On the Local Theta Correspondence

Jialiang ZOU
Supervised by Mr. Gaëtan CHENEVIER

November 4, 2016

Abstract

In this mémoire, We study the local theta correspondence for spherical representations.

First we introduce the Weil representation: Let F be a local field and $(W, \langle , \rangle)$ be a symplectic space over F. Take V_1, V_2 to be two transversal lagrangians of W, such that $W = V_1 \oplus V_2$. The Weil representation is a representation of the metaplectic group $\widehat{Sp}(W)$, which is a double cover of $Sp(W)$, on $\mathcal{S}(V_1)$(the space of Schwartz functions on V_1).

One can restrict the Weil representation to a dual reductive pair inside $\widehat{Sp}(W)$ and study this new representation. The local theta correspondence predicts a bijection between irreducible representations of these two groups for which the tensor product appears as a sub-quotient of this representation. We will focus on one case of the local theta correspondence in this mémoire: the dual reductive pair $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$ inside $\widehat{Sp}(W_1 \otimes W_2)$, where (W_1, Φ_1) is a symmetric bilinear space of dimension divisible by 8, and (W_2, Φ_2) is a symplectic space. Moreover, We only consider the local theta correspondence for the spherical representations of these two groups. For this case, Rallis gives an explicit map in terms of the 'Langlands principle of functoriality'. The main goal of this mémoire is to understand Rallis's result.

Contents

1 Notation

Let F be a non-Archimedian local field.
Let A be the ring of integers of F
Let π be a fixed uniformizer of A.
Let q be the number of elements in the residue field $A/\pi A$.
Let ψ be a addictive character of F.

2 Weil representation

2.1 The Heisenberg group

2.1.1 Basic definitions

Let $(W, \langle , \rangle)$ be a symplectic space over F.

Definition 2.1. *The Heisenberg group associated to W is $H(W) := W \times F$ with the group structure*

$$(w_1, t_1).(w_2, t_2) = \big(w_1 + w_2, t_1 + t_2 + \langle w_1, w_2 \rangle /2\big)$$

Remark 2.1.1. *(i) The center Z of $H(W)$ is simply $\{0\} \times F$, and $H(W)/Z \cong W$, so $H(W)$ is a two-step nilpotent group.*

(ii) A Haar measure of $H(W)$ can be given by the product measure of W and F, and this measure is both left and right invariant, hence $H(W)$ is unimodular.

(iii) If $W \cong W_1 \oplus W_2$, where W_1, W_2 are symplectic spaces, then there is a natural homomorphism

$$\begin{aligned} \kappa : H(W_1) \times H(W_2) &\to H(W) \\ \big((w_1, t_1), (w_2, t_2)\big) &\mapsto (w_1 + w_2, t_1 + t_2) \end{aligned}$$

and $\ker(\kappa) = \{((0, t), (0, -t)) | t \in F\}$

2.1.2 Stone-von Neumann theorem

Given any irreducible admissible representation ρ of $H(W)$, restriction of ρ to the center $Z \cong F$ gives a character of Z. We call this character of Z the central character of ρ. The Stone-von Neumann theorem says that the irreducible admissible representations of $H(W)$ are classified by their central characters.

Theorem 2.2 (Stone-von Neumann theorem)**.** *Let ψ be a character of F, then up to isomorphism, there exists a unique admissible irreducible representation (ρ, S) of $H(W)$ with central character ψ.*

We will prove this theorem step by step.

For any subgroup $A \subset W$, put

$$A^{\perp} = \{w \in W \, : \forall a \in A, \psi(\langle w, a \rangle) = 1\}$$

it is a closed subgroup of W, and if A is closed, we have $A^{\perp\perp} = A$.

Fix a subgroup $A \subseteq W$ such that $A = A^{\perp}$. For example, we can take A to be a lagrangian of W. Let $A_H := A \times F$ and $\psi_A : A_H \to \mathbb{S}^1$ be the character extended to A_H by being trivial on A. Consider the representation (ρ_A, S_A) to be the induced representation from ψ_A, which means

$$\begin{aligned} &S_A = \{f : H(W) \to \mathbb{C} : \forall a \in A_H, f(ah) = \psi_A(a) f(h) \\ &\text{and } f \text{ is fixed by some open L} \subset W\} \\ &\rho_A \text{ acts by the right translation.} \end{aligned}$$

We also consider the compact induction, and in our case they are equal:

Lemma 2.3. *(ρ, S_A) coincides with the compact induction $c - \mathrm{Ind}_{A_H}^{H(W)} \psi_A$.*

Proof. Let $f \in S_A$ be right invariant under a compact open subgroup $L \subset W$. It suffices to show the compactness of $\mathrm{Supp} f$ in $A_H \backslash H(W) = A \backslash W$.

Suppose that $w \in W$ and $f((w,0)) \neq 0$, for any $l \in L \cap A$, we have

$$f((w,0)) = f((w,0)(l,0)) = f((l, \langle w, l \rangle)(w,0)) = \psi(\langle w, l \rangle)\psi_A((l,0)) f((w,0))$$

Thus $\psi(\langle w, l \rangle) = \psi_A((-l,0))$. This pinned down the image of w modulo $(L \cap A)^{\perp}$. However $(L \cap A)^{\perp} = L^{\perp} + A^{\perp} = L^{\perp} + A$, and $L^{\perp}$ is compact since W/L is discrete, hence f is supported on the compact subset $A \backslash (L^{\perp} + A)$. $\square$

Lemma 2.4. *Let $w \in W$, L a compact open subgroup of W. Suppose that $\psi_A = 1$ on $H(A) \cap \big((w,0)(L \times \{0\})(w,0)^{-1}\big)$ (this is always possible by taking L small enough). We can define a function*

$$f_{w,L}(h) = \begin{cases} \psi_A(a) & if \; h = a(w,0)(l,0), h \in A_H(w,0)(L \times \{0\}) \\ 0 & otherwise \end{cases}$$

As w runs over W and L runs over small enough compact open subgroups, these functions generate S_A. In particular, $S_A \neq 0$.

Proof. The hypothesis that $\psi_A = 1$ on $H(A) \cap \big((w,0)(L \times \{0\})(w,0)^{-1}\big)$ guarantees that $f_{w,L}$ is well defined and lies in S_A. Note that $S_A = \bigcup_L S_A^L$, the functions in S_A^L are dertermined by their values on representatives of the double coset $A_H \backslash H/(L \times \{0\})$, which are zero for all but finitely many representatives by the preceding lemma. Our assertion follows at once. $\square$

Lemma 2.5. *The representation (ρ_A, S_A) is irreducible*

Proof. Fix $f \in S_A, f \neq 0$, we want to show that f generates all $f_{w,L}$ under the action of ρ. Fix $w \in W$. By translating f on the right, we may assume that $f((w,0)) \neq 0$. Let L be a compact open subgroup fix f. Fix a Haar measure on A and consider the action of $\mathcal{S}(A)$. For any $\phi \in \mathcal{S}(A), w' \in W$, we have

$$(\rho|_A(\phi)(f))(w',0) = \int_A f((w',0))(a,0)\phi(a)\,\mathrm{d}a$$
$$= \int_A \psi(\langle w',a\rangle)\psi_A(a,0)\phi(a)\,\mathrm{d}a \cdot f((w',0))$$

This resembles a Fourier transform; write $\phi(a) = \psi_A(-a,0)\phi'(a)$, the last term becomes

$$\int_A \psi(\langle w',a\rangle)\psi'(a)\,\mathrm{d}a \cdot f((w',0)) = (\phi'\mu)^\wedge(w'+A)\cdot f((w',0))$$

where ν is some Haar measure on A. Choose ϕ' so that $(\phi'\nu)^\wedge$ is the characteristic function of $w+L+A \subset W/A = \hat{A}$, then $\rho_A|_H(\phi)f$ is f multiplied by the characteristic function of $(A+w+L)\times F$. By taking L small enough, it will be a multiple of $f_{w,L}$.

□

This establishes the existence part of Theorem 2.2.

Let's consider some choices of A which will be used later. Let V_1, V_2 be two transversal lagrangians of W, such that there is a splitting map $W \cong V_1 \oplus V_2$. Consider the representation (ρ_{V_1}, S_{V_1}), by Lemma 2.3, the restriction of function on W to V_2 gives an isomorphism

$$\begin{aligned} S_{V_1} &\to \mathcal{S}(V_2) \\ f &\mapsto f|_{V_2} \end{aligned}$$

Here $\mathcal{S}(V_2)$ means the Schwartz functions on V_2. Under this isomorphism, we can get a representation of $H(W)$ on the Schwartz space $\mathcal{S}(V_2)$, which can be defined as follows.

$$\Big(\rho_{V_1}((x+y,t))\varphi\Big)(y') = \psi\Big(\langle y',x\rangle + \frac{1}{2}\langle y,x\rangle + t\Big)\varphi(y'+y), \forall x \in V_1, y \in V_2$$

This is called the the Schrödinger models of $H(W)$. Moreover, the representation on $\mathcal{S}(V_1)$ and $\mathcal{S}(V_2)$ are intertwined by the Fourier transform

$$\begin{aligned} &\mathcal{S}(V_1) \to \mathcal{S}(V_2) \\ &\quad f \mapsto f^\wedge : f^\wedge(x) = \int_{V_1} f(y)\psi(\langle y,x\rangle)\mathrm{d}y \end{aligned}$$

Next we prove the uniqueness. Let $\mathcal{S}_\psi(H(W))$ be the space consisting of smooth functions f on $H(W)$ such that $f(zh) = \psi(z)f(h)$ for all $z \in Z = \{0\}\times F$

and $h \in H$, and $|f|$ is compactly supported module Z. Restriction of f to W gives rise to an isomorphism.

$$\begin{aligned}\mathcal{S}_\psi(H(W)) &\cong \mathcal{S}(W)\\ f &\mapsto f|_{W\times\{0\}}\end{aligned}$$

$\mathcal{S}_\psi(H(W))$[resp.$\mathcal{S}(W)$] admits two representations:

$$\begin{aligned}\rho_r &= \text{right translation}: \ \ \rho_r(g)f(h) = f(hg)\\ \rho_l &= \text{left translation}: \ \ \rho_l(g)f(h) = f(gh)\end{aligned}$$

so that the $H(W)\times H(W)$ representation $\rho_r\otimes\rho_l$ on $\mathcal{S}_\psi(H(W))$[resp.$\mathcal{S}(W)$] can be defined.

Lemma 2.6. *Let (ρ, S) be a representation of $H(W)$ with central character ψ, let $S^\vee$ denote the smooth dual of S. Then taking coefficients*

$$\begin{aligned}S^\vee\otimes S &\to \mathcal{S}_\psi(H(W))\\ s^\vee\otimes s &\mapsto f_{s^\vee,s}(h) := \langle s^\vee, \rho(h)s\rangle\end{aligned}$$

gives rise to an intertwining operator $c: \rho^\vee\otimes\rho\to\rho_r\otimes\rho_l$ as representation of $H(W)\times H(W)$.

Proof. The crux is to show that for all $s^\vee, s$, the matrix coefficient $f_{s^\vee,s}$ is compactly supported modulo Z, Take a compact open subgroup $L\subset W$, fixing $s^\vee, s$, then for all $l\in L$ we have

$$\begin{aligned}f_{s^\vee,s}((w,0)) &= \langle s^\vee, \rho((w,0))s\rangle = \langle s^\vee, \rho((w,0)(l,0))s\rangle\\ &= \psi(\langle w,l\rangle)\langle\rho((-l,0))s^\vee, \rho((w,0))s\rangle\\ &= \psi(\langle w,l\rangle)\langle s^\vee, \rho((w,0))s\rangle = \psi(\langle w,l\rangle)f_{s^\vee,s}((w,0))\end{aligned}$$

hence $f_{s^\vee,s}((w,0))\neq 0$ implies that $w\in L^\perp$, which is compact. ☐

Lemma 2.7. *ρ_r is isotypic*

Proof. Take two transversal lagrangians V_1, V_2 of W. We have two representations $(\rho,\mathcal{S}(V_1))$, $(\rho,\mathcal{S}(V_2))$ with central character ψ. Apply the same construction to get $(\bar\rho,\mathcal{S}(V_1))$, $(\bar\rho',\mathcal{S}(V_2))$, but this time with central character $\bar\psi$.

Fix a Haar measures on V_1 and V_2. Recall the general fact that $\rho\cong\mathrm{Ind}_{(V_1)_H}^{H(W)}(1\otimes\psi)$ and $\bar\rho\cong\mathrm{Ind}_{(V_1)_H}^{H(W)}(1\otimes\bar\psi)$ are in duality via the pairing

$$(s', s)\mapsto\int_{V_2} s'(y')s(y')\,\mathrm{d}y'$$

Now, use Fourier transform to identify $\bar\rho$ and $\bar\rho'$, the above duality pairing becomes

$$(s, s')\mapsto\int_{V_1\times V_2} s'(x')s(y')\psi(\langle y', x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y'$$

the matrix coefficient now takes the following form: let $x \in V_1, y \in V_2$,

$$
\begin{aligned}
f_{s',s}((x+y),0)) &= \int_{V_1\times V_2} s'(x')\psi\big(\langle y',x\rangle + \frac{\langle y,x\rangle}{2}\big) s(y+y')\psi(\langle y',x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y' \\
&= \int_{V_1\times V_2} s'(x')s(y')\psi\big(\langle y'-y,x\rangle + \frac{\langle y,x\rangle}{2}\big)\psi(\langle y'-y,x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y' \\
&= \psi\big(\frac{\langle x,y\rangle}{2}\big)\int_{V_1\times V_2} s'(x')s(y')\psi(\langle y',x\rangle - \langle y,x'\rangle)\psi(\langle y',x'\rangle)\,\mathrm{d}x'\,\mathrm{d}y'
\end{aligned}
$$

This is the tensor product of two Fourier transforms multiplied by a bicharacter. Hence $\bar{\rho}'\otimes\rho \cong \rho_r\otimes\rho_l$. Since ρ is irreducible, restriction to $1\times H$ shows that ρ_r is a direct sum of copies of ρ. $\square$

Now we can complete the proof of the uniqueness. Let (σ, S) be any smooth and irreducible representation of H with central character ψ. Using the intertwining operator $c : \sigma^\vee\otimes\sigma \to \rho_r\otimes\rho_l$, we can fix $s^\vee \in S^\vee, s^\vee \neq 0$ to embed σ into $\rho_r\otimes\rho_l$. The above lemma implies that $\sigma\cong\rho$.

2.2 Weil representation

Observe that the symplectic group $Sp(W)$ acts on $H(W)$ by

$$
\begin{aligned}
Sp(W)\times H(W) &\to H(W) \\
(g,(w,t)) &\mapsto (g(w),t)
\end{aligned}
$$

The calculation

$$
\begin{aligned}
\big(g,(w_1,t_1).(w_2,t_2)\big) &= \big(g,(w_1+w_2,t_1+t_2+1/2\langle w_1,w_2\rangle)\big) \\
&= \big(g(w_1+w_2),t_1+t_2+1/2\langle w_1,w_2\rangle\big)
\end{aligned}
$$

$$
\begin{aligned}
\big(g,(w_1,t_1)\big).\big(g,(w_2,t_2)\big) &= (g(w_1),t_1).(g(w_2),t_2) \\
&= \big(g(w_1)+g(w_2),t_1+t_2+1/2\langle g(w_1),g(w_2)\rangle\big) \\
&= \big(g(w_1+w_2),t_1+t_2+1/2\langle w_1,w_2\rangle\big)
\end{aligned}
$$

shows that g is really a group homomorphism from $H(W)$ to $H(W)$. Note that this action is trivial on Z.

Definition 2.8. *Fix a model (ρ, S) of $H(W)$, for $g\in Sp(W)$, we can define a new representation ρ^g on the same space S by*

$$\rho^g(h) = \rho(g(h)) \quad for\ all\ h\in H$$

As the acting of g is trivial on Z, this is a representation of $H(W)$ with the same central character. So by Stone-von Neumann theorem, there exists a linear map $M_g : S\to S$ such that

$$M_g\circ\rho = \rho^g\circ M_g$$

By Schur's lemma , M_g is uniquely determined by g up to a scalar in $\mathbb{C}^*$. This gives a projective representation ω of $Sp(W)$

$$\omega : Sp(W) \to \mathrm{GL}(S)/\mathbb{C}^* = \mathrm{PGL}(S)$$

We call this projective Weil representation of $Sp(W)$.

Remark 2.8.1. *We point out some inductive nature of the Projective Weil representation:*

Note that if $W = W_1 \oplus W_2$ is a sum of two symplectic space, let (ρ_1, S_1) and (ρ_2, S_2) be the representation of $H(W_1)$ and $H(W_2)$, we get a representation $(\rho_1 \otimes \rho_2, S_1 \otimes S_2)$ of $H(W_1) \times H(W_2)$. For $\lambda \in \ker(\kappa) = ((0,t),(0,-t))$:

$$(\rho_1 \otimes \rho_2)(\lambda)(v_1 \otimes v_2) = \psi(t)v_1 \otimes \psi(-t)v_2 = v_1 \otimes v_2.$$

So $\rho_1 \otimes \rho_2$ is trivial on $\ker(\kappa)$, hence the representation factor through $H(W)$. Denote this representation as $H(W)$ by $\rho_1 \hat{\otimes} \rho_2$. The restrict of $\rho_1 \hat{\otimes} \rho_2$ to the center of $H(W)$ is also ψ, so by Definition 2.8 we can get a projective representation $\omega_{\rho_1 \otimes \rho_2}$ of $Sp(W)$ on $S_1 \otimes S_2$. Composed with the natural injection map

$$\iota : Sp(W_1) \times Sp(W_2) \hookrightarrow Sp(W)$$

We get a representation of $Sp(W_1) \otimes Sp(W_2)$ on $S_1 \otimes S_2$, denote by $(\omega_{\rho_1 \otimes \rho_2} \circ \iota, S_1 \otimes S_2)$.

Also, under the natural map

$$\begin{gathered} \mathrm{PGL}(S_1) \times \mathrm{PGL}(S_2) \to \mathrm{PGL}(S_1 \otimes S_2) \\ [A] \times [B] \mapsto [A \otimes B] \end{gathered}$$

we can define a projective representation $(\omega_{\rho_1} \otimes \omega_{\rho_2}, S_1 \otimes S_2)$ of $Sp(W_1) \times Sp(W_2)$ on $S_1 \otimes S_2$. These two representations: $(\omega_{\rho_1 \otimes \rho_2} \circ \iota, S_1 \otimes S_2)$ and $(\omega_{\rho_1} \otimes \omega_{\rho_2}, S_1 \otimes S_2)$ are isomorphic.

For such a representation, consider the fiber product

$$\begin{array}{ccc} \widetilde{Sp}(W) & \xrightarrow{\tilde{p}} & Sp(W) \\ \big\downarrow{\scriptstyle \tilde{\omega}} & & \big\downarrow{\scriptstyle \omega} \\ GL(S) & \longrightarrow & PGL(S) \end{array}$$

where

$$\widetilde{Sp}(W) := \{(g, M_g) \in Sp(W) \times GL(S) : M_g \circ \rho = \rho^g \circ M_g\}$$

Then the projective representation can be lift as a ordinary representation of the group $\widetilde{Sp}(W)$

Theorem 2.9 (Weil)**.** *There exist a unique subgroup $\widehat{Sp}(W)$ of $\widetilde{Sp}(W)$ such that $p := \tilde{p}|_{\widehat{Sp}(W)} : \widehat{Sp}(W) \to Sp(W)$ is a two fold covering of $Sp(W)$. Let $\epsilon \in \ker(p)$ be the non-trivial elements, there is a short exact sequence*

$$1 \longrightarrow \{1, \epsilon\} \longrightarrow \widehat{Sp}(W) \xrightarrow{p} Sp(W) \longrightarrow 1$$

Therefore, the restriction of $\widetilde{\omega}$ to $\widehat{Sp}(W)$ lifts the projective representation ω of $Sp(W)$ to an ordinary representation of $\widehat{Sp}(W)$. We denote the group $\widehat{Sp}(W)$ as the metaplectic group

Remark 2.9.1. *(i) The proof can be found in [3]*

(ii) Although our construction of $\widetilde{Sp}(W)$ and $\widehat{Sp}(W)$ depends on the character ψ we fixed, it can be shown that this does not depend on the character. And the lifting to $\widehat{Sp}(W)$ is unique. See [4] for detail of this lifting

We will use the Schrödinger models of $H(W)$ and write explicitly the projective representation of $Sp(W)$.

Let $W = V_1 \oplus V_2$ be as before, and consider (ρ_{V_1}, S_{V_1}). Recall that the restriction of function on W to V_2 gives a isomorphism

$$\begin{aligned} S_{V_1} &\to \mathcal{S}(V_2) \\ f &\mapsto f|_{V_2} \end{aligned}$$

Under this isomorphism, We get a representation of $H(W)$ on the Schwartz space $\mathcal{S}(V_2)$, which can be defines as follow.

$$\Big(\rho((x+y,t))\varphi\Big)(y') = \psi\Big(\langle y', x\rangle + \frac{1}{2}\langle y, x\rangle + t\Big)\varphi(y'+y), \forall x \in V_1, y \in V_2$$

By calculate the intertwining operator for $g \in Sp(W)$, we get a projective representation of $Sp(W)$ on the Schwartz space $\mathcal{S}(V_2)$ defined as follow.

$$\rho\Big(\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}\Big)\varphi(X) = |detA|^{1/2}\varphi({}^tAX)$$

$$\rho\Big(\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\Big)\varphi(X) = \psi(\frac{{}^tXBX}{2})\varphi(X)$$

$$\rho\Big(\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\Big)\varphi(X) = \hat{\varphi}(X)$$

Here we pick a representative elements in each class $[g] \in PGL(S_{V_1})$. As these three kind of elements generate the whole group, this defines a action of $Sp(W)$.

2.3 Dual reductive pairs

2.3.1 The general case

We define the reductive dual pairs of $Sp(W)$ and consider the Weil representation restrict to this reductive pairs.

Definition 2.10. *A reductive dual pair (G_1, G_2) of $Sp(W)$ is a pair of subgroups G_1 and G_2 of $Sp(W)$ such that both of them are reductive groups and*

$$Cent_{Sp(W)}(G_1) = G_2 \quad Cent_{Sp(W)}(G_2) = G_1$$

If $W = W_1 \oplus W_2$ is an orthogonal decomposition of symplectic space, and (G_1^1, G_2^1) and (G_1^2, G_2^2) be dual reductive pairs of $Sp(W_1)$ and $Sp(W_2)$, then $(G_1, G_2) = (G_1^1 \times G_1^2, G_2^1 \times G_2^2)$ is a reductive dual pair in $Sp(W)$. Such pair is said to be reducible. A reductive dual pair (G_1, G_2) arises in this way is said to be irreducible. The irreducible reductive dual pair of $Sp(W)$ is classified by Howe, see [6] for a list. We will construct one kind of reductive dual pair in next section and study it.

We begin with some preliminary about representation of product of two groups.

Lemma 2.11. *Let G_1, G_2 be locally compact totally disconnected groups and let $G = G_1 \times G_2$*

(1) If π_i is an admissible irreducible representation of $G_i, i = 1, 2$, then $\pi_1 \otimes \pi_2$ is an admissible irreducible representation of G

(2) If π is an admissible representation of G, then there exists admissible irreducible representation π_i of G_i such that $\pi \cong \pi_1 \otimes \pi_2$, the isomorphism class of π_i is determined by π

We first proof another lemma that will be used to proof the theorem above.

Lemma 2.12. *A smooth G module W is irreducible if and only if W^K irreducible as $\mathcal{H}(G, K)$ module for all compact open subgroups K of G.*

Proof. This follows from the fact that if U is an $\mathcal{H}(G, K)$ submodule of W^K, then $(\mathcal{H}(G) \cdot U)^K = U$. □

Remark 2.12.1. *The definition of $\mathcal{H}(G, K)$ can be found in Section* 3.1*.*

We begin to proof the theorem

Proof. It is straightforward that

(i) $\mathcal{H}(G_1 \times G_2, K_1 \times K_2) \cong \mathcal{H}(G_1, K_1) \otimes \mathcal{H}(G_2, K_2)$ and

(ii) $(W_1 \otimes W_2)^{K_1 \times K_2} \cong (W_1)^{K_1} \otimes (W_2)^{K_2}$

For every pair of compact open subgroup K_i of G_i and every pair of smooth G_i module W_i. Assertion (1) follows from (ii) and the irreducible criterion.

Conversely, let W be an admissible irreducible G module. Let $K = K_1 \times K_2$, where K_i is a compact open subgroup of $G_i, i = 1, 2$, be such that $W^K \neq 0$. The space W^K is finite dimensional, so there exist a irreducible $\mathcal{H}(G_i, K_i)$ module $W_i^{K_i}$ and an $\mathcal{H}(G, K)$ isomorphism α_K from W^K to $W_1^{K_1} \otimes W_2^{K_2}$. Similar result applies to every open subgroup $K' = K_1' \times K_2'$ of K. There exists $\mathcal{H}(G_i, K_i)$ maps $b_i = b_i(K, K') : W_i^{K_i} \to W_i^{K_i'}$ such that the following diagram is commutative.

$$\begin{tikzcd} W^K \arrow[r, "\alpha_K"] \arrow[d, "incl."] & W_1^{K_1} \otimes W_2^{K_2} \arrow[d, "b_1 \otimes b_2"] \\ W^{K'} \arrow[r, "\alpha_K'"] & W_1^{K_1'} \otimes W_2^{K_2'} \end{tikzcd}$$

Moreover, the map $b_i(K, K')$ can be chosen for every pair of compact open subgroups K, K' of this type in such a way as to form an inductive system. Then $W \cong W_1 \otimes W_2$ where $W_i = \operatorname{ind} \lim_{K_i} W_i^{K_i}$, and W_i is an admissible irreducible representation of $G_i, i = 1, 2$.

The class of W_i is determined by that of W, for the restrictions of W to G_i is W_i isotypic. $\square$

Remark 2.12.2. *The reductive groups over non-Archimedean field satisfies the condition in this lemma.*

Definition 2.13. *Let $H \subset \widetilde{Sp}(W)$ be a closed subgroup. Define $\mathcal{R}(H) = \{\sigma : \sigma$ to be an irreducible representation of H such that there existing a H-nontrivial intertwine map $\alpha : S(V_1) \to \sigma\}$.*

For $G \in Sp(W)$, denote $\widehat{G}$ the inverse image of G in $\widehat{Sp}(W)$. Let (G_1, G_2) be a reductive dual pair in $Sp(W)$, then $\widehat{G_1}$ and $\widehat{G_2}$ commute in $\widehat{Sp}(W)$, We have a natural map

$$\widehat{G_1} \times \widehat{G_2} \to \widehat{G_1} \cdot \widehat{G_2}$$
$$(g_1, g_2) \mapsto g_1 \cdot g_2$$

where product in the right is given in $Sp(W)$. Under this map, an irreducible admissible representation of $\widehat{G_1}.\widehat{G_2}$ can be pulled back to an irreducible admissible representation of $\widehat{G_1} \times \widehat{G_2}$. By lemma 2.2, such a representation has the form $\sigma \otimes \sigma'$ where σ and σ' are irreducible representation of $\widehat{G_1}$ and $\widehat{G_2}$. Thus $\mathcal{R}(\widehat{G_1}.\widehat{G_2})$ defines a correspondence between irreducible admissible representations of $\widehat{G_1}$ and $\widehat{G_2}$.

Conjecture 2.13.1 (Howe duality conjecture)**.** *If (G_1, G_2) ia a reductive dual pair in $Sp(W)$, then $\mathcal{R}(\widehat{G_1}.\widehat{G_2})$ is the graph of a bijection between $\mathcal{R}(\widehat{G_1})$ and $\mathcal{R}(\widehat{G_2})$*

Remark 2.13.1. *(i) This conjecture has been proofed by Wee Teck Gan and Shuichiro Takeda in* 2014

(ii) We will study one case of reductive dual pair and only consider the spherical representation of these groups.

2.3.2 $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$

We will describe one case of reductive pair: $(O(W_1, \Phi_1) \times Sp(W_2, \Phi_2))$ of $Sp(W)$.

Let $W_1 = U \oplus U^*$ be a symmetric bilinear vector space of dimension m, with the symmetric bilinear form Φ_1 given by

$$\langle x_1 + y_1^*, x_2 + y_2^* \rangle_{\Phi_1} = \frac{1}{2}(y_2^*(x_1) + y_1^*(x_2)) \forall x_1, x_2 \in U, y_1^*, y_2^* \in U^*$$

We always assume $\dim(W_1)$ is divisible by 8 in this mémoire.

$W_2 = V_2 \oplus V_2^*$ be symplectic space of $\dim W_2 = 2n$ with symplectic form Φ_2 given by

$$\langle x_1 + y_1^*, x_2 + y_2^* \rangle_{\Phi_2} = \frac{1}{2}(y_2^*(x_1) - y_1^*(x_2)) \ \ \forall x_1, x_2 \in V_2, y_1^*, y_2^* \in V_2^*$$

.

Let $W \cong W_1 \otimes W_2$, we get a symplectic form on $W \cong W_1 \otimes W_2$ as

$$\langle w_1 \otimes w_2, v_1 \otimes v_2 \rangle = \langle w_1, v_1 \rangle_{\Phi_1} . \langle w_2, v_2 \rangle_{\Phi_2}$$

Lemma 2.14. $(O(W_1, \Phi_1), Sp(W_2, \Phi_2))$ *is an irreducible reductive dual pair of* $Sp(W)$.

Proof. We first proof $\mathrm{Cent}_{Sp(W)}(O(W_1, \Phi_1)) = Sp(W_2, \Phi_2)$: let $g \in Sp(W)$ such that $gh = hg$ for every $h \in O(W_1, \Phi_1)$.

Observe that W_1 is irreducible as representation of $O(W_1, \Phi_1)$, so $W_1 \otimes W_2$ is isotropic as a representation of $O(W_1, \Phi_1)$, and every sub-representation of $W_1 \otimes W_2$ is of the form $W_1 \times V$ For some $V \subseteq W_2$.

Pick an element $w_2 \in W_2$, denote the line generated by w_2 as V_2. Consider the space $g(W_1 \otimes V_2)$, as g commute with $O(W_1, \Phi_1)$, the space $g(W_1 \otimes V_2)$ is stabled under the action of $O(W_1, \Phi_1)$, so it is a sub-representation of $W_1 \otimes W_2$. We then have $g(W_1 \otimes V_2) = W_1 \otimes V_2'$ for some line $V_2 \in W_2$. We can pick a $w_2' \in V_2$ and write $g(w_1 \otimes w_2) = g(w_1) \otimes w_2'$ for all $w_1 \in W_1$, then the map

$$\begin{aligned} \vartheta_g : W_1 &\to W_1 \\ w_1 &\mapsto g(w_1) \end{aligned}$$

intertwines the action of $O(W_1, \Phi_1)$ as g commutes with $O(W_1, \Phi_1)$. By Shur's lemma, there exists $\lambda \in \mathbb{C}$ such that $g(w_1) = \lambda w_1$ for every $w_1 \in W_1$. That means $g(w_1 \otimes w_2) = \lambda w_1 \otimes w_2' = w_1 \otimes \lambda w_2'$. So we know that for every $w_2 \in W_2$, there exists a $g(w_2) \in W_2$ such that $g(w_1 \otimes w_2) = w_1 \otimes g(w_2)$ for every $w_1 \in W_1$.

It is easy to check that the map $w_2 \to g(w_2)$ is an action of g on W_2. We then prove $g \in Sp(W_2)$: pick a $w_1 \in W_1$ such that $\Phi_1(w_1, w_1) = 1$, and consider the action of g on $w_1 \otimes W_2$. The symplectic norm on $W_1 \otimes W_2$ restrict to $w_1 \otimes W_2$ is non-degenerated, as $\langle w_1, w_1 \rangle_{\Phi_1} = 1$. ?? So $g \in Sp(W)$ implies $g \in Sp(W_2)$.

The proof of $\mathrm{Cent}_{Sp(W)}(Sp(W_2, \Phi_2)) = O(W_1, \Phi_1)$ is similar. As $W_1 \otimes W_2$ is irreducible as a representation of $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$, this reductive pair is irreducible. □

Moreover, the map

$$\gamma : \widehat{Sp}(W) \to Sp(W)$$

splits on $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$. The splitting map is given as follows

(a) the map $g_1 \mapsto (g_1, s_1(g_1))$ for $g_1 \in O(W_1, \Phi_1)$ where

$$s_1(g_1) = \begin{cases} 1 & \text{if } \det(g_1) = 1 \\ \langle -1 - 1 \rangle_F, & \text{if } \det(g_1) = -1 \end{cases}$$

(b) the map $g_2 \mapsto (g_2, 1)$ for $g_2 \in Sp(W_2, \Phi_2)$

Remark 2.14.1. *The splitting map is more complicated if we do not assume* $8 / \dim(W_1)$.

Under this splitting map, we have an action of $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$ on $\mathcal{S}(W_1 \otimes V_2)$. We will write this action explicitly by using a matrix model.

Fix a basis of W_1 to be $\{e_1, \cdots, e_{m/2}, e_1^*, \cdots, e_{m/2}^*\}$, where $\{e_1, \cdots, e_{m/2}\}$ is a basis of U. Under this basis, we can write elements in $O(W_1, \Phi_1)$ as explicit matrixes.

Also fix a basis of W_2 to be $\{f_1, \cdots, f_n, f_1^*, \cdots, f_n^*\}$, where $\{f_1, \cdots, f_n\}$ is a basis of V_2. Under this basis we can write elements in $Sp(W_2, \Phi_2)$ as explicit matrixes.

Observe that the map

$$\begin{aligned} \varphi : W_1 &\to W_1^* \\ w_1 &\mapsto \varphi(w_1) : v_1 \mapsto \langle v_1, w_1 \rangle_{\Phi_1} \end{aligned}$$

is an isomorphism that identified W_1 with its dual W_1^*.
Under the basis $\{e_1, \cdots, e_{m/2}, e_1^*, \cdots, e_{m/2}^*\}$, the dual basis of $W_1^* = W_1$ can be given by $\{e_1^*, \cdots, e_{m/2}^*, e_1, \cdots, e_{m/2}\}$. Under these basis, φ can be represented by the matrix $\begin{bmatrix} 0 & I_{m/2} \\ I_{m/2} & 0 \end{bmatrix}$, we denote it as A_{Φ_1}. Under the identification

$$f : W_1 \otimes W_2 \cong \mathrm{Hom}_F(W_1^*, W_2),$$

we can identify $W_1 \otimes V_2$ with $\mathrm{Hom}_F(W_1^*, V_2)$. We have

$$W_1 \otimes V_2 \cong Hom_F(W_1^*, V_2) \cong \mathbf{M}_{mn}(F)$$

The last isomorphism is given by writing $g \in Hom_F(W_1^*, V_2)$ as a matrix under the basis $\{e_1^*, \cdots, e_{m/2}^*, e_1, \cdots, e_{m/2}\}$ of W_1^* and the basis $\{f_1, \cdots, f_n\}$ of V_2.

Using this, we can transfer the action of $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$ on $S(W_1 \otimes V_2)$ to $S(\mathbf{M}_{mn}(F))$. This new action is defined as follows:

For $g_1 \in O(W_1, \Phi_1)$

$$\pi(g_1)f(X) = f(g_1^{-1}X)$$

For $g_2 \in Sp(W_2, \Phi_2)$

$$\pi\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} f(X) = (\det A)^{m/2} f(XA)$$

$$\pi\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} f(X) = \psi(\mathrm{Tr}(BX^t A_{\Phi_1} X)/2) f(X)$$

$$\pi\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} f(X) = \int_{M_{m\times n}(F)} f(Y)\psi(-\mathrm{Tr}(Y^t A_{\Phi_1} X))\mathrm{d}Y$$

By Iwasawa decomposition, these three kind of elements generate $Sp(W_2, \Phi_2)$. So these define an action of $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$. We denote this representation as ρ_{mn}

Remark 2.14.2. *By Remark* 2.8.1, *for $n = n_1 + n_2$, we have the restriction of $(\rho_{mn}, S(M_{mn}(F)))$ to $(O(W_1, \Phi_1) \times \mathrm{Sp}_{n_1}(F) \times O(W_1, \Phi_1) \times Sp_{n_2}(F))$ is isomorphic to $(\rho_{mn_1} \otimes \rho_{mn_2}, S(M_{mn_1}(F)) \otimes S(M_{mn_2}(F))$.*

Fix $[Y_0] \in M_{mn_2}(F)$ and $G \in \mathrm{Sp}_{n_2}(F) \subset \mathrm{Sp}_n(F)$ the map

$$\begin{aligned} \gamma : S(\boldsymbol{M}_{mn}(F)) &\to S(\boldsymbol{M}_{mn_1}(F)) \\ f &\to \gamma(f) : X \mapsto \rho_{mn}(G) f[X|Y_0] \end{aligned}$$

intertwines the action of $Sp_{n_1}(F)$ on both side. This inductive nature of ρ_{mn} will be used later.

We will study this representation in this mémoire. For simplicity, we assume that $m \geq 2n$. We sometimes write ρ_{mn} as ρ for brevity if this will not cause ambiguity.

3 local theta correspondence

We will study the spherical quotient of the representation ρ in this Section. We give some preliminaries at first.

3.1 Hecke algebra and Satake isomorphism

Let G be a connected, reductive, algebraic group scheme over F. Assume that G is split over F, then G is the general fibre of a group scheme (also denote G)

over A with reductive special fibre. Fix a torus T and a Borel subgroup B of G such that $T \subset B \subset G$, and define the Weyl group of T as $W = N_G(T)/T$. We know that $G(A)$ is a maximal compact subgroup of G. Denote $G(A)$ by K and let N be the unipotent radical of B, then $B = T \rtimes N$.

3.1.1 Root datum of reductive group

We introduce some fact on root and coroot of G.

Definition 3.1. *The character and co-character of T are defined by*

$$X^\bullet(T) = \mathrm{Hom}(T, \mathbb{G}_m)$$
$$X_\bullet(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

These are free abelian groups of rank $l = \dim(T)$, which are paired into

$$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$$

Consider the adjoint action of T on $Lie(G)$: the lie algebra of G. As T is split, this action can be diagonalized. The character of T appearing in this action is called the root of G, denote as Φ. We also have the coroot $\check{\Phi}$ of G lying inside $X_\bullet(T)$.

The positive root Φ^+ is defined to be the characters appearing in the adjoint action of T on $Lie(B)$. We have the decomposition $\Phi = \Phi^+ \cup (-\Phi^+)$. It determines a root basis $\triangle \subset \Phi^+$ consisting of positive indecomposable roots.

The root basis determines a Weyl chamber P^+ in $X_\bullet(T)$

$$\begin{aligned} P^+ =& \{\lambda \in X_\bullet(T) | \langle \lambda, \alpha \rangle \geq 0, \forall \alpha \in \Phi^+\} \\ =& \{\lambda \in X_\bullet(T) | \langle \lambda, \alpha \rangle \geq 0, \forall \alpha \in \triangle\} \end{aligned}$$

Let $2\rho = \sum_{\Phi^+} \alpha$ in $X_\bullet(T)$, then for $\lambda \in P^+$, the half integer $\langle \lambda, \rho \rangle$ is non-negative.

There is a partial ordering in P^+. We say $\lambda \geq \mu$ if the difference $\lambda - \mu$ can be written as a sum of positive roots with coefficients in $\mathbb{N}$. If $\check{\alpha} \in \check{\triangle}$ then $\langle \check{\alpha}, \rho \rangle = 1$, so $\lambda \geq \mu$ implies $\langle \lambda - \mu, \rho \rangle$ is non-negative half integers. Index the root basis $\triangle$ to be $\{\alpha_1, \alpha_2, ...\alpha_n\}$, and define the fundamental co-characters as $\varepsilon_i \in X_\bullet(T), i = 1, \cdots, n$ such that $\langle \varepsilon_i, \lambda_j \rangle = \delta_{ij}, j = 1, \cdots, n$.

Let $\hat{G}$ be the complex dual group of G. This is a connected, reductive group over $\mathbb{C}$ whose root datum is the dual of G. If we fixed a maximum torus $\hat{T}$ and a Borel subgroup $\hat{B}$ of G such that $\hat{T} \subset \hat{B} \subset \hat{G}$. Then we have an isomorphism

$$X_\bullet(\hat{T}) \cong X^\bullet(T) \tag{1}$$

which sends the positive coroots corresponding to $\hat{B}$ to positive roots corresponding to B. We know that the element $\lambda \in P^+ \subset X^\bullet(\hat{T})$ indexes the finite dimensional irreducible representation of $\hat{G}$: λ is the highest weight for $\hat{B}$ acting on V_λ. Let $\chi_\lambda = \mathrm{Trace}(V_\lambda)$ in $\mathbb{Z}[X^\bullet(\hat{T})]$, then χ_λ is fixed by the Weyl group, so it lies in the subgroup $\mathbb{Z}[X^\bullet(\hat{T})]^W$. We have an isomorphism

$$\mathrm{R}(\hat{G}) \cong \mathbb{Z}[X^\bullet(\hat{T})]^W \tag{2}$$

Combine Equation 1, we have

$$\mathrm{R}(\hat{G}) \cong \mathbb{Z}[X_\bullet(T)]^W \tag{3}$$

Example 3.1.1. *For $GL_n(F)$, fixed T to be the set of diagonal matrix and B to be the set of upper triangular matrixes. Let*

$$e_i \in X_\bullet(T) : e_i \begin{pmatrix} t_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & t_n \end{pmatrix} = t_i$$

$$\check{e}_i \in X^\bullet(T) : \check{e}_i(t) = \begin{pmatrix} 1 & & & \\ & \cdot & & \\ & & t & \\ & & & 1 \end{pmatrix} \text{the } i \times i \text{ position is } t$$

then we have $\langle e_i, \hat{e}_j \rangle = \delta_{ij}$ and

$$\begin{aligned} X_\bullet(T) &= \langle e_1, e_2, ..., e_n \rangle \\ X^\bullet(T) &= \langle \hat{e}_1, \hat{e}_2, ..., \hat{e}_n \rangle \end{aligned}$$

the roots and coroots are given by

$$\begin{aligned} &\Phi = \{e_i - e_j, 0 \le i, j \le n\} \\ &\Phi^+ = \{e_i - e_j, 0 \le i < j \le n\} \\ &\check{\Phi} = \{\hat{e}_i - \hat{e}_j, 0 \le i, j \le n\} \\ &\check{\Phi}^+ = \{\hat{e}_i - \hat{e}_j, 0 \le i < j \le n\} \\ &\triangle = \{e_1 - e_2, e_2 - e_3, ..., e_{n-1} - e_n\} \\ &\check{\triangle} = \{\hat{e}_1 - \hat{e}_2, \hat{e}_2 - \hat{e}_3, ..., \hat{e}_{n-1} - \hat{e}_n\} \end{aligned}$$

and the map ι between Φ and $\check{\Phi}$ is given by $\iota(e_i) = \hat{e}_i$. The Weyl chamber is given by

$$P^+ = \{m_1\hat{e}_1 + m_2\hat{e}_2 + ... + m_n\hat{e}_n | m_1 \ge m_2 \ge ... \ge m_n\}$$

and the partial order on it is defined by

$$x_1\hat{e}_1 + x_2\hat{e}_2 + ... + x_n\hat{e}_n \ge y_1\hat{e}_1 + y_2\hat{e}_2 + ... + y_n\hat{e}_n$$

if

$$x_1 \geq y_1, x_1 + x_2 \geq y_1 + y_2, ..., x_1 + x_2 + ..x_{n-1} \geq y_1 + y_2 + ... + y_{n-1}$$

and

$$x_1 + x_2 + ..x_n = y_1 + y_2 + ... + y_n$$

The basis of P^+ *is* $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ *where*

$$\varepsilon_1 = \hat{e}_1, \varepsilon_2 = \hat{e}_1 + \hat{e}_2, ..., \varepsilon_n = \hat{e}_1 + \hat{e}_2 + ...\hat{e}_n$$

ε_i *correspondent to* $V_{\varepsilon_i} = \bigwedge^i \mathbb{C}^n$ *under the isomorphism given by Equation* 3.

3.1.2 Hecke algebra

Definition 3.2. *Let* G *be a connected reductive group over* F*, the Hecke algebra* $\mathcal{H}(G)$ *of* G *is defined to be the space of all compactly supported functions* $\phi : G \to \mathbb{C}$*. The product is defined by convolution:*

$$(\phi * \psi)(g) = \int_G \phi(h)\psi(h^{-1}g)dh$$

Also, we can define the Hecke algebra of G with respect to a open compact subgroup K:

Definition 3.3. *Let* G *be a connected reductive group over* F *and* K *be a open compact subgroup of* G*. The Hecke algebra* $\mathcal{H}(G, K)$ *is defined to be the space of all compactly supported functions* $\phi : G \to \mathbb{C}$ *such that* $\phi(kgk^{'}) = \phi(g)$ *for all* $k, k^{'} \in K$*. The product is defined by*

$$(\phi * \psi)(g) = \int_G \phi(h)\psi(h^{-1}g)dh$$

Remark 3.3.1. *Let* ε_K *be the characteristic function of* K*, then we have* $\mathcal{H}(G, K) = \varepsilon_K * \mathcal{H}(G) * \varepsilon_K$*.*

We will study the Hecke algebra of G and K defined in Section 3.1. In this case, K is a maximal compact subgroup of G, and $\mathcal{H}(G, K)$ is called the spherical Hecke algebra of G.

3.1.3 Satake isomorphism

The construction we are going to expound is due to Satake, it is the p-adic counterpart of a well-known construction of Harish-Chandra in the set-up of real reductive Lie groups.

Definition 3.4. *For any* $f \in \mathcal{H}(G, K)$*, define the Satake transform to be*

$$\mathcal{S}f(t) = \delta^{1/2}(t) \int_N f(tn)dn. = \delta^{-1/2}(t) \int_N f(nt)dn.$$

for $t \in T$, here δ is the modular character of B defined by

$$\delta(t) = \det(ad(t)|Lie(N)) = \rho(t)^2$$

Remark 3.4.1. *The support of the function*

$$\theta : N \to \mathbb{C}$$
$$n \to f(tn)$$

is $\{t^{-1}supp(f) \cap N\}$*. As* N *is closed in* G*,* $\{t^{-1}supp(f) \cap N\}$ *is compact in* N*. This shows* $\mathcal{S}f$ *is well defined.*

It is easy to check that δ is trivial on $T' = T \cap K$: the maximum compact subgroup of T. Moreover, since $f \in \mathcal{H}(G, K)$, $f(nt) = f(tn) = f(n)$ for all $t \in T'$, we see that $\mathcal{S}f$ is a function on T which is left and right invariant by the action of T'.

Theorem 3.5. *The Satake transform gives a ring isomorphism*

$$\mathcal{S} : \mathcal{H}(G, K) \cong \mathcal{H}(T, T')^W$$

The proof can be found in $[1, Theorem\, 4.1]$

Remark 3.5.1. *(i) The theorem implies the commutativity of the Hecke algebra* $\mathcal{H}(G, K)$*, as* $\mathcal{H}(T, T')$ *is obviously commutative.*

(ii) We explain idea of proving the theorem: For $\tau \in X_\bullet(T)$ *and* c_τ *be the characteristic function of* $K\tau(\pi)K$*. By calculation, One get* $\mathcal{S}(c_\tau) = q^{\langle \tau, \rho \rangle}\chi_\tau + \sum_{\mu \leq \tau} a_\tau(\mu)\chi_\mu$*, and the number of* $\mu \in P^+$ *such that* $\mu \leq \tau$ *is finite. As* $G = \bigcup_{\tau \in X_\bullet} K\tau(\pi)K$ *is a open cover of* G*, there exist a finite set* $\Lambda \subset X_\bullet(T)$ *such that support of* f *is inside* $K\tau K$ *for* $\tau \in \Lambda$*. So* $\mathcal{S}(f)$ *is a finite sum of the function* χ_μ *for* $\mu \in X_\bullet(T)$*, hence compact support on* T*. The isomorphism of these two ring also come from these calculation.*

We give a further description of $\mathcal{H}(T, T')^W$:

$$0 \to T' \to T \to X_\bullet(T) \to 0$$

is an exact sequence of locally compact groups, where the last arrow γ is given by $\gamma(t)$, and $\gamma(t)$ is the unique co-character of T satisfies

$$\langle \gamma(t), \chi \rangle = \mathrm{ord}(\chi(t))$$

for all $\chi \in X^\bullet(T)$ and $t \in T$. The choice of a uniformizer π of F gives the split of this sequence, which maps $\lambda \in X_\bullet(T)$ to the elements $\lambda(\pi)$ in T.

Lemma 3.6. *We have*

$$\beta : \mathcal{H}(T, T') \cong \mathbb{C}[X_\bullet(T)]$$
$$\nu_\lambda \to [\lambda]$$

Here ν_λ *denotes the characteristic function of* $\lambda(\pi)T' = T'\lambda(\pi)$

Proof. The bijection is given by the exact sequence above, and as $\nu_\lambda * \nu_\mu = \nu_{\lambda+\mu}$, this is a ring isomorphism. □

Combine Equation 3, Lemma 3.5 and Lemma 3.6, we have

$$\mathcal{H}(G,K) \cong \mathcal{H}(T,T')^W \cong \mathbb{C}[X_\bullet(T)]^W \cong \mathrm{R}(\hat{G}) \otimes \mathbb{C}$$

Let $\{\varepsilon_1, \varepsilon_2, ..., \varepsilon_n\}$ be a basis of $X_\bullet(T)$, we have

$$\mathbb{C}[X_\bullet(T)]^W \cong \mathbb{C}[\varepsilon_1, ..., \varepsilon_n, \varepsilon_1^{-1}, \cdots, \varepsilon_n^{-1}]^W$$

. So we get the the structure of $\mathcal{H}(G,K)$ as a polynomial algebra.

Example 3.6.1. *We give an example for the case $G = \mathrm{GL}_n(F)$ and $K = \mathrm{GL}_n(A)$. From Example* 3.1.1 *we have:*

$$\mathcal{H}(G,K) \cong \mathbb{C}[X_\bullet(T)]^W \cong \mathbb{C}[\hat{e}_1, \cdots, \hat{e}_n, \hat{e}_1^{-1}, \cdots, \hat{e}_n^{-1}]^W \cong \mathbb{C}[\varepsilon_1, \cdots, \varepsilon_n, \varepsilon_n^{-1}],$$

and ε_i corresponds to $\bigwedge^i \mathbb{C}^n$, ε_n corresponds to $\det$ *and ε_n^{-1} corresponds to* $\det^{-1}$ *in* $\mathrm{R}(\hat{G}) \otimes \mathbb{C}$.

3.2 Spherical representation

We will define the spherical representation of G and classify these in terms of the character of Its spherical Hecke algebra.

Definition 3.7. *Let G and K be in Section* 3.1*, for an admissible representation (π, V) of G, we call it spherical if it contains a K fixed vector*

Theorem 3.8. *(i) If (π, V) is an irreducible admissible representation and V^K is nonzero , then V^K is an irreducible $\mathcal{H}(G,K)$ module.*

(ii) If (π, V) and (σ, W) are irreducible admissible representations, and if $V^K \cong W^K$ as $\mathcal{H}(G,K)$modules, then π and σ are isomorphic representations.

Proof. We prove (i) first. If V is irreducible, it is sufficient to show that $\mathcal{H}(G,K)u = V^K$ for a given nonzero $u \in V^K$. Let $v \in V^K$, since V is irreducible, we may find $\psi \in \mathcal{H}(G)$ such that $\pi(\psi)u = v$. Consider $\phi = \varepsilon_K * \psi * \varepsilon_K \in \mathcal{H}(G,K)$. We have $\pi(\varepsilon_K)u = u$ and $\pi(\varepsilon_K)v = v$ since $u, v \in V^K$. Now

$$\pi(\phi)u = \pi(\varepsilon_K)\pi(\psi)\pi(\varepsilon_K)u = \pi(\varepsilon_K)\pi(\psi)u = \pi(\varepsilon_K)v = v$$

proves that $v \in \mathcal{H}(G,K)u$.

We then prove (ii). Suppose V and W are irreducible G modules such that V^K and W^K are isomorphic as $\mathcal{H}(G,K)$ modules.

Let $\lambda : V^K \to W^K$ denote an isomorphism. Let $l : W^K \to \mathbb{C}$ be a nonzero linear functional and let $\omega \in W^K$ be a vector such that $l(w) \neq 0$. We claim that there exists $\hat{\omega} \in \hat{W}^K$ such that $l(x) = \langle x, \hat{w} \rangle$ for $x \in W^K$. Indeed, we can extend the functional l to an arbitrary functional $\hat{w}_1$, then take $\hat{w} = \hat{\sigma}(\varepsilon_K)(\hat{w}_1)$.

$$\langle x, \hat{w} \rangle = \frac{1}{\mathrm{vol}(K)} \int_K \langle x, \hat{\sigma}(k) w_1 \rangle \, \mathrm{d}k = \frac{1}{\mathrm{vol}(K)} \int_K \langle \sigma(k) x, w_1 \rangle \, \mathrm{d}k = l(x).$$

Similarly we may find $\hat{v} \in \hat{V}^K$ such that $l(\lambda x) = \langle x, \hat{v} \rangle$ for $x \in V^K$, let $v \in V^K$ be the unique vector such that $\lambda(v) = w$. We will show that if $\phi \in \mathcal{H}$, then

$$\langle \pi(\phi) v, \hat{v} \rangle = \langle \sigma(\phi) w, \hat{w} \rangle.$$

If $\phi \in \mathcal{H}(G, K)$, we have

$$\langle \pi(\phi) v, \hat{v} \rangle = l(\lambda(\pi(\phi) v) = l(\sigma(\phi)\lambda(v)) = l(\sigma(\phi) w) = \langle \sigma(\phi) w, \hat{w} \rangle.$$

The general case follows from the following consideration. Let $\phi \in G$, and let $\phi' = \varepsilon_K * \phi * \varepsilon_K$, then

$$\langle \pi(\phi') v, \hat{v} \rangle = \langle \pi(\varepsilon_K) \pi(\phi) \pi(\varepsilon_K) v, \hat{v} \rangle = \langle \pi(\phi) \pi(\varepsilon_K) v, \hat{\pi}(\varepsilon_K) \hat{v} \rangle = \langle \pi(\phi) v, \hat{v} \rangle.$$

and similarity $\langle \sigma(\phi') w, \hat{w} \rangle = \langle \sigma(\phi) w, \hat{w} \rangle$. Thus the general case follows from the special case we have proved.

Now let $L \subset K$ be a smaller compact open subgroup. Since V^L and W^l are finite dimensional simple $\mathcal{H}(G, L)$ modules we conclude that $V^L \cong W^L$ as $\mathcal{H}(G, L)$ modules. The isomorphism are uniquely determined up to scalar by Schur's lemma and the scalar is determined if we require that the isomorphism agree with λ on $V^K \subset V^L$. Now if L' is another compact open subgroup of K, then the isomorphism λ_L and λ'_L must agree on $V^L \cap V^{L'}$ because they agree with $\lambda_{L \cap L'}$ on $V^{L \cap L'} \supset V^L \cap V^{L'}$. Therefore these isomorphism may be patched together to get a $\mathcal{H}(G)$ isomorphism $V \to W$. It is a G module isomorphism since $\pi(g) v = \pi(\phi) v$ if ϕ is any function supported on a sufficiently small neighborhood of g such that $\int_G \phi = 1$, so the action of $\mathcal{H}(G)$ determines the action of G on any admissible module. □

Recall that $\mathcal{H}(G, K)$ is a finite dimensional commutative algebra, so the irreducible module of it is one dimensional. It gives a character of $\mathcal{H}(G, K)$.

Proposition 3.9. *Let V be an irreducible spherical representation of G, then V^K is one dimensional.*

Let $f \in \mathrm{H}_K$, we know that f maps V to V^K, so f is finite operator, hence it is in the trace class. Moreover we have

Proposition 3.10. *Let χ be the character of $\mathcal{H}(G, K)$ induced by the action on V^K, for $f \in \mathcal{H}(G, K), \chi(f) = \mathrm{Trace}\, f_{|V}$*

Proof. Recall a statement from linear algebra: suppose that $W \subseteq V$ is a finite dimensional vector space over $\mathbb{C}$. If X is a linear operator on V and the image of X lies in W, then X is of a finite operator and is in the trace class. We have $\operatorname{Trace} X_{|V} = \operatorname{Trace} X_{|W}$. Applying this for the action of f on V, we get $\operatorname{Trace} f_{|V} = \operatorname{Trace} f_{|V^K} = \chi$. □

As we have the Satake isomorphism

$$\mathcal{S} : \mathcal{H}(G, K) \cong \mathrm{R}(\hat{G}) \otimes \mathbb{C} \tag{4}$$

the complex characters of $\mathcal{H}(G, K)$ give the complex characters of $R(\hat{G}) \otimes \mathbb{C}$. On the other hand, we know that the complex characters

$$\omega : \mathrm{R}(\hat{G}) \otimes \mathbb{C} \to \mathbb{C}$$

are indexed by the semi-simple conjugacy class s in the dual group $\hat{G}$. The value of the character ω_s on χ_λ is given by

$$\omega_s(\chi_\lambda) = \chi_\lambda(s) = \mathrm{Trace}(s|V_\lambda)$$

We call s the Satake parameter of the character of $\mathcal{H}(G, K)$ and have the following lemma:

Lemma 3.11. *For any irreducible spherical representation π of G, the map $\pi \to s(\pi)$ gives a bijection between the set of isomorphic classes of spherical representations of G and the set of semi-simple conjugacy classes $s(\pi)$in $\hat{G}$*

We will construct the spherical representation by the principle series:

Let χ be a complex unramified character of T, we extend it to B by letting it be trivial on N and still denote it as χ. Define the induced representation from χ to be $\big(Ind_B^G(\delta^{1/2}\chi), I(\chi)\big)$ i.e. $I(\chi)$ is the space consisted of the locally constant functions $f : G \to C$ such that

$$f(tng) = \delta^{1/2}(t)\chi(t)f(g) \;\; for \; t \in T, n \in N, g \in G$$

and the group G acts by right translation:

$$\rho(g)f(g') = f(g'g) \;\; for \; g, g' \in G$$

here δ is the modular character of B.

Lemma 3.12. *If χ is a unramified character of T, then $I(\chi)$ is spherical, and $dim(I(\chi)^K) = 1$.*

Proof. Recall the Cartan decomposition of G: $G = TNK$, then $f \in I(\chi)^K$ implies that $f(tnk) = f(tn) = \delta^{1/2}(t)\chi(t)f(1)$. So f is determined by its value $f(1)$, hence $I(\chi)^K$ is one dimensional. □

Denote ϑ_χ to be the character of $\mathcal{H}(G,K)$ obtained by its action on $I(\chi)$

Definition 3.13. *For $\forall\chi$ unramified character of T, The Satake Fourier transform of f is defined as:*

$$\mathcal{FS}(f)(\chi) = \int_T \mathcal{S}(f)(t)\chi(t)dt$$

Lemma 3.14. *$\mathcal{FS}(f)(\chi) = \vartheta_\chi(f)$.*

Proof. Pick a spherical vector f' such that $f'(1) = 1$, then $f'(tnk) = \delta^{1/2}(t)\chi(t)$. For $f \in \mathcal{H}(G,K)$

$$\vartheta_\chi(f) = f * f'(1) = \int_G f(g)f'(g)dg$$

Choose a the left (also right) Haar measure on G related to the Iwasawa decomposition as follows:

$$\int_G f(g)\, dg = \int_K \int_T \int_N f(tnk)\, dk\, dt\, dn \tag{5}$$

$$\int_G f(g)\, dg = \int_K \int_T \int_N f(knt)\, dk\, dt\, dn \tag{6}$$

Here we choose a Haar measure on K, N, T such that $\int_K dk = 1, \int_{N\cap K} dn = 1, \int_{T\cap K} dt = 1$. We have

$$\int_G f(g)f'(g)dg = \int_K \int_T \int_N f(tnk)f'(tnk)\, dk\, dt\, dn =$$
$$\int_K \int_T \int_N f(tn)\delta^{1/2}(t)\chi(t)\, dk\, dt\, dn = \int_T \chi(t)\delta^{1/2}(t) \int_N f(tn)\, dt\, dn =$$
$$\int_T \chi(t)\mathcal{S}f(t)\, dt = \mathcal{FS}(f)(\chi)$$

□

Denote the set of all the complex unramified character of T as $\Lambda(T)$. For $\chi \in X^\bullet(T)$ and $s \in \mathbb{C}$, we can define a complex character χ_s of T by $\chi_s(t) = |\chi(t)|^s$. This defines a map:

$$\Psi : X^\bullet(T) \otimes \mathbb{C} \to \Lambda(T)$$

Let $M = \{m \in X^\bullet(T) \otimes \mathbb{C} | \langle m, \alpha\rangle \in \mathbb{Z}, \forall \alpha \in X_\bullet(T)\}$, then the kernel of Ψ is given by $\frac{2\pi i}{\log q}M$. We have the following lemma.

Lemma 3.15. *The above map defines an isomorphism :*

$$\Psi' : X^\bullet(T) \otimes \mathbb{C} / \big(\frac{2\pi i}{\log q}M\big) \to \Lambda(T)$$

If we choose a basis of $X^\bullet(T)$ to be $\{\chi_1, \chi_2, ...; \chi_n\}$, then the map Ψ is defined as follows: if $s = \sum_i s_i\chi_i \in X^\bullet(T)\otimes\mathbb{C}$, for $s_i \in \mathbb{C}, i = 1, 2, .., n$, then $\Psi(s) \in \Lambda(T)$ is the character: $\Psi(s)(t) = \prod_i |\chi_i(t)|^{s_i}$.

Next we will calculate $\vartheta_\chi(f)$ for $\chi \in \Lambda(T)$ and $f \in \mathcal{H}(G, K)$. Let

$$
\begin{aligned}
s \in X^\bullet(T) \otimes \mathbb{C} \dashrightarrow \chi = \Psi(s) \in \Lambda(T) \\
\varpi \in \mathbb{C}[X_\bullet(T)]^W \dashrightarrow f = \mathcal{S}^{-1}\varpi \in \mathcal{H}(G, K)
\end{aligned}
$$

then by calculation, we have

$$\vartheta_\chi(f) = q^{-\langle\varpi,\rho\rangle}q^{-\langle\varpi,s\rangle}$$

Let $W.\left(\frac{2\pi i}{logq}M\right) = \hat{W}$ be the group of affine transformation of $X^\bullet(T) \otimes \mathbb{C}$ generated by the group W and the group of translations defined by $\frac{2\pi i}{logq}M$, this is a semi-product of these two groups, then the above argument shows that $\mathrm{Spec}(\mathcal{H}(G, K)) \cong (X^\bullet(T) \otimes \mathbb{C})/\hat{W}$, under the pairing:

$$
\begin{aligned}
\mathcal{H}(G, K) \times \{(X^\bullet(T) \otimes \mathbb{C})/\hat{W}\} \to \mathbb{C} \\
(f, \chi) \to \vartheta_\chi(f)
\end{aligned}
$$

3.3 Satake isomorphism for $O(W_1, \Phi_1)$ and $Sp(W_2, \Phi_2)$

In this section, we will calculate explicitly the Satake isomorphism for $O(W_1, \Phi_1)$ and $Sp(W_2, \Phi_2)$.

Fix a maximum torus T_1, a Borel subgroup B_1 of $G_1 = O(W_1, \Phi_1)$ as follows:

$$T_1 = \{\left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_{m/2} & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_{m/2}^{-1} \end{array}\right) |t_i \in F^*, i = 1, \cdots, m/2\}$$

$$B_1 = \{\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \cdot \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} | \text{ } A \text{ upper triangular and } B \text{ antisymmetric}\}$$

The unipotent radical N_1 of B_1 is

$$N_1 = \{\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \cdot \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} |A \text{ unipotent upper triangular and B antisymmetric}\}$$

Choose $\{\epsilon_i \in X^\bullet(T_1)|i = 1, 2, \cdots, m/2\}$ such that

$$\epsilon_i \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_n & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_n^{-1} \end{array}\right) = t_i$$

and $\{\hat{\epsilon}_i \in X_\bullet(T_1) | i = 1, 2, \cdots, m/2\}$ such that

$$\hat{\epsilon}_i(t) = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \cdot & & & & 0 & & \\ & & t & & & & & \\ & & & \cdot & & & & \\ \hline & & & & 1 & & & \\ & & 0 & & & \cdot & & \\ & & & & & & t^{-1} & \\ & & & & & & & \cdot \end{array}\right)$$

where t is in the $i \times i$ position. This gives the basis of $X^\bullet(T_1)$ and $X_\bullet(T_1)$, with the pairing $\langle \epsilon_i, \hat{\epsilon}_j \rangle = \delta_{ij}$. The root and coroot are given by

$$\begin{aligned}
&\Phi(G_1) = \{\pm\epsilon_i \pm \epsilon_j, 0 \leq i, j \leq m/2, \} \\
&\Phi^+(G_1) = \{\epsilon_i \pm \epsilon_j, 0 \leq i < j \leq m/2\} \\
&\check{\Phi}(G_1) = \{\pm\hat{\epsilon}_i \pm \hat{\epsilon}_j, 0 \leq i, j \leq m/2\} \\
&\check{\Phi}^+(G_1) = \{\hat{\epsilon}_i \pm \hat{\epsilon}_j, 0 \leq i < j \leq m/2\} \\
&\Delta(G_1) = \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{\frac{m}{2}-1} - \epsilon_{m/2}, \epsilon_{m/2-1} + \epsilon_{m/2}\} \\
&\check{\Delta}(G_1) = \{\hat{\epsilon}_1 - \hat{\epsilon}_2, \hat{\epsilon}_2 - \hat{\epsilon}_3, \cdots, \hat{\epsilon}_{m/2-1} - \hat{\epsilon}_{m/2}, \hat{\epsilon}_{m/2-1} + \hat{\epsilon}_{m/2}\}
\end{aligned}$$

The map between Φ and $\check{\Phi}$ is given by $\iota(\pm\epsilon_i \pm \epsilon_j) = \pm\hat{\epsilon}_i \pm \hat{\epsilon}_j$.
$2\rho = (m-2)\epsilon_1 + (m-4)\epsilon_2 + \cdots + 2\epsilon_{m/2-1}$.
The fundamental co-character is

$\{\varepsilon_1 = \hat{\epsilon}_1, \varepsilon_2 = \hat{\epsilon}_1 + \hat{\epsilon}_2, \cdots, \varepsilon_{(m/2-2)} = \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3 + \cdots \hat{\epsilon}_{(m/2-2)},$
$\varepsilon_{(m/2-1)} = 1/2(\hat{\epsilon}_1 + \hat{\epsilon}_2 + \cdots \hat{\epsilon}_{(m/2-1)} - \hat{\epsilon}_{m/2}), \varepsilon_m/2 = 1/2(\hat{\epsilon}_1 + \hat{\epsilon}_2 + \cdots \hat{\epsilon}_{(m/2-1)} + \hat{\epsilon}_{m/2})\}$

The Wely group $W(G_1)$ is generated by

$$\begin{aligned}
w_{ij} &: \epsilon_i \leftrightarrow \epsilon_j \\
w_i &: \epsilon_i \leftrightarrow -\epsilon_i
\end{aligned}$$

and it is isomorphic to $\mathcal{S}_{m/2} \rtimes (\mathbb{Z}/2\mathbb{Z})^{m/2}$

Remark 3.15.1. *The Weyl group of $SO(W_1, \Phi_1)$ contains w_{ij} and the even sign changes, it is isomorphic to $\mathcal{S}_{m/2} \rtimes (\mathbb{Z}/2\mathbb{Z})^{m/2-1}$, but $O(W_1, \Phi_1)$ is not connected, its Weyl group is twice the Weyl group of $SO(W_1, \Phi_1)$.*

Taking $\hat{\epsilon}_i$ as variable X_i, we have

$$\mathcal{H}(G_1, K_1)) \simeq \mathbb{C}[X_1, \ldots, X_{m/2}, X_1^{-1}, \ldots, X_{m/2}^{-1}]^{W(G_1)} \tag{7}$$

Fix a maximum torus T_2, a Borel subgroup $\subset B_2$ of $G_2 = Sp(W_2, \Phi_2)$ as follows:

$$T_2 = \{ \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_n & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_n^{-1} \end{array} \right) | t_i \in F^*, i = 1, \cdots, n \}$$

$$B_2 = \{ \left(\begin{array}{cc} A & 0 \\ 0 & (A^t)^{-1} \end{array} \right) \cdot \left(\begin{array}{cc} I & B \\ 0 & I \end{array} \right) | A \ upper \ triangular \ and \ B \ symmetric \}$$

The unipotent radical N_2 of B_2 is

$$N_2 = \{ \left[\begin{array}{cc} A & 0 \\ 0 & (A^t)^{-1} \end{array} \right] \cdot \left(\begin{array}{cc} I & B \\ 0 & I \end{array} \right) | A \ unipotent \ upper \ triangular \ and \ B \ symmetric \}$$

Choose $\{\epsilon_i \in X^\bullet(T_2) | i = 1, 2, .., n\}$ such that

$$\epsilon_i \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_n & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_n^{-1} \end{array} \right) = t_i$$

and $\{\hat{\epsilon}_i \in X_\bullet(T_2) | i = 1, 2, .., n\}$ such that

$$\hat{\epsilon}_i(t) = \left(\begin{array}{cccc|cccc} 1 & & & & & & & \\ & \cdot & & & & 0 & & \\ & & t & & & & & \\ & & & \cdot & & & & \\ \hline & & & & 1 & & & \\ & & 0 & & & \cdot & & \\ & & & & & & t^{-1} & \\ & & & & & & & \cdot \end{array} \right)$$

where t is in the $i \times i$ position. This gives a basis of $X^\bullet(T_2)$ and $X_\bullet(T_2)$ with the pairing $\langle \epsilon_i, \hat{\epsilon}_j \rangle = \delta_{ij}$. The roots and coroots are given by

$$\begin{aligned} \Phi(G_2) &= \{\pm\epsilon_i \pm \epsilon_j, \pm 2\epsilon_i, 0 \le i, j \le n, \} \\ \Phi^+(G_2) &= \{\epsilon_i \pm \epsilon_j, 2\epsilon_i, 0 \le i < j \le n\} \\ \check{\Phi}(G_2) &= \{\pm\hat{\epsilon}_i \pm \hat{\epsilon}_j, \pm\hat{\epsilon}_i, 0 \le i, j \le n\} \\ \check{\Phi}^+(G_2) &= \{\hat{\epsilon}_i \pm \hat{\epsilon}_j, \hat{\epsilon}_i, 0 \le i < j \le n\} \\ \Delta(G_2) &= \{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \cdots, \epsilon_{n-1} - \epsilon_n, 2\epsilon_n\} \\ \check{\Delta}(G_2) &= \{\hat{\epsilon}_1 - \hat{\epsilon}_2, \hat{\epsilon}_2 - \hat{\epsilon}_3, \cdots, \hat{\epsilon}_{n-1} - \hat{\epsilon}_n, \hat{\epsilon}_n\} \end{aligned}$$

The map ι between Φ and $\check{\Phi}$ is given by $\iota(\pm\epsilon_i \pm \epsilon_j) = (\pm\hat{\epsilon}_i \pm \hat{\epsilon}_j)$ and $\iota(\pm 2\epsilon_i) = \pm\hat{\epsilon}_n$.
$2\rho = 2n\epsilon_1 + 2(n-1)\epsilon_2 +, \cdots, +4\epsilon_{(n-1)} + 2\epsilon_n$.
The fundamental co-character is

$$\{\varepsilon_1 = \hat{\epsilon}_1, \varepsilon_2 = \hat{\epsilon}_1 + \hat{\epsilon}_2, \cdots \varepsilon_{n-1} = \hat{\epsilon}_1 + \hat{\epsilon}_2 + \hat{\epsilon}_3 + \cdots + \hat{\epsilon}_{n-1}, \varepsilon_n = 1/2(\hat{\epsilon}_1 + \hat{\epsilon}_2 + \cdots + \hat{\epsilon}_n)\}$$

The Wely group $W(G_2)$ is generated by

$$w_{ij} : \epsilon_i \leftrightarrow \epsilon_j$$
$$w_i : \epsilon_i \leftrightarrow -\epsilon_i$$

and it is isomorphic to $\mathcal{S}_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n$.

Taking $\hat{\epsilon}_i$ as variable X'_i, we have

$$\mathcal{H}(G_2, K_2) \simeq \mathbb{C}[X'_1, \ldots, X'_n, (X'_1)^{-1}, \ldots, (X_n)'^{-1}]^{W(G_2)} \tag{8}$$

3.4 Statement of the local theta correspondence

We will study the Howe duality conjecture for the spherical representations in this subsection.

Definition 3.16. *Let $H \subseteq$ be a closed subgroup of $\widetilde{Sp}(W)$. Define $\mathcal{R}^s(H)$ to be the set of σ such that σ is an irreducible spherical representation of H, and there exists a H-nontrivial intertwining map: $\alpha : S(V_1) \to \sigma$.*

Recall that the spherical representation of G_1 and G_2 is parameterized by the spectrum of $\mathcal{H}(G_1, K_1)$ and $H(G_2, K_2)$. We can identify $\mathcal{R}^s(G_1)$ as a subset inside $\mathrm{Spec}(\mathcal{H}(G_1, K_1))$ and $\mathcal{R}^s(G_2)$ as a subset inside $\mathrm{Spec}(\mathcal{H}(G_1, K_1))$. As the spherical representation of $G_1 \times G_2$ is the tensor product of spherical representation of G_1 and G_2, we can identify $\mathcal{R}^s(G_1 \times G_2)$ as a subset inside $\mathrm{Spec}(\mathcal{H}(G_1, K_1)) \times \mathrm{Spec}(\mathcal{H}(G_2, K_2))$.

We will prove the following theorem

Theorem 3.17 (The Main Theorem)**.** *$\mathcal{R}^s(O(W_1, \Phi_1) \times Sp(W_2, \Phi_2))$ defines a graph of bijection between $R^s(O(W_1, \Phi_1))$ and $\mathcal{R}^s(Sp(W_2, \Phi_2))$. Under the isomorphism given by Equation 7 and 8, the bijection is given by a closed immersion β from Spec$(\mathcal{H}(G_2, K_2))$ to Spec$(\mathcal{H}(G_1, K_1))$ as follows:*

$$\begin{cases} X_1 & \to & X'_1 \\ X_2 & \to & X'_2 \\ \vdots & & \vdots \\ X_n & \to & X'_n \\ X_{n+1} & \to & p^{m/2-n-1} \\ X_{n+2} & \to & p^{m/2-n-2} \\ \vdots & & \vdots \\ X_{m/2} & \to & p^0 \end{cases}$$

So, if $\chi \in Spec(\mathcal{H}(G_1, K_1))$ and $\mu \in Spec(\mathcal{H}(G_2, K_2))$ such that $I_1(\chi) \otimes I_2(\mu)$ is a sub-quotient of $S(\boldsymbol{M}_{m\times n}(F))$, then $\chi = \mu \circ \phi$.

Remark 3.17.1. *Rigorously saying, β defines a surjective map*

$$\mathbb{C}[X_1, \cdots, X_{m/2}, X_1^{-1}, \cdots, X_{m/2}^{-1}] \to \mathbb{C}[X'_1, \cdots, X'_n, (X'_1)^{-1}, \cdots, (X'_n)^{-1}]$$

and restricting it to $\mathbb{C}[X_1, \cdots, X_{m/2}, X_1^{-1}, \cdots, X_{m/2}^{-1}]^{W_1}$ gives the surjection map

$$\mathbb{C}[X_1, \cdots, X_{m/2}, X_1^{-1}, \cdots, X_{m/2}^{-1}]^{W_1} \to \mathbb{C}[X'_1, \cdots, X'_n, (X'_1)^{-1}, \cdots, (X'_n)^{-1}]^{W_2}$$

3.5 Proof of the local theta correspondence

3.5.1 Construct the intertwining operator

We begin to prove the main theorem in this section.

We firstly construct the intertwining operator from $S(\mathbf{M}_{mn}(F))$ to the unramified principle series of $G_1 \times G_2$.

Fix the basis of $X^\bullet(T_1)$ introduced in section 3.3, we have an isomorphism:

$$\mathbb{C}^{m/2} \cong X^\bullet(T_1) \otimes \mathbb{C}$$

$$\mu = (s_1, s_2, ...s_{(m/2)}) \mapsto \sum_{i=1}^{m/2} s_i \epsilon_i$$

where $\sum_{i=1}^{m/2} s_i \epsilon_i$ is the character of T_1 defined as follows

$$x_1 = \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_{m/2} & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_{m/2}^{-1} \end{array}\right) \longrightarrow \prod |t_i|^{s_i}$$

Then we can define $(\mathrm{Ind}_{B_1}^{O(W_1,\Phi_1)}(\delta_1^{1/2}\mu), I_1(\mu))$ as the induced representation from $\delta_1^{1/2}\mu$, where $\delta_1(t_1) = |t_1|^{m-2}|t_2|^{m-4}....|t_{m/2}|^0$ is the modular character with respect to B_1

Fix the basis of $X^\bullet(T_2)$ introduced in section 3.3, we have an isomorphism:

$$\mathbb{C}^n \cong X^\bullet(T_2) \otimes \mathbb{C}$$

$$\chi = (s'_1, s'_2, .., s'_n) \mapsto \sum_{i=1}^{n} s'_i \epsilon'_i$$

where $\sum_{i=1}^{n} s_i'\epsilon_i'$ is the character of T_2 defined as follows:

$$x_2 = \left[\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_n & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_n^{-1} \end{array}\right] \longrightarrow \prod |t_i|^{s_i'}$$

We also can define $(\mathrm{Ind}_{B_2}^{Sp(W_2,\Phi_2)}(\delta_2^{1/2}\chi), I_2(\chi))$ to be the induced representation from $\delta_2^{1/2}\chi$, here $\delta_2(t_2) = |t_1|^{2n}|t_2|^{2n-2}....|t_n|^2$ is the modular character with respect to B_2.

Let μ and χ be two unramified characters of T_1 and T_2 given above, and $I_1(\chi)$ and $I_2(\mu)$ be the corresponding induced representations. By Frobenius reciprocity.

$$\begin{aligned} &\mathrm{Hom}_{O(W_1,\Phi_1)\times Sp(W_2,\Phi_2)}(\mathcal{S}(\mathbf{M}_{mn}(F)), I_1(\mu)\otimes I_2\chi) \\ &\cong \mathrm{Hom}_{B_1\times B_2}(\mathcal{S}(\mathbf{M}_{mn}(F)), \delta_1^{1/2}\mu\otimes\delta_2^{1/2}\chi) \end{aligned}$$

To construct the intertwining operator from ρ to the principle series, we just need to construct some densities on $\mathbf{M}_{mn}(F)$ lying in the space:

$$\mathrm{Hom}_{B_1\times B_2}(\mathcal{S}(\mathbf{M}_{mn}(F)), \delta_1^{1/2}\mu\otimes\delta_2^{1/2}\chi)$$

Let Y_n be the set of $n\times n$ upper triangular matrixes,

$$Y_n = \{D(a_{11},...,a_{nn})U(z_{ij}) | a_{ii}\in F^*, z_{ij}\in F^* \ i<j\}$$

where

$$D(a_{11},...,a_{nn}) = \begin{pmatrix} a_{11} & & 0 \\ & \cdot & \\ & & \cdot \\ 0 & & a_{nn} \end{pmatrix} \ \textit{and}\ U(z_{ij}) = \begin{pmatrix} 1 & & z_{ij} \\ & \cdot & \\ & & \cdot \\ 0 & & 1 \end{pmatrix} (n\times n\ \textit{matrix})$$

A right Haar measure of Y_n can be defined by

$$\int_{Y_n} f(Y_n)d_rY_n = \int_D\int_U f(U(z_{ij})D(a_{11},...,a_{nn}))\prod d^\times a_{ii}\prod dz_{ij}$$

For $\sigma = \{(\sigma_1,\sigma_2,...,\sigma_n) | \sigma_i\in\mathbb{C}\ \}$ we can define a map Z_σ

$$\begin{aligned} Z_\sigma :&S(M_{m\times n}(F)) \longrightarrow \mathbb{C} \\ &f \longrightarrow \int_{Y_n} f\begin{pmatrix} Y_n \\ 0 \end{pmatrix}\prod |a_{ii}|^{\sigma_i} d_rY_n \end{aligned}$$

Theorem 3.18. *For σ such that $Re(\sigma_i) \gg 0, \forall i$, the map Z_σ converges for all f and defines a map in*

$$Hom_{B_1 \times B_2}(\mathcal{S}(\boldsymbol{M}_{m\times n}(F)), \delta_1^{1/2}\mu \otimes \delta_2^{1/2}\chi)$$

where

$$\begin{aligned}\mu =&(\sigma_1 - m/2 + n, \sigma_2 - m/2 + n - 1, \sigma_3 - m/2 + n - 2, ..., \sigma_n - m/2 + 1,\\ &n + 1 - m/2, n + 2 - m/2, .., 1, 0)\\ \chi =&(-\sigma_1 + m/2 - n, -\sigma_2 + m/2 - n + 1, -\sigma_3 + m/2 - n + 2, ..., -\sigma_n + m/2 - 1)\end{aligned}$$

hence by Frobenious reciprocity, the map

$$f \longrightarrow \{(g_1, g_2) \to Z_\sigma(\rho(g_1, g_2)(f))\}$$

defines a $O(W_1, \Phi_1) \times Sp(W_2, \Phi_2)$ intertwining map from $S(\boldsymbol{M}_{m\times n}(F))$ to $I_1(\mu) \otimes I_2(\chi)$

Proof. We first check that $N_1 \times N_2$ acts trivially on Z_σ. As N_1 is given by

$$N_1 = \{\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \cdot \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} |A \text{ unipotent upper triangular and B antisymmetric}\}$$

For

$$n_1 = \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}.$$

we have

$$(n_1)^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

where A_1 is a $n \times n$ unipotent upper triangular matrix, and A_3 is a $(m-n) \times (m-n)$ matrix.

$$\begin{aligned} Z_\sigma(\rho(n_1, 1)(f)) = \int_{Y_n} f\Big(\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} . \begin{bmatrix} \frac{Y_n}{0} \end{bmatrix}\Big) \\ \prod |a_{ii}|^{\sigma_i} d_r Y_n = \int_{Y_n} f\Big(\begin{bmatrix} \frac{A_1.Y_n}{0} \end{bmatrix}\Big) \prod |a_{ii}|^{\sigma_i} d_r Y_n = Z_\sigma(f) \end{aligned}$$

the last equality comes from the fact that $A_1 \in U(z_{ij})$. For

$$n_1 = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

write

$$(n_1)^{-1} = \begin{pmatrix} I & B_1 \\ 0 & B_2 \end{pmatrix}$$

where B_1 is $n \times (m-n)$ matrix and B_2 is $(m-n)\times(m-n)$ matrix, then

$$Z_\sigma(\rho(n_1,1)(f)) = \int_{Y_n} f\left(\begin{bmatrix} I & B_1 \\ 0 & B_2 \end{bmatrix} \cdot \begin{bmatrix} Y_n \\ \hline 0 \end{bmatrix} \right)$$

$$\prod |a_{ii}|^{\sigma_i} d_r Y_n = \int_{Y_n} f\left(\begin{bmatrix} Y_n \\ \hline 0 \end{bmatrix} \right) \prod |a_{ii}|^{\sigma_i} d_r Y_n = Z_\sigma(f)$$

For N_2 the proof is the same.

We then calculate the action of $T_1 \times T_2$ on $Z_\sigma(f)$: for

$$t_1 = \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_{m/2} & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_{m/2}^{-1} \end{array} \right)$$

$$Z_\sigma(\rho(t_1,1)(f)) = \int_{Y_n} f\left(\left[\begin{array}{ccc|ccc} t_1^{-1} & & & & & \\ & \cdot & & & 0 & \\ & & t_{m/2}^{-1} & & & \\ \hline & & & t_1 & & \\ & 0 & & & \cdot & \\ & & & & & t_{m/2} \end{array} \right] \cdot \begin{bmatrix} Y_n \\ \hline 0 \end{bmatrix} \right)$$

$$\prod |a_{ii}|^{\sigma_i} d_r Y_n = \int_{Y_n} f\left(\begin{array}{c} \begin{bmatrix} t_1^{-1} & & & \\ & t_2^{-1} & & \\ & & \cdot & \\ & & & t_n^{-1} \end{bmatrix} \cdot Y_n \\ \hline 0 \end{array} \right) \prod |a_{ii}|^{\sigma_i} d_r Y_n =$$

$$\prod_{i=1}^{i=n} |t_i|^{n+1-2i} \prod_{i=1}^{i=n} |t_i|^{\sigma_i} Z_\sigma(f) = \prod_{i=1}^{i=n} |t_i|^{n+1-2i+\sigma_i} Z_\sigma(f)$$

and for

$$t_2 = \left(\begin{array}{ccc|ccc} t_1 & & & & & \\ & \cdot & & & 0 & \\ & & t_n & & & \\ \hline & & & t_1^{-1} & & \\ & 0 & & & \cdot & \\ & & & & & t_n^{-1} \end{array} \right) \in T_2$$

$$Z_\sigma(\rho(1,t_2)(f)) = \int_{Y_n} \prod_{i=1}^{i=n} |t_i|^{m/2} f\left(\left[\frac{Y_n}{0} \right] \begin{bmatrix} t_1 & & \\ & \cdot & \\ & & t_n \end{bmatrix} \right)$$

$$\prod |a_{ii}|^{\sigma_i} d_r Y_n = \int_{Y_n} \prod_{i=1}^{i=n} |t_i|^{m/2} \prod_{i=1}^{i=n} |t_i|^{-\sigma_i} f\left(\left[\frac{Y_n}{0} \right] \right) \prod |a_{ii}|^{\sigma_i} d_r Y_n =$$

$$\prod_{i=1}^{i=n} |t_i|^{m/2-\sigma_i} Z_\sigma(f)$$

We get

$$\mu \otimes \delta_1^{1/2}(t) = \prod_{i=1}^{i=n} |t_i|^{(n+1-2i+\sigma_i)}$$

$$\chi \otimes \delta_2^{1/2}(t') = \prod_{i=1}^{i=n} |t_i|^{(m/2-\sigma_i)}$$

this implies

$$\mu(t) = \prod_{i=1}^{i=n} |t_i|^{(\sigma_i+n-m/2-i)} . \prod_{j=(n+1)}^{m/2} |t_j|^{j-m/2}$$

$$\chi(t) = \prod_{i=1}^{i=n} |t_i|^{(-\sigma_i-n+m/2+i)}$$

As μ and $-\mu$ lies in the same orbit upon the action of $\widetilde{W}_1$ on $X^\bullet(T_1)\otimes C$. Replace μ by $-\mu$, we get the following commutative diagram:

$$\begin{array}{ccc} X^\bullet(T_2)\otimes\mathbb{C} & \xrightarrow{\beta'} & X^\bullet(T_1)\otimes\mathbb{C} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} \\ X^\bullet(T_2)\otimes\mathbb{C}/\widetilde{W}_2 & \xrightarrow{\hat{\beta}} & X^\bullet(T_1)\otimes\mathbb{C}/\widetilde{W}_1 \end{array}$$

where β' is defined by $\beta'(s_1,\cdots,s_n) = (s_1,\cdots,s_n, m/2-n-1, m/2-n-2,\cdots,1,0)$. The corresponding map from $\mathcal{H}(G_1,K_1)$ to $\mathcal{H}(G_1,K_1)$ is then given by β:

$$\begin{cases} X_1 & \to & X_1' \\ X_2 & \to & X_2' \\ \vdots & & \vdots \\ X_n & \to & X_n' \\ X_{n+1} & \to & p^{m/2-n-1} \\ X_{n+2} & \to & p^{m/2-n-2} \\ \vdots & & \vdots \\ X_{m/2} & \to & p^0 \end{cases}$$

$\square$

Next we show that this map gives all the intertwining operators from $S(M_{mn}(F))$ to the spherical representations $I^1(\psi_1) \otimes I^2(\psi_2)$.

Lemma 3.19. *If $Z_\sigma(f)$ is injective on the $K_1 \times K_2$ invariants on $S(\boldsymbol{M}_{mn}(F))$,i.e if $f \in S(M_{mn}(F))^{K_1 \times K_2}$, $Z_\sigma(f) = 0$ for every σ such that $Re(\sigma_i) \gg 0, \forall i$ implies $f = 0$. Then Z_σ gives all the intertwining operator from $S(M_{mn}(F))$ to the spherical representation $I^1(\psi_1) \otimes I^2(\psi_2)$. i.e. if there is an intertwining operator from $S(M_{mn}(F))$ to the spherical reprsentation $I^1(\psi_1) \otimes I^2(\psi_2)$, then $\psi_1 = \hat{\beta}(\psi_2)$ or $\psi_1 = \psi_2 \circ \beta$*

Proof. Define the idea I in $\mathcal{H}(G_1, K_1) \otimes \mathcal{H}(G_2, K_2)$ which is generated by $g \otimes 1 - 1 \otimes \beta(g)$ for every $g \in \mathcal{H}(G_1, K_1)$, then $\operatorname{Spec}(I)$ is the graph of the map $\hat{\beta}$. Observe that for $f \in S(M_{mn}(F))^{K_1 \times K_2}$,$\gamma \in I$ and $\sigma \in \mathbb{C}^n$ such that $Z_\sigma(f)$ converges, $Z_\sigma(\rho(\gamma)(f)) = \rho(\gamma)(Z_\sigma(f)) = 0$. The last equality comes from the fact that $Z_\sigma(f)$ is a spherical vector in $I_\sigma(\mu) \otimes I_\sigma(\chi)$ and it is killed by I. By the assumption that Z_σ is injective, we get $\gamma(f) = 0$. Let

$$\delta : S(\mathbf{M}_{mn}(F)) \to I^1(\psi_1) \otimes I^2(\psi_2)$$

As taking the $K_1 \times K_2$ invariants is an exact functor, we have

$$\delta : S(\mathbf{M}_{mn}(F))^{K_1 \times K_2} \to I^1(\psi_1)^{K_1} \otimes I^2(\psi_2)^{K_2}$$

is a surjective $\mathcal{H}(G_1, K_1) \otimes \mathcal{H}(G_2, K_2)$ module morphism. So $\gamma(I_1(\psi_1)^{K_1} \otimes I_2(\psi_2)^{K_2}) = 0$ for every $\gamma \in I$. This implies $(\psi_2, \psi_1) \in \operatorname{Spec}(I)$, so $\psi_1 = \hat{\beta}(\psi_2)$. $\square$

3.5.2 Jacquet module

For a reductive group G and a parabolic subgroup $P = MN$, we can form a representation of G by parabolic induction from P, i.e. for an admissible representation (ρ, W) of M, we can extend it to P by letting N act trivially on

W, denote this new representation by (ρ', W). We then construct an induced representation $\left(\mathrm{Ind}_P^G W, I(\rho')\right)$ from ρ', where

$$I(\rho') = \{f : G \to V \text{ smooth} | f(pg) = \rho'(p) f(g), \forall h \in P\}$$

and G acts on $I(\rho')$ by the right translation. By Frobenious reciprocity, we have

$$\mathrm{Hom}_G(V', \mathrm{Ind}_P^G V) \cong \mathrm{Hom}_P(V'|_P, V)$$

for V' an admissible representation of G and V a representation extended from M to P. As we know that N acts trivially on V,

$$\mathrm{Hom}_P(V'|_P, V) \cong \mathrm{Hom}_M(V'_N, V)$$

where V'_N is the coinvariant of N. More explicitly, let $V'[N]$ be the subspace generated by $\rho(n)v - v$ for all $n \in N$, then $V'_N = V'/V'[N]$. Hence the structure of V'_N as a M module determines the embedding of V' into the induced representation $\mathrm{Ind}_P^G V$.

Definition 3.20. *For a representation (ρ, V) of G, and P, M, N be in above, the space $V_N = V/V[N]$ is a $M \cong P/N$ module. Denote the representation of M on V_N as ρ_N, then (ρ_N, V_N) is called the Jacquet module of (ρ, V) associated to P*

Remark 3.20.1. *the map $V \to V_N$ defines a functor from the representation of G to the representation of M.*

Lemma 3.21. *The Jacquet functor is exact, i.e. if*

$$0 \to U \to V \to W \to 0$$

is exact as G modules, then

$$0 \to U_N \to V_N \to W_N \to 0$$

is exact as M modules.

Proof. It is standard fact that

$$U_N \to V_N \to W_N \to 0$$

is exact. We will proof $U_N \to V_N$ is injective, this comes from the following lemma about another criterion for $V[N]$, which implies $U[N] = U \cap V[N]$. $\square$

Remark 3.21.1. *In cohomology language, the lemma says that $H_1(N, V) = 0$.*

Assume for the moment N to be any locally compact group such that the compact open subgroup of N forms a basis of the neighbourhood of identity , and possessing arbitrarily large compact subgroups as well. This means that if X is any compact subset of N, then there exists a compact open subgroup N_0 containing X. This condition is satisfied in our case where N is the unipotent radical of P.

Lemma 3.22. *Suppose N satisfies the condition above. For a compact open subgroup $N_0 \subseteq N$, let $V'(N_0)$ to be the subspace containing the element $\{v \in V | \int_{N_0} \rho(n)v\, dn\}$. Define $V'(N)$ to be $\bigcup V(N_0)$, the union over all compact open groups of N. Then $V'(N) = V[N]$.*

Proof. We first prove $v \in V[N]$ implies $v \in V'(N)$,. As v is a finite combination of element of the form $\rho((n_0)v') - v'$. We just need to prove that element of the form $(\rho(n_0)v' - v')$ is in $V'(N)$. Take a compact open subgroup N_0 contain n_0 (this is guaranteed by the assumption). Then $\int_{N_0} \rho(n)(\rho(n_0)v' - v')\,\mathrm{d}n = \int_{N_0} \rho(n)v' - \int_{N_0} \rho(n)v' = 0$.

For the opposite direction, suppose $v \in V'(N)$, which means $\int_{N_0} \rho(n)v\mathrm{d}n = 0$ for a N_0 which is open compact in N. As V is smooth, there exist a compact open subgroup $N_1 \subset N$ such that $v \in V^{N_1}$. Assume $N_1 \subset N_0$ (We can always choose such N_1), then

$$\int_{N_0} \rho(n)v\,\mathrm{d}n = c * \sum_{N_0/N_1} \rho(n)v = 0$$

so

$$v = d * \sum_{N_0/N_1} (\rho(n)v - v)$$

here c equals the measure of N_1 and d equals to $-1/d'$ where d' is the number of element in N_0/N_1. □

3.5.3 Some invariant theory

We present some results coming form classical invariant theory, and this result will be used later.

We first give a list of maximum parabolic groups inside $\mathrm{Sp}(W_2)$. Recall that we have $W_2 = V_2 \oplus V_2^*$ for V_2, V_2^* be two traversal lagrangians of W, let $V_2^* = (V_2^*)^0 \supseteq (V_2^*)^1 \supseteq (V_2^*)^2 \cdots \supseteq (V_2^*)^{n-1} \supseteq (V_2^*)^n = (0)$ be a flag of codimension one subspaces. Denote

$$(P_2)_k = \{g \in Sp(W_2) | g \text{ stable } (V_2^*)^k\}$$

Up to conjugacy, $\{(P_2)_k, k = 0, 1 \cdots n-1\}$ gives all the maximum parabolic subgroups of $Sp(W_2)$. The nilpotent radical of $(P_2)_k$ is given by

$$(N_2)_k = \{g \in Sp(W_2) | g \text{ fix pointwise } (V_2^*)^k, {(V_2^*)^k}^\perp/(V_2^*)^k \text{ and } W_2/{(V_2^*)^k}^\perp\}$$

and the Levi factor of $(P_2)_k$ is isomorphic to

$$\mathrm{GL}((V_2^*)^k) \times \mathrm{Sp}({(V_2^*)^k}^\perp/(V_2^*)^k, \Phi_2)$$

Here $\perp$ means the perpendicular complement with respect to Φ_2, the symplectic form on W_2 naturally gives a symplectic form on ${(V_2^*)^k}^{\perp}/(V_2^*)^k$, which we also denote as Φ_2 in above.

Under the basis $\{f_1, f_2, .., f_n, f_1^*, f_2^*, .., f_n^*\}$ of W_2 given in the subsection 2.3.2, taking $(V_2^*)^k = \{f_{k+1}^*, f_{k+2}^*, \cdots, f_n^*\}$, we can write elements in $(N_2)_k$ as

$$\left\{ \left[\begin{array}{cc|cc} I & A & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & B & I & 0 \\ B^t & C & -A^t & I \end{array} \right] \in \mathrm{Sp}_n(F) \right\} \tag{9}$$

where A is a $k \times (n-k)$ matrix. Also the Levi factor of $(P_2)_k$ is given by

$$\left\{ \left[\begin{array}{cc|cc} A & 0 & 0 & 0 \\ 0 & B & C & 0 \\ \hline 0 & D & E & 0 \\ 0 & 0 & 0 & (A^t)^{-1} \end{array} \right] \in \mathrm{Sp}_n(F) \right\}$$

where $A \in \mathrm{GL}_k(F)$ and $\begin{bmatrix} B & C \\ D & E \end{bmatrix} \in \mathrm{Sp}_{n-k}(F)$

Consider the following map

$$\begin{aligned} \Psi : M_{mn}(F) &\to \mathrm{Sym}_{nn}(F) \\ X &\mapsto X^T A_{\Phi_1} X \end{aligned}$$

where $\mathrm{Sym}_{nn}(F)$ is the set of $n \times n$ symmetric matrix over F. It is easy to check that this map is invariant by the action of $O(W_1, \Phi_1)$, so this defines a map

$$\begin{aligned} \Psi^* : S(\mathrm{Sym}_{nn}(F)) &\longrightarrow S(\mathbf{M}_{mn}(F))^{G_1} \\ f &\longrightarrow \delta^*(f) : X \mapsto f(X^T A_{\Phi_1} X) \end{aligned}$$

Remark 3.22.1. *(i) The geometry meaning of the map Ψ is given as follows: recall that after taking a basis, we have the isomorphism $\boldsymbol{M}_{mn}(F) \cong W_1 \otimes V_2 \cong (W_1)^n$. Under this isomorphism, $X \in M_{mn}(F)$ can be represent by $[\xi_1, \xi_2, ...\xi_n]$, where $\xi_i \in W_1$. Then $X^T A_{\Phi_1} X$ is the Gram matrix of these n vectors, i.e. let $Y = X^T A_{\Phi_1} X$, then $Y_{ij} =< \xi_i, \xi_j >_{\Phi_1}$. In the following context, we sometimes refer $X \in \boldsymbol{M}_{mn}(F)$ as $[\xi_1, \xi_2, ...\xi_n], \xi_i \in W_1$ without mention the isomorphic above, if this will not cause ambiguity.*

(ii) The map Ψ^* intertwines the action of $GL_n(F)$ on both sides, where the actions of $GL_n(F)$ are given by the post multiplication.

Definition 3.23. *For $t = 1, 2, \cdots, n$, define the characteristic variety Λ_t of the* *parabolic $(P_2)_t$ to be the set:*

$$\Psi^{-1}\left\{\left(\begin{array}{c|c} X & 0 \\ \hline 0 & 0 \end{array}\right) |X \text{ an arbitrary } t\times t \text{ symmetric matrix}\right\}$$
$$= \left\{[\xi_1|\cdots,|\xi_n]|\langle \xi_i, \xi_j\rangle_{\Phi_1} = 0 \text{ for} \left\{\begin{array}{l} (n-t-1)\leq i\leq n \\ (n-t-1)\leq j\leq n \end{array}\right.\right.$$
$$\text{and } \langle \xi_i, \xi_j\rangle_{\Phi_1} = 0 \text{ for } \left.\left\{\begin{array}{l} 1\leq i\leq n-t-1 \\ 1\leq j\leq n-t-1 \end{array}\right.\right\}$$

Lemma 3.24. *The space $\{f \in S(M_{mn}(F))|f$ vanish on Λ_t $\}$ coincides with the Jacquet space $S(M_{mn}(F))[(N_2)_t \cap (N_2)_0]$, i.e. the vector space spanned by $\rho(n_2)f - f$ for all $n_2 \in (N_2)_t \cap (N_2)_0$ and $f \in S(M_{mn}(F))$*

Proof. We prove this by two steps.

In step 1 we prove a lemma about the property of the Jacquet space $S(M_{mn}(F))[(N_2)_t \cap (N_2)_0]$.

Lemma 3.25. *For $f \in S(M_{mn}(F)), f \in S(M_{mn}(F))[(N_2)_t \cap (N_2)_0]$ if and only if there exist a compact open subgroup $K \subseteq (N_2)_t \cap (N_2)_0$ such that*

$$\int_K \rho(k)f\, dk = 0$$

Proof. The "only if" part deduced from direct calculation. We prove the "if" part: if there exist a open compact group $K \subset (N_2)_t \cap (N_2)_0$ such that

$$\int_K \rho(k)f \mathrm{d}\, k = 0$$

then f is span of the function $\rho(k)g - g$ for $k \in K$ and $g \in S(M_{mn}(F))$. This comes from the property that K is compact: Consider $V = S(M_{mn}(F))$ as a representation space of K. As K is compact, the invariant and coinvariant of K coincide, i.e. $V = V^K \oplus V_K$ where V^K is the K invariant of V, and V_K is spanned by $\rho(k)(f) - f$ for $k \in K$ and $f \in S(M_{mn}(F)$. And

$$\phi = \int_K \mathrm{d}\, k : f \mapsto \int_K \rho(k)f\mathrm{d}\ k$$

is an project operator from V to V^K, so $f \in \ker(\phi)$ implies $f \in V_K$ □

Step 2: With the above lemma, we just need to prove that for $f \in S(M_{mn}(F))$ such that f vanishing on Λ_t, there exists a compact open subgroup $K \subset (N_2)_t \cap (N_2)_0$ such that $\int_K \rho(k)f\, dk = 0$. For $X \in M_{mn}(F)$,

$$(\phi_K f)(X) = \int_K \rho(M)f(X)dk = \int_K \psi(\mathrm{Trace}(M'X^tA_{\Phi_1}X))f(X)dk$$

Here $M \in K \subseteq (N_2)_t \cap (N_2)_0$ *is of the form* $\begin{bmatrix} I & 0 \\ M' & I \end{bmatrix}$ *and* $M' = \begin{bmatrix} 0 & B \\ B^t & C \end{bmatrix}$ *by Formula* 9.

Note that for $X \in M_{mn}(F)$

$$\eta_X : (N_2)_t \cap (N_2)_0 \to \mathbb{C}^* \tag{10}$$

$$M \mapsto \psi(\mathrm{Trace}(M' X^A_{\Phi_1} X)) \tag{11}$$

defines a character of $(N_2)_t \cap (N_2)_0$ *as an addictive group.*

If $X \in \Lambda_t$

$$M' X^A_{\Phi_1} X = \left[\begin{array}{c|c} 0 & B \\ \hline B^t & C \end{array}\right] \cdot \left[\begin{array}{c|c} Y & 0 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

η_X *is trivial on* $(N_2)_t \cap (N_2)_0$. *If* $X \notin \Lambda_t$, *then* η_X *is a non-trivial on* $(N_2)_t \cap (N_2)_0$.

For $X \in \Lambda_t$, $f(X) = 0$, *so* $\phi_K f(X) = 0$. *For* $X \notin \Lambda_t$, *choose a* $K_X \subset (N_2)_t \cap (N_2)_0$ *such that* η_X *is not trivial on* K_X. *Then*

$$\int_{K_X} \psi(\mathrm{Trace}(M' X^t A_{\Phi_1} X)) f(X)\, dk = f(X) \int_{K_X} \psi(\mathrm{Trace}(M' X^t A_{\Phi_1} X))\, dk = 0$$

The last equality comes from the fact that the average of a nontrivial character on a compact group is 0. *Also, as the character is a continuous map, we can find a neighbourhood* U_X *of* X *such that for* $X' \in U_X$, $\eta_{X'}$ *defines a nontrivial character of* K_X. *By the fact that the support of* f *is compact, we can find a* K *such that for* $X \in supp(f)$ *and* $X \notin \Lambda_t$, η_X *is non-trivial on* K, *so* $(\phi_K f)(X) = 0$

□

We then prove a property of the space $S(M_{mn}(F))^{K_1 \times K_2}$.

Lemma 3.26. *let* $f \in S(M_{mn}(F))^{K_1 \times K_2}$ *and suppose that for each* $g_2 \in Sp(W_2, \Phi_2)$, $\rho(g_2) f$ *vanishes on* Λ_0. *Then* $f = 0$

Proof. We prove this by induction on n: For $n = 1$, up to conjugacy, there is only one maximum parabolic subgroup of $\mathrm{Sp}_1(F) \cong \mathrm{SL}_2(F)$: the Borel subgroup. As $\rho(g_2) f$ vanishes on Λ_0, by Lemma 3.24, $\pi(G_2)(f) \in S(M_{mn}(F))[N_2]$, so the Jacquet module $\pi(G_2)(f)$ is zero. Combine the fact that $\pi(G_2)(f)$ is a cyclic module generated by f, $\pi(G_2)(f)$ is cuspidal as a representation of G_2, but we know that a cuspidal representation of G_2 can not have a K_2 fixed vector, while $f \in \pi(G_2)(f)$ is. So $\pi(G_2) f$ is identically 0 on $M_{mn}(F)$.

We assume that it has been proved for $k \leq n-1$. For $X = [\xi_1, \cdots, \xi_t, \cdots, \xi_n] \in \Lambda_t$, Freeze the component $[\xi_{t+1}, \cdots, \xi_n]$ and consider the $Sp_t(F)$ intertwining map introduced in 2.14.2

$$\begin{aligned} &S(M_{mn}(F)) \to S(M_{mt}(F)) \\ &\qquad f \to f_t : [X_1, \cdots, X_t] \mapsto f[X_1, X_2, \cdots, X_t, \xi_{t+1}, \cdots, \xi_n] \end{aligned}$$

here $X_1, \cdots, X_t$ are arbitrary vectors, we see that f_t is invariant by $\rho_{mt}(K_1^t \times K_2^t)$, and f vanishes on Λ_0 of $Sp_t(F)$. By induction hypothesis, f is zero for $X_1, \cdots, X_t$ arbitrary vectors. Hence $\rho(G_2)f$ vanishes on Λ_t for each t, By lemma 3.24, $\rho(G_2)(f) \in S(\mathbf{M}_{mn}(F))_{(N_2)_t \cap N_2} \in S(\mathbf{M}_{mn}(F))_{(N_2)_t}$, So $\rho(G_2)(f)$ is cuspidal as a representation of $\mathrm{Sp}_n(F)$, so it can not have K_2 fixed vectors. Thus $\rho(G_2)(f)$ is identically zero. □

Remark 3.26.1. *The lemma shows that the $K_1 \times K_2$ invariants functions are totally determined by its restriction to Λ_0.*

Next we give a geometric description of the set Λ_0. Recall that the group $O(W_1, \Phi_1) \times GL_n(F)$ acts on the set $M_{mn}(F)$ by

$$X \longrightarrow g_1^{-1}.X.g_2 \quad \text{for } (g_1, g_2) \in O(W_1, \Phi_1) \times GL_n(F)$$

This action preserve Λ_0, so it defines an action of $O(W_1, \Phi_1) \times GL_n(F)$ on Λ_0. The orbit structure is given as follows:

Lemma 3.27. *Let $\Lambda_0^i = \{X \in \Lambda_0 | rank(X) = i\}$, then Λ_0^i(if nonempty) is an orbit under $O(W_1, \Phi_1) \times GL_n(F)$, and Λ_0 is a disjoint union of the form Λ_n^i for $i = 0, 1, \cdots, n$*

Proof. $X \in \mathbf{M}_{mn}(F) = [\xi_1, \xi_2, ...\xi_n] \in \Lambda_0^i$ if and only if $< \xi_i, \xi_j >_{\Phi_1} = 0$ and the rank of $[\xi_1, \xi_2, ...\xi_n]$ is i. By right multiplication of a matrix g_2 in $GL_n(F)$, we can make the first i vectors be linear independent and the last be 0. Write as $[\xi_1, \cdots, \xi_n] \cdot g_2 = [\xi'_1, \cdots, \xi'_i, 0, \cdots, 0]$. As the subspace spanned by $\xi'_i, i = 1, \cdots, t$ is totally isotropic, by Witt's theorem, there exists a $g_1 \in O(W_1, \Phi_1)$ such that

$$g_1(\xi'_j) = e_j, \quad i = 1, \cdots, t$$

So we have

$$g_1 \cdot [\xi_1, \cdots, \xi_n] \cdot g_2 = g_1 \cdot [\xi'_1, \cdots, \xi'_i, 0, \cdots, 0] = [e^1, \cdots, e^i, 0, \cdots, 0]$$

This means:

$$g_1.X.g_2 = \left(\begin{array}{c|c} I_i & 0 \\ \hline 0 & 0 \end{array}\right)$$

So we have proved that Λ_0^i is an orbit of $O(W_1, \Phi_1) \times GL_n(F)$ on

$$\left(\begin{array}{c|c} I_i & 0 \\ \hline 0 & 0 \end{array}\right)$$

And as $n \leq m$, the rank of an $m \times n$ matrix is less or equal to $\min(m, n) = n$, So $\Lambda_0 = \bigcup_{i=0}^{i=n} \Lambda_0^i$. □

Remark 3.27.1. *It is easy to check that Λ_0^n is dense in Λ_0 in Zariski topology.*

3.5.4 Injectivity on $K_1 \times K_2$ invariants

To finish the proof of the main theorem, We prove the map Z_σ is injective on $S(M_{mn}(F))^{K_1\times K_2}$,i.e. if $f \in S(M_{mn}(F))^{K_1\times K_2}$ and $Z_\sigma(f) = 0$ for every σ such that $Re(\sigma_i) \gg 0\,\forall i$, then $f = 0$. By Lemma 3.19, this implies that Z_σ gives all the intertwining operators from $S(M_{mn}(F))$ to the spherical representations. and by Lemma 3.26, it is sufficient to prove the following lemma:

Lemma 3.28. *Let $f \in S(M_{mn}(F))^{K_1\times K_2}$, if*

$$Z_\sigma(f) \equiv 0$$

for all σ such that $Re(\sigma_i) \gg 0 (i = 1, 2...n)$. Then $\rho(g_2)f$ vanish on Λ_0 for all $g_2 \in Sp_n(F)$

Proof. As Λ_0^n is dense in Λ_0, it suffices to prove $\rho(g_2)f$ vanishes on Λ_0^n. Then we asserts that from the hypothesis of this Lemma, all we need to prove is that $f\left(\frac{X}{0}\right) = 0$ for X an arbitrary $n\times n$ matrix. Indeed if we let $(g_1, g_2) \in G_1\times G_2$, the we write $g_1 = k_1 \cdot p_1$ with $k_1 \in K_1, p_1 \in P_1$ and $g_2 = k_2 \cdot t_2 \cdot n_2$ with $k_2 \in K_2, t_2 \in T_2, n_2 \in N_2$. Then we have

$$\begin{aligned}\rho(g_1, g_2)(f)\left(\frac{I_n}{0}\right) &= \rho(p_1, p_2)(f)\left(\frac{I_n}{0}\right)\\ &= \det(t_2)^{m/2} f\left(\frac{X}{0}\right) \text{ with } X \text{ some } n \times n \text{ matrix.}\end{aligned}$$

We can further reduce this to the case of $GL_n(F) \times \mathrm{GL}_n(F)$ duality: Define

$$f'(X) = f\left(\frac{X}{0}\right) \quad \text{for} \quad X \text{ a } n\times n\text{matrix}$$

and consider the embedding of $GL_n(F)$ to $O(W_1, \Phi_1)$ by

$$\eta(X_1) = \left(\begin{array}{c|c} M & 0 \\ \hline 0 & (M^T)^{-1}\end{array}\right) \text{for } X_1 \in GL_n(F)$$

where $M = \left(\begin{array}{c|c} X_1 & 0 \\ \hline 0 & I_{(m/2)-n}\end{array}\right)$, and the embedding of $GL_n(F)$ to $Sp(W_2, \Phi_2)$ by

$$\eta'(X_2) = \left(\begin{array}{c|c} X_2 & 0 \\ \hline 0 & (X_2^T)^{-1}\end{array}\right) \text{for } X_2 \in GL_n(A)$$

Observe that these embedding send $\mathrm{GL}_n(A)$ to K_1 and K_2. Take $k_1, k_2 \in \mathrm{GL}_n(A)$, we have $f(\eta(k_1) \cdot X' \cdot \eta'(k_2)) = f(X')$ for $X' \in \mathbf{M}_{mn}(F)$. For $X' = \left(\frac{X}{0}\right)$ where X a $n \times n$ matrix, it is the same as $f'(k_1^{-1} \cdot X \cdot k_2) = f'(X)$. So the above lemma is equivalent as the following lemma:

Lemma 3.29. *Consider the action of $GL_n(F) \times GL_n(F)$ on $S(M_{nn})(F)$ by $\gamma(g_1, g_2)(\phi)(X) = \phi(g_1 \cdot X \cdot g_2^{-1})$. If $f \in S(M_{nn})(F)$ such that*

$$\phi(k_1 \cdot X \cdot k_2) = \phi(X)$$

for all $k_1, k_2 \in GL_n(A)$ and all $X \in M_{nn}(F)$. Then suppose

$$\int_{Y_n} \phi(Y_n) \prod_{i=1}^{n} |a_{ii}|^{\sigma_i} d_r Y_n = 0$$

for all $\sigma \in \mathbb{C}^n$ such that $Res(\sigma_i) \gg 0 (i = 1, 2, ...n)$. Then $f = 0$

Proof. The proof of this can be found in [11] Lemma5.2. We give a proof when the support of f lies in $\mathrm{GL}_n(F)$. Suppose that, then $f \in \mathcal{H}(G, K)$ when restricted to $\mathrm{GL}_n(F)$, here we denote $G = \mathrm{GL}_n(F), K = \mathrm{GL}_n(A)$. Recalling Example 3.1.1, fix $\{\hat{e}_1, \cdots, \hat{e}_n\}$ to be a basis of $X^\bullet(T)$, then $\sigma = (\sigma_1, \cdots, \sigma_n)$ represents $\chi_\sigma \in (X^\bullet(T) \otimes \mathbb{C})/\hat{W}$ where $\chi_\sigma = \beta(\sum_{i=1}^n \sigma_i \hat{e}_i)$, here β denote the natural projection map

$$\beta : (X^\bullet(T) \otimes \mathbb{C}) \to (X^\bullet(T) \otimes \mathbb{C})/\hat{W}$$

We have the following formula

$$\int_{Y_n} \phi(Y_n) \prod_{i=1}^{n} |a_{ii}|^{\sigma_i} d_r Y_n = \vartheta_{\chi_\sigma}(f)$$

Identify $(X^\bullet(T) \otimes \mathbb{C})/\hat{W}$ as $\mathrm{Spec}(\mathcal{H}(G, K))$, and $U = \{\sigma : \mathrm{Res}(\sigma_i) \gg 0 (i = 1, 2, ...n)\}$ is open and dense in $(X^\bullet(T) \otimes \mathbb{C})/\hat{W}$. So $f \in \mathcal{H}(G, K)$ and f vanishes on U implies $f = 0$ on G. Combine the fact that G is dense in $\mathbf{M}_{nn}(F)$ and f is continuous, we get $f = 0$ in $\mathbf{M}_{nn}(F)$.

□

□

Remark 3.29.1. *The map above gives an $B_1 \times B_2$ intertwining map of $S(M_{nn})(F)$ to $\mu_\sigma \otimes \chi_\sigma$, where B_1 and B_2 is the upper triangle matrix of $\mathrm{GL}_n(F)$. By Frobenious reciprocity, this defines a $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ intertwining map from $S(\boldsymbol{M}_{nn}(F))$ to some spherical representation of $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$. The lemma above shows that the $GL_n(A) \times GL_n(A)$ invariants of $S(\boldsymbol{M}_{nn}(F))$ is determined by this intertwines map. This gives the spherical local theta correspondence for $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$ acting on $S(\boldsymbol{M}_{nn}(F))$.*

References

[1] P. Cartier, *Representations of p-adic groups: a survey.* Proceedings of Symposia in Pure Mathematics Vol. 33(1979), part 1, pp.111-155

[2] C.Moeglin,M-F Vigneras and J-L Waldspurger, *Correspondances de Howe sur un corps p-adique.* Lecture notes in Math ,vol. 1291,Springer-Verlag,New York , 1987

[3] A.Weil, *Sur certain groupes d'opérateurs unitaires.* Acta.Math, 111: 143-211, 1964

[4] Stephen S.Kudla, *Notes on the local theta correspondence.*1996

[5] Wen-wei Li, *The Weil representation and Its character.*2010

[6] Howe Roger E, *Transcending classical invariant theory.* Journal of the American Mathematical Society, American Mathematical Society, 2 (3): pp.535-552 ,1989b

[7] D. Bump, *Hecke Algebra.* lecture note, 2010

[8] D.Flath, *Decomposotion of representations into tensor products.* Proceedings of Symposia in Pure Mathematics Vol. 33(1979), part 1, pp.179-183

[9] Alexander Kirillov, *An introduction to Lie Groups and Lie Algebras.* Cambridge Studies in advanced mathematics 113

[10] Gaëtan Chenevier and Jean Lannes, *Formes automorphes et voisins de Kneser des réseaux de Niemeier*

[11] Stephen Rallis, *Langlands' Functoriality and Weil representation.* American Journal of Mathematics. Vol. 104. No.3(Jun. 1982), pp. 469-515