \mathbb{N}$ and an element $\hat{\xi} \in \mathscr{F}(\hat{A})$, does there exist some $\xi \in \mathscr{F}(A)$ such that $\xi \equiv \hat{\xi} \ mod \ I^n$?

4.2.4 Reduction to solutions of a system of equations

It is natural to put some restriction condition on the functor $\mathscr{F}$. In particular, we will require it to be well behaved with respect to inductive limits.

Definition 4.2.5. Let A be a ring and consider a functor

$$\mathscr{F} : A\text{-}algebras \longrightarrow Sets$$

We say that $\mathscr{F}$ is *locally of finite presentation* if for every filtering inductive system of A-algebras $\{B_i\}$, the canonical map

$$\varinjlim \mathscr{F}(B_i) \longrightarrow \mathscr{F}(\varinjlim B_i)$$

is bijective.

Assume that the functor which classifies the structures we are interested in is locally of finite presentation. Any A-algebra B can be written as

$$B = \varinjlim B_i \quad \text{with } B_i \text{ finitely presented over } A$$

If $\xi \in \mathscr{F}(B) = \varinjlim(\mathscr{F}(B_i))$, then there exists some $\xi_i \in \mathscr{F}(B_i)$ such that $B_i \longrightarrow B$ induces

$$\mathscr{F}(B_i) \longrightarrow \mathscr{F}(B)$$

$$\xi_i \mapsto \xi$$

If $B_i = \frac{A][T_1,\ldots,T_n]}{(f_1,\ldots,f_m)}$, we have a canonical bijection

$$Hom_A(B_i, C) = \{(c_1, \ldots, c_n) \in C^n : f_i(c) = 0 \text{ for every } i\}$$

for every A-algebra C. Summarizing, we obtained

Corollary 4.2.6. *Let $\mathscr{F}$: A-algebras $\longrightarrow$ Sets be locally of finite presentation. If B is any A-algebra and $\xi \in \mathscr{F}(B)$, there exist:*

1. *a finite system of polynomials $f = (f_1, \ldots, f_m) \in A[T_1, \ldots, T_n]^m$.*
2. *a functorial rule which associates to every solution of f in an A-algebra C an element of $\mathscr{F}(C)$.*

3. *a solution of f in B, so that the functorial rule applied to that solution yields ξ.*

Proof. With the notation used above, take f as the polynomials which identify B_i. Then the functorial rule is just

$$\mathscr{F}(\psi) : \mathscr{F}(B_i) \longrightarrow \mathscr{F}(C)$$

$$\xi_i \mapsto \mathscr{F}(\psi)(\xi_i)$$

where $\psi : B_i \longrightarrow C$ is the A-homomorphism that corresponds to the given solution of f. Finally, the solution of f in B is given by the images of the residue classes $T_1 + (f_1, \dots, f_m), \dots, T_n + (f_1, \dots, f_m)$ in B_i via the canonical map. □

With this in mind, it is clear that in order to answer positively to *Question* 4.2.3 it suffices to answer positively to the following question:

Question 4.2.7. Let $f = (f_1, \dots, f_m) \in A[T_1, \dots, T_n]^m$ and let $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n) \in \hat{A}^n$ be a solution. Given some $N \in \mathbb{N}$, does there exist a solution $y = (y_1, \dots, y_n) \in A^n$ of f such that

$$y_i \equiv \hat{y}_i \ mod\ I^N \text{ for every } i = 1, \dots, n\ ?$$

This is exactly the meaning of the following result.

Corollary 4.2.8. *Let (A, I) be a pair and assume that Question 4.2.7 has a positive answer for every system of polynomial equations $f = (f_1, \dots, f_m)$ in $A[T_1, \dots, T_n]$, for every solution $\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$ in $\hat{A}^n$ and for every integer N. If $\mathscr{F}$ is a functor locally of finite presentation and if $\hat{\xi} \in \mathscr{F}(\hat{A})$, then for every $N \in \mathbb{N}$ there exists some $\xi \in \mathscr{F}(A)$ such that*

$$\xi \equiv \hat{\xi} \ mod\ I^N$$

Proof. By **Corollary** 4.2.6 there exist a system of polynomials

$$f = (f_1, \dots, f_m) \in A[T_1, \dots, T_n]^m$$

and a solution

$$\hat{y} = (\hat{y}_1, \dots, \hat{y}_n) \in \hat{A}^n$$

of f such that the functiorial rule mentioned in **Corollary** 4.2.6 gives $\hat{y} \mapsto \hat{\xi}$. By our assumptions, there exists a solution of f

$$y = (y_1, \dots, y_n) \in A^n$$

such that

$$y_i \equiv \hat{y}_i \ mod\ I^N \text{ for every } i = 1, \dots, n$$

Let $\xi \in \mathscr{F}(A)$ be the element that corresponds to y via the functorial rule. As $y \equiv \hat{y} \ mod\ I^N$, they induce the same element $\eta \in \mathscr{F}(A/I^N)$, i.e.

$$\xi \equiv \hat{\xi} \ mod\ I^N$$

□

Then the new problem is to find out when *Question* 4.2.7 has a positive answer. This is what Artin does in the main result of [Art69].

Theorem 4.2.9. *Let R be a field or an excellent Dedekind domain and let A be the henselization of an R-algebra of finite type at a prime ideal. Let I be a proper ideal of A. Given an arbitrary system of polynomial equations*

$$f(T) = (f_1(T_1, \dots, T_n), \dots, f_m(T_1, \dots, T_n))$$

in $A[T] = A[T_1, \dots, T_n]$, a solution

$$\hat{y} = (\hat{y}_1, \dots, \hat{y}_n)$$

in $\hat{A}^n$ and an integer N, there exists a solution

$$y = (y_1, \dots, y_n)$$

in A^n such that

$$y_i \equiv \hat{y}_i \text{ mod } I^N \text{ for every } i = 1, \dots, n$$

Then **Corollary** 4.2.8 provides us the following important result:

Theorem 4.2.10. *Keep the notation and the assumptions of **Theorem** 4.2.9. Let $\mathscr{F}$ be a functor which is locally of finite presentation. Given any $\hat{\xi} \in \mathscr{F}(\hat{A})$ and any $N \in \mathbb{N}$, there exists an element $\xi \in \mathscr{F}(A)$ such that*

$$\xi \equiv \hat{\xi} \text{ mod } I^N$$

Moreover, Artin conjectured in [Art69] that *Question* 4.2.7 has an affirmative answer for any excellent henselian ring.

4.3 Popescu's Theorem

4.3.1 Artin's second conjecture

In his later work [Art82], Artin conjectured an even stronger result. In this section we will briefly discuss this. The reason why this second conjecture was introduced is the following: let $(f_1, \dots, f_m)$ be an ideal in $A[T_1, \dots, T_n]$. Then a solution of the system of equations f given by such ideal in $\hat{A}$ corresponds uniquely to a morphism $\psi : A[T_1, \dots, T_n]/(f_1, \dots, f_m) \longrightarrow \hat{A}$ that fits into the diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & \hat{A} \\ & \frac{A[T_1,\dots,T_n]}{(f_1,\dots,f_m)} \arrow[ur, "\psi"'] & \end{tikzcd}$$

Assume that ψ factorizes through a smooth A-algebra D of finite type. As we are assuming that A is henselian, it can be shown that there exists an A-morphism $D \longrightarrow A$

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \hat{A} \\
\frac{A[T_1,\ldots,T_n]}{(f_1,\ldots,f_m)} \arrow[r] \arrow[ur, "\psi"] & D \arrow[u] \arrow[ul, dashed]
\end{tikzcd}$$

Then it is clear that the solution of f in $\hat{A}$ lifts to a solution in A. Therefore, one can ask whether the smooth A-algebra of finite type D always exists when A is an excellent henselian ring. This can be regarded as a property of the canonical morphism $A \longrightarrow \hat{A}$. Then Artin's second conjecture takes the following shape:

Let $\phi : A \longrightarrow B$ be a regular morphism between noetherian rings. Is B a direct limit of smooth A-algebras of finite type?

4.3.2 Popescu's Theorem

An answer to Artin's second conjecture was given by D. Popescu with a series of articles published in the period from 1985 to 1990. He gave a characterization of regular homomorphisms between noetherian rings.

Theorem 4.3.3 (Popescu)**.** *Let $f : A \longrightarrow B$ be an homomorphism of noetherian rings. Then f is regular if and only if B is a filtered colimit of smooth A-algebras.*

In the remaining part of this section we discuss a generalization of **Theorem** 4.2.9 to the nonlocal case. We give the result that can be found in [StacksProj, Tag 0AH5].

Proposition 4.3.4. *Let (A, I) be an henselian pair with A noetherian. Assume that one of the following hypothesis is verified:*

1. *$A \longrightarrow \hat{A}$ is a regular ring map.*
2. *A is a noetherian G-ring.*
3. *(A, I) is the henselization of a pair (B, J), where B is a noetherian G-ring.*

Let $f_1, \ldots, f_m \in A[T_1, \ldots, T_n]$ and let $\hat{y} = (\hat{y}_1, \ldots, \hat{y}_n)$ be a solution of the system in $\hat{A}$. Let N be any positive integer. Then there exist a solution $y = (y_1, \ldots, y_n)$ of the system of polynomials equations $f_1, \ldots, f_m$ in A such that

$$y_i \equiv \hat{y}_i \text{ mod } I^N \text{ for every } i = 1, \ldots, n$$

Proof. By [StacksProj, Tag 0AH3], we see that *3.* $\Rightarrow$ *2.* and by [StacksProj, Tag 0AH2] we see that *2.* $\Rightarrow$ *1.* Then it suffice to show the proposition assuming that we are in situation *1.*

Popescu's theorem (**Theorem** 4.3.3) implies that we can find a smooth A-algebra B and elements $b_1, \ldots, b_n \in B$ such that the canonical morphism

$$A \longrightarrow \hat{A}$$

factors through B and such that

$$f_i(b_1, \ldots, b_n) = 0 \text{ for all } i = 1, \ldots, n$$

Let $\sigma : B \longrightarrow \hat{A} \longrightarrow A/I^N$. By [StacksProj, Tag 07M7], there exists an étale ring map $A \longrightarrow A'$ which induces an isomorphism

$$A/I^N \cong A'/I^N A'$$

and an A-algebra homomorphism $\tau : B \longrightarrow A'$ which lifts σ. By [StacksProj, Tag 09XI] there exists an A-homomosphism $\rho : A' \longrightarrow A$. Then the elements

$$a_i = \rho(\tau(b_i)) \text{ for every } i = 1, \ldots, n$$

have the desired property. □

In particular, applying **Corollary** 4.2.8, we can reformulate **Theorem** 4.2.10.

Theorem 4.3.5. *Let (A, I) be an henselian pair with A noetherian and assume that one of the hypothesis of **Proposition** 4.3.4 is satisfied. Let $\mathscr{F}$ be a functor which is locally of finite presentation*

$$A\text{-}algebras \longrightarrow Sets$$

Given any $\hat{\xi} \in \mathscr{F}(\hat{A})$ and any $N \in \mathbb{N}$, there exists an element $\xi \in \mathscr{F}(A)$ such that

$$\xi \equiv \hat{\xi} \ mod \ I^N$$

Chapter 5

Proper base change over henselian pairs

In this chapter we describe how to adapt Artin's proof of [Art69, Theorem 3.1] to the nonlocal case.

5.1 The statement of the theorem

Let X be a proper finitely presented scheme over $S = Spec(A)$, where (A, I) is a pair. Denote $\acute{E}t_f(Z)$ the category of finite étale coverings of the scheme Z. Then we have a functor

$$\acute{E}t_f(X) \longrightarrow \acute{E}t_f(X_0)$$

induced by $- \times_S S_0$, where $S_0 = Spec(A/I)$ and $X_0 = X \times_S S_0$. It is a very important fact that if (A, I) is an henselian local ring, then the functor above is an equivalence. This was first proved by Artin, Grothendieck and Verdier in [SGAIV, Exposé XII]. A new (simpler) proof was later given by Artin in [Art69].
In this chapter we describe how it is possible to generalize the proof due to Artin to the case where (A, I) is an henselian pair.

Theorem 5.1.1. *Let (A, I) be an henselian pair. Let $S = Spec(A)$ and let $f : X \longrightarrow S$ be a proper finitely presented morphism. Let $X_0 = X \times_S S_0$, where $S_0 = Spec(A/I)$. Then*

$$\acute{E}t_f(X) \longrightarrow \acute{E}t_f(X_0)$$

$$Z \mapsto Z \times_S S_0$$

is an equivalence of categories.

The main ingredients for the proof are **Proposition** 4.3.4 and **Theorem** 4.3.5. We will see that the original proof of Artin adapts very well to the case of our interest.

5.2 The proof of the theorem

5.2.1 A reduction step

In this first section we reduce to the case where A is the henselization of a finitely presented $\mathbb{Z}$-algebra.

Lemma 5.2.2. *Let $S = Spec(A)$ and let $g : X \longrightarrow S$ be a proper morphism of finite presentation. Then the functor*

$$\mathscr{F} : A\text{-}Algebras \longrightarrow Sets$$

$$B \mapsto \{\textit{finite étale coverings of } Spec(B) \times_S X\}/\textit{isomorphism}$$

*is locally of finite presentation (**Definition** 4.2.5).*

Proof. See the beginning of the proof of [Art69, Theorem 3.1]. □

Lemma 5.2.3. *Let $S = Spec(A)$ and let $g : X \longrightarrow S$ be a proper morphism of finite presentation. Let $Z_1 \to X$ and $Z_2 \to X$ be two finite étale covers of X. Then the functor*

$$\mathscr{G} : A\text{-}algebras \longrightarrow Sets$$

$$B \mapsto Hom_{X \times_S Spec(B)}(Z_1 \times_S Spec(B), Z_2 \times_S Spec(B))$$

is locally of finite presentation.

Proof. The lemma is a straightforward consequence of **Theorem** 3.4.3. □

Let (A, I) be an henselian pair and write A as a direct limit $\varinjlim A_i$, where each A_i is a subalgebra of A that is finitely generated over $\mathbb{Z}$. Let $(A_i^h, (I \cap A_i)^h)$ be the henselization of $(A_i, (I \cap A_i))$ for each i. Then by **Proposition** 2.2.4 $\varinjlim(A_i^h, (I \cap A_i)^h)$ is an henselian pair. It is easy to see that

$$(A, I) = \varinjlim(A_i^h, (I \cap A_i)^h)$$

Write $S_i = Spec(A_i^h)$ for every index i. Then

$$S = \varprojlim S_i$$

By **Theorem** 3.4.6 we know that X comes from a finitely presented scheme X_{i_0} for some index i_0, i.e. $X \cong X_{i_0} \times_{S_{i_0}} S$. Moreover, by **Thereom** 3.4.8 X_{i_0} is also proper over S_{i_0}. As the functor

$$\mathscr{F} : A_{i_0}^h - Algebras \longrightarrow Sets$$

$$B \mapsto \{\text{finite étale coverings of } Spec(B) \times_{S_{i_0}} X_{i_0}\}/\text{isomorphism}$$

is locally of finite presentation, we have that

$$\mathscr{F}(A) = \varinjlim \mathscr{F}(A_i^h)$$

Therefore, every finite étale cover of X comes from a finite étale cover of $X_i = S_i \times_{S_{i_0}} X_{i_0}$ for a suitable index i.

Remark 5.2.4. All schemes $X_{i_0} \times_{S_{i_0}} S_i$ and $X \cong X_{i_0} \times_{S_{i_0}} S$ are quasi-compact and quasi-separated, as they are proper over affine schemes.

Let $Z \to X$ and $W \to X$ be two finite étale covers of X. Then we can assume without loss of generality that they come from two finite étale covers $Z_{i_0} \to X_{i_0}$, $W_{i_0} \to X_{i_0}$. Then by **Lemma** 3.4.3 we see that

$$\varinjlim Hom_{X_i}(Z_i, W_i) = Hom_X(Z, W)$$

It is then clear that we can reduce to the case where (A, I) is the henselization of a pair (B, J), where B is finitely generated over $\mathbb{Z}$. In particular, B is a G-ring.

5.2.5 End of the proof

Lemma 5.2.6. *The functor in **Theorem** 5.1.1 is essentially surjective.*

Proof. Consider a finite étale cover $X_0' \longrightarrow X_0$. Label $\hat{A}$ the completion of A with respect to the ideal I and let $\hat{S} = Spec(\hat{A})$, $\hat{X} = X \times_S \hat{S}$. Notice that $\hat{A}$ is a complete separated ring by Krull's theorem (see [AM, Theorem 10.17]). By [EGA IV.4, Theorem 18.3.4], we have that the functor

$$\acute{E}t_f(\hat{X}) \longrightarrow \acute{E}t_f(X_0)$$

$$Z \mapsto Z \times_S S_0$$

is an equivalence of categories. Then there exists some $\hat{X}' \longrightarrow \hat{X} \in \mathscr{F}(\hat{A})$ such that

$$\hat{X}' \times_{\hat{S}} S_0 \cong X_0'$$

Keeping the notation of **Lemma** 5.2.2, we have that

$$\begin{tikzcd} \mathscr{F}(A) \arrow[dr] & & \mathscr{F}(\hat{A}) \arrow[dl] \\ & \mathscr{F}(A/I) & \end{tikzcd}$$

and by **Theorem** 4.3.5 we get that there exists some finite étale cover $X' \longrightarrow X$ which is congruent modulo I to $\hat{X}' \longrightarrow \hat{X}$, i.e.

$$X' \times_S S_0 \cong X_0'$$

□

It remains only to show that the functor in **Theorem** 5.1.1 is fully faithful.

Lemma 5.2.7. *The functor in **Theorem** 5.1.1 is fully faithful.*

Proof. Let X' and X'' be two finite étale schemes over X and let $\phi \in Hom_X(X', X'')$. The morphism ϕ corresponds uniquely to its graph $\Gamma_\phi : X' \longrightarrow X' \times_X X''$, which is an open immersion as both X' and X'' are of finite type over X and as X'' is étale over X (see [SGAI, Corollaire 3.4]). Also notice that Γ_ϕ is a closed immersion (see [Liu, Exercise 3.3.10]). If we assume that X' is connected and nonempty, ϕ corresponds uniquely to a connected component of $X' \times_X X''$ of degree one over X'. The degree of such a component can be measured at any point of X'. We conclude therefore by applying the next lemma to a component of $X' \times_X X''$. □

Lemma 5.2.8. *X is nonempty and connected if and only if the same is true for X_0.*

Proof. We are given the following cartesian square

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ S_0 \arrow[r] & S \end{tikzcd}$$

If X is connected and nonempty, then $f(X) \subseteq S$ is an nonempty closed subset of S (as f is proper). Let J be an ideal of A that identifies $f(X)$. Let $f(x) = p \in V(J)$ be a closed point of S. As I is contained in the Jacobson radical of A, the prime ideal p lies in S_0. Then

$$\begin{tikzcd} \{x\} \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\ & X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ & S_0 \arrow[r] & S \end{tikzcd}$$

In particuar, X_0 is nonempty. Furthermore, as this argument can be used for any connected component of X, if X is disconnected then also X_0 is disconnected.
Conversely, assume that X_0 is disconnected. Label C_0 a nonempty connected component of X_0. As the scheme X_0 is quasi-compact, C_0 is open and closed in X_0. Therefore, $C_0 \longrightarrow X_0$ is a finite étale morphism. By **Lemma** 5.2.6, there exists a finite étale morphism $C \longrightarrow X$ which induces $C_0 \longrightarrow X_0$. As C_0 is connected and nonempty, the same is true for C. The morphism $C \longrightarrow X$ is therefore of degree 1 at every point of C. As it is also finite and étale, it is both an open and a closed immersion, i.e. C is a connected component of X. If $C = X$, we would get $C_0 = X_0$, a contradiction. Then X is disconnected. Finally, it is clear that if X_0 is nonempty, X is nonempty too. □

Then **Theorem** 5.1.1 follows immediately from **Lemma** 5.2.6 and **Lemma** 5.2.7.

Chapter 6

Proper base change and henselian couples

In this chapter we introduce the notion of *henselian couple* as it is defined in [EGA IV.4]. We prove that, in the affine case, it coincides with the notion of henselian pair and that **Theorem** 5.1.1 allows us to prove that a couple (X, X_0) which lies over an henselian pair by means of a proper morphism is henselian. Finally, we discuss a conjecture which appears in [SGAIV, Exposé XII, Remarks 6.13]. A proof in the affine case was provided by the works of R. Elkik in [Elk] and of O. Gabber in [Gab].

6.1 Henselian couples

Let (A, I) be an henselian pair. As an immediate consequence of the definition we see that, for every finite morphism $= Spec(B) = X \longrightarrow Spec(A)$, we have a bijection

$$Id(B) = Of(X) = Of(X_0) = Id(B/IB) \quad \text{where } X_0 = X \times_{Spec(A)} Spec(A/I)$$

Here $Of(Z)$ denotes the set of subsets of Z which are both open and closed.
This fact suggests the following definition (see [EGA IV.4, Définition 18.5.5]), which is meant to generalize the notion of henselian pair to the non-affine setting.

Definition 6.1.1. Let X be a scheme and let X_0 be a closed subscheme. We say that (X, X_0) form an *henselian couple* if for every finite morphism $Y \longrightarrow X$ we have a bijection

$$Of(Y) = Of(Y_0)$$

where $Y_0 = Y \times_X X_0$.

Remark 6.1.2. If X is locally noetherian, it is a consequence of [EGAI, Proposition 6.1.4] and [EGAI, Corollaire 6.1.9] that connected sets in $Of(X)$ (resp. $Of(X_0)$) are in bijection with $\Pi_0(X)$ (resp. $\Pi_0(X_0)$), the set of connected components of X (resp. X_0).

Remark 6.1.3. It is a consequence of **Lemma** 2.1.5 that (X, X_0) is an henselian couple if and only if $(X_{red}, (X_0)_{red})$ is an henselian couple as well.

Proposition 6.1.4. *1. Let (X, X_0) be an henselian couple and let $Y \longrightarrow X$ be a finite morphisms. Let $Y_0 = Y \times_X X_0$. Then (Y, Y_0) is an henselian couple.*

2. Let $X = \coprod X_i$ be a disjoint union of schemes. Let X_0 be a closed subscheme of X with $X_0 = \coprod X_{i,0}$, where each $X_{i,0}$ is a closed subscheme of X_i. Then (X, X_0) is an henselian couple if and only if each couple $(X_i, X_{i,0})$ is henselian.

Proof. 1. This follows immediately from the definition. In fact, if $Z \longrightarrow Y$ is a finite morphism, then $Z \longrightarrow X$ is finite as well. Moreover, $Z_0 = Z \times_Y Y_0 \cong Z \times_X X_0$. Therefore,

$$Of(Z) = Of(Z_0)$$

2. ($\Rightarrow$) This is an immediate consequence of *1.*
($\Leftarrow$) A morphism $g : Z \longrightarrow X$ is finite if and only if each restriction $g_i : Z_i = g^{-1}(X_i) \longrightarrow X_i$ is finite. Set $Z_{i,0} = g_i^{-1}(X_{0,i})$ for every index i. Then we have a bijection between the set of open and closed subsets U (resp. U_0) of Z (resp. Z_0) and the collections of open and closed subsets (U_i) (resp. $(U_{i,0})$) of Z_i (resp. $Z_{i,0}$). □

Remark 6.1.5. It is immediate to observe that if (A, I) is a pair and $(\mathit{Spec}(A), \mathit{Spec}(A/I))$ is an henselian couple, then I is contained in the Jacobson radical of A. In fact, if $m \subseteq A$ is a maximal ideal, then we have a bijection

$$Of(\mathit{Spec}(A/m)) = Of(\mathit{Spec}(A/m \otimes_A A/I))$$

In particular, $\mathit{Spec}(A/m \otimes_A A/I)$ can not be the empty scheme. Therefore, as it is a closed subscheme of $\mathit{Spec}(A/m)$, we must have an equality $\mathit{Spec}(A/m) = \mathit{Spec}(A/m \otimes_A A/I)$, whence $I \subseteq m$. Moreover, if $Z \longrightarrow \mathit{Spec}(A)$ is a finite morphism, then $Z = \mathit{Spec}(B)$ is affine and the corresponding morphism $A \longrightarrow B$ is finite. Then we have bijections

$$Id(B) = Of(\mathit{Spec}(B)) = Of(\mathit{Spec}(B/IB)) = Id(B/IB)$$

We have just showed that an affine henselian couple is an henselian pair. The converse was observed at the beginning of the paragraph.

We shall now see an application of **Theorem** 5.1.1.

Lemma 6.1.6. *Let (A, I) be an henselian pair with A noetherian and let X be a proper A-scheme. Set $S = \mathit{Spec}(A)$, $S_0 = \mathit{Spec}(A/I)$ and let $X_0 = X \times_S S_0$. Then (X, X_0) is an henselian couple.*

Proof. First of all, notice that since the base scheme is noetherian, X is finitely presented over A. Therefore, the hypothesis of **Theorem** 5.1.1 are verified. Now let $Y \longrightarrow X$ be a finite morphism and label Y_0 the fiber product of Y and X_0 over X. In particular, Y is proper and finitely presented over $\mathit{Spec}(A)$, whence we have an equivalence of categories

$$\acute{E}t_f(Y) \longrightarrow \acute{E}t_f(Y_0)$$

which implies that the set of connected components of Y is in bijection with the set of connected components of Y_0. □

Lemma 6.1.7. *Let X be a scheme and let X_0 be a closed subscheme. Let A be a noetherian ring and assume that X is proper over $Spec(A)$. Also assume that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Put $J = ker(B = \mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0))$. If (B,J) is an henselian pair, then (X, X_0) is an henselian couple.*

Proof. Let (A^h, I^h) be the henselization of the couple (A, I) given by **Theorem** 2.3.6. Then we have the following diagram

$$\begin{tikzcd} & (X, X_0) \arrow[d, "f"] \\ (Spec(A^h), Spec(A^h/I^h)) \arrow[r, "\gamma"] & Spec(A, Spec(A/I)) \end{tikzcd}$$

which induces the following diagram of pairs:

$$\begin{tikzcd} & (B, J) \\ (A^h, I^h) \arrow[ur, dashed, "\psi"] & (A, I) \arrow[l] \arrow[u] \end{tikzcd}$$

The morphism ψ is the one induced by the universal property of (A^h, I^h). As

$$Hom_{Rings}(A^h, B) = Hom_{Schemes}(X, Spec(A^h))$$

the homomorphism ψ identifies a unique morphism of schemes $\phi : X \longrightarrow Spec(A^h)$. Thus we get the following commutative diagram

$$\begin{tikzcd} & X \arrow[dl, "\phi"'] \arrow[d, "f"] \\ Spec(A^h) \arrow[r, "\gamma"] & Spec(A) \end{tikzcd}$$

Moreover, by *Remark* 2.3.8, we get that

$$\gamma^{-1}(Spec(A/I)) = Spec(A^h \otimes_A A/I) = Spec(A^h/I^h)$$

whence

$$X \times_{Spec(A^h)} Spec(A^h/I^h) = X_0$$

Therefore, the couple (X, X_0) lies over the henselian couple $(Spec(A^h), Spec(A^h/I^h))$. Furthermore, A^h is a noetherian ring (see [StacksProj, Tag 0AGV]). Finally, as f is a proper morphism and γ is separated, we get that ϕ is proper as well by [Liu, Proposition 3.3.16]. Then we can conclude that (X, X_0) is an henselian couple by the previous lemma. $\square$

The previous lemma tells us that, under some appropriate hypothesis, if the pair

$$(\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$$

is henselian, then (X, X_0) is an henselian couple. It is natural to ask if the converse is true, i.e. if given an henselian couple (X, X_0) the associated pair is henselian. An answer is provided by the next lemma.

Lemma 6.1.8. *Let X quasi-compact and quasi-separated scheme and let $i : X_0 \longrightarrow X$ be a closed immersion such that (X, X_0) is an henselian couple.*
Then $(B, J) = (\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$ is an henselian pair.

Proof. By [StacksProj, Tag 09XI], it is sufficient to show that for every étale ring map $B \longrightarrow C$ together with a B-morphism $\sigma : C \longrightarrow B/J$, there exists a B-morphism $C \longrightarrow B$ which lifts σ. Let $\phi : B \longrightarrow C$ be an étale ring map and let $\sigma : C \longrightarrow B/J$ be a B-morphism, i.e. $\sigma = \pi \circ \phi$, where $\pi : B \longrightarrow B/J$ is the canonical map.
Consider the cartesian diagram

$$\begin{tikzcd} X_C = X \times_{Spec(B)} Spec(C) \arrow[r] \arrow[d] & X \arrow[d] \\ Spec(C) \arrow[r] & Spec(B) \end{tikzcd}$$

As $Spec(C) \longrightarrow Spec(B)$ is étale and separated, the morphism $X_C \longrightarrow X$ is étale and separated as well. Then, by [EGA IV.4, Proposition 18.5.4], we have a bijection

$$\Gamma(X_C/X) \longrightarrow \Gamma(X_C \times_X X_0/X_0)$$

between the sections of $X_C \longrightarrow X$ and those of $X_C \times_X X_0 \longrightarrow X_0$.
Remark 1. The universal property of $X_C \times_X X_0$ tells us that

$$\Gamma(X_C \times_X X_0/X_0) = Hom_X(X_0, X_C)$$

Remark 2. Let $\mathscr{J} \subseteq \mathscr{O}_X$ be the sheaf of ideals associated to X_0. Then we have a short exact sequence of $\mathscr{O}_X$-modules

$$0 \longrightarrow \mathscr{J} \longrightarrow \mathscr{O}_X \longrightarrow i_*\mathscr{O}_{X_0}(X_0) \longrightarrow 0$$

Applying the global sections functor, we get an exact sequence

$$0 \longrightarrow J = \mathscr{J}(X) \longrightarrow \mathscr{O}_X(X) = B \longrightarrow \mathscr{O}_{X_0}(X_0)$$

Hence, we have an homomorphism

$$B/J \longrightarrow \mathscr{O}_{X_0}(X_0)$$

Therefore, we get a morphism of schemes

$$X_0 \longrightarrow Spec(\mathscr{O}_{X_0}(X_0)) \longrightarrow Spec(B/J)$$

Also notice that the diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d] \\
Spec(B/J) \arrow[r] & Spec(B)
\end{tikzcd}$$

is commutative.

Now consider the diagram

$$\begin{tikzcd}
X_0 \arrow[dr, dashed, "\exists!\ \tilde{\alpha}"] \arrow[drr] \arrow[ddr] \arrow[ddd] & & \\
& X_C = X \times_{Spec(B)} Spec(C) \arrow[r] \arrow[d] & X \arrow[d] \\
& Spec(C) \arrow[r] & Spec(B) \\
Spec(B/J) \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Label $\tilde{\alpha} : X_0 \longrightarrow X_C$ the X-morphism provided by the universal property of X_C and let $\alpha : X \longrightarrow X_C$ be the corresponding X-morphism in $\Gamma(X_C/X)$.

Consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[rr, bend left, "id_X"] \arrow[dr] & X_C \arrow[r] \arrow[d] & X \arrow[d] \\
& Spec(C) \arrow[r] & Spec(B) \\
& & Spec(B/J) \arrow[ul] \arrow[u]
\end{tikzcd}$$

and the corresponding commutative diagram in *Rings*:

$$\begin{tikzcd}
B \arrow[rr, bend left, "id_B"] & \mathscr{O}_{X_C}(X_C) \arrow[l, "\alpha"'] & B \arrow[l] \\
& C \arrow[ul, "\psi"] \arrow[u] \arrow[dr, "\sigma"'] & B \arrow[l, "\phi"'] \arrow[u, "id_B"'] \arrow[d, "\pi"] \\
& & B/J
\end{tikzcd}$$

It is then clear that ψ is the B-morphism we were looking for. This concludes the proof of the lemma. $\square$

Corollary 6.1.9. *Let (X, X_0) be an henselian couple. Assume that X is proper over a noetherian ring A and that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Then (X, X_0) is proper over an henselian pair.*

Proof. As X is proper over $Spec(A)$, it is a quasi-compact and quasi-separated scheme. Hence, by **Lemma** 6.1.8, $(\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$ is an henselian pair. Therefore, by the same construction described in **Lemma** 6.1.7, we get that (X, X_0) is proper over (A^h, I^h). $\square$

Corollary 6.1.10. *Let (X, X_0) be a couple and assume that X is proper over a noetherian ring A and that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Then (X, X_0) is an henselian couple if and only if $(\mathscr{O}_X(X), ker(\mathscr{O}_X(X) \longrightarrow \mathscr{O}_{X_0}(X_0)))$ is an henselian pair.*

6.2 A remark on a property of a subclass of henselian couples

In [SGAIV] the proposition which follows is proved (see [SGAIV, Exposé XII, Proposition 6.5]). We will restate it for the reader's convenience. For a proof and for the exact definitions of the objects which are involved, we refer to the original text.

Proposition 6.2.1. *Let $h : Y \longrightarrow X$ be a morphism between quasi-compact and quasi-separated schemes.*

1. *Let I be a set with at least two points. The following statements are equivalent:*

   (a) *For every sheaf of sets $\mathscr{F}$ over X, the canonical map*

   $$H^0(X, \mathscr{F}) \longrightarrow H^0(Y, h^*\mathscr{F})$$

   is injective (resp. bijective).

   (b) *For every finite morphism $X' \longrightarrow X$, let $h' : Y' \longrightarrow X'$ be the morphism induced by h. The canonical map*

   $$H^0(X', I_{X'}) \longrightarrow H^0(Y', I_{Y'})$$

   induced by h' is injective (resp. bijective).

   (c) *With the same notation as in (b), the map $U \mapsto (h')^{-1}(U) : Of(X') \longrightarrow Of(Y')$ is injective (resp. bijective).*

   (d) *(If X is locally noetherian) With the same notation as in (b), the map $U \mapsto (h')^{-1}(U) : \Pi_0(X') \longrightarrow \Pi_0(Y')$ is surjective (resp. bijective). Equivalently, if X' is non-empty (resp. connected and non-empty), the same is true for Y'.*

   Moreover, if these conditions are satisfied, for every sheaf of groups $\mathscr{F}$ over X, the functor $\mathscr{P} \mapsto h^*\mathscr{P}$ from the category of torsors over $\mathscr{F}$ to the category of torsors over $h^*\mathscr{F}$ is faithful (resp. fully faithful). A fortiori, in the case respé, the canonical map

   $$H^1(X, \mathscr{F}) \longrightarrow H^1(Y, h^*\mathscr{F})$$

   is injective. Finally, under the same assumptions, the functor

   $$\acute{E}t_f(X) \longrightarrow \acute{E}t_f(Y)$$

   $$X' \mapsto Y' = X' \times_X Y$$

   is faithful (resp. fully faithful).

2. *Let $\mathbb{L}$ be a non-empty subset of the set of prime numbers $\mathbb{P}$. The following statements are equivalent:*

   (a) *For every sheaf of groups $\mathscr{F}$ ind-$\mathbb{L}$-finite over X, the canonical map*

   $$H^i(X, \mathscr{F}) \longrightarrow H^i(Y, h^*\mathscr{F})$$

   is bijective for $i = 0, 1$.

(b) *For every finite morphism $X' \longrightarrow X$, label h' the map induced by h via base change. For every ordinary $\mathbb{L}$-group $\mathscr{G}$, the canonical map*

$$H^i(X', \mathscr{G}_{X'}) \longrightarrow H^i(Y', h^*\mathscr{G}_{Y'})$$

is bijective for $i = 0, 1$.

(c) *If $\mathbb{L} = \mathbb{P}$, with the same notations as in (b), the inverse image functor induced by h' induces an equivalence between the category $\acute{E}t_f(X')$ and the category $\acute{E}t_f(Y')$.*

(d) *If X is noetherian and if $\mathbb{L} = \mathbb{P}$, with the same notations as in (b), if X' is non-empty, the same is true for Y'. If y' is a geometric point of Y' and x' is its image in X', we have two bijections*

$$\Pi_0(Y') \longrightarrow \Pi_0(X')$$

and

$$\Pi_1(Y', y') \longrightarrow \Pi_1(X', x')$$

induced by the canonical maps.

3. *Let $\mathbb{L}$ be a non-empty subset of the set of prime numbers $\mathbb{P}$ and let $n \in \mathbb{N}$. The following statements are equivalent:*

(a) *For every $\mathbb{L}$-torsion sheaf $\mathscr{F}$ on X, the homomorphism*

$$H^i(X, \mathscr{F}) \longrightarrow H^i(Y, h^*\mathscr{F})$$

is an isomorphism if $i \leq n$ and is a monomorphism if $i = n + 1$.

(b) *If $n \geq -1$, with the same notation as in (a), the homomorphism is injective if $i = 0$ and it is surjective if $i \leq n$.*

(c) *If $n \geq -1$, for every finite morphism $X' \longrightarrow X$, let $h' : Y' \longrightarrow X'$ be the morphism induced by h. For every $l \in \mathbb{L}$ and for every $\nu > 0$, the canonical homomorphism*

$$H^i(X', (\mathbb{Z}/l^\nu\mathbb{Z})_{X'}) \longrightarrow H^i(Y', (\mathbb{Z}/l^\nu\mathbb{Z})_{Y'})$$

is ijective for $i = 0$ and surjective for $i \leq n$

Remark 6.2.2. Notice that, if $h : Y \longrightarrow X$ is a closed immersion, condition *1.* in **Proposition** 6.2.1 is equivalent to say that (X, Y) is an henselian couple.

In [SGAIV, Exp. XII, Remarks 6.13] the authors conjectured the following statement:

If (X, X_0) is an henselian couple, then conditions 2. and *3.* in **Proposition** 6.2.1 *are satisfied with $\mathbb{L} = \mathbb{P}$ and for every n.*

A positive answer to this question is provided in the affine case by the works of R. Elkik in [Elk] and of O. Gabber in [Gab]. In particular, the fact that condition *2.* is satisfied is a consequence of the footnote at page 326 in [Gab]. Notice that it is a particular case of **Theorem** 5.1.1. The fact that condition *3.* is satisfied is a consequence of [Gab, Corollary 1].

Notice that every henselian couple (X, X_0) which arises as in **Lemma** 6.1.6 satisfies conditions *2.* and *3.* in **Proposition** 6.2.1 with $\mathbb{L} = \mathbb{P}$ and for every n as well. In fact, **Theorem** 5.1.1 immediately implies that condition *2.(c)* in **Proposition** 6.2.1 is satisfied. The fact that condition *3.* is satisfied can be seen as a consequence of [Gab, Corollary 1] also in this case. Then, applying **Corollary** 6.1.9, we get the following proposition:

Proposition 6.2.3. *Let (X, X_0) be an henselian couple. Assume that X is proper over a noetherian ring A and that $X_0 = X \times_{Spec(A)} Spec(A/I)$ for some ideal $I \subseteq A$. Then conditions 2. and 3. in **Proposition** 6.2.1 are satisfied with $\mathbb{L} = \mathbb{P}$ and for every n.*

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