Comparison theorem between algebraic De Rham cohomology and infinitesimal cohomology

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1 Introduction

The study of differential forms sparked by the study of integrals of different kinds dates back to at least the times of Euler and was a motivator for generations of mathematicians afterwards. However it was until the work of de Rham who showed there is an isomorphism between the singular cohomology groups of a smooth manifold and, what we now call, its de Rham cohomology groups, that the relationship between these objects and the intrinsic topological properties of the manifold was firmly established and a modern and more algebraic treatment was allowed.

Later on a completely algebraic analog was developed so it was possible to apply this theory to the theory of schemes. It was Grothendieck who then showed that this algebraic de Rham theory was compatible with what was already known by proving that for an affine nonsingular scheme X over $\mathbb{C}$, the de Rham cohomology of this space is the same as the singular cohomology of its associated analytic space X^{an}. [Gro66]

Now turning towards a theory that would work for schemes over fields of positive characteristic, Grothendieck and Berthelot began working on crystalline cohomology and its characteristic zero version, infinitesimal cohomology through the general theory of a topoi over a site.

The concept of (Koszul) connections over a scheme was extended to the concept of stratifications and an equivalence between these and certain sheaves over the infinitesimal site called crystals is stated. This is a fundamental fact that helped Grothendieck to prove in [GGK68] the following theorem:

Theorem 1.1. *Let K be of characteristic 0 and let X be a smooth scheme over K, then there is a canonical isomorphism $H^\bullet(X/K_{inf}, \mathcal{O}^{inf}_{X/K}) \cong H^\bullet_{dR}(X/S)$*

The idea is that it is possible to replace the use of differential forms with the differential behaviour reflected on the stratifications of the space. As Grothendieck concludes after stating this result, here lies the importance of this theorem, it allows us to study de Rham cohomology in more general context where it is known to be problematic, for example non smoothness and positive characteristic

The present work intends to present a general introduction to the theory of algebraic de Rham cohomology. The goal of this project is to display the classical comparison theorem between algebraic de Rham cohomology with the infinitesimal cohomology.

Through this memoire we will only assume a basic knowledge of scheme theory and of category theory. The appendices at the end will try to recall all the necessary definitions and results from homological algebra and the general theory of topos.

To do so we will present a brief introduction to the theory of algebraic differential forms on a scheme and we will compute the cohomology of some classical and illustrative examples.

Afterwards we will introduce the concept of connections on an $\mathcal{O}_X$-module over a scheme X, which similarly to differential forms, is an algebraic analog of a classical theory in differential varieties.

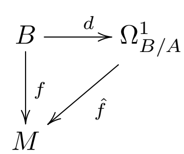
The next section will introduce the infintesimal site using the general language of topos and sites. We will establish some important properties of this site and its relationship with connections. For the main theorem we will follow a recent paper by [BdJ11] that will tie up all the previous concepts to prove the comparison theorem for infinitesimal cohomology and algebraic de Rham cohomology.

2 Algebraic DeRham cohomology

Through this section we will follow [Liu02], [Har77] and [Ked] Let $A \to B$ be a morphism of rings so that B is an A-algebra. We can define an analog for differential forms in an algebraic setting by defining formally $\Omega^1_{B/A}$ as the B-module generated by elements db where $b \in B$ with the following relations:

1. $da = 0$ for $a \in A$
2. $d(bb') = bdb' + b'db$ (Leibniz' rule)
3. $d(b + b') = db + db'$

This module $\Omega^1_{A/B}$ comes equipped with a morphism $d : B \to \Omega^1_{B/A}$ and the corresponding universal property: If $f : B \to M$ is a morphism of A-algebras satisfying the Leibniz' rule, then there exists a map $\hat{f} : M \to \Omega^1_{B/A}$ such that the following diagram commutes:



Definition 2.1. *Let M be a B-module, the set of all morphisms $B \to M$ satisfying the Leibniz rule are called the A-derivations of B into M and we denote by $Der(B, M)$ the set of all of them.*

Classical and useful examples are the following

Example. *Let f be the identity $Id : A \to A$, then $\Omega^1_{A/A} = 0$*

Example. *If $S^{-1} \subseteq A$ is a multiplicative then the canonical morphism $A \to S^{-1}A$ induces $\Omega^1_{S^{-1}A/A} = 0$*

Proof. If $a \in S^{-1}A$ then there is $a' \in A$ such that $a'a \in A$, so $a'd(a) = d(a'a) = 0$ and since a' is invertible, we have $d(a) = 0$ □

Example. *Let A, $B = A[x_1, \ldots, x_n]$, and $\phi : A \to B$ the inclusion, then $\Omega^1_{B/A}$ is the free B-module generated by the dx_i's*

The following results are helpful for computing the modules $\Omega_{B/A}$

Proposition 2.1. *A morphism of A-algebras $f : B \to C$ induces morphisms $\alpha : \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$ and $\beta : \Omega^1_{C/A} \to \Omega^1_{B/A}$*

Proof. The morphism α is given by the rule $\alpha(db \otimes c) := cdf(b)$ and the morphism β is given by the rule $\beta(dc) := dc \in \Omega^1_{C/B}$ □

Proposition 2.2. *Let A' be an A-algebra and let $B' := B \otimes_A A'$, then $\Omega_{B'/A'} \cong B' \otimes_B \Omega^1_{B/A}$ as B'-modules*

Proof. The morphism $d \otimes Id_{A'} : B' \to \Omega^1_{B/A} \otimes B'$ satisfies the universal property of $\Omega^1_{B'/A'}$ since for every A'-module M and every derivation $f : B' \to M$ we have a derivation $B \to M$ given by $b \to f(b) \otimes 1) \in M$, and by the universal property of $\Omega^1_{B/A}$ there is a morphism $\hat{f} : \Omega^1_{B/A} \to M$ which induces a morphism $\Omega^1_{B/A} \otimes B' \to M$ by the rule $db \otimes b \otimes a' \to a'\hat{f}(db)$ □

Proposition 2.3. *Let $A \to B$ be a morphism of A-algebras then there is an exact sequence:*

$$\Omega^1_{B/C} \otimes_B C \xrightarrow{\alpha} \Omega^1_{C/A} \xrightarrow{\beta} \Omega^1_{C/B} \to 0$$

Proof. Applying the functor $Hom_C(\_, M)$ for some C-module M, we would only have to check the exactness of the sequence

$$0 \to Hom_C(\Omega^1_{C/B}, M) \to Hom_C(\Omega^1_{C/A}, M) \to Hom_C(\Omega^1_{B/A} \otimes_B C, M)$$

but this follows from the identity $Hom_C(\Omega^1_{B/A} \otimes_B C, M) = Hom_B(\Omega^1_{B/A}, M)$ since the sequence

$$0 \to Der_B(C, M) \to Der_A(C, M) \to Der_A(B, M)$$

□

Proposition 2.4. *Let $S^{-1} \subseteq B$ be a multiplicative subset, then $\Omega^1_{S^{-1}B/A} \cong S^{-1}\Omega^1_{B/A}$*

Proof. If we take $C = S^{-1}(B)$ then the previous proposition gives us the sequence

$$\Omega^1_{B/A} \otimes S^{-1}B \to \Omega^1_{S^{-1}B/A} \to \Omega^1_{S^{-1}B/B} = 0$$

□

More generally we can define the sheaf of differentials over a scheme X relative to Y given a morphism $X \to Y$ in the following way:

Definition 2.2. *Let $\Delta : X \to X \times_Y X$ be the diagonal morphism, which defines a closed subscheme isomorphic to X in an open subset of $X \times_Y X$. To this subscheme $\Delta(X)$ corresponds a sheaf of ideals $\mathcal{I}$. We define the sheaf of differentials as $\Omega^1_{X/Y} := \Delta^*(\mathcal{I}/\mathcal{I}^2)$.*

Remark. *These two definitions are compatible in the case where X and Y are affine schemes*

Definition 2.3. *Let us denote $\mathcal{O}_X$ by $\Omega^0_{X/Y}$ and $\Omega^i_{X/Y}$ by $\bigwedge^i \Omega^i_{X/Y}$.*

Remark. *If $\mathcal{O}_X$ is generated by $\{x_i\}_{i \in I}$ then the $\Omega^n_{X/Y}$ are generated as $\mathcal{O}_X$-modules by the elements $dx_{i_0} \wedge \cdots \wedge dx_{i_n}$ for $i_j \in \{1, \ldots, n\}$*

To ease notation we write $dx_{i_0\dots i_n}$ instead of $dx_{i_0} \wedge \dots \wedge dx_{i_n}$ for $i_j \in \{1, \dots, n\}$.In this way, every element of $\Omega^n_{X/Y}$ can be written as a finite sum of the form $\underset{i_0,\dots,i_n}{\Sigma} f dx_{i_0\dots i_n}$ for some section f of $\mathcal{O}_X$

Definition 2.4. *Let $X \to Y$ be a morphism of schemes, if we denote $\mathcal{O}_X$ by $\Omega^0_{X/Y}$ and $\Omega^i = \bigwedge^i \Omega^1_{X/Y}$ for all $i \geq 2$, then the algebraic de Rham complex $\Omega^\bullet_{X/Y}$ is the sequence defined as*

$$\Omega^0_{X/Y} \xrightarrow{d^0} \Omega^1_{X/Y} \xrightarrow{d^1} \Omega^2_{X/Y} \to \dots \to \Omega^i_{X/Y} \to \dots$$

Where the differentials $d^i : \Omega^i_{X/Y} \to \Omega^{i+1}_{X/Y}$ are calculated by $d^i(\Sigma f dx_{i_0\dots i_n}) := \Sigma df \wedge dx_{i_0\dots i_n}$

We now can define the algebraic deRham cohomology $H^\bullet_{dR}(X/Y)$ as the hypercohomology associated to the algebraic deRham complex, that is $\mathbb{H}^\bullet(X, \Omega^\bullet_{X/Y})$

Proposition 2.5. *(Algebraic Poincare Lemma) Let K be a field of characteristic 0 and $B = K[x_1 \dots x_n]$ ($SpecB = \mathbb{A}^n_K$) as in the previous example, then the de Rham complex $B \to \Omega^1_{B/K} \to \dots \to \Omega^n_{B/K} \to \dots$ is exact.*

Proof. Following [Har75, Section 7.2], one can proceed by induction on n.
If n=0 the result is clear, so now let $\omega \in \Omega^p_B$, to get a form $\rho \in \Omega^{p-1}_B$ such that $d\rho = \omega$, we write $\omega = \omega_1 dx_1 + \omega_2$, where ω_1 and ω_2 are forms not involving dx_1. So now if we put $\rho_1 = \int \omega_1 dx_1$ we see that $d\rho_1 = \omega_1 dx_1 + \omega_3$ where dx_1 doesn't appear in ω_3. So putting $\omega - d\rho_1$ in the place of ω we see that it is possible to reduce to the case where ω is of the form $\Sigma f_\sigma dx_i \wedge \dots \wedge dx_p$ where $i > 1$ and so the problem is reduced to the case $K[x_2 \dots x_n]$ which is true by induction. □

Remark. *If K had positive characteristic p, then this no longer holds as $d(x^p) = px^{p-1}d(x) = 0$ and so the sequence can no longer be exact*

There is an important result due to Grothendieck that compares the algebraic de Rham cohomology of an affine nonsingular scheme to the usual de Rham cohomology of the associated analytic space. We have the following:

Proposition 2.6. *Let X be a scheme of finite type over $\mathbb{C}$, then the analytification of the scheme X, denoted by $(X^{an}, \mathcal{O}^{an}_X)$ is a complex variety with a morphism of ringed spaces $\phi : X^{an} \to X$ such that the map on the underlying space is the inclusion of closed points.*

Proof. We following [Har75, Section 1.6].
Let U_i be an affine cover of X and let let $f_i : U_i \to \mathbb{A}^n_i$ be a closed embedding. Then the polynomials that generate the ideal that determines the embeddings f_i define a complex variety U^{an}_i. The variety resulting of the glueing of the U^{an}_i (compatible with the gluing of the U_i in X) is the space X^{an}. □

Theorem 2.1. *Let X be an affine nonsingular scheme over $\mathbb{C}$, then $H(X^{an}, \mathbb{C}) \cong H_{dR}(X/\mathbb{C})$, where $H(X^{an}, \mathbb{C})$ denotes the Betti cohomology of X^{an}*

Proof. [Gro66] □

Another useful result due to Faltings for calculating algebraic de Rham cohomology is the degeneration of the Hodge to de Rham complex:

Theorem 2.2. *Let X be a smooth proper scheme over a field k of chacateristic 0, then there is a spectral sequence $E_1^{p,q} = H^p(X, \Omega^q) \Rightarrow H_{dR}^{p+q}(X)$ that degenerates at the E_1 page, giving an isomorphism $\bigoplus_{p+q=n} H^p(X, \Omega^q_{X/K}) \cong H^n(X, K)$*

We present the following classical examples and a detailed computation of their cohomology groups. Through the rest of this section we suppose K is a field of characteristic 0.

Example. *Let $A = K[x, y]/(y^2 - x^3)$ and $X = Spec(A)$ be a K-scheme where K is of characteristic 0. Then its de Rham cohomology groups are $H^0(X) = K$, $H^1(X) = K$ and $H^2(X) = 0$.*

Proof. First we notice that by the relation $y^2 - x^3$ gives us that

$$K[x, y]/(y^2 - x^3) = \{f(x) + yg(x) \mid f(x), g(x) \in K[x]\}$$

The relation $y^2 - x^3$ above gives us a relation $2ydy - 3x^2dx$ in $\Omega^1_{X/K}$, that is

$$\Omega^1_{X/K} = (Ady \oplus Adx)/(2ydy - 3x^2dx)$$

and so we can write the elements of $\Omega^1_{X/K}$ as the elements of the form $(f(x)+yg(x))dx+h(x)dy$ where $f, g, h \in K[x]$
We calculate Imd^0 directly, that is

$$Imd^0 = \{d(f(x) + yg(x)) \mid f, g \in K[x]\} = \{f'(x)dx + yg'(x)dx + g(x)dy\}$$

By the form of the elements of Imd^0 we get that

$$Kerd^0 = \{f(x) + yg(x) \mid f'(x) = 0, g(x) = 0\} = \{c \mid c \in K\}$$

This means that $H^0(X) = Kerd^0 = K$
We have that $\Omega^2_{X/K} = \Omega^1_{X/K} \wedge \Omega^1_{X/K}$ and so the elements are

$$\begin{aligned} &((f_1(x) + yg_1(x))dx + h_1(x)dy) \wedge (f_2(x) + yg_2(x))dx + h_2(x)dy \\ &= (f_1h_2(x) + yg_1h_2(x))dx \wedge dy + (f_2h_1(x) + yg_2h_1(x))dy \wedge dx \end{aligned}$$

And so we can write all elements of $\Omega^2_{X/K}$ just as $f(x)dx \wedge dy$ for some $f(x) \in K[x]$
Continuing we have

$$\begin{aligned} Imd^1 = \{d((f(x) + yg(x))dx + h(x)dy)\} &= \{g(x)dy \wedge dx + h'(x)dx \wedge dy\} \\ &= \{(g(x) - h'(x))dx \wedge dy\} \end{aligned}$$

This means that

$$Kerd^1 = \{(f(x) + yg(x))dx + h(x) \mid (g(x) - h'(x))dx \wedge dy = 0\}$$

and so we must have $g(x) - h'(x) = x^2p(x)$ for some $p(x) \in K[x]$
So

$$Kerd^1 = \{(f(x) + y(x^2p(x) + h'(x)))dx + h(x)dy\}$$

but after some inspection we can realize that the elements $yx^n dx$ are integrable for $n \geq 3$, more precisely we have that

$$d(y^3x^{n-2}) = (n-2)y^3x^{n-3}dx + 3x^{n-2}y^2dy = 3yx^n dx + (n-2)yx^n dx = (n+1)yx^n dx$$

Then

$$H^1(X) = Kerd^1/Imd^0$$
$$= \{(f(x) + y(x^2p(x) + h'(x)))dx + h(x)dy\}/\{(f'(x) + yh'(x))dx + h(x)dy\}$$

and by the discussion above we get that $H^1(X)$ is generated by Kx^2ydx
Finally

$$H^2(X) = \Omega^2_{X/K}/Imd^1 = \{h(x)dx \wedge dy \mid h(x) \in K[x]\}/\{(g(x) - f'(x))dx \wedge dy \mid f, g \in K[x]\}$$

which clearly must be 0 as every polynomial $h(x)$ is of the form $g(x) - f'(x)$ for some $h, g, f \in K[x]$. □

Remark. *The previous example displays the misbehaviours of the algebraic de Rham cohomology in the case of nonsmooth schemes, as the first Betti cohomology group of the complex curve $y^2 - x^3 = 0$ is 0.*

Example. *Let $X = Spec(K[x, x^{-1}])$ over $Spec(K)$. Then its cohomology groups are $H^0_{dR}(X) \cong K$, $H^1_{dR}(X) \cong K$, $H^i_{dR}(X) = 0\ \forall i \geq 2$*

Proof. If we let $S = \{x^k \mid k \in \mathbb{N}\}$ then we have the isomorphism $S^{-1}K[x] \cong K[x, x^{-1}]$. This means we can describe $K[x, x^{-1}]$ as

$$K[x, x^{-1}] = \{\frac{f(x)}{x^k}\}$$

Again by this isomorphism and by proposition 2.4 we have

$$\Omega^1_{S^{-1}K[x]/K} = \{\frac{f(x)dx}{x^k} \mid f_i \in K[x], k \in \mathbb{N}\}$$

Since we have $d(f(x)) = f'(x)dx$ and $d((x^k)^{-1}) = -kdx(x^{k+1})^{-1}$ then we have the following description for the submodule Imd^0

$$Imd^0 = \{\frac{f'(x)dx}{x^k} + \frac{-kdx}{x^{k+1}} \mid k \in \mathbb{N}, f(x) \in K[x]\}$$

And so for $Kerd^0$ we have :

$$Kerd^0 = \{\frac{f(x)}{x^k} \mid f'(x) = 0, k = 0\} = \{k \mid k \in K\} \cong K$$

Since $S^{-1}\bigwedge^i \Omega^1_{K[x]/K} \cong \bigwedge^i S^{-1}\Omega^1_{K[x]/K}$ then

$$\bigwedge^i S^{-1}\Omega^1_{K[x]/K} = 0, \forall i \geq 2$$

So the first de Rham cohomology group is the quotient

$$H^1_{dR}(X) = \Omega^1_{S^{-1}K[x]/K}/Imd^0 = \{\frac{f(x)}{x^k}\}/\{\frac{f'(x)dx}{x^k} + \frac{-kdx}{x^{k+1}}\}$$

From this we can see that $\frac{dx}{x}$ is a generator of this group, that is

$$H^1_{dR}(X) \cong K\frac{dx}{x}$$

□

Through the following example we will use the following characterization of $\mathbb{P}^n_A$

Proposition 2.7. *Let A be a ring and consider the A-schemes $U_i = Spec(A[x_i^{-1}x_j])$ for $0 \leq i, j \leq n$ then the A-schemes U_i can be glued along the principal open sets $U_{ij} = D(x_i^{-1}x_j) \subseteq U_i$ since $U_{ij} \cong U_{ji}$ for $i, j \in \{1, \ldots, n\}$. The resulting scheme is isomorphic to $ProjA[x_1, \ldots, x_n] = \mathbb{P}^n_A$*

Proof. ([Liu02, Example 3.34]) □

Example. *Let $X = \mathbb{P}^1_K$ over K, then its de Rham cohomology groups are $H^0_{dR} \cong K$, $H^i_{dR} = 0$ for $i \geq 0$.*

Proof. Let $U_0 = Spec(K[x])$, $U_1 = Spec(K[x^{-1}])$ be an affine cover of X, then the cohomology groups of U_0, U_1 and $U_0 \cap U_1 = Spec(K[x, x^{-1}])$ will be enough to compute the cohomology of the whole space by Ĉech cohomology (see A.18).
The only thing we're missing is the first cohomology group of $H^1(U_1)$.
We can see that

$$\Omega^1(U_1) = \{f(x^{-1})d(x^{-1}) \mid f(x^{-1}) \in K[x^{-1}]\}$$

and that

$$Imd^0 = \{f'(x)d(x^{-1}) \mid f(x^{-1}) \in K[x^{-1}]\}$$

So the first de Rham cohomology group is generated by $d(x^{-1})/x$, that is:

$$H^1_{dR}(U_1) := \Omega^1(U_1)/Imd^0 \cong Kd(x^{-1})/x$$

So we have the following data:

$$H^0_{dR}(U_0) \cong H^0_{dR}(U_1) \cong K$$
$$H^1_{dR}(U_0) = 0, H^1_{dR}(U_1) \cong K$$
$$H^0_{dR}(U_0 \cap U_1) \cong K$$
$$H^1_{dR}(U_0 \cap U_1) \cong K$$

Now, the sequence becomes :

$$0 \to H^0_{dR}(X) \to H^0_{dR}(U_0) \oplus H^0_{dR}(U_1) \to H^0_{dR}(U_0 \cap U_1) \to H^1_{dR}(X) \to$$
$$H^1_{DR}(U_0) \oplus H^1_{dR}(U_1) \to H^1_{dR}(U_0 \cap U_1) \to 0 \to \dots$$

Where the first nonzero map is given by $f \mapsto (res_{U_0}(f), res_{U_1}(f))$, the second map by $(f, g) \mapsto res_{U_0 \cap U_1}(f) - res_{U_0 \cap U_1}(g)$, the map $H^0_{dR}(U_0 \cap U_1) \to H^1_{dR}(X)$ is the connection map given by the snake lemma and the following morphisms are the induced by the cohomology functors.
Replacing what we know about the cohomology:

$$0 \to H^0_{dR}(X) \to K \oplus K \to K \to H^1_{dR}(X) \to K \to K \to 0 \to \dots$$

From this we get that $H^0_{dR}(X) \cong K$, that $H^1_{dR}(X) = 0$ and that $H^i_{dR}(X) = 0$ for all $i \geq 0$. □

Example. *Let $X = Spec(K[x, y]/(y^2 - x^3 - ax - b)$ be an elliptic curve (with $\Delta \neq 0$), then its de Rham cohomology is $H^0_{dR}(X) = K$, $H^1_{dR}(X) \cong K \oplus K$, $H^2_{dR}(X) = K$ and $H^i_{dR}(X) = 0$ for all $i \geq 2$*

Proof. Let $A = K[x, y]/(y^2 - P(x))$ where $P(x) = x^3 + ax + b$, so every element of A can be written as follows:

$$A = \{f(x) + yg(x) \mid f(x), g(x) \in K[x]\}$$

Since X is an elliptic curve, we know that $P(x)$ has no repeated roots and so there are polynomials R, S such that $RP + SP' = 1$. With this relation we get that

$$dx = (1)dx = (RP + SP')dx = (RPdx + SP'dx)$$

if we use the relation $ydy = 1/2P'(x)dx$ we get $RPdx + 2Sydy = Ry^2dx + 2Sydy$ so

$$y(Rydx + 2Sdy) = dx$$

Let us call $\alpha = Rydx + 2Sdy$.
Then we get then:

$$y\alpha = dx, \frac{1}{2}\alpha P'(x) = dy$$

Using this, we write the image Imd^0 as follows:

$$Imd^0 = \{d(f(x) + yg(x))\} = \{f'(x)dx + g'(x)ydx + g(x)dy\}$$

From here we see that

$$Kerd^0 = H^0_{dR} = \{f(x) + yg(x) \mid g(x) = 0 = f'(x)\} \cong K$$

We can also write the submodule Imd^0 , using the identities above and the relation $y^2 = P(x)$, as

$$Imd^0 = \{(\frac{1}{2}P'g(x) + g'(x)P)\alpha + f'(x)y\alpha\}$$

By this and by the observation that we can write

$$\Omega^1_{A/K} = \{(f(x) + yg(x))\alpha \mid f(x), g(x) \in K[x]\}$$

We can now calculate $H^1_{dR}(X)$ by noticing that the degree of α term must be at least 2 and that we can write any term of degree ≥ 2 as a linear combination of $(\frac{1}{2}P'(x)g(x) + g'(x)P)$, in particular the leading term kx^n of a polynomial $f(x)$ of degree at least 2 can be written as

$$kx^n = (\frac{3}{2} + (n-2))\frac{k}{(3/2 + (n-2))}x^n, n \geq 2$$

□

This shows that the first de Rham cohomology group is

$$H^1_{dR}(X) \cong K\alpha \oplus Kx\alpha$$

3 Connections

Through this section we will follow [Del70], [DM] and [Lee06, Chapter 5]
The next step towards a comparison theorem is the study of the concept of a connection for a morphism of schemes $f : X \to S$ and its relationship with modules over crystals of $\mathcal{O}_{X/S}$-modules.
To begin with this let us give a brief reminder of connections and parallel transport in smooth manifolds.

Remark. *Given a smooth manifold M, $\pi : E \to M$ a vector bundle over M, a connection is a map $\nabla : \mathcal{T}(M) \times \mathcal{E}(M) \to \mathcal{E}(M)$, where $\mathcal{T}(M)$ are the smooth sections of the tangent bundle and $\mathcal{E}(M)$ are the smooth sections over the vector bundle E, that satisfy the following:*

1. *∇ is $C^\infty(M)$-linear on the first entry*
2. *∇ is $\mathbb{R}$-linear on the second entry*
3. *∇ satisfies $\nabla(X, fY) = f\nabla(X, Y) + X(f)Y$ for all $f \in C^\infty(M)$*

If we wanted to measure the change of a vector field along a curve $\gamma : I \to M$ with respect to a connection ∇ over the tangent bundle, we can define the covariant derivative $D_t(V) : \mathcal{T}(\gamma) \to \mathcal{T}(\gamma)$ which satisfies:

1. *It is $\mathbb{R}$-linear*

2. *Satisfies the product rule* $D_t(fV) = \dot{f}V + fD_t(V)$

3. *For any extension* $\hat{V}$ *of* V*, we have* $D_t(V) = \nabla_{\dot{\gamma}d}(\hat{V})$

So if we put $t_0 \in M$*,* $v_0 \in T_{\gamma(t_0)}$*, then there exists a parallel vector field* V *along* γ *such that* $V(t_0) = V_0$ *which induces a map* $P_{t_0,t_1} : T_{\gamma(t_0)}(M) \to T_{\gamma(t_1)}(M)$ *by setting* $P_{t_0,t_1}(V_0) = V(t_1)$
It is possible see ([Lee06, Exercise 4.12]) that one can recover the covariant differential along γ *by the operators* P_{t_0,t_1} *by setting* $D_t(V)(t_0) := \frac{d}{dt}P^{-1}_{t_0,t_1}(V(t_0)) \mid_t = t_0$

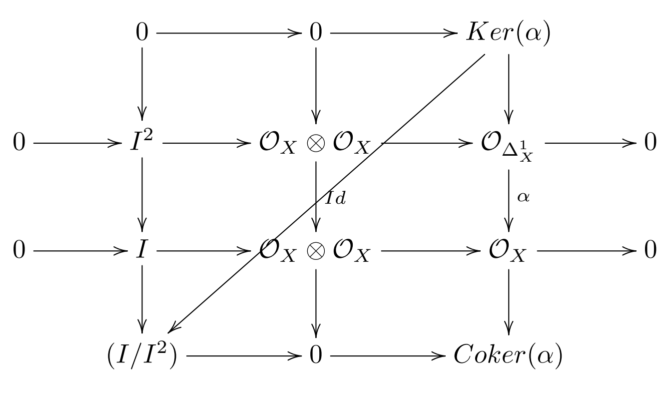
With this motivation in mind, we can see what it means to have a connection on a smooth variety.
The main obstacle in the construction is, again, to come up with the correct notion of 'infinitely close', this time for two points. So, if X is a complex variety over S let $\Delta_X = X \times X$ denote the diagonal with associated sheaf of ideals $\mathcal{I}$ and Δ^1_X the infinitesimal neighborhood of first order, i.e. $\mathcal{O}(X) \otimes \mathcal{O}(X)/\mathcal{I}^2$.
For two points $x, y \in X$ it would be good to consider them 'infinitely' close to each other if they have the same linear order.
To be more precise, we say that $x, y \in X(S)$ are infinitely close of first order if the morphism $(x, y) \in (X \times X)(S)$ can be factorized through Δ^1_X. If V is a locally free $\mathcal{O}_X$-module of finite rank then a connection γ over V is given by isomorphisms $\gamma_{x,y} : x^*V \to y^*V$ of $\mathcal{O}_S$-modules, indexed by pairs of points x, y infinitely close of first order, such that:

1. If $f : S' \to S$ is a morphism then $f^*(\gamma_{x,y}) = \gamma_{xf,yf}$

2. $\gamma_{x,x} = Id$

This definition is completely analogous to the morphisms $P_{t_0 t_1}$ given before and thus it shouldn't be a surprise that we can recover an operator that behaves like a connection on a smooth manifold.
First, notice that $p_1, p_2 : \Delta^1_X \to X$ are, by definition, two points infinitely close of first order. So a connection gives an isomorphism $\gamma_{p_1,p_2} \in Hom_{\mathcal{O}_{\Delta^1_X}}(p_1^*V, p_2^*V)$, which by adjunction corresponds to a morphism $D_\gamma : V \to p_{1*}p_2^*V$.

Remark. *The sheaf* $p_{1*}p_2^*V$ *is canonically isomorphic to* $\mathcal{O}_{\Delta^1_X} \otimes_{\mathcal{O}_X} V$ *where the action on* Δ^1_X *is on the right. This isomorphism is given by considering the following diagram:*



Where the diagonal morphism is given by the snake's lemma

If we denote $p_V : \mathcal{O}_{\Delta^1_X} \otimes_{\mathcal{O}_X} V \to V$ the morphism defined by $p_V(f \otimes v) = v$ then $p_V(D(v)) = v$, so, if we put $j^1 : V \to \mathcal{O}_{\Delta^1_X} \otimes_{\mathcal{O}_X} V$ defined, on sections s of V and f of $\mathcal{O}_X$, by $j^1(fv) = 1 \otimes f \otimes v$, we can define an operator determined by the following rule on sections $\nabla(v) = j^1(v) - D(v)$.
This construction will model the notion of connection we will use:

Definition 3.1. *Let $g : X \to S$ a morphism of schemes, a (Koszul) connection over $\mathcal{E} \in \mathcal{O}_X$-mod, relative to S, is a $g^{-1}\mathcal{O}_S$-linar morphism :*

$$\nabla : \mathcal{E} \to \Omega^1_{(X/S)} \otimes_{\mathcal{O}_X} \mathcal{E}$$

Such that $\nabla(fv) = df \otimes v + f\nabla(v)$, where f and v are sections of $\mathcal{O}(X)$ and $\mathcal{E}$ respectively.

Remark. *Given two connections $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2)$ it is possible to build new connections, namely:*

1. *The direct product connection $\nabla : \mathcal{E}_1 \oplus \mathcal{E}_2 :\to \Omega^1_{(X/S)} \otimes_{\mathcal{O}_X} (\mathcal{E}_1 \oplus \mathcal{E}_2)$ given by the formula on sections :*

$$\nabla(v_1 + v_2) := \nabla_1(v_1) + \nabla_2(v_2)$$

2. *The tensor product connection $\nabla : \mathcal{E}_1 \otimes \mathcal{E}_2 \to \Omega^1_{(X/S)} \otimes_{\mathcal{O}_X} (\mathcal{E}_1 \otimes \mathcal{E}_2)$, given by the formula on sections:*

$$\nabla(v_1 \otimes v_2) = \nabla_1(v_1) \otimes v_2 + \nabla_2(v_2) \otimes v_1$$

3. *The Hom connection $\nabla : Hom(\mathcal{E}_1, \mathcal{E}_2) \to \Omega^1_{(X/S)} \otimes_{\mathcal{O}_X} Hom(\mathcal{E}_1, \mathcal{E}_2)$ given by the formula on sections:*

$$\nabla(f)(v_1) = \nabla_2(f(v_1)) - f(\nabla_1(v_1))$$

Before going further, the following examples mean to illustrate the idea behind the connections

Example. *Let $X = Spec(\mathbb{C}[X])$, $S = Spec(\mathbb{C})$ and $\mathcal{E} = \widetilde{\mathbb{C}[x]} = \mathcal{O}_X$.*
Then the map $\nabla : \mathcal{E} \to \Omega^1_{(X/S)} \otimes_{\mathcal{O}_X} \mathcal{E}$, defined on sections by $\nabla(fv) = df \otimes v$, is a connection.
It is immediate to check that ∇ satisfies the connection condition.
We think about this connection as way to associate to a section f of $\mathcal{O}_X$ and a section v of a vector field, the derivative of f in the direction of v.

Example. *Let* $X = \mathbb{P}^1_K$ *and let* $g(x)dx$ *be a 1-differential form, then we define the connection* $\nabla(f(x)) := df + f(x)g(x)dx$

Definition 3.2. *A connection is integrable or flat if* $\nabla^2 = 0$

Remark. *A connection is equivalent to a* $\mathcal{P}^1$*-linear isomorphism* $\epsilon : \mathcal{P}^1 \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^1$ *which is the identity modulo* $\Omega^1_{(X/S)}$.

Proof. The equivalence can be seen by setting

$$\nabla \mapsto Id + \nabla$$

and

$$\epsilon \mapsto \nabla(s) := \epsilon((1 \otimes 1) \otimes v) - v \otimes (1 \otimes 1)$$

where v is a section of $\mathcal{E}$. Indeed, let ∇ be a connection, then we write

$$\epsilon((a \otimes b) \otimes v) = (v \otimes (a \otimes b)) + \nabla(v) \otimes (1 \otimes 1)$$

and the associated connection

$$\nabla'(v) \otimes (1 \otimes 1) = \epsilon(v \otimes (1 \otimes 1)) - v \otimes (1 \otimes 1) = v \otimes (1 \otimes 1) + \nabla(v) \otimes (1 \otimes 1) - v \otimes (1 \otimes 1) = \nabla(v) \otimes (1 \otimes 1)$$

for all sections v of $\mathcal{F}$ □

An immediate generalization of a connection is that of an n-connection or stratification structure, if a connection encoded the data of parallel transport up to first order closeness, then a stratification should encode the same information but for higher order closeness.

Definition 3.3. *Let* $\mathcal{P}^n = (\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X)/\mathcal{I}^n$ *be the structure sheaf of the nth neighborhood of the diagonal, a stratification is a family of isomorphisms of* $\mathcal{P}^n$*-modules* $\{\epsilon_n : \mathcal{P}^n \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n\}_{n \in \mathbb{N}}$ *such that :*

1. $\epsilon_0 = id$
2. ϵ_n *is compatible with restrictions* $\mathcal{P}^{n+1} \to \mathcal{P}^n$.
3. $p_{12}^*\epsilon_n \circ p_{23}^*\epsilon_n = p_{13}^*$ *for all* $n \in \mathbb{N}$ *where* p_{ij} *are the projections of the n-th neighborhood of the scheme* $X \times_S X \times_S X$

4 The infinitesimal site

Through this section we follow [BO78] and [BdJ11]
Through this section we introduce the general ideas behind the infinitesimal site, which will be the setting in which we will be able to proof our main theorem.

Definition 4.1. *Let $X \to S$ be an S-scheme, the category $Inf(X/S)$ consists as objects, morphisms $f : U \to T$ where $U \subset X$ is a Zariski open subset and f is a closed S-immersion such that its associated ideal is nilpotent.*
The morphisms between $U \to T$ and $U' \to T'$ are S-morphisms where $u : U \to U'$ is an open immersion and $t : T \to T'$ makes the following diagram commute.

$$\begin{array}{ccc} U & \xrightarrow{u} & U' \\ \downarrow & & \downarrow \\ T & \xrightarrow{t} & T' \end{array}$$

On this category it is possible to define the following site:

Definition 4.2. *A set of morphisms $\{(U_i \to T_i) \to (U \to T)\}$ of $Inf(X/S)$ is a covering family of $(U \to T) \iff T = \bigcup T_i$ and $U_i = U \times_T T_i$*

In the affine case this simply means that covers of the object $(U \to T) \in Inf(X/S)$ consists of morphisms of the form $(U_i \to T_i)$ where $U_i = Spec(R/I) \times_{Spec(R)} Spec(R_i)$ where $Spec(R_i) = T_i$, that is $U_i = Spec((R/I) \otimes_R R_i)$ and $T = \bigcup T_i$.
Before going further, let us prove that these sets of morphisms form a site:

Proposition 4.1. *The set of morphisms of the form $\{(U_i \to T_i) \to (U \to T)\}$ induce the structure of a site on the category $Inf(X/S)$*

Proof.

a) Let $(U_i \to T_i)$ be any element of a covering family of $(U \to T)$ and let $(U' \to T') \to (U \to T)$ be any morphism. Then the object $((U \times_T T_i) \times_U U' \to T_i \times_T T')$ is the corresponding pullback.
The only interesting detail to check is that the morphisms $(U \times_T T_i) \times_U U' \to T_i$ and $(U \times_T T_i) \times_T U' \to T'$ are open immersions, but this is given by the change of base of the open immersions $U_i \to U$ and $U' \to U$.

b) Let $(U_i \to T_i) \to (U \to T) \leftarrow (U' \to T')$ where $\{(U_i \to T_i)\}_i \in Cov(Inf(X/S))$. To see that $U_i \times_U U' = U' \times_{T'} (T_i \times_T T')$ it only remains to check that the object in the left hand side satisfies the universal property of the right hand side using the commutativity of the diagrams and the fact that $U_i = U \times_T T_i$. The calculations are immediate.

c) If $\{U_i \to T_i\}_i$ is a covering family of $(U \to T)$ and $\{U_{ij} \to T_{tj}\}_j$ is a covering for $(U_i \to T_i)$, then a general categorical argument shows that if $U_i = U \times_T T_i$ and $U_{ij} = U_i \times_{T_i} T_{ij}$ then $U_{ij} = U \times_T T_{ij}$ $\forall i, j$

d) The morphism $(U \to T) \xrightarrow{Id} (U \to T)$ forms a cover, as the identity on U is open and $U = U \times_T T$

□

The next natural step is to consider the topos associated to this site.

Definition 4.3. *Let $(X/S)_{inf} := Sh(Inf(X/S))$, the associated topos of abelian groups.*

In fact it is possible to give a more explicit description of these sheaves as follows:

Proposition 4.2. *An object $F \in (X/S)_{inf}$ is determined by a family of abelian sheaves $F_{(U \to T)}$ on T, indexed over all $(U \to T) \in Inf(X/S)$ and a family of transitive morphisms $\gamma_{T,T'} : t^{-1}F_{U \to T} \to F_{U' \to T'}$ for every morphism $(u,t) : (U \to T) \to (U' \to T')$, such that $\gamma_{T,T'}$ is an iso when $T \to T'$ is an open immersion.*

Proof. Given a sheaf $F \in (X/S)_{inf}$, an object $(U \to T) \in Inf(X/S)$ and an open $T' \subset T$ one can associate to this sheaf, a sheaf $F_{(U \to T)}$ on T, calculated on T' as $F_{(U \to T)}(T') = F((U \cap T') \to T')$. It is immediate to see that this sheaf $F_{(U \to T)}$ is a sheaf on T for every object on $Inf(X/S)$ as all the required properties are inherited by $F \in (X/S)_{inf}$.
Let $(u,t) : (U \to T) \to (U' \to T')$ a morphism and $V' \subseteq T'$, then if we put $V = t^{-1}(V') \subseteq T$, we get morphisms $((U \cap V) \to V) \to (U \to T) \to (U' \to T')$ which induce a map of sets $F(U,T) \to F(U \cap V, V)$ that corresponds by the argument in the previous paragraph, to a morphism $F_{U \to T}(T) \to F_{U \to T}(V) = t_* F_{U' \to T'}(V')$. Using the adjunction between $(\_)_*$ and $(\_)^{-1}$ we get a morphism $\gamma_{T,T} : t^{-1}F_{U \to T} \to F_{U',T'}$. It should be clear that, if $T \to T'$ is open, then the above map must be an isomorphism, as $t^{-1}(V') = V' \cap T$ for every $V' \subseteq T'$.
Thus, from a family of sheaves $F_{U \to T}$ for every $U \to T$ it is possible to recover a sheaf $F \in (X/S)_{inf}$ by setting $F(U \to T) = F_{U \to T}(T)$. □

By this characterization, sheaves on the site $Inf(X/S)$ can be thought as sheaves on T 'relative to U' in the sense that they reflect behaviour around the closed U in T in a compatible way.

Example. *Let $F_{(U \to T)}(T') = \mathcal{O}_T(T')$. For a morphism $t : T \to T'$ we write $\gamma_{T,T'}$ as morphism associated to $t^\sharp : \mathcal{O}_{T'} \to \mathcal{O}_T$ by the general adjunction $(\_)_* \dashv (\_)^{-1}$.*
We will call $\mathcal{O}_{(X/S)} \in (X/S)_{inf}$, the structure sheaf on $Inf(X/S)$ to the sheaf obtained from the shaves $\mathcal{O}_T$
Analogous to this sheaf, we can obtain sheaves $\mathcal{O}_X \in (X/S)_{inf}$ and $\mathcal{J}_{(X/S)}$ associated to the sheaves $G_{U \to T} = \mathcal{O}_U$ and $H_{U \to T} = Ker(\mathcal{O}_U \to \mathcal{O}_T)$, respectively.
These sheaves are compatible in the sense that there is a short exact sequence:

$$0 \to \mathcal{J}_{(X/S)} \to \mathcal{O}_{(X/S)} \to \mathcal{O}_X \to 0$$

Definition 4.4. *A crystal of $(\mathcal{O}_{(X/S)})$-modules is a sheaf of abelian groups over $Inf(X/S)$ such that $F_{U \to T}$ are all $\mathcal{O}_T$-modules and the morphisms $t^* F_{U \to T'} \to F_{U \to T}$ are linear isomorphisms over $\mathcal{O}_T$*

The following result will give us a relationship between stratifications and crystals

Proposition 4.3. *Let $X \to S$ be a smooth S-scheme and $\mathcal{E} \in \mathcal{O}_X$-mod. Then the following is equivalent:*

1. *A stratification structure $(\epsilon_n)_{n \in \mathbb{N}}$ over $\mathcal{E}$.*

2. *A family of $\mathcal{O}_X$-morphisms $\theta_n : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{P}^n$ satisfying :*

$$\begin{array}{ccc} \mathcal{E} \otimes \mathcal{P}^{m+n} & \longrightarrow & \mathcal{E} \otimes \mathcal{P}^m \otimes \mathcal{P}^n \\ {\scriptstyle \theta_{n+m}} \uparrow & & \uparrow {\scriptstyle \theta_m \otimes id_{\mathcal{P}^n}} \\ \mathcal{E} & \xrightarrow{\theta_n} & \mathcal{E} \otimes \mathcal{P}^n \end{array}$$

3. *A morphism $\Theta : colim_n Hom_{\mathcal{O}_X}(\mathcal{P}^n, \mathcal{O}_X) \to End_{\mathcal{O}_X}(\mathcal{E})$*

4. *A crystal on $(X/S)_{inf}$*

Proof. [BO78, Prop. 2.11] □

Remark. *The $\mathcal{O}_X$-module $colim_n Hom_{\mathcal{O}_X}(\mathcal{P}^n, \mathcal{O}_X)$ is called the ring of differential operators of finite order of X/S. It can also be described as the set of $f^{-1}\mathcal{O}_S$-linear maps $h : \mathcal{O}_X \to \mathcal{O}_X$ such that the induced morphism $\hat{h} : \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_S} \mathcal{O}_X \to \mathcal{O}_X$ annihilates $I^{n+1} \otimes \mathcal{O}_X$, where I is the ideal of the diagonal $\mathcal{P}_{X/S}$ generated as an $\mathcal{O}_X$-module by the elements of the form $1 \otimes b$, where $b \in \mathcal{O}_X$.*
A more detailed treatment of these modules can be found in [BO78, Chapter 2]

Lemma 1. *Let $(U \to T) \in Inf(X/S)$ and $g : U \to U'$ be an open immersion such that $g(U)$ is contained in an affine subset of U'. Then there is a T' and a morphism $T \to T'$ making the following diagram commute:*

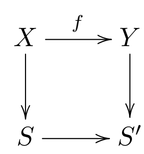
$$\begin{array}{ccc} U & \xrightarrow{g} & U' \\ \downarrow {\scriptstyle \pi} & & \downarrow {\scriptstyle \pi'} \\ T & \longrightarrow & T' \end{array}$$

Proof. It's possible to suppose $U = Spec(A)$, $T = Spec(R)$ and $U' = Spec(A')$, so if we write $R' = R \times_A A'$, $T' = Spec(R \times_A A')$ and J be the ideal associated to the closed immersion $U \to T$ then the ideal $J' := \{(r, 0) \mid r \in J\}$ is also a nilpotent ideal of R' which corresponds to the kernel of $\pi'^{\sharp}$ □

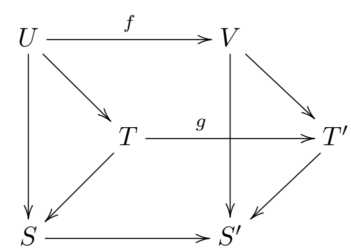
Remark. *The infinitesimal site doesn't have, in general, a final object.*
This can be seen using the previous lemma, since it is possible to enlarge T and get a non isomorphic object $U' \to T'$ and a morphism $(U \to T) \to (U' \to T')$

The advantage of working with the topos $(X/S)_{inf}$ instead of with the site $Inf(X/S)$ is, as is common in general category theory, categories of functors inherit some properties of the target category.
For example, despite the last remark it is possible to define global sections of a sheaf $F \in (X/S)_{inf}$ by defining $\Gamma(X, F) := hom_{(X/S)_{inf}}(*_X, F)$ where $*_X$ is the final object in $(X/S)_{inf}$.
Let $f : X \to Y$ be a morphism and $S \to S'$ a morphism making the following diagram

commute:



One would like to get a morphism of topoi $f_* : (X/S)_{inf} \to (Y/S')_{inf}$. This is possible to do in the following way:
First one start by defining a functor f^* on the representable objects $(V \to T') \in Inf(Y/S')$ we write, if $f(U) \subseteq V$, $(f^*(V \to T'))(U \to T) := \{g : T \to T'\}$ which make the following diagram commutes:



In the case that $f(U) \nsubseteq V$ then $(f^*(V \to T'))(U \to S) := \emptyset$. This determines a sheaf over $Inf(Y/S')$
Now we can make use of the lemma 4 in the following way: Writting $\phi := f^*$, we get presheaves ϕ_* and ϕ^*.

Proposition 4.4. *The sheaf $\phi_*(\mathcal{G})$ is a sheaf for every $\mathcal{G} \in (Y/S')_{inf}$*

Proof. Let $\{U_i \to T_i\}$ be a covering of $U \to T$ and $s_i \in f_*(\mathcal{G})(U_i \to T_i)$ a compatible family of sections, a section $s \in f_*(\mathcal{G})(T)$ compatible with the s_i is then a family of morphisms $s_{T'} : f^*(T)(T') \to \mathcal{G}$, so if $h \in f^*(T)(T')$, we get morphisms $h_i : h^{-1}(T_i) \to T_i$, and the family $s_i(h_i)$ defines elements in $\mathcal{G}(h^{-1}(T_i))$, but since $\mathcal{G}$ is a sheaf, there is a global section compatible with these morphisms. This section defines then a morphism $s_{T'} : f^*(T)(T') \to \mathcal{G}$ which is compatible with the family s_i. □

The sheafification ϕ^{-1} associated to the presheaves defined by ϕ^* is a left adjoint to the functor ϕ_* since the functors ϕ_* and ϕ^* are adjoints.
In order to define then a morphism of topoi (f^{-1}, f_*) it only remains to see that the functor f^{-1} preserves inverse limits, the proof for this can be found in [BO78] Proposition 5.9

5 The main theorem

Through this section we follow [BdJ11]
Through this section $\mathcal{F}$ will denote a quasi-coherent crystal with associated connection

(M, ∇). X will be an affine S-scheme $Spec(A)$ and D will be the completion by J of P of any sequence of the form

$$J \to P \twoheadrightarrow A$$

$\mathcal{F}(n) := \mathcal{F}(X \times \cdots \times X)$ where the product is taken n-times, and $D(n)$ the completion by $J \otimes \cdots \otimes J$ of $P \otimes \cdots \otimes P$ with the tensor product taken (n+1)-times.

Proposition 5.1. *There is a correspondence between quasicoherent crystals $\mathcal{F}$ and a $\mathcal{O}_S$-module M with an integral connection ∇*

Proof. To a quasicoherent crystal $\mathcal{F}$ we associate the module $M := \mathcal{F}(X)$ and the connection given by the morphism $\nabla : \mathcal{F} \to \mathcal{F} \otimes_{X/S} \Omega^1_{X/S}$ induced by the projection morphisms $\alpha_i : pr_i^* \mathcal{F}_T \to \mathcal{F}_{T \times T}$ indexed on objects (U, T). More specifically we can calculate this on any section $s \in \mathcal{F}_T(T)$ as $\nabla(s) = \alpha_1(s \otimes 1) - \alpha_2(1 \otimes s)$.
Conversely for a module M we assign the sheaf $\mathcal{F}$ calculated on affine objects as $\mathcal{F}(U = Spec(A), T = Spec(B)) = B \otimes_D M$ where the map $D \to B$ is any map such that the diagram commutes

$$\begin{array}{ccc} D & \longrightarrow & A \\ \downarrow & & \downarrow {\scriptstyle Id_A} \\ B & \longrightarrow & A \end{array}$$

It can be checked the definition of $\mathcal{F}$ is independent of the choice of lift $D \to B$, and this in fact forms sheaf. □

Definition 5.1. *The kernel pair of a morphism $f : X \to Y$ in a category $\mathcal{C}$ is the fiber product $X \times_Y X \rightrightarrows X$ of $f : X \to Y$ with itself*

Definition 5.2. *An effective epimorphism in a category $\mathcal{C}$ is a morphism $f : X \to Y$ such that it is a coequalizer of its kernel pair $X \times_Y X \rightrightarrows X$*

Lemma. *If $\mathcal{C}$ is a topos and if $X \to *$ is an effective epimorphism, then for any abelian sheaf $\mathcal{F}$, $R\Gamma(*, \mathcal{F})$ is computed by a bicomplex of the form $R\Gamma(X, \mathcal{F}) \to R\Gamma(X \times X, \mathcal{F}) \to R\Gamma(X \times X \times X, \mathcal{F}) \to \dots$*

Proof. [BdJ11, Remark 2.5] □

Lemma. *Let $\mathcal{F}$ be a quasicoherent $\mathcal{O}_{X/S}$-module such that $R^1\mathcal{F}(X \times \cdots \times X) = 0$, then $\mathcal{F}(\bullet)$ computes $\Gamma R(X/S, \mathcal{F})$*

Proof. [BdJ11, Lemma 2.4] □

Lemma. *There is a quasi-isomorphism between $M \otimes^{\wedge}_D \Omega^*_D$ and $M(n) \otimes^{\wedge}_{D(n)} \Omega^*_{D(n)/D}$*

Proof. The morphisms $D \to D(n)$ induce isomorphisms $M(n) \cong M \otimes^{\wedge}_D D(n)$ which in turn induce a filtration $M(n) \otimes^{\wedge}_D \Omega_{(X/S)^*}$ of $M(n)$ and a filtration $M \otimes^{\wedge}_D \Omega^i_D \otimes^{\wedge}_D \Omega^*_{D(n)/D}$. So the result is reduced to see that the map $D \to (D(n) \to \Omega^1_{D(n)/D} \to \Omega^2_{D(n)/D} \to \dots$ is a quasi-isomorphism, but this is true since $D(n)$ is a formal power series ring over D and its de Rham cohomology is D at 0 and 0 everywhere else by the Poincare lemma of algebraic de Rham cohomology. □

Given this correspondence we denote $M(n) := \mathcal{F}(n) := \mathcal{F}(X \times \cdots \times X)$ where the product is taken n+1 times.
We can see now that a module M with an integrable connection ∇ induces a sequence of the form $M(n) \to M(n) \otimes^{\wedge}_{D(n)} \Omega^{\bullet}_{D(n)}$ for every $n \geq 0$ and so we get a double complex of the form

$$\begin{array}{ccccccc}
M & \longrightarrow & M \otimes^{\wedge}_{D} \Omega^1_D & \longrightarrow & M \otimes^{\wedge}_{D} \Omega^2_D & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(1) & \longrightarrow & M(1) \otimes^{\wedge}_{D(1)} \Omega^1_{D(1)} & \longrightarrow & M(1) \otimes^{\wedge}_{D(1)} \Omega^2_D & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots
\end{array}$$

Lemma. *The complex* $M \otimes^{\wedge}_{D} \Omega^i_D \to M(1) \otimes^{\wedge}_{D(1)} \Omega^i_{D(1)} \to M(2) \otimes^{\wedge}_{D(2)} \Omega^i_{D(2)} \to \dots$ *computes* $R\Gamma(X/S, \mathcal{F} \otimes^{\wedge}_{X/S} \Omega^i_{X/S})$

Proof. This is clear since $M(n) \otimes^{\wedge}_{D} \Omega^i_D = \mathcal{F}(X \times \cdots \times X) \otimes^{\wedge}_{D} \Omega^i_D$ and since the Ω^i_D are quasicoherent we can use the lemma 5 and this shows what we wanted. □

Lemma. *The complex* $\Omega^1_D \to \Omega^1_{D(1)} \to \Omega^1_{D(2)}$ *is homotopic to zero as a* $D(\bullet)$*-cosimplicial module.*

Proof. [BdJ11, Lemma 2.5] □

Lemma. *For all* $i > 0$*, the cosimplicial module* $M \otimes_D \Omega^i_D \to M(1) \otimes_{D(1)} \Omega^i_{D(1)} \to M(2) \otimes_{D(2)} \Omega^i_{D(3)} \to \dots$ *is homotopy equivalent to 0.*

Proof. This follows from the previous lemma noticing that taking completions and tensor and wedge products preserves being homotopical to zero □

Theorem 5.1. *Let* $\mathcal{M}$ *be a crystal and* ∇ *its corresponding connection, then there is a natural quasi-isomorphism* $R\Gamma(X/S, \mathcal{M}) \cong (M \to M \otimes_D \Omega^1_D \to M \otimes_D \Omega^2_D \to \dots)$

Proof. If we take the double complex with terms $M^{n,m} := M(n) \otimes^{\wedge}_{D(n)} \Omega^m_{D(n)}$ where the horizontal morphisms are the ones given by the sequence $M \to M(1) \to M(2) \dots$ and the vertical ones the induced by the deRham complex.
By a previous lemma we have that $M^{\bullet,m}$ is quasi-isomorphic to $M \otimes^{\wedge}_{D} \Omega^{\bullet}_D$ independently from the first term and so we see that $Tot(M^{\bullet,\bullet})$ computes the cohomology of $M \otimes^{\wedge}_{D} \Omega^*_D$.
By 5 we have that the top row of the double complex computes the cohomology of $\mathcal{F}$ and by the previous lemma we have that the next rows are all homotopical to zero.
We conclude now that the top row and the first column are quasi-isomorphic □

Corollary 1. *Let* X *be a smooth* K*-scheme, then there is a canonical isomorphism* $H^n((X/K)_{\mathrm{inf}}, \mathcal{O}^{inf}_{X/K}) \cong H^n_{dR}(X/K)$

Proof. [BdJ11] □

A Homological algebra

Through this section we will follow [Wei95]
In this section we will give a brief introduction to the basic theory of homological algebra.

Definition A.1. *A category $\mathcal{C}$ is abelian if it satisfy the following conditions:*

1. *For every $A, B \in \mathcal{C}$, $Hom_{\mathcal{C}}(A, B) \in \mathbb{Z} - mod$*
2. *It has all finite direct sums (equivalently all direct products)*
3. *Every morphism has a kernel and a cokernel*
4. *Every monomorphism is a kernel and every epimorphism is a cokernel*

Example. *Let R be a ring, not necesessarely commutative, then the category $R - Mod$ of R-modules is an abelian category*

Example. *Let $(X, \mathcal{O}_X)$ be a scheme, then the category of $\mathcal{O}_X$-modules is an abelian category*

Definition A.2. *A sequence $\cdots \to A_i \xrightarrow{f_i} A_{i+1} \xrightarrow{f_{i+1}} A_{i+2} \to \dots$ in an abelian category $\mathcal{C}$ is called exact if for every i we have $Ker f_{i+1} = Im f_i$.*
An exact sequence of the form $0 \to A \to B \to C \to 0$ is called a short exact sequence.

Definition A.3. *We say that a complex $\mathcal{A}^\bullet$ is bounded below if there is a $n\mathbb{Z}$ such that $A^i = 0 \; \forall i \leq n$*

Definition A.4. *A functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ with $\mathcal{A}$ and $\mathcal{B}$ abelian categories is called additive if it commutes with products, that is $\mathcal{F}(A \Pi_{\mathcal{A}} A') = \mathcal{F}(A) \Pi_{\mathcal{B}} \mathcal{F}(A')$*

Remark. *All through this section all functors $\mathcal{F}$ between abelian categories are assumed to be additive*

Definition A.5. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor between abelian categories, then $\mathcal{F}$ is called exact if for every exact sequence $\cdots \to A_i \to A_{i+1} \to \dots$ in $\mathcal{C}$ the sequence $\cdots \to \mathcal{F}(A_i) \to \mathcal{F}(A_{i+1}) \to \dots$ in $\mathcal{D}$ is exact.*
Similarly we say that $\mathcal{F}$ is left (resp. right) exact if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ we get an exact sequence $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$ (resp $\mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$).

Example. *Let $\mathcal{C} = \mathcal{D} = \mathbb{Z} - mod$ and let $\mathcal{F}(A) = Hom(M, \_)$ for a fixed object $M \in \mathcal{C}$, then $\mathcal{F}$ is a left exact functor and it is right exact when A is a projective object.*

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ an exact sequence in $\mathcal{C}$, and the associated short exact sequence in $\mathcal{D}$

$$0 \to Hom(M, A) \xrightarrow{f^*} Hom(M, B) \xrightarrow{g^*} Hom(M, C)$$

Since f is a monomorphism we have that for any two $h, h' \in Hom(M, A)$ such that $f \circ h = f \circ h'$ we have that $h = h'$.

If $h \in Hom(M,B)$ is such that $g^*(h) = 0$ then $Imh \subseteq Kerg = Imf$ and so the associated $\hat{h} : M \to A$ satisfies $f^*(\hat{h}) = h$ so $Imf^* = Kerg^*$
If we let M be a projective object then any morphism $h : M \to C$ can be lifted to a morphism $\hat{h} : M \to B$ and so $Img^* = Hom(M,C)$ □

Remark. *Similarly for $\mathcal{C}$ and $\mathcal{D}$ as before, the functor $\mathcal{F} = Hom(\_, M)$ is left exact and exact whenever M is an injective object.*

Our motivation is that for a left (right) exact functor $\mathcal{F}$ to be able to continue the image of short exact sequences on the right (left) and obtain a measure of the exactness of the sequence under the functor. In other words, we would like to have family of objects $D^i \in \mathcal{D}$ and a family of morphism $\to \mathcal{F}(C) \to D^0 \to D^1 \to \dots$ such that this continued sequence is now exact.

Definition A.6. *An abelian category $\mathcal{C}$ has enough injectives (resp. projectives) if for every object $A \in \mathcal{C}$ there is a monomorphism (resp. epimorphism) $A \to I$ (resp. $P \to A$) where I is an injective object (resp. P is a projective object).*

Example. *The category $\mathcal{C} = \mathbb{Z}-mod$ of abelian groups has enough injectives. For example, if $M \cong \mathbb{Z}^{(I)}/K$ for some set I and subgroup K of $\mathbb{Z}^{(I)}$, then we have a monomorphism $M \to \mathbb{Q}^{(I)}/K$*

Example. *The category $\mathcal{O}_X - Mod$ of $\mathcal{O}_X$-modules over a scheme X has enough injectives*

Definition A.7. *A sequence $A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \to \dots$ in an abelian category $\mathcal{C}$ is called a complex if $d^{i+1} \circ d^i = 0$, for every index i. We denote such a complex as $\mathcal{A}^\bullet$.*

Definition A.8. *Let $\mathcal{A}^\bullet, \mathcal{B}^\bullet$ be two complexes on $\mathcal{C}$, then a morphism of complexes $f^\bullet : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ is a collection of morphisms $f^i : A^i \to B^i$ such that they commute with the d^i's, that is $f^{i+1} \circ d^i_A = d^i_B \circ f^i$*

Remark. *If $\mathcal{C}$ is an abelian category, then the complexes on $\mathcal{C}$ together with morphisms of complexes form an abelian category $C^\bullet(\mathcal{C})$*

Definition A.9. *Let $\mathcal{A}^\bullet$ be a complex, then for every $i \geq 0$, the i-th cohomology group of $\mathcal{A}^\bullet$ is the group $H^i(\mathcal{A}^\bullet) := Kerd^i/Imd^{i-1}$*

Remark. *If $\mathcal{C}$ is an abelian category and $\mathcal{A}^\bullet = A^0 \to A^1 \to \dots \in \mathcal{C}$ is an exact sequence, then $H^0(\mathcal{A}^\bullet) = A^0$ and $H^i(\mathcal{A}^\bullet) = 0$ for all $i \geq 1$.*

Definition A.10. *Let $\mathcal{C}$ be an abelian category and $A \in \mathcal{C}$, then an injective (resp. projective) resolution of A is an exact sequence of the form $0 \to A \to I^0 \to I^1 \to \dots$ $(\dots \to P^1 \to P^0 \to A \to 0)$ for I^i injective objects (resp P^i projective objects) for all $i \in \mathbb{N}$.*

Remark. *If an abelian category $\mathcal{C}$ has enough injectives (resp. projectives) then every object $A \in \mathcal{C}$ has an injective(resp. projective) resolution*

Proof. By induction: for $n = 0$, then by hypothesis there is a monomorphism $A \to I^0$ where I^0 is an injective. Now suppose this is true for n. Again by hypothesis there is a

monomorphism $coker(I^{n-1} \to I^n) \to I^{n+1}$ where I^{n+1} is an injective object, then the induced morphism $I^n \to I^{n+1}$ is a monomorphism and the sequence $I^{n-1} \to I^n \to I^{n+1}$ is exact by hypothesis.
The proof for projective resolutions is analogous □

Example. *Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathcal{F} \in \mathcal{O}_X - mod$, then the i-th sheaf cohomology group of X with coefficients in $\mathcal{F}$ is the group $H^i(X, \mathcal{F})$*

Definition A.11. *Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a left exact functor between abelian categories where $\mathcal{C}$ has enough injectives. Then the right derived functors $R^i(\mathcal{F}) : \mathcal{C} \to \mathcal{D}$ are the functors defined on objects by $R^i(\mathcal{F})(A) := H^i(F(\mathcal{I}^\bullet))$ where $\mathcal{I}^\bullet$ is an injective resolution of A.*

Remark. *The definition of right derived functor for an object A does not depend on the choice of the injective resolution $A \to \mathcal{I}^\bullet$.*

Lemma 2. *Let $\mathcal{C}$ be an abelian category with enough injectives and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$. If $A \to \mathcal{I}^\bullet$ and $B \to \mathcal{J}^\bullet$ are two injective resolutions for objects A and $f : A \to B$ a morphism, then there is a morphism $R^i(\mathcal{F})(A) \to R^i(\mathcal{F})(B)$ for every $i \in \mathbb{N}$.*

Proof. By induction on n: If n=0 then we have a morphism $A \xrightarrow{f} B \to J^0$ and a monomorphism $A \to I^0$, so by the injectivity of J^0 there is a morphism $I^0 \to J^0$ extending $A \xrightarrow{f} B \to J^0$. We can now suppose there are morphisms f^i for all $i \leq n$, Since $d_B^n \circ f^n \circ d_A^{n-1} = d_B^n \circ d_B^{n-1} \circ f^{n-1} = 0$, then $d_B^n \circ f^n$ factorizes through $Cokerd^{n-1}$, meaning that there is a morphism $Cokerd^n \to J^{n+1}$ and so by the injectivity of J^{n+1} we can extend this map to a map $f^{n+1} : I^{n+1} \to J^{n+1}$

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \xrightarrow{d_A^0} & \dots & \longrightarrow & I^n & \xrightarrow{d_A^n} & \dots & \longrightarrow \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & & & \downarrow{\scriptstyle f^n} & & & \\
0 & \longrightarrow & B & \longrightarrow & J^0 & \xrightarrow{d_B} & \dots & \longrightarrow & J^n & \xrightarrow{d_B^n} & \dots & \longrightarrow
\end{array}$$

This morphism of complexes induce a morphism on the cohomology groups □

Lemma 3. *Let I be an injective object in $\mathcal{C}$ an abelian category with enough injectives. Then $R^i(\mathcal{F})(I) = 0$ for all $i \geq 1$*

Proof. Let I be an injective object in $\mathcal{C}$, then we consider the injective resolution $0 \to I \xrightarrow{Id} I \to 0 \to \dots$ and so if $i \geq 1$, $R^i(\mathcal{F})(I) = 0/0 = 0$ and $R^0(\mathcal{F})(I) = \mathcal{F}(I)$. □

Proposition A.1. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in $\mathcal{C}$ an abelian category with enough injectives and let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, then there is a morphism $\delta^i : R^i(\mathcal{F})(C) \to R^{i+1}(\mathcal{F})(A)$ for every $i \geq 0$ such that the sequence*

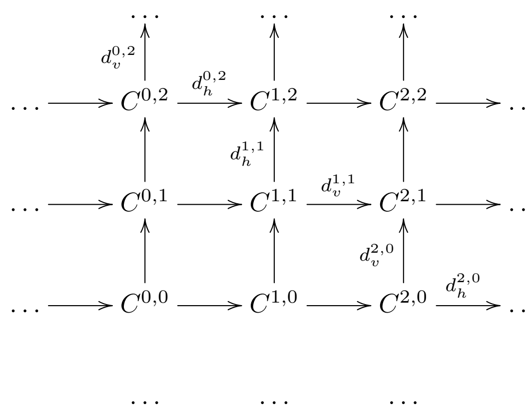
$$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to R^1(\mathcal{F})(A) \to \dots \to R^i(\mathcal{F})(C) \to R^{i+1}(\mathcal{F})(A) \to \dots$$

is an exact sequence

A.1 Spectral Sequences

Spectral sequences are a useful tool for computing cohomology groups, through this section we define them and outile their usefulness.

Definition A.12. *A differential bigraded complex is a collection of objects* $\{C^{p,q}\}$ *and maps* $d_h^{p,q} : C^{p,q} \to C^{p+1,q}$ *and* $d_v^{p,q} : C^{p,q} \to C^{p,q+1}$ *such that* $d_h \circ d_h = 0 = d_v \circ d_v$ *and* $d_h^{p,q+q} \circ d_v^{p,q} + d_v^{p+1,q} \circ d^{p,q} = 0$ *as in the following diagram*



Definition A.13. *Let* $\{C^{\bullet,\bullet}, d_h, d_v\}$ *be a double complex, when it exists, we can associate it the the chain complex given by* $Tot(C^{\bullet,\bullet}) := \bigoplus_{p+q=n} C_n^{p,q}$ *with maps* $d_h + d_v : Tot(C^{\bullet,\bullet})_n \to Tot(C^{\bullet,\bullet})_{n-1}$

Definition A.14. *A cohomological spectral sequence is a family of objects* $E_r^{p,q}$ *which we will call the pages and maps* $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ *such that* $d_r \circ d_r = 0$ *and such that the cohomology* $H^{p,q}(E_r^{\bullet,\bullet}) := Ker(d^{p,q} : E_r^{p,q} \to E_r^{p+s,q+1-s})/Im(d^{p-s,q+s-1} : E_r^{p-s,q+s-1} \to E_r^{p,q})$ *is isomorphic to* $E_{r+1}^{p,q}$

Definition A.15. *We say a spectral sequence* $\{E_r^{p,q}, d_r\}$ *is bounded if for every n, there are only finitely many zero terms* $E_r^{p,q}$ *such that* $p+q=n$*. This implies that there exists a page* $E_r^{p,q}$ *such that* $E_r^{p,q} = E_{r+1}^{p,q}$*, we will denote this page by* $E_\infty^{p,q}$

Definition A.16. *We say that a spectral sequence* $\{E_r^{p,q}, d_r\}$ *converges to* $H^{p,q}$ *if there exists a finite of* H^n

$$0 = F^1 H^n \subseteq \dots F^p H^n \subseteq F^{p+1} H^n \subseteq \dots F^k H^n = H^n$$

Such that $E_\infty^{p,q} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$.
When the spectral sequence coverges to H^n *we denote it by* $E_r^{p,q} \Rightarrow H^n$

Definition A.17. *We say that a spectral sequence* $\{E_r^{p,q}, d_r\}$ *collapses if there is a page* $E_r^{p,q}$ *such that* $E_r^{p,q} = 0$ *iff* $p \neq k$ *or* $q \neq k$ *for some fixed integer k.*
If a collapsing spectral sequence converges to $H^\bullet$ *then* $E_r^{p,q} = H^{p+q}$

Example. *Let* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ *and* $\mathcal{G} : \mathcal{D} \to \mathcal{E}$ *be two functors between abelian categories such that* $\mathcal{D}$ *has enough injectives and* $\mathcal{F}(I)$ *is injective for all injectives* $I \in \mathcal{C}$*. Then* *for every object $A \in \mathcal{C}$ there is a spectral sequence given by $E_2^{p,q} := (R^p(\mathcal{G})(R^q(\mathcal{F}(A))))$ such that it converges to $R^{p+q}(\mathcal{G}(\mathcal{F}))(A)$.*

$$
\begin{array}{ccccc}
\vdots & \vdots & \vdots & \vdots & \\
2 & R^2\mathcal{G}(\mathcal{F}(A)) & R^2\mathcal{G}(R^1\mathcal{F}(A)) & R^2\mathcal{G}(R^2\mathcal{F}(A)) & \dots \\
1 & R^1\mathcal{G}(\mathcal{F}(A) & R^1\mathcal{G}(R^1\mathcal{F}(A)) & R^1\mathcal{G}(R^2\mathcal{F}(A)) & \dots \\
0 & \mathcal{F}(A) & R^1\mathcal{F}(A) & R^2\mathcal{F}(A) & \dots \\
 & 0 & 1 & 2 & \dots
\end{array}
$$

This is called the Grothendieck spectral sequence [Wei95]

As an application of Grothendieck spectral sequences we have the following spectral sequence, called the Ĉech-to derived exact sequence

Definition A.18. *Let U_i be an open covering of X and $\mathcal{F}$ an abelian sheaf on X, then we define the following*

1. $U_{i_0 \dots i_n} = U_{i_0} \cap \dots \cap U_{i_n}$
2. $\check{C}^n(U_i, \mathcal{F}) := \Pi \mathcal{F}(U_{i_0 \to i_n})$
3. $d^n : \check{C}^n(U_i, \mathcal{F}) \to \check{C}^{n+1}(U_i, \mathcal{F})$ *is the map given by* $(s_{i_0 \dots i_n}) \mapsto \Sigma_j \mathcal{F}(p_j)(s_{i_0 \dots \hat{i_j} \dots i_n})$ *where p_j are the projections* $p_j : U_{i_0 \dots i_n} \to U_{i_0} \dots U_{i_j} \hat{\dots} i_n$
4. *The complex* $\check{\mathcal{C}}^\bullet := \check{C}^0 \xrightarrow{d^0} \check{C}^1 \to \dots$ *is called the Ĉech complex (of U_i and $\mathcal{F}$)*
5. $\check{H}^0(U_i, \mathcal{F}) := Ker(\Pi \mathcal{F}(U_i))$
6. $\check{H}^n(U_i, \mathcal{F}) := H^n(\check{\mathcal{C}}^\bullet)$

Example. *Let U_i be an open cover of X. The Ĉech-to derived spectral sequence is the Grothendieck spectral sequence given by the functors*

$$R^p(\check{H}^0(U_i, \_) \circ R^q(i) : Sh(X) \to Ab$$

Where $Sh(X)$is the category of abelian sheaves on X and $i : Ab(X) \to Psh(X)$ is the inclusion functor. Then this sequence converges

$$E_2^{p,q} := \check{H}^p(U_i, R^q(i)) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Where $H^i(X, \mathcal{F})$ are the sheaf cohomology groups.

When the cover consists only of two open subsets the corresponding sequence degenerates and it is called the Mayer-Vietoris exact sequence. The following is an alternative proof of the existence of this sequence in the case of sheaves of $\mathcal{O}_X$-modules.

Example 1. *Let $(X, \mathcal{O}_X)$ be a scheme and let $\mathcal{F} \in \mathcal{O}_X - mod$, if $X = U_1 \cup U_2$ then there is an exact sequence*

$$0 \to \mathcal{F}(X) \to \mathcal{F}(U) \oplus \mathcal{F}(V) \to \mathcal{F}(U \cap V) \to H^1(X, \mathcal{F}) \to$$
$$H^1(U, \mathcal{F}) \oplus H^1(V, \mathcal{F}) \to H^1(U \cap V, \mathcal{F}) \to \dots$$

Proof. If $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution, this follows from the fact that $\mathcal{F}$ is a sheaf and injective $\mathcal{O}_X$-modules are flasque sheaves. □

A.2 Hypercohomology

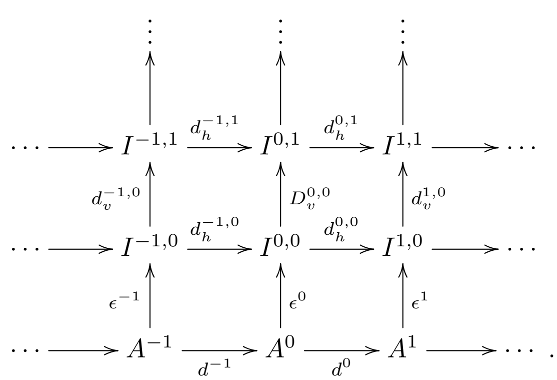
We can generalize the definition of derived functors to complexes of sheaves; these are called hyper-derived functors. Hypercohomology then corresponds to the right hyper-derived functors of the global sections functor.

As in the classical case, the right hyper-derived functors can always be defined for bounded below complexes when the functor considered is left exact and its domain is an abelian category with enough injectives.

In what follows, let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, such that $\mathcal{C}$ has enough injectives, and $F : \mathcal{C} \to \mathcal{D}$ a left exact functor. Denote $\mathrm{Ch}^+(\mathcal{C})$ the category of bounded below complexes and consider $(A^\bullet, d^\bullet) \in \mathrm{Ch}^+(\mathcal{C})$.

Proposition A.2. *There exists a double complex $(I^{\bullet,\bullet}, d_h, d_v)$ consisting of injective objects and a morphism of complexes $\epsilon^\bullet : A^\bullet \to I^{\bullet,0}$ with the following properties:*

- *$I^{p,q} = 0$ for $q < 0$.*
- *If $A^p = 0$ then the complex $I^{p,\bullet}$ is zero.*
- *The complex $I^{p,\bullet}$ is an injective resolution of A^p.*



From this double complex $I^{\bullet,\bullet}$ we can obtain the associated total complex

$$I^n = \oplus_{p+q=n} I^{p,q}\,,\; d = \oplus_{p+q=n} d_h^{p,q} + (-1)^p d_v^{p,q}.$$

Note that this is a bounded below complex formed of injective objects. We have an induced morphism of complexes $i^n : A^n \xrightarrow{\epsilon^n} I^{n,0} \to I^n$, since by definition $D_h^{n,0} \circ \epsilon^n = \epsilon^{n+1} \circ d^n$ and $D_v^{n,0} \circ \epsilon^n = 0$. This morphism $i^\bullet : A^\bullet \to I^\bullet$ induces an isomorphism $H^n(A^\bullet) \cong H^n(I^\bullet)$ on cohomology groups for every n; such a morphism is called a quasi-isomorphism.

Definition A.19. *Let $i^\bullet : A^\bullet \to I^\bullet$ be a quasi-isomorphism, where $I^\bullet$ is a bounded below injective complex. We set*

$$R^n F(A^\bullet) := H^n(F(I^\bullet)).$$

Moreover, this construction is functorial: if $j^\bullet : B^\bullet \to J^\bullet$ is a quasi-isomorphism with $J^\bullet$ a bounded below injective complex and $f^\bullet : A^\bullet \to B^\bullet$ is a morphism of complexes, then there exists a canonical morphism $R^n F(f^\bullet) : R^n F(A^\bullet) \to R^n F(B^\bullet)$, where the derived functors are computed respectively using the quasi-isomorphisms $i^\bullet$ and $j^\bullet$. We call

$$R^n F : Ch^+(\mathcal{C}) \to \mathcal{D}$$

the (right) hyper-derived functors of F.

We have discussed the existence of a complex $I^\bullet$ satisfying the conditions of the previous definition, so indeed $R^n F$ is defined for every bounded below complex. As expected, the definition is independent of the choice of this complex, and thus the groups $R^n F(A^\bullet)$ are well defined (up to canonical isomorphism).

Remark. *We can also define the right hyper-derived functors using complexes of acyclic objects relative to the functor considered.*

If $A^\bullet$ is a complex such that $A^n = 0$ for every $n \neq 0$ and $A^0 = A$ then $R^n F(A^\bullet) = R^n F(A)$ is the right derived functor of F evaluated at the object A. In this sense, hyper-derived functors are a generalization of derived functors.
When $\mathcal{C}$ is the category of sheaves of abelian groups over a topological space X and $F = \Gamma(X, \_)$ is the functor of global sections, the groups $R^n(\mathcal{F}^\bullet)$ are called the hypercohomology groups of the complex of sheaves $\mathcal{F}^\bullet$, and we denote them

$$\mathbb{H}^n(X, \mathcal{F}^\bullet) = R^n \Gamma(\mathcal{F}^\bullet).$$

B Sites and Topoi

Through this section we'll follow the results found in [Mil80],[BO78],[Sta16]
It is convenient to introduce general results on the theory of sites and topoi.
A site will be a a category along with some data that will model the behaviour of open coverings of topological spaces in a more abstract context. While topoi have arised in different branches of mathematics in different presentations, we will only focus on Grothendieck topoi, that is, categories of sheaves over some site.
Through this section one can suppose all categories $\mathcal{C}, \mathcal{D}$ have all finite limits.

To understand sheaves in this more general context, let us begin by recalling the definition of a sheaf on a topological space:

Definition B.1. *Let X be a topological space and $\mathcal{O}(X)$ its category of open subsets with morphisms $Mor(U,V) = *$ iff $U \subseteq V$ and empty otherwise . A sheaf of sets $\mathcal{F}$ on X is a contravariant functor $\mathcal{F} : \mathcal{O}(X) \to Sets$ such that it satisfies the following two conditions:*
To ease notation one can denote the morphisms $\mathcal{F}(U \to V)$ as $res_{V,U}$

a) If $\{U_i\}$ is a covering family of an open subset $U \subseteq X$ and $s, t \in \mathcal{F}(U)$ are such that $\forall\, U_i\ res_{U,U_i}(s) = res_{U,U_i}(t)$ then $s = t$.

b) If $s_i \in \mathcal{F}(U_i)$ is such that $res_{U_i,U_i\cap U_j}(s_i) = res_{U_j,U_i\cap U_j}(s_j)\ \forall\ i,j$ then there exists $s \in \mathcal{F}(U)$ such that $res_{U,U_i}(s) = s_i$

Remark. *Let us notice that these two conditions can be encoded into the condition that the first arrow must be an equalizer in the following diagram:*

$$\mathcal{F}(U) \to \Pi_i \mathcal{F}(U_i) \rightrightarrows \Pi_{i,j}\mathcal{F}(U_i \cap U_j)$$

If one were to study these phenomena in a broarder setting, it would be necessary to have a substitute of $\mathcal{X}$. To be able to state the conditions a) and b) it is then necessary for such a category to have a notion of covering for every object.
Even more, this category must at least admit certain fibered products by condition b). It is then reasonable to ask that a covering family of an object U on a category $\mathcal{C}$ must be a family of morphisms $\{U_i \to U\}_i$ with the following conditions

1. $Id_U : U \to U$ must be a covering family for every object U
2. The product $Y \times_U U_i$ must exist for every arrow $Y \to U$ and the family $\{Y \times_U U_i \to Y\}$ must be a covering family of Y.
3. If $\{U_i \to U\}$ is a covering of U and $\{U_{ij} \to U_i\}$ is a covering of U_i for every i, then $\{U_{ij} \to U\}$ is also a covering family of U.

Definition B.2. *A category $\mathcal{C}$ together with covering families $\{U_i \to U\}_i$ for every object U is called a site*

The following examples are classical and illustrative of the essence of sites:

Example. *Let $\mathcal{C} = \mathcal{O}(X)$ be the category of open subsets of a topological space, if one considers the families $\{U_i \to U\}$ as a covering in $\mathcal{C}$ when $U = \bigcup U_i$, then it is easy to see that this is a site.*

Example. *Let $\mathcal{C} = AffSch$, the category of affine schemes. the Zariski site is given by coverings of the form $\{Spec(R[a_i^{-1}]) \to Spec(R)\}_{i\in\{1,\ldots,n\}}$ such that there are $\{b_i \in R\}$ such that $\Sigma b_i a_i = 1$, the morphisms induced by the canonical ring morphisms $\{R \to R[a_i^{-1}]\}$. These coverings satisfy the conditions :*

1. *Let $a = 1$, then $R[a^{-1}] = R$ so $\{Spec(R) \to Spec(R)\}$ is a cover*
2. *Let $\phi : Spec(S) \to Spec(R)$ and $\{Spec(R[a_i^{-1}]) \to Spec(R)\}_i$ a covering, then $Spec(R[a_i] \otimes_R S) = Spec(S[\phi^\sharp(a_i^{-1})])$, and if $1 = \Sigma b_i a_i$ then $1 = \Sigma \phi^\sharp(b_i a_i)$*
3. *If $\{Spec(R[a_i^{-1}] \to Spec(R)\})$ is a covering family of R, and for every i, $\{Spec(R[a_i^{-1}][b_{ij}^{-1}]) \to Spec(R[a_i^{-1}])\}_j$ is a covering family of $R[a_i^{-1}]$, we can see that $R[a_i^{-1}][b_{ij}^{-1}] \cong R[(a_i b_{ij})^{-1}]$ and so there is a number k_i with $a_i^{k_i} \in (a_i b_{ij})_j$ and so if $1 = \Sigma c_i a_i$ then $1 = (\Sigma c_i a_i)^k$ for a large enough k.*

Example. *Another classical example is that of the big etale site. Let X be a noetherian scheme and let $\mathcal{C}$ the category of X-schemes with morphisms given by etale morphisms. Then a covering family is given by a family of etale morphisms $\{\Phi_i : U_i \to U\}_i$ such that $\bigcup Im(\Phi_i) = U$. These coverings form a site called the etale site of X. (See [Mil80])*

Now that the structure of the space has been established, it is possible to talk about sheaves on these spaces. The definition is straightforward:

Definition B.3. *Let $\mathcal{C}$ be a site and $\mathcal{F} : \mathcal{C} \to Sets$ a contravariant functor. We say that $\mathcal{F}$ is a sheaf if for every object U and every covering, $\{U_i \to U\}$, the first arrow of the following diagram is an equalizer:*

$$\mathcal{F}(U) \to \Pi_i \mathcal{F}(U_i) \rightrightarrows \Pi_{i,j} \mathcal{F}(U_i \times_U U_j)$$

It is possible to distinguish certain sheaves over the sites of the two previous examples, for example the structure sheaf $\mathcal{O}_{Zar}$ is simply the associated sheaf to the forgetful functor $\mathcal{F} : Rng \to Sets$.
An important property of sheaves over topological spaces is that of functoriality in the sense that it is possible to define a direct and inverse image functors once a morphism of topological spaces $X \to Y$. It is then important to establish a noteion of morphism of sites.

Definition B.4. *Let $\mathcal{C}, \mathcal{D}$ be two sites, a functor $u : \mathcal{C} \to \mathcal{D}$ is continuous if, for every covering family $\{U_i \to U\}$ of any object U of $\mathcal{C}$, the family $\{u(U_i) \to u(U)\}$ is a covering family of $u(U)$ and if $V \to U$ is any morphism, then $u(V \times_U U_{U_i}) \to u(V) \times_{u(U)} f(U_i)$ is an isomorphism.*

Given a morphism of sites $u : \mathcal{C} \to \mathcal{D}$ and a presheaf $\mathcal{F}$ over $\mathcal{D}$, it is possible to define a presheaf on $\mathcal{C}$ by setting $u_*\mathcal{F}(U) := \mathcal{F}(u(U))$. It turns out that if $\mathcal{F}$ is a sheaf then the associated presheaf $u_*\mathcal{F}$ is also a sheaf.
Now it is necessary to define the dual concept of the u_* construction:

Definition. *Let $V \in D$, denote by $\mathcal{I}_V^u$ the category with objects $\{(U, \phi) \mid \phi : V \to u(U)\}$ and obvious morphisms.*
Given a sheaf $\mathcal{F}$ on $\mathcal{C}$, for every object $V \in \mathcal{D}$ there are functors $\mathcal{F}_V : \mathcal{I}_V^{op} \to Sets$ defined by $\mathcal{F}_V(U, \phi) = \mathcal{F}(U)$.
The sheaf associated to the presheaf over $\mathcal{D}$, $V \mapsto colim_{I_V^{op}} \mathcal{F}_V$ is denoted by u^{-1}

The concept of a morphism between sites is the following

Definition B.5. *A morphism of sites $f : \mathcal{D} \to \mathcal{C}$ is a continuous functor $u : \mathcal{C} \to \mathcal{D}$ such that u^{-1} is an exact functor, that is, preserves arbitrary finite limits.*

It is possible to see that in the site $\mathcal{O}(X)$ from an example above, a continuous map $f : X \to Y$ defines a continuous functor $u : \mathcal{O}(Y) \to \mathcal{O}(X)$ with u^{-1} an exact functor. We can now define the concept of a topos:

Definition B.6. *A category $\mathcal{T}$ is called a topos if it is equivalent to a category of the form $Sh(\mathcal{C})$ where $\mathcal{C}$ has a site structure.*

Remark. *It is possible for two topoi $\mathcal{T} \cong Sh(\mathcal{C})$ and $\mathcal{T}' \cong Sh(\mathcal{D})$ to be equivalent while $\mathcal{C}$ different from $\mathcal{D}$.*

Example. *Let $\mathcal{C} = \{*\}$ be the category with one point and identity morphism. The covering given by the only morphism forms a site. The topos associated to the category $\mathcal{C}$ is equivalent to the category of sets. Every presheaf is a sheaf that corresponds to a choice of a set and a sheaf morphism corresponds to a map of sets.*

Example. *The topos associated to the category $\mathcal{C} = \mathcal{O}(X)$ is just the category of sheaves over the space X.*

Definition B.7. *A morphism of topoi $\mathcal{T} \to \mathcal{S}$ is a pair of functors (f_*, f^{-1}) where $f_* : \mathcal{T} \to \mathcal{S}$ and $f^{-1} : \mathcal{S} \to \mathcal{T}$, such that $f^{-1} \dashv f_*$ and f^{-1} preserving finite inverse limits.*

It is clear that a morphism of sites immediately induces a morphism of topoi but not every morphism of topoi is induced in such a way, in fact this will be the case for the morphism of infinitesimal topoi induced by a scheme morphism.
The following lemmas are general results which will be useful in the future:

Lemma 4. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories and $PSh(\mathcal{C})$ and $Psh(\mathcal{D})$ their corresponding categories of presheaves. For every functor $\phi : \mathcal{C} \to Psh(\mathcal{D})$, there exists an adjoint pair $\phi_* \dashv \phi^*$ with $\phi_* : Psh(\mathcal{D}) \to Psh(\mathcal{C})$ and $\phi^* : Psh(\mathcal{C}) \to Psh(\mathcal{D})$ such that $\phi_* |_{\mathcal{C}} = \phi$*

Proof. Let $\phi_*(\mathcal{G})(T) := Hom_{Psh(\mathcal{D})}(\phi(T), \mathcal{G})$ for every $\mathcal{G} \in Psh(\mathcal{D})$ and every $T \in \mathcal{C}$.
For every $T' \in \mathcal{D}$ consider the category $\phi(T') := \{\widetilde{T} \to \phi(T) \mid T \in \mathcal{C}\}$, we define $\phi^*(G)(T') := \varinjlim_{\phi(T')} \mathcal{G}(T')$ for $\mathcal{G} \in Psh(\mathcal{C})$.
The details on the adjunction and the property $\phi_* |_{\mathcal{C}} = \phi$ can be seen in [BO78] □

Lemma 5. *The final object of a topos $\mathcal{T}$ is the sheafification of the preseheaf $U \mapsto \{*\}$*

6 References

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