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Universität Duisburg-Essen

ALGANT Master's Thesis

On the Néron-Ogg-Shafarevich Criterion for K3 Surfaces

Submitted By:
Yukihide Nakada

Supervisor:
Prof. Bruno Chiarellotto

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Chapter 0

Introduction

The study of families of varieties and their degenerations is a well-established segment of both algebraic and complex geometry. Wherever one can define the notion of a family of varieties over a space, one may ask how varieties vary across this family and whether it extends to singularities of the space.

While we will be concerned with arithmetic algebraic geometry, we first describe the classical complex-analytic situation for its motivation and intuition. A family of complex-analytic varieties, broadly speaking, is a space $\mathfrak{X}$ with a morphism $f : \mathfrak{X} \to S$ to some base space S such that each fiber $X_t := f^{-1}(t)$ is a complex-analytic variety, with some conditions like flatness and properness to guarantee that the fibers are well-behaved. One can imagine the fibers as varieties continuously varying as we move across S.

But not all natural families are well-behaved throughout all of S. For example, consider the family

$$\begin{array}{c} \operatorname{Spec}\mathbb{C}[x,y,t]/(xy-t) \\ \Big\downarrow f \\ \operatorname{Spec}\mathbb{C}[t] = \mathbb{A}^1_{\mathbb{C}}. \end{array}$$

Over a nonzero point $t_0 \neq 0$ in $\mathbb{A}^1_{\mathbb{C}}$, the fiber is the smooth curve $xy = t_0$. Over $t = 0$, though, the fibers 'degenerate' into the singular curve $xy = 0$, the union of two intersecting lines. Given such a family, how is the geometry of the singular fiber related to that of the smooth ones? How bad can the singular fiber be?

One can switch perspectives and instead start with a family over a subset of the base space and consider the various ways in which this family can be extended to the whole space. The most basic setting is a smooth family of complex manifolds $f : \mathfrak{X}^* \to \Delta^*$ over the punctured unit disk. The extension of this family to Δ is represented by a completion $\mathfrak{X} \to \Delta$ to a proper flat (but not necessarily smooth!) family over the disk with $\mathfrak{X}$ Kähler. As a foreshadowing of the arithmetic analogue, let us call $X_0 = f^{-1}(0)$ the special fiber and all of the other fibers $X_t = f^{-1}(t)$ for $t \neq 0$ the generic fibers. In general [Kem+73, Theorem p. 53] tells us that up to base change, blow up, and blow down we can suppose that the singularities in the special fiber are of the form $x_1 x_2 \ldots x_n = t$, as in the above example. When we restrict ourselves to certain kinds of varieties, we can obtain more concrete classifications of the special fibers.

As the varieties vary across the base space, so do their associated invariants. The cohomology of the fibers is arguably the most lucrative invariant to study in this context, letting us tap into the rich Hodge theory of complex manifolds. For example, Deligne's theory of mixed Hodge structures (see [PS10, Part II]), which provides a theory of Hodge structures for singular varieties, expands the borders of Hodge theory to include the special fiber, and Steenbrink ([Ste76]) showed that the well-known map between the cohomology of the generic and special fibers could be interpreted as a morphism of mixed Hodge structures once an appropriate 'limit mixed Hodge structure' was attached to the cohomology of the generic fiber. We can further incorporate the monodromy operator on cohomology induced by the nontrivial loops in Δ^*, and there is a powerful theory cataloguing the interaction between these structures and others like them.

So much for the transcendental story. What is the arithmetic counterpart? We have a choice which radically determines the tools at our disposal. We may choose to work over discrete valuation rings of either equal or mixed characteristic. For the former, consider a curve X defined over a curve C defined over $\mathbb{F}_p$. As above, choose a distinguished point $0 \in C$ and consider the localization of the completion at this point. We obtain a scheme $\tilde{X}$ over the discrete valuation ring $k[[t]]$ of formal power series of mixed characteristic. The special fiber over its closed point k corresponds to fiber above 0, and the generic fiber over $k((t))$ corresponds to the generic fiber over a point $t \neq 0$.

Our interest, however, will be the mixed characteristic situation. In mixed characteristic, the most basic 1-dimensional space in which arithmetic takes place is the spectrum of a discrete valuation ring $\mathcal{O}_K$, for instance the p-adic integers $\mathbb{Z}_p$. Of course, despite being 1-dimensional as schemes, they are very small as topological spaces! They consist of only a closed point corresponding to the maximal ideal and the generic point corresponding to the zero ideal. Nevertheless we imagine it as the affine line with all but one of its closed points removed and use it as the analogue of our base space Δ.

A family of varieties over $\mathcal{O}_K$ is thus a flat and proper scheme (or algebraic space) $\mathfrak{X}$ over $\mathcal{O}_K$. Let K denote the generic point of $\mathcal{O}_K$ and k the closed point. Given a scheme over $\mathcal{O}_K$, we call its fiber over K the *generic fiber* and the fiber over k its *special fiber*. As before, we may switch perspectives and ask when a 'family' of varieties extends to the special fiber and in what way. We use the term 'family' very liberally: since the entire punctured disk Δ^* is replaced by the generic point, the 'family' consists of a single generic fiber X defined over the generic point K. As such, the question of degeneration asks, given a variety X/K, is there a proper and flat scheme/algebraic space $\mathcal{X}/\mathcal{O}_K$ with generic fiber X?

We call such a space $\mathcal{X}$ a *model* of X over $\mathcal{O}_K$. We may also ask when X has a *smooth* model, a model which is smooth as well as proper or, equivalently, a model with smooth special fiber. This is the arithmetic analogue of a family $\mathfrak{X}^* \to \Delta^*$ extending smoothly to a family $\mathfrak{X} \to \Delta$. When X has a smooth model, we say that X has *good reduction*. Thus the question of good reduction is:

Given a smooth variety X over K, when does there exist a smooth and proper model $\mathcal{X}$ of X over $\mathcal{O}_K$?

This question is the object of our thesis.

The problem of good reduction is inextricably linked to the invariants associated to our varieties, such as their fundamental groups and cohomology. In a point of divergence from the transcendental story, we have a substantive choice of cohomology theory. Suppose that $\mathcal{O}_K$ has mixed characteristic with $\operatorname{char} k = p$. Then given a variety X over the generic fiber K we may consider its l-adic étale cohomology groups, whose theory was developed to attack the Weil conjectures, or its p-adic étale cohomology groups, whose theory was driven by Grothendieck's dream of a 'mysterious functor' connecting it to other cohomology theories such as algebraic de Rham cohomology and crystalline cohomology, giving rise the p-adic Hodge theory developed by Fontaine and others. These cohomology groups, and especially their associated structures (Galois representations, action by Frobenius, monodromy), are exceptionally sensitive to the reductions of our variety.

Learning about the cohomology of a given geometric scenario, while sometimes extremely difficult to lay the foundations for, has comparably few obstructions once those foundations have been laid. The C_{st}-conjecture, extremely involved in its proof but quite general in its scope, is a quintessential example of this phenomenon. On the other hand, gleaning geometric information from the algebraic information encoded in cohomology is often very difficult. It is for this reason that we have very general results about the cohomology of smooth and proper varieties with good reduction, but only a handful of results, under very specific circumstances, concerning what we can learn about the models of a variety from its cohomology.

These latter kinds of results have only been established for the simplest types of varieties. Serre and Tate proved ([ST68]) that the ramification of the l-adic Galois representation associated to an abelian variety determined whether it had good reduction, a result they called the Néron-Ogg-Shafarevich criterion. Coleman-Iovita ([CI99]) and Breuil ([Bre00]) proved analogously that an abelian variety had good reduction if and only if its associated representation was crystalline, completing the picture for abelian varieties. Oda ([Oda95]) and Andreatta-Iovita-Kim ([AIK15]) proved l- and p-adic analogues for curves, respectively, involving fundamental groups instead of cohomology.

One of the first steps towards such a good reduction criterion for K3 surfaces was Matsumoto's potential good reduction criterion for K3 surfaces in [Mat14]. He showed, similarly to the case of abelian varieties, that a K3 surface (with the extra assumption of a potentially semi-stable model) has potential good reduction if and only if its l-adic cohomology groups are unramified or its p-adic cohomology group is crystalline. This was refined by Matsumoto and Liedtke in [LM18] through a study of the Galois action on such models obtained after a finite extension. They were able to show that a smooth model could be guaranteed after a finite *unramified* extension and found that even when a K3 surface did not have good reduction its smooth model after a finite extension always descended to a model over the original field, called its RDP model, whose special fiber has worst canonical singularities.

On the other hand, they produced examples of K3 surfaces which obtained good reduction only after a nontrivial extension. It followed that a good reduction criterion for K3 surfaces had to impose cohomological restrictions in addition to the l- and p-adic representations being unramified or crystalline.

Chiarellotto, Lazda, and Liedtke provided these additional l- and p-adic criteria in [CLL17]. Given a K3 surface X over K, they studied in detail the minimal resolution of singularities Y/k of the special fiber of the model guaranteed by [LM18] and the Galois action on the Weyl group generated by reflections about its exceptional divisors. They found that a K3 surface had good reduction if and only if, in addition to the l- and p-adic representations of X being respectively unramified and crystalline, there was a comparison between these cohomology groups and the cohomology of Y; specifically, if they were isomorphic in the l-adic case and if $\mathbb{D}_{\mathrm{cris}}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_p)) \cong H^2_{\mathrm{cris}}(X/W)$ in the p-adic case. The goal of this thesis is to explain and flesh out the details of their proof of this Néron-Ogg-Shafarevich criterion for K3 surfaces and the preliminary results by Matsumoto and Liedtke-Matsumoto which paved the way.

The thesis is organized as follows:

- Chapter 1 is a minimalist dash through the tools we will need in the proof of the Néron-Ogg-Shafarevich criterion for K3 surfaces. We define our objects of study, K3 surfaces alongside varieties with log structure, and the theory we will use to study them, namely p-adic Hodge theory, p-adic cohomology theories, and various forms of descent.

  We have tried to gather and explain most of the algebraic geometry beyond the standard curriculum necessary to understand the proof. However, for the sake of space, we occassionally reference the bibliography for substantial topics beyond those we can justifiably call common knowledge. Chief among them are algebraic spaces, rigid analytic spaces, [and the geometry of surfaces]. For algebraic spaces, we used [Ols10] as our primary reference. The book [Bos14] is a good reference for formal and rigid analytic geometry.

- In Chapter 2 we fix the details of our problem and explain the preparatory results which provided the groundwork for the full good reduction criterion. The bulk of it is dedicated to the proof of Matsumoto's cohomological criterion for potential good reduction in [Mat14]. This established a cohomological criterion for deciding whether a K3 surface has good reduction after possibly a finite extension and will be the basis of the proof of the full Néron-Ogg-Shafarevich criterion.

  The chapter concludes with some refinements to Matsumoto's result. We describe the strengthing of the result in the subsequent paper by Liedtke and Matsumoto ([LM18]), which refined the l-adic criterion and assured good reduction after a finite *unramified* extension, and the results in [CLL17] which allowed the p-adic criterion to be reintegrated into this strengthened version of the theorem. We also describe the RDP model associated to a K3 surface, which will be a central player in both the statement and the proof of the good reduction criterion for K3 surfaces.

- Chapter 3 finally chronicles the proof of the Néron-Ogg-Shafarevich criterion in [CLL17] largely in parallel to the original exposition. Our primary contribution is omitted details and motivation.

  Given a K3 surface X which has good reduction after an extension L/K with Galois group G, we use descent to translate the existence of a smooth model over K to the regularity of the rational G-action on the smooth model. This is in turn expressed as the triviality of a certain element in non-abelian cohomology with respect to the action of G on the Weyl group generated by reflections about the exceptional divisors of its RDP model.

On the other hand, comparison theorems between the cohomology of the special and generic fibers of a smooth model relates this non-abelian cohomology set to non-abelian cohomology with respect to the action of G on l- and p-adic cohomology groups associated to the minimal resolution of singularities Y of the special fiber of the RDP model of X. Via a correspondence between forms and non-abelian cohomology, we can convert this into a cohomological relationship between the cohomology groups associated to X and Y respectively, mediated by an element in non-abelian cohomology.

The cohomological criteria in the Néron-Ogg-Shafarevich criterion, in essence, impose that this element in non-abelian cohomology is trivial. A technical result on the kernel of a map in non-abelian cohomology guarantees that this assumption implies the triviality of the element in non-abelian cohomology with respect to the action of G on the Weyl group described above, which in turn implies that X must have good reduction.

Chapter 1

Preliminary Notions

In this chapter we catalogue, without proof, the preliminary material necessary for approaching the proof of the Néron-Ogg-Shafarevich criterion for K3 surfaces.

1.1 K3 Surfaces

The original l-adic Néron-Ogg-Shafarevich criterion for good reduction of abelian varieties was proven by Serre and Tate in [ST68]. The p-adic analogue involving crystalline instead of unramified representations was proved by Coleman and Iovita in [CI99] and by Breuil in [Bre00], and Oda and Andretta-Iovita-Kim proved l- and p-adic analogues for curves, respectively. Another class of algebraic varieties which are simultaneous rich and accessible are *K3 surfaces*.

Definition 1.1. Let k be a field of arbitrary characteristic. A *K3 surface over* k is a complete, non-singular variety X of dimension 2 over k such that

$$\Omega^2_{X/k} \cong \mathcal{O}_X \qquad \text{and} \qquad H^1(X, \mathcal{O}_X) = 0.$$

That is, it has trivial canonical bundle and the first cohomology class of $\mathcal{O}_X$ is trivial. Algebraic K3 surfaces are always projective, since any smooth complete surface is projective, see [Huy, Remark 1.2]. The Hodge diamond of a K3 surface is the following:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Denote by $\mathrm{NS}(X)$ the *Néron-Severi group*

$$\mathrm{NS}(X) := \mathrm{Pic}(X)/\,\mathrm{Pic}^0(X)$$

where $\mathrm{Pic}^0(X)$ is the subgroup of line bundles algebraically equivalent to 0 (equivalently, the connected component of the Picard variety $\mathrm{Pic}(X)$) and by $\mathrm{Num}(X)$ the quotient

$$\mathrm{Num}(X) := \mathrm{Pic}(X)/\,\mathrm{Pic}^\tau(X)$$

where $\mathrm{Pic}^\tau(X)$ the subgroup of numerically trivial line bundles. Then we have

Proposition 1.1 ([Huy], Proposition 2.4)**.** *For a K3 surface X the natural projections*

$$\mathrm{Pic}(X) \twoheadrightarrow \mathrm{NS}(X) \twoheadrightarrow \mathrm{Num}(X)$$

are isomorphisms.

Other results concerning K3 surfaces and their invariants are under the hood of many of the results that we reference in this thesis, but the above is all we need directly. In particular we do not need the rich theory of complex K3 surfaces. For a comprehensive reference, see [Huy].

1.2 Log Geometry

1.2.1 Introduction and Basic Definitions

In this section we give a cursory introduction to log geometry with the aim of describing log K3 surfaces. Log structures on schemes are a formalism to study the behavior of schemes which are not smooth but whose singularities nonetheless exhibit good behavior. A prototypical example from complex geometry is that of a normal crossing divisor: if X is a complex manifold and $D \subset X$ a hypersurface, then we say that D is a *normal crossing divisor* if it is locally of the form $z_1 \cdots z_r = 0$ for some integer r. Manifolds with normal crossing divisors have a well-behaved theory of divisors and cohomology. The theory of log structures formalizes this framework for the category of schemes and provides a robust language for working with schemes with mild singularities. Our primary references will be [Kat89], [Ogu18], and [Nak00].

Notation. Our monoids will be commutative with a unit element, and a homomorphism of monoids will be assumed to preserve the unit element. Every monoid M comes associated with a group $M^{\mathrm{gp}} := \{ab^{-1} \mid a, b \in M\}$, with the usual equivalence relation that $ab^{-1} = cd^{-1}$ if $sad = sbc$ for some $m \in M$. We use the notation $X_{\text{ét}}$ for the étale site of X.

Definition 1.2. 1. Let X be a scheme. A *pre-logarithmic* or *pre-log* structure on X is a sheaf of monoids M on $X_{\text{ét}}$ endowed with a homomorphism of sheaves of monoids $\alpha : M \to (\mathcal{O}_X, \cdot, 1)$. A morphism $(X, M, \alpha) \to (Y, N, \beta)$ is a pair of morphisms (f, h) where $f : X \to Y$ is a morphism of schemes and $h : f^{-1}(N) \to M$ is a morphism making the diagram

$$\begin{tikzcd} f^{-1}(N) \arrow[r, "h"] \arrow[d, "f^{-1}\beta"'] & M \arrow[d, "\alpha"] \\ f^{-1}(\mathcal{O}_Y) \arrow[r, "f^\sharp"] & \mathcal{O}_X \end{tikzcd}$$

commute; we omit the morphisms α and β when it is clear from context.

2. A pre-log structure is called a *logarithmic structure* or *log structure* if α induces an isomorphism

$$\alpha^{-1}(\mathcal{O}_X^*) \xrightarrow{\sim} \mathcal{O}_X^*.$$

A morphism between two log schemes is a morphism as pre-log schemes.

3. If (M, α) is a pre-log structure on X, we can endow X with a log structure M^a, called its *associated log structure*, as the pushout of the diagram

$$\begin{tikzcd} \alpha^{-1}(\mathcal{O}_X^*) \arrow[r, "i"] \arrow[d, "\alpha"] & M \\ \mathcal{O}_X^* & \end{tikzcd}$$

in the category of monoids on $X_{\text{ét}}$, endowed with a logarithmic structure via

$$\begin{aligned} M^a &\to \mathcal{O}_X \\ (a, b) &\mapsto \alpha(a)b \text{ for } a \in M, b \in \mathcal{O}_X^*. \end{aligned}$$

Since $\mathcal{O}_X^*$ is in fact a group and not simply a monoid, the pushout is explicitly described locally as $M \oplus \mathcal{O}_X^* / \sim$ where $\sim$ is the relation $(a, b) \sim (a', b') \Leftrightarrow$ there exists $h_1, h_2 \in \alpha^{-1}\mathcal{O}_X^*$ such that $ai(h_1) = a'i(h_2)$ and $b\alpha(h_2) = b'\alpha(h_1)$.

4. Let $f : X \to Y$ be a morphism of schemes. For a log structure N on Y, we define a log structure $f^*(N)$ on X, called the *inverse image* or *pull-back* of N, to be the log structure associated to the pre-log structure on $f^{-1}(M)$ given by the composition

$$f^{-1}(M) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X.$$

To formulate it another way, α is a logarithmic structure if it induces a surjection $\alpha^* : M^* \to \mathcal{O}_X^*$ and has trivial kernel - thus it is an isomorphism on units which additionally has trivial kernel globally.

Remark. We can of course work with other sites such as the Zariski, fppf, or fpqc sites, but in the context of semi-stable reduction and (strict) normal crossing divisors the étale topology is the most appropriate setting.

Example 1.1. A standard example of a log structure is provided by a scheme with a normal crossing divisor.

Definition 1.3. Let X be a locally Noetherian scheme. A *strict normal crossing divisor* on X is an effective Cartier divisor $D \subseteq X$ such that for every $p \in D$ the local ring $\mathcal{O}_{X,p}$ is regular and there exists a system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ such that D is cut out by $x_1, \ldots, x_r$ in $\mathcal{O}_{X,p}$.

Now suppose X is a locally Noetherian scheme, let D be a normal crossing divisor, and define

$$M := \{g \in \mathcal{O}_X \mid g \text{ is invertible outside } D\} \subseteq \mathcal{O}_X.$$

Then (X, M) is a log scheme.

Example 1.2. Here is a more sophisticated example. We first begin with a

Definition 1.4. Let k be an arbitrary field. A variety Y over k is a *normal crossing variety* if Y is geometrically connected, the irreducible components of Y are geometrically irreducible and have equal dimension d, and Y as a scheme over k is étale-locally isomorphic to $\operatorname{Spec} k[x_0, \ldots, x_d]/(x_0 \cdots x_r)$ where $0 \leq r \leq d$ depends on the étale local neighborhood.

Let Y be a normal crossing variety and suppose there are m connected components of its singular locus. Number them $D_1, \dots, D_m$. We first endow $\operatorname{Spec} k$ with a log structure via the morphism

$$\mathbb{N}^m \ni e_i = (0, \dots, \overset{i}{1}, \dots, 0) \mapsto 0 \in k$$

and denote this 'log point' by $\operatorname{Spec} k^{\log}$. We can use this log structure to endow Y with a log structure, as follows:

1. *At smooth points of* Y: Étale locally on the neighborhood of a smooth point of Y, give Y a log structure via the pull-back of the log structure on $\operatorname{Spec} k^{\log}$.

2. *At singualr points of* Y: Étale locally on the neighborhood of a point of D_i isomorphic to $\operatorname{Spec} k[x_0, \dots, x_d]/(x_0 \cdots x_r)$, endow Y with a log structure via the pull-back of the log structure on $\operatorname{Spec} k[x_0, \dots, x_d]/(x_0 \cdots x_r)$ which is associated to the pre-log structure given by

$$\begin{tikzcd} \mathbb{N}^{\oplus i-1} \oplus \mathbb{N} \oplus \mathbb{N}^{\oplus m-i} \arrow[r, "0"] \arrow[d, "\mathrm{id} \oplus \mathrm{diag} \oplus \mathrm{id}"'] & k \arrow[d] \\ \mathbb{N}^{\oplus i-1} \oplus \mathbb{N}^{\oplus r+1} \oplus \mathbb{N}^{\oplus m-i} \arrow[r] & k[x_0, \dots, x_d]/(x_0 \cdots x_r). \end{tikzcd}$$

   where the bottom arrow takes the first and last components to 0 and, on the middle component, is defined by $\mathbb{N}^{\oplus r+1} \ni e_j \mapsto x_{j-1} \in k[x_0, \dots, x_d]/(x_0 \cdots x_r)$.

We denote by $Y^{\log}/\operatorname{Spec} k^{\log}$ or $Y^{\log}$ the log scheme endowed with the log structure above, and call it a *normal crossing log variety* or *NCL variety* for short. We call it *simple* if the underlying scheme Y is a simple normal crossing variety, i.e., its irreducible components are geometrically irreducible and smooth.

1.2.2 Logarithmic Differentials

Logarithmic differentials are defined using generators and relations as follows:

Definition 1.5. Let $\alpha : M \to \mathcal{O}_X$ and $\beta : N \to \mathcal{O}_Y$ be pre-log structures and let $f : (X, M) \to (Y, N)$ be a morphism. Then the sheaf of differentials $\Omega^1_{X/Y}(\log(M/N))$ is an $\mathcal{O}_X$-module defined to be the quotient

$$\Omega^1_{X/Y}(\log(M/N)) := \Omega^1_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M^{\mathrm{gp}})/R$$

where R is the $\mathcal{O}_X$-module generated (locally) by sections of the forms

$$(d\alpha(a), 0) - (0, \alpha(a) \otimes a) \text{ for } a \in M$$
$$(0, \otimes 1 \otimes a) \text{ for } a \in \operatorname{Im}(f^{-1}(N) \to M)$$

The higher differentials are defined as

$$\Omega^n_{X/Y}(\log(M/N)) := \bigwedge^n \Omega^1_{X/Y}(\log(M/N)).$$

When there is no risk of confusion, we write $\Omega^n_{X/Y}$ for logarithmic differentials instead.

Remark. An alternative definition in terms of derivations and universal objects, paralleling the classical construction of the sheaf of differentials for schemes, can be found in [Ogu18, Section IV.1].

1.2.3 Log K3 Surfaces

As an application of the construction of normal crossing varieties in Example 1.2 above, we define the logarithmic analogue of K3 surfaces.

Definition 1.6. Let $X^{\log}/\operatorname{Spec} k^{\log}$ be a NCL variety of pure dimension 2. We say that it is a *normal crossing log K3 surface* (or *NCL K3 surface* for short) if the underlying scheme X is proper over k, $H^1(X, \mathcal{O}_X) = 0$, and $\Omega^2_{X/k} \cong \mathcal{O}_X$. We call it a *simple normal crossing log K3 surface* (or *SNCL K3 surface* for short) if it is simple as a normal crossing variety.

Among the SNCL K3 surfaces are the following three types:

Definition 1.7. Let X be a proper surface over a field k and let $\overline{k}$ be a fixed algebraic closure. Consider the following conditions:

I: X is a smooth K3 surface over k;

II: $X \times_k \overline{k} = X_1 \cup X_2 \cup \cdots \cup X_N$ is a chain of smooth surfaces with X_1, X_N rational and others elliptic ruled. In addition, the double curves $X_h \cap X_{h'}$ are rulings if $|h - h'| = 1$ and empty otherwise; in other words, the dual graph of $X \times_k \overline{k}$ is a segment with endpoints X_1 and X_N.

III: $X \times_k \overline{k} = X_1 \cup X_2 \cup \cdots \cup X_N$ is a chain of smooth surfaces with each X_i rational and the double curves on X_i are rational and form a cycle on X_i (there are no triple points). The dual graph of $X \times_k \overline{k}$ is a triangulation of the sphere S^2.

It follows that $H^1(X, \mathcal{O}_X) = 0$ by a spectral sequence argument. We say that X is a *combinatorial Type I (Type II or Type III, respectively) K3 surface* if X satisfies I (II or III, respectively), X has a log structure as in Example 1.2, and $\Omega^2_{X/k} \cong \mathcal{O}_X$.

In fact, all SNCL K3 surfaces belong to one of the above species:

Proposition 1.2 ([Nak00], Proposition 3.4)**.** *Let $X^{\log}/\operatorname{Spec} k^{\log}$ be a SNCL K3 surface. Then $X \times_k \overline{k}$ is a combinatorial Type I, II, or III K3 surface.*

This proposition reduces the general study of SNCL K3 surfaces into a largely combinatorial one. The usefulness of this classification is evident in, for example, Matsumoto's proof of the criterion for potential good reduction of K3 surfaces in Chapter 2.

1.3 p-adic Hodge Theory

1.3.1 Motivation and Definitions

The main reference for this section is [Ber00]. We have simplified the discussion a little bit, but it is broadly the same.

Let $\mathcal{O}_K$ be a Henselian DVR with residue field k and fraction with K, let $K_0 = \operatorname{Frac}(W(k))$ be the fraction field of its Witt ring $W(k)$, and $G_K = \operatorname{Gal}(\overline{K}/K)$ the absolute Galois group of K. A prototypical example is $k = \mathbb{F}_p$, $\mathcal{O}_K = \mathbb{Z}_p$, and $K = \mathbb{Q}_p$. Let X be a smooth and proper variety over K. One of the broad directions of arithmetic geometry, in which the paper by Chiarellotto, Lazda, and Liedtke belongs, is to relate the behavior of X to its various cohomology groups and their structures. One of the most fruitful cohomology groups in this setting are the étale cohomology groups $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ for l an arbitrary. It's well-known these are in fact $\mathbb{Q}_l$-vector spaces and come equipped with a natural G_K-action. In other words, $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is an l-adic representation of G_K:

Definition 1.8. An l-adic representation V of G_K is a finite-dimensional $\mathbb{Q}_l$-vector space with a continuous linear action of G_K.

The behavior of this representation for $l = p$ and $l \neq p$ diverge radically - the former has an analytic flavor, while the latter is largely algebraic. The goal of p-adic Hodge theory, imagined by Fontaine and developed by many others, is to provide the tools to study the link between p-adic Galois representations, the geometry of X, and other cohomology theories (crystalline, de Rham).

1.3.2 Period Rings

Let V be a p-adic representation of dimension d. Generally speaking, period rings, introduced by Fontaine, are topological $\mathbb{Q}_p$-algebras B equipped with a continuous, linear action of G_K and some additional structure, such that once we cut out the G_K-structure by taking invariants to obtain the B^{G_K}-module

$$\mathbb{D}_B(V) := (B \otimes_{\mathbb{Q}_p} V)^{G_K}$$

we have an interesting invariant with the additional structures of V. It is assumed that B is G_K-regular, i.e., that if $\mathbb{Q}_p \cdot b$ is stable under G_K for some $b \in B$, then in fact $b \in B^*$. It follows easily that this forces B^{G_K} to be a field, and so $\mathbb{D}_B(V)$ is a vector space.

It is easy to show that $\dim_{B^{G_K}} \mathbb{D}_B(V) \leq d$.

Definition 1.9. V is *B-admissible* if $\dim_{B^{G_K}} \mathbb{D}_B(V) = d$.

This is equivalent to $B \otimes_{\mathbb{Q}_p} V \cong B^d$ as $B[G_K]$-modules, and in this case

$$B \otimes_{B^{G_K}} D_B(V) \cong B \otimes_{\mathbb{Q}_p} V.$$

The period rings which are relevant for our study are the rings $B_{\text{cris}}, B_{\text{st}}$, and B_{dR}, for crystalline, semi-stable, and de Rham, respectively. They are equipped with the structure of an F-isocrystal, the structure of an F-isocrystal and a monodromy endomorphism, and a filtration, respectively. We say that a p-adic representation V is *de Rham* if it is B_{dR}-admissible, and similarly for B_{cris} and B_{st}. We have that

$$B_{\text{cris}} \subseteq B_{\text{st}} \subseteq B_{\text{dR}}$$

and

$$\begin{aligned} B_{\text{dR}}^{G_K} &= K \\ B_{\text{st}}^{G_K} &= K_0 \\ B_{\text{cris}}^{G_K} &= K_0 \end{aligned}$$

is $B_{\text{dR}}^{G_K} = K$ or $\hat{K}$? The nomenclature is justified by the following remarkable results:

Theorem 1.1 (The C_{dR}-conjecture)**.** *Let X be a smooth and proper variety over K. Then $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is B_{dR}-admissible, and*

$$\mathbb{D}_{\mathrm{dR}}(V) := \mathbb{D}_{B_{\mathrm{dR}}}(V) \cong H^i_{\mathrm{dR}}(X/K)$$

as filtered K-vector spaces. In particular,

$$B_{\mathrm{dR}} \otimes_K H^i_{\mathrm{dR}}(X/K) \cong B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p).$$

Theorem 1.2 (The C_{cris}-conjecture)**.** *Let X be a smooth and proper variety over K with good reduction, that is, a smooth and proper model $\mathcal{X}$ over $\mathcal{O}_K$. Let X_k denote the special fiber. Then $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is $\mathbb{B}_{\mathrm{cris}}$-admissible, and*

$$\mathbb{D}_{\mathrm{cris}}(V) := \mathbb{D}_{B_{\mathrm{cris}}}(V) \cong H^i_{\mathrm{cris}}(X_k/W)$$

as F-isocrystals. In particular,

$$B_{\mathrm{cris}} \otimes_{K_0} H^i_{\mathrm{cris}}(X_k/W) \cong B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p).$$

Theorem 1.3 (The C_{st}-conjecture)**.** *Let X be a smooth and proper variety over K with semi-stable reduction, that is, a proper and flat model $\mathcal{X}$ over $\mathcal{O}_K$ with special fiber X_k a normal crossing divisor. Let X_k be endowed with its natural log structure. Then $V = H^i(X_{\overline{K}}, \mathbb{Q}_p)$ is C_{st}-admissible, and we have an isomorphism*

$$D_{\mathrm{st}}(V) := D_{B_{\mathrm{st}}}(V) \cong H^i_{\text{log-cris}}(X_k/W)$$

compatible with the Frobenius endomorphism and the monodromy operator. In particular,

$$B_{\mathrm{st}} \otimes_{K_0} H^i_{\text{log-cris}}(X_k/W) \cong B_{\mathrm{st}} \otimes_K H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p).$$

The C_{st}-conjecture implies the C_{cris} and C_{dR}-conjectures. The C_{st}-conjecture was proven in full by Tsuji, with Nizioł, Faltings, and Beilinson subsequently contributing alternate proofs.

The above results say that a variety with a specific quality (proper and smooth, good reduction, semi-stable reduction) carries with it a particular type of representation on its cohomology groups $V = H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. The p-adic versions of the Néron-Ogg-Shafarevich criterion, and related results, can be thought of as converses to these results - they answer whether we can detect that a variety has a specific quality from the data of the representation. For abelian varieties, we have a perfect parallel to the l-adic Néron-Ogg-Shafarevich criterion proven by Serre and Tate: Coleman-Iovita and Breuil showed that an abelian variety A has good reduction if and only if V is crystalline, and that A has semi-stable reduction if and only if V is semi-stable. In the case of curves, Andretta, Iovita, and Kim proved in [AIK15] that a curve had good reduction if and only if a certain non-abelian p-adic unipotent fundamental group is crystalline. The p-adic results in the paper by Chiarellotto, Lazda, and Liedtke on K3 surfaces are of the same kind.

1.4 p-adic Cohomology Theories

1.4.1 Crystalline Cohomology

One of the first attempts to devise a well-behaved p-adic cohomology theory is crystalline cohomology, developed by Grothendieck and Berthelot. Our primary reference will be the survey [Cha98]. The definition, intuitively, is sheaf cohomology in which the Zariski open subsets of our variety are enriched with 'infinitesimal thickenings'. Despite this sheaf-theoretic definition, one of the crucial results of the theory is that crystalline cohomology can be computed via differential methods.

The algebraic structure faciliating crystalline cohomomology is that of a *divided power structure*, or *PD-structure* for short. It is a set of functions γ_n for all $n \geq 0$ on a ring which morally behaves like "$\gamma_n(x) = x^n/n!$". More precisely,

Definition 1.10. Let $I \subseteq A$ be an ideal of a ring. A *PD-structure* over I is a set of functions $\gamma_n : I \to A$ for $n \in \mathbb{N}$ such that

- $\gamma_0(x) = 1$ and $\gamma_1(x) = x$ for all $x \in I$;
- $\gamma_n(x) \in I$ for all $n \geq 1$ and $x \in I$;
- $\gamma_n(x + y) = \sum_{i+j=n} \gamma_i(x)\gamma_i(y)$ for all $x, y \in I$;
- $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$ for all $\lambda \in A$ and $x \in I$;
- $\gamma_n(x)\gamma_m(x) = \binom{m+n}{n}\gamma_{m+n}(x)$ for all $m, n \in \mathbb{N}$ and $x \in I$;
- $\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(x)$.

Example 1.3. The most important example in our circumstance is the following. Let k be a perfect field of characteristic $p > 0$ and let $W(k)$ denote the ring of Witt vectors over k. Let $I = (p)$. Then the valuation of $p^n/n!$ is nonzero for $n \geq 0$ and in fact is positive for $n \geq 1$, so $p^n/n! \in W(k)$ for all $n \geq 0$. The ideal (p) paired with the maps $\gamma_n(p) = p^n/n!$ defines a PD-structure on $W(k)$.

Note that if an ideal I which is killed by some $n \in \mathbb{N}$ can be equipped with a PD-structure, then it is necessarily a nil-ideal. Indeed, $x^n = n!\gamma_n(x) = 0$. In particular, $\operatorname{Spec} A$ and $\operatorname{Spec} A/I$ have the same underlying topological space. Their different structure sheaves differentiate, intuitively, their spaces of functions. The structure sheaf of $\operatorname{Spec} A/I$ encodes the behavior in an "infinitesimal thickening" of $\operatorname{Spec} A$, analogous to keeping the higher terms of a Taylor expansion around a point.

With this extra structure we may define the *crystalline site*. One can think of it as the Zariski site in which each open is equipped with an 'infinitesimal thickening'. Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ its ring of Witt vectors, and let W_n denote the ring $W_n(k) := W/p^n$ for $n \geq 0$. Let X be a k-scheme.

Definition 1.11 (The Crystalline Site)**.** Let $n \in \mathbb{N}$. We define the *crystalline site* $\mathrm{Cris}(X/W_n)$ to be the site consisting of the following data:

- The objects are commutative diagrams

$$\begin{tikzcd} U \arrow[r, hook, "i"] \arrow[d] & V \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W_n \end{tikzcd}$$

where $U \subseteq X$ is a Zariski open subset and $i : U \hookrightarrow V$ is a *PD-thickening*. That is, it is a closed immersion of W_n-schemes such that its ideal of definition

$$I = \ker(\mathcal{O}_V \to \mathcal{O}_U)$$

is equipped with a PD-structure δ compatible with the canonical PD-structure γ on W_n described in Example 1.3, i.e., $\delta(pa) = \gamma_n(p)a^n$ for $pa \in I$.

A morphism $(U, V, \delta) \to (U', V', \delta')$ is intuitively a morphism in the usual category of Zariski opens which is compatible with all the other structures. More precisely, it is an open immersion $U \hookrightarrow U'$ with a morphism $V \to V'$ compatible with the PD-structures.

- A covering is a family of morphisms $(U_i, V_i, \delta_i) \to (U, V, \delta)$ such that $V_i \to V$ is an open immersion and $V = \bigcup_i V_i$.

The associated topos, known as the *crystalline topos*, is denoted $(X/W_n)_{\text{cris}}$.

One can show (see [BO78, Proposition 5.1]) that to define a sheaf on $\text{Cris}(X/W_n)$ is equivalent to providing for each PD-thickening (U, V, δ) a Zariski sheaf $\mathscr{F}_V$ on V and for every morphism $u : (U', V', \delta') \to (U, V, \delta)$ a map $p_U : u^{-1}(\mathscr{F}_V) \to \mathscr{F}_{V'}$ satisfying some basic compatibilities. The most important example for our purposes is the assignment

$$(U, V, \delta) \mapsto \mathcal{O}_V$$

which one can show satisfies the conditions above. It thus defines a sheaf on the crystalline site, known as the *structure sheaf* of X/W_n and denoted $\mathcal{O}_{X/W_n}$.

Definition 1.12 (Crystalline Cohomology)**.** 1. We define

$$H^i_{\text{cris}}(X/W_n) := H^i((X/W_n)_{\text{cris}}, \mathcal{O}_{X/W_n})$$

2. We define the crystalline cohomology of X over $W = W(k)$ to be

$$H^i(X/W) := \varprojlim_n H^i(X/W_n)$$

It is difficult to compute this from the definition. Berthelot and Grothendieck proved that it can be computed using de Rham, methods, as follows. Let $j : X \hookrightarrow Z$ be a closed immersion into a scheme smooth over W_n. *A priori* the ideal of definition does not admit a PD-structure, so we extend Z to a scheme $\tilde{Z}$ universally so that it does: we have a morphism $\tilde{j} : X \hookrightarrow \tilde{Z}$ which *is* a PD-thickening, equipped with a morphism $\tilde{Z} \to Z$ such that the diagram

$$\begin{tikzcd} & \tilde{Z} \arrow[dr] & \\ X \arrow[ur, hook, "\tilde{j}"] \arrow[rr, "j"] & & Z \end{tikzcd}$$

commutes. If $\tilde{J}$ is the ideal of definition, then $\mathcal{O}_{\tilde{Z}}/\tilde{J} = \mathcal{O}_X$.

There exists on $\mathcal{O}_{\tilde{Z}}$ a unique integrable connection

$$d : \mathcal{O}_{\tilde{Z}} \to \mathcal{O}_{\tilde{Z}} \otimes_{\mathcal{O}_Z} \Omega^1_{Z/W}$$

such that $d\gamma_n(x) = \gamma_{n-1}(x) \otimes dx$ for all $x \in \tilde{J}$. This corresponds to the intuition that $d(x^n/n!) = x^{n-1}/(n-1)!\,dx$. The complex $\mathcal{O}_{\tilde{Z}} \otimes \Omega^{\bullet}_{Z/W}$ is a complex of abelian sheaves on $\tilde{Z}$. Since $\tilde{Z}$ is a PD-thickening of X they coincide as topological spaces, and thus we may consider it a sheaf on X itself. Then we have the following remarkable result:

Theorem 1.4 (Berthelot)**.** *There exists a canonical isomorphism*

$$H^i(X/W_n) \to \mathbb{H}^i(X, \mathcal{O}_{\tilde{Z}} \otimes_{\mathcal{O}_Z} \Omega^{\bullet}_{Z/W})$$

Corollary 1.1. *If Z/W_n is a smooth lifting of X, then $\tilde{Z} = Z$ and consequently the crystalline cohomology of X/W_n is canoincally isomorphic to the de Rham cohomology $H^i_{\mathrm{dR}}(Z/W_n)$ of Z.*

Assuming still that k is perfect of characteristic p, suppose that it is the residue field of a DVR V with special fiber K of characteristic 0. Let X_k be the special fiber of a formal V-scheme $\mathcal{X}$. We have the following comparison between de Rham and crystalline cohomology:

Theorem 1.5 ([BO83], Theorem 2.4)**.** *There is a canonical isomorphism*

$$H^i_{\mathrm{dR}}(\mathcal{X}/V) \otimes_V K \xrightarrow{sim} H^i_{\mathrm{cris}}(X_k/W) \otimes_W K.$$

1.4.2 Log-Crystalline Cohomology

If instead of working with schemes and morphisms of schemes we go through the above constructions in the category of log-schemes, with the necessary modifications such as considering étale instead of Zariski coverings in the definition of the crystalline site, we obtain a notion of log-crystalline cohomology as outlined in [Kat89, Section 5].

1.4.3 Rigid Cohomology

Crystalline cohomology provides a good p-adic Weil cohomology when X is proper and smooth. Otherwise the theory breaks down - in particular the crystalline cohomology groups, while defined, become infinite-dimensional. There is also a p-adic theory for smooth affine varieties, called *Monsky-Washnitzer cohomology*. Both are subsumed under a p-adic cohomology developed by Berthelot known as *rigid cohomology*, which is a Weil cohomology for schemes that are neither smooth nor proper.

Remark. For the sake of space, we assume that the reader knows the basic definitions of rigid analytic geometry. [Bos14] is a good reference.

1.4.4 Basic Definitions

Our reference for these definitions is [Ber97, Section 1] and [Le 07, Chapter 2].

Let k be a field of characteristic p; all of this works if k is not perfect, but to fix ideas we assume that k is perfect so that it has essentially one Cohen ring, the ring of Witt vectors $W = W(k)$. Let K denote the generic fiber of W. As usual, we have in our minds $k = \mathbb{F}_p$, $W = \mathbb{Z}_p$, and $K = \mathbb{Q}_p$.

Let X be a k-scheme, and suppose we have a locally closed immersion of k-schemes $i : X \hookrightarrow P_k$ where P_k is the special fiber of a formal W-scheme P. The formal scheme P also has a "generic fiber" P_K which is a quasi-separated rigid analytic variety over K equipped with a *specialization map* $P \to P_K$; if P is the formal completion of a proper W-scheme $\tilde{P}$, then P_K is the rigid analytic space $\tilde{P}_K^{\mathrm{an}}$ associated to the generic fiber $\tilde{P}_K$. Thus we're in the following situation:

$$\begin{tikzcd} X \arrow[d, hook, "i"] & & \\ P_k \arrow[r] \arrow[d] & P \arrow[d] & P_K \arrow[l, "\mathrm{sp}"'] \arrow[d] \\ k \arrow[r] & W & K \arrow[l] \end{tikzcd}$$

Remark. This is only a picture to keep everything in orrder. This diagram is not meant to be commutative - these morphisms do not even live in the full-fledged category of any of these objects such as the category of rigid analytic spaces! Their common category is rather the category of ringed spaces.

Definition 1.13. The *tube* of X in P is

$$]X[_P := \mathrm{sp}^{-1}(X) \subseteq P_K$$

where X is idenfied as a subset of P via the above morphisms.

Suppose $j : X \hookrightarrow \overline{X}$ is an open immersion into a closed subscheme $\overline{X} \hookrightarrow P_k$ of the special fiber. Let Z be the complement of X in $\overline{X}$. Then certainly $]X[_P$ and $]Z[_P$ form an open covering of $]\overline{X}[_P$, but this is not an admissible covering in general. A *strict neighborhood*, morally, is an enlarged open neighborhood $V \subseteq]\overline{X}[_P$ of $]X[_P$ which makes it an admissible covering. Technically,

Definition 1.14. A *strict neighborhood* of $]X[_P$ in $]\overline{X}[_P$ is an open subset $V \subseteq]\overline{X}[_P$ such that V and $]Z[_P$ for an admissible covering of $]\overline{X}[_P$.

The example which motivates the above intuition is the following:

Example 1.4. Let X/k be the special fiber of an affine W-scheme X' and suppose we have an open immersion $X' \hookrightarrow \mathbb{A}^n_W$. Let $\overline{X'}$ be the closure of X' in $\mathbb{P}^n_W \supseteq \mathbb{A}^n_W$, and $\widehat{\overline{X'}}$ be the formal completion. Then one can show that $]X[_{\widehat{\overline{X'}}}$ is the intersection of the generic fiber X'^{an}_K with the closed unit ball inside $\mathbb{A}^{n,\mathrm{an}}_K$. The strict neighborhoods of $]X[_{\widehat{\overline{X'}}}$ in $\widehat{\overline{X'}}_K$ is the intersection of X'^{an}_K with the open balls of radius $\lambda > 1$ in $\mathbb{A}^{n,\mathrm{an}}_K$. Thus, in this instance, the strict neighborhoods are slight open enlargements of the tube inside $\widehat{\overline{X'}}_K$.

We need one more construction to define rigid cohomology. Let P be a formal W-scheme and let X be a k-scheme with an open immersion $j : X \hookrightarrow \overline{X}$ into a closed subscheme $\overline{X} \subseteq P_k$ of the special fiber.

Definition 1.15. The endofunctor $j^\dagger$ on the category of abelian sheaves on $]\overline{X}[_P$ is defined to be

$$j^\dagger E := \varinjlim_V j_{V*} j_V^{-1} E$$

where the limit is taken over all strict neighborhoods of $]X[_P$ and j_V is the inclusion of V into $]\overline{X}[_P$.

For example, in the case of Example 1.4, we are essentially pulling back the sheaf E and taking a limit over all open balls of radius $\lambda > 1$.

We now have the necessary concepts to define rigid cohomology:

Definition 1.16. Let k be a separated k-scheme, $j : X \hookrightarrow \overline{X}$ a compactification of X over k. Suppose there exists a formal W-scheme P with a closed immersion $\overline{X} \hookrightarrow P$ such that P is smooth at the points of X. Then the *rigid cohomology of X relative to K* is

$$H^*_{\mathrm{rig}}(X/K) := H^*(]\overline{X}[_P, j^\dagger \Omega^\bullet_{]\overline{X}[})$$

1.4.5 Basic Facts

As one would hope, this construction is independent of the choice of $\overline{X}$ and P, and is functorial in X. In addition, Nagata compactification always provides the necessary compactification.

The promised identification between crystalline and rigid cohomology for proper smooth varieties is the following, proven by Berthelot:

Theorem 1.6 ([Ber97], Proposition 1.9). *Suppose X is smooth and proper over k, and let W be the Cohen ring of k. Then there a canonical isomorphism*

$$H^*_{\mathrm{rig}}(X/K) \xrightarrow{\sim} H^*_{\mathrm{cris}}(X/W) \otimes K.$$

The strategy is to show that we can choose the same liftings of X in both constructions. There are also interactions between the de Rham and rigid cohomology of the generic and special fibers of a smooth W-scheme, respectively. We will not go over the technical construction here, but one may define rigid cohomology with compact support $H^*_{c,\mathrm{rig}}(X_k/K)$ for X_k the special fiber of a W-scheme X. It is proven in [BCF04, Theorem 6.6] that there is a canonical functorial K-linear map

$$\mathrm{cosp}^* : H^*_{c,\mathrm{rig}}(X_k/K) \to H^*_{\mathrm{dR},c}(X_K/K)$$

called the *cospecialization map* where X_K is the generic fiber of X. There is a Poincaré duality for these cohomologies compatible with this cospecialization morphism, inducing a *specialization map* $\mathrm{sp} : H^*_{\mathrm{dR}}(X_K/K) \to H^*_{\mathrm{rig}}(X_k/K)$, see [CCM13, Introduction].

1.5 Forms and Descent

1.5.1 Introduction

In this section we describe the necessary facts about descent, namely of Galois descent, descent of F-isocrystals, and faithfully flat descent. Our primary references will be [BLR90, Chapter 6], [Con], and [CLL17, Section 2].

Informally, descent answers when an 'object over' a scheme S' arises as the pullback of an object over a scheme S via the pullback along a morphism $p : S' \to S$. In other words, suppose we have a category $F(X)$ of 'objects over X' for every scheme X and a pullback functor $p^* : F(Y) \to F(X)$ for every morphism $p : X \to Y$. Then given a morphism $p : S' \to S$, descent describes the image of the functor p^*. For example, we may take $F(X) = \mathbf{QCoh}(X)$ the category of quasi-coherent sheaves over X and p^* being the standard pullback, or $F(X) = \mathbf{Sch}(X)$ the category of schemes over X and p^* the base-change functor. Descent is useful for answering when information that we get about a scheme after base change 'descends' to information about our original scheme.

In this short section we stick to the down-to-earth explanation in [BLR90, Chapter 6]. One can formalize descent more precisely, abstractly, and generally in the language of fibered categories and stacks. This can be found in Grothendieck's foundational exposition in FGA or in modern treatments such as [Fan+05].

1.5.2 Descent Data for Quasi-Coherent Sheaves

To fix ideas, let $F(X) = \mathbf{QCoh}(X)$ be the category of quasi-coherent sheaves over X, fix a morphism $p : S' \to S$, and let p^* be the standard pullback: this is Grothendieck's original setting. Let $S'' := S' \times_S S'$ and let $p_i : S'' \to S'$ be the projections, so that we have a diagram

$$\begin{tikzcd} S'' = S' \times_S S' \arrow[r, "p_1"] \arrow[d, "p_2"'] & S' \arrow[d, "p"] \\ S' \arrow[r, "p"] & S \end{tikzcd}$$

Definition 1.17. Let $\mathscr{F}'$ be a quasi-coherent S'-module. Then a *covering datum* for $\mathscr{F}'$ is an S''-isomorphism $\varphi : p_1^*\mathscr{F}' \to p_2^*\mathscr{F}'$. The pairs $(\mathscr{F}', \varphi)$ of quasi-coherent S'-modules with covering data constitute a category, with a morphism $(\mathscr{F}', \varphi) \to (\mathscr{G}', \psi)$ consisting of an S'-morphism $f : \mathscr{F}' \to \mathscr{G}'$ such that the diagram

$$\begin{tikzcd} p_1^*\mathscr{F}' \arrow[r, "\varphi"] \arrow[d, "p_1^* f"'] & p_2^*\mathscr{F}' \arrow[d, "p_2^* f"] \\ p_1^*\mathscr{G}' \arrow[r, "\psi"] & p_2^*\mathscr{G}' \end{tikzcd}$$

commutes.

The essential example of an S'-module with the covering data is the pullback $p^*\mathscr{F}$ of a quasi-coherent S'-module $\mathscr{F}$. Indeed, it comes equipped with a canonical isomorphism

$$p_1^*(p^*\mathscr{F}) \cong (p \circ p_1)^*\mathscr{F} = (p \circ p_2)^*\mathscr{F} \cong p_2^*(p^*\mathscr{F}).$$

In other words, the pullback functor $p^* : \mathscr{F} \mapsto p^*\mathscr{F}$ is a functor from the category of quasi-coherent S-modules to the category of S'-modules with covering data. When p is faithfully flat and quasi-compact, we have the following result:

Proposition 1.3 ([BLR90], Proposition 6.1)**.** *Let p be faithfully flat and quasi-compact. Then the functor $\mathscr{F} \mapsto p^*\mathscr{F}$ from the category of quasi-coherent S-modules to the category of S'-modules with covering data is fully faithful.*

The proof first uses the quasi-compactness of p to reduce to the case where S and S' are both affine and modules over S and S' are associates to modules over their respective rings, then applies faithfully-flat base change.

As an S'-module with covering data, $\mathscr{F}' = p^*\mathscr{F}$ comes equipped with extra information. Let $S''' := S' \times_S S' \times_S S'$ and let $p_{ij} : S''' \to S''$ be the projection onto the factors with indices i, j for $i < j$ and $i, j = 1, 2, 3$. Then the following diagram commutes

$$\begin{array}{ccccc} p_{12}^* p_1^* \mathscr{F}' & \xrightarrow{p_{12}^*\varphi} & p_{12}^* p_2^* \mathscr{F}' = p_{23}^* p_1^* \mathscr{F}' & \xrightarrow{p_{23}^*\varphi} & p_{23}^* p_2^* \mathscr{F}' \\ \| & & & & \| \\ p_{13}^* p_1^* \mathscr{F}' & & \xrightarrow{\qquad p_{13}^*\varphi \qquad} & & p_{13}^* p_2^* \mathscr{F}' \end{array}$$

because all of the isomorphisms are the canonical ones.

Definition 1.18. For an S'-module with $\mathscr{F}'$ with covering datum φ, we call the commutativity of the above diagram the *cocycle condition* for φ, and call a covering datum φ which satisfies the cocycle condition a *descent datum* for $\mathscr{F}'$. These form a category in the natural way. The descent datum is called *effective* if the pair $(\mathscr{F}', \varphi)$ is isomorphic to the pullback $p^*\mathscr{F}$ of some S-module $\mathscr{F}$ equipped with its natural descent datum.

Example 1.5. The nomenclature is explained in the following example, which shows that covering data and the cocycle condition generalize the gluing construction of sheaves. Let S be a quasi-separated scheme and let $S = \bigcup_{i \in I} S_i$ be a finite affine open covering. Let $S' := \bigsqcup_{i \in I} S_i$ with $p : S' \to S$ the canonical projection; note that it is faithfully flat and quasi-compact. Then an S'-module is a collection of S_i-modules $\mathscr{F}_i$ and the pullback p^* is the restriction of an S-module $\mathscr{F}$ to the open covering.

What is a covering datum $\varphi : p_1^*\mathscr{F}' \to p_2^*\mathscr{F}'$? We have

$$S'' = S' \times_S S' = \bigsqcup_{i,j \in I} S_i \times_S S_j = \bigsqcup_{i,j \in I} S_i \cap S_j$$

with the projection p_1 on $S_i \cap S_j$ being the inclusion into S_i and p_2 the inclusion into S_j. Then the functors p_i^* are the restriction functors $\mathscr{F}_i \mapsto \mathscr{F}_i|_{S_i \cap S_j}$, and thus the covering datum consists of a collection of isomorphisms

$$\varphi_{ij} : \mathscr{F}_i|_{S_i \cap S_j} \to \mathscr{F}_j|_{S_i \cap S_j}$$

and it is easy to see that the cocycle condition is the data of an isomorphism

$$\varphi_{ik}|_{S_i \cap S_j \cap S_k} = \varphi_{jk}|_{S_i \cap S_j \cap S_k} \circ \varphi_{ij}|_{S_i \cap S_j \cap S_k}.$$

Thus a descent datum φ consists of compatibilities of the sheaves $\mathscr{F}_i$ on S_i along their overlaps satisfying the usual cocycle condition.

The classical result that a collection of sheaves $\mathscr{F}_i$ on the open cover S_i with this data can always be glued into a sheaf on S translates to the statement that all descent datum on S' with respect to the morphism $p : S' \to S$ is effective: any S'-module $\mathscr{F}'$ with descent datum, which is really a collection of S_i-modules $\mathscr{F}_i$ on an open covering, glues together into an S-module $\mathscr{F}$, which is to say, $p^*\mathscr{F} = \mathscr{F}'$.

In other words, the functor $p^* : \mathscr{F} \mapsto p^*\mathscr{F}$ to the category of S-modules to the category of S'-modules with descent data is essentially surjective. Since this functor is fully faithful by Proposition 1.3, p^* constitutes an equivalence of categories.

Grothendieck showed that the above is true in vast generality:

Theorem 1.7 (Grothendieck)**.** *Let $p : S' \to S$ be faithfully flat and quasi-compact. Then the functor $p^* : \mathscr{F} \mapsto p^*\mathscr{F}$ from the category of quasi-coherent S-modules to the category of S'-modules with descent data is an equivalence of categories.*

1.5.3 Descent Data for Schemes over a Base

As before, fix a morphism of schemes $p : S' \to S$, but now let $F(S) = \mathbf{Sch}(S)$ be the category of schemes over S and let the pullback functor $p^* : X \mapsto p^*X$ be the base change of X/S to S'. The functoriality of base change provides an analogous notion of descent datum on an S'-scheme X': it is an S'-scheme X' with an $S'' := S' \times_S S'$-morphism $\varphi : p_1^*X' \to p_2^*X'$ satisfying the obvious cocycle condition. As before, $p^* : X \mapsto p^*X$ is a functor from the category of S-schemes to the category of S'-schemes with descent data. This functor, however, fails to be essentially surjective in general, i.e., not all descent data are effective, even when p is faithfully flat and quasi-compact. However, we do have the following. If U' is an open subscheme of an S'-scheme X with descent datum φ, we say that U' is stable under φ if φ restricts to a descent datum $p_1^*U' \xrightarrow{\sim} p_2^*U'$ on U'. Then we have

Theorem 1.8 ([BLR90], Theorem 6.6)**.** *Let $p : S' \to S$ be faithfully flat and quasi-compact. Then*

1. *The functor $p^* : X \mapsto p^*X$ mapping to the category of S'-schemes with descent data is fully faithful;*
2. *Assume S and S' are affine. Then a descent datum φ on an S'-scheme X' is effective if and only if X' can be covered by quasi-affine (or affine) open subschemes which are stable under φ.*

1.5.4 Galois Descent

We finally come to the examples of descent which we will encounter in the sequel.

Definition 1.19. Let $p : S' \to S$ be a finite and faithfully flat morphism of schemes. We say that p is a *Galois covering* if there is a finite group Γ of S-automorphisms of S' such that the morphism

$$\begin{aligned} \Gamma \times S' &\to S'' \\ (\sigma, x) &\mapsto (\sigma x, x) \end{aligned}$$

is an isomorphism. Here $\Gamma \times S'$ is the disjoint union of copies of S' parametrized by Γ.

This is an algebraic analogue of Galois covers in algebraic topology: S'' intuitively consists of pairs of elements in S' mapped to the same element in S via p, and the isomorphism $\Gamma \times S' \cong S''$ suggests that all such pairs are Galois conjugates of each other.

Example 1.6. If K'/K is a Galois extension with Galois group Γ, then $p : \operatorname{Spec} K' \to \operatorname{Spec} K$ is a Galois covering.

Similarly, if $R \subseteq R'$ is an extension of DVRs with R Henselian, R'/R finite étale over R, and the residue extension of R'/R is Galois, then $p : \operatorname{Spec} R' \to \operatorname{Spec} R$ is a Galois cover.

Let X' be an S'-scheme with a Γ-action compatible with that on S', i.e., such that the diagram

$$\begin{tikzcd} X' \arrow[r, "\sigma"] \arrow[d] & X' \arrow[d] \\ S' \arrow[r, "\sigma"] & S' \end{tikzcd}$$

commutes for all $\sigma \in \Gamma$.

Proposition 1.4. *Such a Γ-action is equivalent to a descent datum on X' with respect to the Galois cover $p : S' \to S$.*

This is simply a matter of defining the right projection maps and using the associativity condition to check the necessary compatibilities and the cocycle condition.

One can use Theorem 1.8 to relate the Galois action to effectiveness of descent. It can be shown that if S and S' are affine and X' is quasi-separated, then the descent data is effective if and only if the Γ-orbit of each $x \in X'$ is contained in a quasi-affine open subscheme of X'. Details, as well the explicit correspondence between the Γ-action and descent data, can be found in [BLR90, Example 6.2.B]. The example above where K'/K is a Galois extension and our Galois cover is the morphism $p : \operatorname{Spec} K' \to \operatorname{Spec} K$ has a classical predecessor in Galois descent for vector spaces. Throughout, let L/K be a finite extension of fields.

Definition 1.20. For an L-vector space V, a K-subspace W such that a K-basis of W is an L-basis of V is called a *K-form* of V.

This is the easier definition to work with, but the connection to descent is more transparent with the following coordinate-free rendition; their equivalence is easy to show.

Definition 1.21. For an L-vector space V, a *K-form* is a K-subspace W for which the L-linear map

$$\begin{aligned} W \otimes_K L &\to V \\ a \otimes w &\mapsto aw \end{aligned}$$

is an isomorphism of L-vector spaces.

Thus a K-form for an L-vector space V is a preimage of the extension of scalars functor $\mathbf{Vect}_K \to \mathbf{Vect}_L$, $W \mapsto W \otimes_K L$.

Now suppose L/K is Galois with Galois group G. Then a K-form W for an L-vector space V induces a Galois action on $V \cong W \otimes_K L$ via the Galois action on the second. This action is certainly G-semilinear, i.e., for every $\sigma \in G$ the action σ is σ-semilinear. This describes the form completely, in the sense of the following elementary result:

Theorem 1.9 ([Con], Theorem 2.14)**.** *Let V be an L-vector space. We have a bijection*

$$\begin{aligned} \{K\text{-forms on } V\} &\leftrightarrow \{G\text{-semilinear actions on } V\} \\ W &\mapsto W \otimes_K L \\ V^G &\leftarrow\!\shortmid V \end{aligned}$$

Since all K-vector spaces naturally induce L-vector spaces with a G-semilinear action, we can interpret this in the following way:

Theorem 1.10 ([GW10], Theorem 14.83)**.** *Let L/K be a finite Galois extension with Galois group G. Then the functor*

$$\begin{aligned} \{K\text{-vector spaces}\} &\to \{L\text{-vector spaces with a } G\text{-semilinear action}\} \\ W &\mapsto W \otimes_K L \end{aligned}$$

is an equivalence of categories with quasi-inverse $V^G \leftarrow\!\shortmid V$.

Phrased in this way and in light of the above result, which states that G-actions are descent data for Galois covers, we see that this is a special case of Galois descent for the Galois cover $p : \operatorname{Spec} K' \to \operatorname{Spec} K$.

1.5.5 Descent of F-isocrystals

The above discussion and equivalence of categories continue to hold when our vector spaces are additionally equipped with the structure of an F-isocrystal (though with a very slight change of perspective in the definition). In this discussion, assume $K = K_0$ is completely and absolutely unramified with characteristic $p > 0$, L/K is a finite unramified Galois extension with Galois group G, and let V be an F-isocrystal over K.

If V/K is an F-isocrystal, then $V \otimes_K L$ is an F-isocrystal over L with a compatible G-semilinear action. Let F-**Isoc**(K) be the category of F-isocrystals over K and G-F-**Isoc**(L) the category of F-isocrystals over L with a compatible G-semilinear action. As a narrowing of the equivalence in Theorem 1.10, we have

Proposition 1.5. *The functor*

$$\begin{aligned} F\text{-}\mathbf{Isoc}(K) &\to G\text{-}F\text{-}\mathbf{Isoc}(L) \\ V &\mapsto V \otimes_K L \end{aligned}$$

is an equivalence of categories, with quasi-inverse $W^G \leftarrow\!\shortmid W$.

Forms have an analogous definition when equipped with the structure of an F-isocrystal.

Definition 1.22. 1. An *L-form of V* is an F-isocrystal W/K such that there exists an isomorphism $V \otimes_K L \cong W \otimes_K L$ as F-isocrystals over L.

2. Two L-forms of V are said to be *equivalent* if they are isomorphic as F-isocrystals over K.

We denote the class of L-forms by $E(L/K, V)$.

The set $E(L/K, V)$ is pointed with the distinguished element being the class of V with its identity morphism. If $W \in E(L/K, V)$, both sides of the assumed isomorphism

$$f : V \otimes_K L \to W \otimes_K L$$

have a natural G-action, but this isomorphism will not be G-equivariant in general. If the isomorphism were G-equivariant, then $f^{-1} \circ \sigma \circ f = \sigma$ for all $\sigma \in G$. This motivates the function

$$\begin{aligned} \alpha_f &: G \to \mathrm{Aut}_{L,F}(V \otimes_K L) \\ \alpha_f(\sigma) &= f^{-1} \circ \sigma(f) \end{aligned}$$

as a measure of the failure of the isomorphism to be G-equivariant; here $\sigma(f)$ is the natural action of G on $\mathrm{Aut}_{L,F}(V \otimes_K L)$ by conjugation. The map α_f is a 1-cocycle for this natural action of G on $\mathrm{Aut}_{L,F}(V \otimes_K L)$. We have the following result:

Proposition 1.6. *The map $f \mapsto \alpha_f$ induces a bijection*

$$E(L/K, V) \to H^1(G, \mathrm{Aut}_{L,F}(V \otimes_K L))$$

of pointed sets.

The inverse is provided by Proposition 1.5. Given a representative $\alpha : G \to H^1(G, \mathrm{Aut}_{L,F}(V \otimes_K L))$ of a 1-cocycle, we obtain a new action of G on $V \otimes_K L$ via the *homomorphism*

$$\begin{aligned} \rho^\alpha &: G \to \mathrm{Aut}_{L,F}(V \otimes_K L) \\ \rho^\alpha(\sigma)(v) &= \alpha(\sigma)(\sigma(v)). \end{aligned}$$

This new G-semilinear action on the F-isocrystal $V \otimes_K L$ corresponds, by Proposition 1.5, to an F-isocrystal

$$V^\alpha := (V \otimes L)^{\rho^\alpha}$$

over K. It is a matter of checking minor details about compatibilities and independence of representatives to see that this is a well-defined map in the inverse direction and that it is the desired inverse.

Remark. Like classical Galois descent, it it is probably possible to formalize this in the language of descent theory, perhaps through a fibered category over schemes with additional constraints.

1.5.6 Forms of G_k-modules

There is an l-adic analogue in which the F-isocrystal structure (which naturally occurs in the context of crystalline cohomology) is replaced by a Galois representation (which appear alongside étale cohomology), and extension of scalars is replaced by restricting the representation.

Let l be any prime, including p. Suppose k'/k is a finite Galois extension with Galois group G, and let $G_{k'}$ and G_k denote the absolute Galois groups of k' and k, respectively.

Definition 1.23. Let V be a finite-dimensional $\mathbb{Q}_l$ representation equipped with an l-adic Galois representation $\rho : G_k \to \mathrm{GL}(V)$.

1. A *k'-form of V* is an l-adic Galois representation $\psi : G_k \to \mathrm{GL}(W)$ such that there exists a $G_{k'}$-equivariant and $\mathbb{Q}_l$-linear isomorphism $V|_{G_{k'}} \cong W|_{G_{k'}}$.
2. Two k'-forms of V are said to be *equivalent* if they are isomorphic as G_k-representations.

As above, we denote the set of equivalence classes of k'-forms by $E(k'/k, V)$. It is a pointed set.

There is a natural G_k-action on the $G_{k'}$-equivariant automorphisms $\mathrm{Aut}_{G_{k'}}(V)$ by conjugation, and since $G_{k'}$ acts trivially by assumption, this induces a $G_k/G_{k'} \cong G$-action on $\mathrm{Aut}_{G_{k'}}(V)$. A k'-form W provides a $G_{k'}$-equivariant isomorphism

$$f : V \to W$$

which is not G_k-equivariant in general, and its failure is measured again by the 1-cocycle

$$\alpha_f : G \to \mathrm{Aut}_{G_{k'}}(V)$$
$$\alpha_f(\sigma) = f^{-1} \circ \sigma(f)$$

where σ acts by conjugation as before. Then we have

Proposition 1.7 ([CLL17], Proposition 2.11)**.** *This map induces a bijection*

$$E(k'/k, V) \to H^1(G, \mathrm{Aut}_{G_{k'}}(V))$$

of pointed sets.

The inverse, similar to the case of F-isocrystals, is obtained by twisting the natural G-action by the cocycle. If $\alpha : G \to \mathrm{Aut}_{G_{k'}}(V)$ represents a 1-cocycle, we can twist the given representation ρ by α to obtain a new representation

$$\rho^\alpha(g)(v) := \alpha(g)(\rho(g)(v)).$$

It is easy to show that this is a k'-form of V using the $G_{k'}$-equivariance of the action, and one can show that $\alpha \mapsto \rho^\alpha$ is inverse to the above construction.

Chapter 2

Potential Good Reduction

We fix our setting once and for all. Let $\mathcal{O}_K$ be a Henselian DVR with perfect residue field of charactersitic $p \geq 0$ and fraction field K of characteristic 0. Geometrically, $\operatorname{Spec} k$ and $\operatorname{Spec} K$ are the special and generic fibers of $\operatorname{Spec} \mathcal{O}_K$, respectively. We write $G_K = \operatorname{Gal}(\overline{K}/K)$ for the absolute Galois group inside a fixed algebraic closure, and if $p > 0$ we write $W = W(k)$ for the ring of Witt vectors of k and $K_0 = \operatorname{Frac} W$ for its fraction field.

The various types of models are defined as follows:

Definition 2.1. Let X be a smooth and proper variety over K.

1. A *model of X over $\mathcal{O}_K$* is an algebraic space that is flat and proper over $\operatorname{Spec} \mathcal{O}_K$ and whose generic fiber is isomorphic to X.
2. We say that X has *good reduction* if there exists a model of X that is smooth over $\mathcal{O}_K$.
3. We say that X has *potential good reduction* if there exists a finite extension L/K such that $X_L := X \otimes_K L$ has good reduction over L.

Our quest towards establishing a cohomological criterion for good reduction of K3 surfaces starts with the more modest, but crucial, step of finding a criteria for *potential* good reduction.

To have some hope of establishing criteria for (potential) good reduction a helpful starting handhold is a bare and basic model that we can work with, and all of the results that follow are predicated on the existence of basic models of K3 surfaces. More precisely, if X is a K3 surface over K, then a *Kulikov model* is a flat and proper algebraic space $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$ whose relative canonical is trivial, with generic fiber X and special fiber a strict normal crossing divisor. We will often be assuming the following:

Assumption. $(*)$ A K3 surface X/K satisfies $(*)$ if there exists a finite extension L/K such that X_L admits a Kulikov model.

This is the arithmetic analogue of classical models of semistable degenerations of complex K3 surfaces. Kulikov ([Kul77]), Persson ([Per77]), and Persson-Pinkham ([PP81]) showed that a degeneration of K3 surfaces (which can always be assumed to be semistable by [Kem+73, Theorem p. 53]) always always has such a model and classified the special fibers, see [Kul77, Theorem II]. In mixed charactersitic, when such a model exists, Nakajima classified the possible special fibers in [Nak00], a result which we detailed in Section .

Remark. Assumption $(*)$ essentially follows from potential semi-stable reduction of K3 surfaces, which is expected; see [LM18, Proposition 3.1].

The most comprehensive result on the potential good reduction of K3 surfaces is

Theorem 2.1 ([CLL17], Theorem 1.1)**.** *Suppose that $p > 0$, and let X be a K3 surface over K that satisfies $(*)$. Then, the following are equivalent:*

(1) X has good reduction after a finite and unramified extension of K.

(2) The G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for one $l \neq p$.

(3) The G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for all $l \neq p$.

(4) The G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.

This chapter will be devoted to an overview of the proof of this theorem.

2.1 Matsumoto's result on potential good reduction

The first ingredient for the criterion was discovered by Matsumoto in [Mat14], and it is the foundation of Theorem 2.1. His result was the following:

Theorem 2.2 ([Mat14], Theorem 1.1)**.** *Let X be a K3 surface over K which admits an ample line bundle L satisfying $p > L^2 + 4$. Assume that one of the following holds:*

1. For some prime $l \neq p$, $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified.

2. (K is of characteristic 0 and) $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.

Then X has potential good reduction with an algebraic space model, that is, for some finite extension K'/K there exists an algebraic space smooth and proper over $\mathcal{O}_{K'}$ with generic fiber isomorphic to $X_{K'}$.

We give a sketch of the proof in the following two sections. The argument has both a geometric and arithmetic component. On the geometric side, which is the content of Section 2.2, we deduce the existence of nice models under fairly general conditions, possibly after a finite extension. More specifically, we show that under the above assumption that X admits an ample line bundle L with $p > L^2 + 4$ we can find, after perhaps a finite extension, an algebraic space model whose generic fiber is isomorphic to X and whose special fiber is a simple normal crossing log K3 surface (SNC log K3 surface for short) in the sense of Nakkajima [Nak00]. Note that a K3 surface in our usual sense is a type of SNC K3 surface.

Our assumptions, however, control the cohomology of the generic fiber only. We thus reach for more arithmetic arguments in Section 2.3 to prove comparison theorems between the cohomology of the special and generic fibers, generalizations of well-known comparison theorems to the algebraic space case. Through these comparison theorems our assumptions provide restrictions on the behavior of the special fiber of the above model, and via Nakkajima's classification of SNC log K3 surfaces we deduce that the special fiber must in fact be a K3 surface.

2.2 Models of K3 Surfaces

We begin by presenting Matsumoto's result that assumption ($*$) is satisfied when X can be equipped with an ample line bundle satisfying $p > L^2 + 4$. This foreign constraint on X is primarily enforced to guarantee the existence of an initial model for our K3 surface using [Sai04, Corollary 1.9]. The precise result is the following:

Proposition 2.1 ([Mat14], Section 3)**.** *Let X be a K3 surface over K which admits an ample line bundle satisfying $p > L^2 + 4$. Then there exists a finite extension L/K and an algebraic space model $\mathcal{X}$ of X_L with special fiber a SNC log K3 surface. In particular, X satisfies* ($*$)*.*

Proof. We give a brief summary of the argument. First, in order to have a model to work with, we use a result of Saito which, vaguely speaking, guarantees the existence of a projective semi-stable model of a scheme U of dimension n, after perhaps a finite extension, assuming that it can be constructed as a sequence of schemes $U = U_n \to U_{n-1} \to \cdots \to U_0 = \operatorname{Spec} K$ where U_{i+1} is the complement of a curve of small genus over U_i by a divisor ([Sai04, Corollary 1.9]). Matsumoto applies this to the sequence $X' \to \mathbb{P}^1 \to \operatorname{Spec} K$, where X' is a scheme birational to X and the morphism $X' \to \mathbb{P}^1$ is a morphism induced by our ample line bundle; an ample line bundle can equip X with a variety of morphisms and embeddings, but Matsumoto narrows it down to three possibilities and addresses them in turn. This give us a projective strictly semi-stable model $\mathcal{X}'$ of X' over $\mathcal{O}_K$, after replacing K by a finite extension.

Using Kawamata's minimal model theorem for schemes of relative dimension 2 over a DVR ([Kaw94, Theorem 5.7]), we may refine our model to a flat projective scheme $\mathcal{X}''$ over $\mathcal{O}_K$ with the following properties:

1. The generic fiber is smooth and birational to X', and hence to X;

2. the irreducible components of the special fiber are geometrically normal;

3. The relative canonical divisor $K_{\mathcal{X}''/\mathcal{O}_K}$ is nef and $\mathbb{Q}$-cartier; and

4. $(\mathcal{X}'', X''_k)$ has at worst log terminal singularities.

In his study of such models for K3 surfaces ([Mau14, Section 4.3]), Maulik showed that these conditions imply that the generic fiber X''_K is isomorphic to X and that $K_{\mathcal{X}''/\mathcal{O}_K} = 0$. Kawamata's classification of log terminal singularities arising from such models classifies the possible singularities into one of two types: (1) a semistable singularity (i.e., étale locally of the form $\mathcal{O}_K[x,y,z]/(xy-\pi)$ or $\mathcal{O}_K[x,y,z]/(xyz-\pi)$); (2) an isolated non-smooth point which is a rational double point in the special fiber X''_k.

By a result of Artin ([Art74a, Theorem 2]) the latter sort of singularities can be resolved in the category of algebraic spaces, potentially after replacing K by another finite extension. We thus obtain an algebraic space $\mathcal{X}'''$ over $\mathcal{O}_K$ with at worst strictly semistable singularities. Its special fiber X'''_k is an SNC surface, and Matsumoto shows that in fact it is an SNC log K3 surface, that is, $\bigwedge^2 \Omega^1_{X'''_k/k}(\log)$ is trivial and $H^1(X'''_k, \mathcal{O}_{X'''_k}) = 0$; for definitions, see Section . ∎

As described in Section , Nakkajima proved ([Nak00, Proposition 3.4]) that X_k''' comes in one of the following three shapes:

- Type I: A smooth K3 surface;
- Type II: A union of surfaces $Z_1, \dots, Z_m$ with Z_1 and Z_m rational and others elliptically ruled. The double curves $Z_h \cap Z_{h'}$ are rulings if $|h-h'| = 1$ and empty otherwise (it is essentially a sequence of elliptically ruled surfaces).
- Type III: A union of rational surfaces whose dual graph of the configuration is a triangulation of the sphere S^2.

Theorem 2.1 thus follows if, under the hypotheses, X_k''' cannot be of Type II or III.

2.3 Comparison Theorems in Cohomology

To eliminate the possibility that X_k''' is of Type II or III, Matsumoto proves p- and l-adic comparison theorems between the cohomology of the generic and special fibers of an algebraic space model. They allow us to link information about the cohomology of the generic fiber to the cohomology of the irreducible components of the special fiber, and we will see that the restriction that the representation be crystalline or unramified (in the p- and l-adic scenarios, respectively) forces X_k''' to be of Type I.

We will need the following definition, which is completely analogous to that for schemes:

Definition 2.2. An algebraic space X over $\mathcal{O}_K$ is said to be *semistable semistable* of purely of dimension n if it is étale-locally isomorphic to $\mathcal{O}_K[x_1, \dots, x_{n+1}]/(x_1 \dots x_r - \pi)$, where π is a uniformizer of $\mathcal{O}_K$. It is *strictly semistable* if moreover each irreducible component of the special fiber is smooth.

Given a strictly semistable algebraic space X over $\operatorname{Spec} \mathcal{O}_K$, which use the following standard notation:

- X_K and X_k are the generic and special fibers of X, and $X_{\overline{K}}$ and $X_{\overline{k}}$ are the corresponding geometric fibers.
- Z_h for $h = 1, \dots, m$ are the irreducible components of X_k, which are smooth by assumption. In particular all of the connected components of $(Z_h)_{\overline{k}}$ are smooth and hence irreducible.
- $X_k^{(p)}$ is the disjoint union of the smooth $(n-p)$-dimensional subspaces $Z_H = Z_{h_0} \cap \dots \cap Z_{h_p}$ for subsets $H = \{h_0, \dots, h_p\} \subseteq \{1, \dots, m\}$ of cardinality $p+1$. $X_{\overline{k}}^{(p)}$ is constructed similarly from the irreducible components of $X_{\overline{k}}$.

The p-adic comparison theorem can be thought of as the arithmetic analogue of classical results on semistable degenerations $\pi : \mathfrak{X} \to \Delta$ over the complex unit disk (see [Mor84, Section 1]). If $\mathfrak{X}_0 := \pi^{-1}(0) = \sum Z_i$ is the special fiber consisting of smooth irreducible components Z_i and $\mathfrak{X}_0^{(p)}$ is defined as above, then we have a spectral sequence

$$E_1^{p,q} = \check{H}^q(\mathfrak{X}_0^{(p)}, \mathbb{Q}) \Rightarrow \check{H}^{p+q}(\mathfrak{X}_0, \mathbb{Q})$$

which degenerates at E_2.

In the p-adic setting, we have

Proposition 2.2 ([Mat14], Proposition 2.2)**.** *Assume K is of characteristic 0. Let X be a proper strictly semistable algebraic space over $\mathcal{O}_K$ whose fibers are 2-dimensional schemes. Let $W = W(k)$ and $K_0 = \mathrm{Frac}\, W$.*

1. *We have a p-adic spectral sequence*

$$E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H^{q-2i}_{\mathrm{crys}}(X_k^{(p+2i)}/W)(-i) \Rightarrow H^{p+q}_{\mathrm{logcrys}}(X_k/W)$$

Moreover this spectral sequence is compatible with the monodromy operator in the following sense: there is an endomorphism N on the Hyodo-Steenbrink complex $W_n A^\bullet$ defined by Mokrane [Mok93] which induces a map

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H^{q-2i}_{\mathrm{crys}}(X_k^{(p+2i)}/W)(-i) & \Longrightarrow & H^{p+q}_{\mathrm{logcrys}}(X_k/W) \\ \downarrow N & & \downarrow N \\ E_1^{p+2,q-2} = \bigoplus_{i \geq \max\{0,-p-2\}} H^{q-2i}_{\mathrm{crys}}(X_k^{(p+2i)}/W)(-i+1) & \Longrightarrow & H^{p+q}_{\mathrm{logcrys}}(X_k/W) \end{array}$$

2. *The spectral sequence degenerates at E_2 modulo torsion.*

3. *The morphism N induces the following isomorphisms on E_2 terms modulo torsion:*

$$N : {E_2^{-1,3}}_{\mathbb{Q}} \xrightarrow{\sim} E_2^{1,1}(-1)_{\mathbb{Q}}, N^2 : {E_2^{-2,4}}_{\mathbb{Q}} \xrightarrow{\sim} E_2^{2,0}(-2)_{\mathbb{Q}}$$

4. *Assume moreover taht X_k is liftable to a semistable scheme over K_0. (This liftability assumption is satisfied, for example, if X_k is a projective SNC log K3 surface of Type II or III.) Then we have an isomorphism*

$$H^2_{\mathrm{logcrys}}(X_k/W) \otimes_W K \cong (D_{\mathrm{pst}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \otimes_{K_0^{ur}} \overline{K})^{G_K}$$

compatible with the operator N. In particular if $H^2_{\text{ét}}$ is crystalline (so that monodromy acts trivially on the right hand side) then the operator N on the right hand side of the spectral sequence in (1) is zero modulo torsion.

Note that (1)-(3) are statements for the special fiber only. The only result which compares the generic and special fibers is (4), which is an extension of the C_{st} conjecture to algebraic spaces.

The l-adic case is a generalization to algebraic spaces of the theory surrounding the weight spectral sequence for proper schemes with strict semistable reduction (which can, in turn, be viewed as the l-adic analogue of the nearby cycles spectral sequence for smooth and proper complex analytic spaces whose special fiber is a strict normal crossing divisor, see [Ill94]). For a brief survey of this, see [Ill15, Section 6.4].

Proposition 2.3 ([Mat14], Proposition 2.3)**.** *Let X be as in the previous proposition (with no assumption on* $\operatorname{char} K$*). Let $l \neq p$ be a prime. Let $\Lambda = \mathbb{Z}/l^n\mathbb{Z}, \mathbb{Z}_l$, or $\mathbb{Q}_l$.*

1. *We have an l-adic spectral sequence*

$$E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H_{\text{ét}}^{q-2i}(X_{\overline{k}}^{(p+2i)}, \Lambda(-i)) \Rightarrow H_{\text{ét}}^{p+q}(X_{\overline{K}}, \Lambda)$$

compatible with the action of G_K. Moreover this spectral sequence is compatible with the monodromy operator in the following sense: Let T be an element of the inertia group I_K such that $t_l(T)$ is a generator of $\mathbb{Z}_l(1)$ (where $t_l : I_K \to \mathbb{Z}_l(1)$ is the canonical surjection). Then the endomorphism $N = T - 1$ of the complex $R\psi\Lambda$ of nearby cycles induces a map

$$\begin{array}{ccc} E_1^{p,q} = \bigoplus_{i \geq \max\{0,-p\}} H_{\text{ét}}^{q-2i}(X_{\overline{k}}^{(p+2i)}, \Lambda(-i)) & \Longrightarrow & H^{p+q}(X_{\overline{K}}, \Lambda) \\ \big\downarrow {\scriptstyle 1 \otimes t_l(T)} & & \big\downarrow {\scriptstyle N} \\ E_1^{p+2,q-2} = \bigoplus_{i-1 \geq \max\{0,-p-2\}} H_{\text{ét}}^{q-2i}(X_{\overline{k}}^{(p+2i)}, \Lambda(-i+1)) & \Longrightarrow & H^{p+q}(X_{\overline{K}}, \Lambda) \end{array}$$

of spectral sequences.

2. *The spectral sequence degenerates at E_2 modulo torsion.*

3. *Let $\Lambda = \mathbb{Q}_l$. The morphism N induces the following isomorphisms on E_2 terms*

$$N : E_2^{-1,3} \xrightarrow{\sim} E_2^{1,1}(-1), N^2 : E_2^{-2,4} \xrightarrow{\sim} E_2^{2,0}(-2)$$

Here, in contrast to the p-adic case, it is the spectral sequence which compares the special and generic fibers. We omit the proofs, which consist of an extension of the techniques in the classical proofs to algebraic spaces.

2.4 Criteria for potential good reduction

We now put the results together to prove the criteria for potential good reduction. Recall that it suffices to show that the given hypotheses on cohomology rule out the possibility that the special fiber is of Type II or III.

Note that if H^2 of the generic fiber is crystalline or unramified then the $E_2^{1,1}$ and $E_2^{2,0}$ terms of their respective spectral sequences vanish. To see this, recall that if H^2 is crystalline then the monodromy operator N is 0 by Proposition 2.2 (4) and similarly if H^2 is unramified then N in the l-adic setting is 0 since $T \in I_K$ acts trivially by assumption and $N = T - 1$. But N and N^2 are isomorphisms on $E_2^{1,1}$ and $E_2^{2,0}$, respectively, so they must be zero.

Thus it suffices to show that when the special fiber is of Type II or Type III then either $E_2^{1,1}$ or $E_2^{2,0}$ is nonzero. Since the proof in the l-adic and p-adic cases are more or less identical, we use the l-adic notation like Matsumoto. Let $\Lambda = \mathbb{Q}_p$.

Type II. By definition $E_1^{0,1} = H^1(X_k''^{(0)}, \Lambda)$ and $E_1^{1,1} = H^1(X_k''^{(1)}, \Lambda)$. Each component of $X_k''^{(1)}$, consisting of the double curves, is an elliptic curve, so $H^1(X_k''^{(1)}, \Lambda)$ is the direct sum of Λ^2 for each double curve. For the cohomology of $X_k''^{(0)}$, recall that each non-rational component of $X_k''^{(0)}$ is an elliptic ruled surface, i.e. isomorphic to $E \times \mathbb{P}_k^1$, and the Künneth formula ([Mil13, Theorem 22.4]) gives that the cohomology of such surfaces is Λ^2. Hence $H^1(X_k''^{(0)}, \Lambda)$ is similarly the direct sum of Λ^2 for each non-rational component.

Looking at the explicit description of the d_1 term in the spectral sequence, we see that $E_2^{1,1} = \operatorname{coker}(\rho : E_1^{0,1} \to E_1^{1,1})$ where ρ is induced by restriction on the de Rham-Witt complex (see [Mok93, Proposition 4.12, Corollaire 4.14]). Its explicit description is not important, since we can conclude that the cokernel is nonzero for dimension reasons: if the special fiber is the union of $m+1$ surfaces, then there are $m-1$ non-rational components and m double curves, so that

$$E_2^{1,1} = \operatorname{coker}(\rho : \Lambda^{\oplus 2m-2} \to \Lambda^{\oplus 2m}) \neq 0.$$

Type III. In this case, it is clear that $E_1^{1,0} = H^0(X_k''^{(1)}, \Lambda)$ and $E_1^{2,0} = H^0(X_k''^{(2)}, \Lambda)$ are the direct sum of Λ for each double curve and triple point, respectively. Since the dual graph of the configuration is a triangulation of the sphere S^2, working through the definitions shows that $E_2^{2,0} = \operatorname{coker}(\mathrm{res} : E_1^{1,0} \to E_1^{2,0})$ is isomorphic to $H^2_{\mathrm{sing}}(S^2, \Lambda)$. Since this latter singular cohomology group is nonzero, $E_2^{2,0}$ is nonzero, as needed.

2.5 Refinements

Liedtke and Matsumoto ([LM18]) refined the l-adic result by showing that only an *unramified* extension was required and that, even in the case where X had only potential good reduction, the smooth model after a finite extension always induces a singular model of X with manageable singularities, called *canonical* or *RDP singularities.*

Definition 2.3. Let S be a surface over an arbitrary field F. We say that it has *at worst canonical singularities* (or *RDP singularities*) if it is geometrically normal, Gorenstein, and if the minimal resolution of singularities $f : \tilde{S} \to S$ satisfies $f^*\omega_S \cong \omega_{\tilde{S}}$. Here ω_S and $\omega_{\tilde{S}}$ denote the dualizing sheaves of S and $\tilde{S}$, respectively.

For a classical account of canonical singularities on surfaces, see [Rei87].
Namely, studying Galois actions on models, Liedtke and Matsumoto proved

Theorem 2.3 ([LM18], Theorem 6.1)**.** *Let X be a K3 surface over K that safisfies $(*)$. If the G_K representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for some $l \neq p$ then*

1. *there exists a model of X that is a projective scheme over $\mathcal{O}_K$, whose special fiber is a K3 surface with at worst RDP singularities;*

2. *there exists an integer N, independent of X and K, and a finite unramified extension L/K of degree $\leq N$ such that X_L has good reduction over L.*

This model will be crucial to proving the good reduction criterion, so we describe it here.

To be more precise, the "RDP model" model provided by this theorem is not unique, but we obtain a canonical model after we fix a polarization $\mathcal{L}$ on X, and the model that we will primarily be concerned with. Let L/K be a finite, unramified extension with Galois group G over which X_L has good reduction, say with model $\mathcal{Y} \to \operatorname{Spec} \mathcal{O}_L$; we can assume that L/K is unramified by (2).

We can find a birational map $\mathcal{Y} \dashrightarrow \mathcal{Y}^+$ such that the specialization $\mathcal{L}^+_{k_L}$ of $\mathcal{L}_L$ to the special fiber $\mathcal{Y}^+_{k_L}$ is big and nef by [LM18, Proposition 4.5]. Then the projective scheme

$$P(X_L, \mathcal{L}_L) := \operatorname{Proj}\left(\bigoplus_{n\geq 0} H^0(\mathcal{Y}^+, \mathcal{L}^{+,\otimes n})\right)$$

(see [LM18, Proposition 4.6].) This depends only on X_L and $\mathcal{Y}_L$ upto canonical isomorphism and not on the choice of $\mathcal{Y}$ or $\mathcal{Y}^+$ by [LM18, Proposition 4.7]. Moreover the G-action on X_L extends to a *regular* G-action on $P(X_L, \mathcal{L}_L)$ by [LM18, Proposition 5.1], so that we may take the quotient

$$P(X, \mathcal{L}) := P(X_L, \mathcal{L}_L)/G.$$

This is a flat projective $\mathcal{O}_K$-scheme again depending on X and $\mathcal{L}$ up to *canonical* isomorphism whose special fiber $P(X, \mathcal{L}))_k$ has at worst canonical singularities, providing us our desired canonical RDP model.

Furthermore, the minimal resolution of singularities $Y \to P(X, \mathcal{L})_k$ of its special fiber is a K3 surface over K by [LM18, Proposition 4.6], depends only on X, and is unique up to canonical isomorphism.

To summarize, we have the following configuration of objects over their respective bases:

$$
\begin{tikzcd}
P(X_L, \mathcal{L}_L)_{k_L} \arrow[rr] \arrow[dr] & & P(X, \mathcal{L})_k \arrow[drr] & & & \\
& P(X_L, \mathcal{L}_L) \arrow[rrr] & & & P(X, \mathcal{L}) \arrow[ddddd] & \\
\mathcal{Y}_{k_L} \arrow[uu, squiggly] \arrow[dr, hook] \arrow[ddd] & & Y \arrow[uu, squiggly] \arrow[ddd] & & & \\
& \mathcal{Y} \arrow[uu] \arrow[ddd] & & & & \\
& & & X_L \arrow[ull, hook] \arrow[rr] \arrow[ddd] & & X \arrow[ddd] \\
k_L \arrow[rr, hook] \arrow[dr, hook] & & k \arrow[drr] & & & \\
& \mathcal{O}_L \arrow[rrr] & & & \mathcal{O}_K & \\
& & & L \arrow[ull, hook] \arrow[rr] & & K \arrow[ul, hook]
\end{tikzcd}
$$

where the squiggly arrows are minimal resolutions of singularities. All of these constructions are compatible with base change, which is to say that all squares are base change squares. If X has good reduction over K so that $L = K$ in the above construction, it follows from [MM64] that Y is the special fiber of any smooth model.

Definition 2.4. We call Y the *canonical reduction* of X and $P(X, \mathcal{L})$ the *(canonical) RDP model* of $(X, \mathcal{L})$.

We return to the proof of Theorem 2.1. Theorem 2.3 (2) will let us close the circle of implications of the first three conditions of the theorem. To incorporate the p-adic criterion, we need two results relating field extensions and crystalline representations. The first is

Lemma 2.1 ([CLL17], Lemma 2.6)**.** *Let $\rho : G_K \to \mathrm{GL}(V)$ be a p-adic Galois representation. Let $K \subseteq K'$ be a finite extension and denote by $\rho' : G_{K'} \to \mathrm{GL}(V)$ the restriction of ρ to $G_{K'}$.*

(1) Assume that K'/K is unramified. Then ρ is a crystalline G_K-representation if and only if ρ' is a crystalline $G_{K'}$-representation.

(2) Assume that K'/K is totally ramified and Galois, say with Galois group $H = \mathrm{Gal}(K'/K)$. Then the following are equivalent:

(a) ρ is crystalline;

(b) ρ' is crystalline and the induced H-action on

$$(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_{K'}}$$

is trivial.

(3) The following are equivalent:

(a) ρ is crystalline;

(b) ρ is potentially crystalline and the I_K-action on $\mathbb{D}_{\mathrm{pst}}(V)$ is trivial.

Proof. We may assume without loss of generality that $K = \widehat{K}$ since all of the invariants stay the same. Let $d := \dim_{\mathbb{Q}_p} V$.

(1) If ρ is crystalline then so is ρ'. Indeed, if $M = (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_K}$, we have by construction an inclusion of $B_{\mathrm{cris}}^{G_{K'}} = K_0'$ vector spaces

$$M \otimes_{K_0} K_0' \hookrightarrow (B_{\mathrm{cris}} \otimes V)^{G_{K'}}.$$

If ρ is crystalline then $\dim_{K_0} M = d$ and hence $\dim_{K_0'} M \otimes K_0' = d$. Thus

$$d = \dim_{K_0'}(M \otimes_{K_0} K_0') \leq \dim_{K_0'}(B_{\mathrm{cris}} \otimes V)^{G_{K'}} \leq d$$

and thus $\dim_{K_0'}(B_{\mathrm{cris}} \otimes V)^{G_{K'}} = d$, which means that ρ' is crystalline.

Conversely, suppose ρ' is crystalline. Replacing $K \subseteq K'$ by its Galois closure, we may assume that K'/K is Galois, say with Galois group H. We have the equality of sets

$$\left((V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_{K'}}\right)^H = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K}.$$

Indeed, if a set M has a G_K-action, then $M^{G_{K'}}$ has an $G_K/G_{K'} = H$ action, and thus taking H-invariants of $M^{G_{K'}}$ is equivalent to taking G_K-invariants of M.

Since K'/K is unramified, K_0'/K_0 is also Galois with Galois group H. Thus by Galois descent the map $W \mapsto W^H$ is a bijective dimension-preserving correspondence between K_0'-vector spaces with an H-action and K_0-vector spaces. Furthermore, since ρ' is crystalline, $(V \otimes B_{\mathrm{cris}})^{G_{K'}}$ is a K_0'-vector space of dimension d. Thus $\left((V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_{K'}}\right)^H = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K}$ is a K_0-vector space of dimension d, which means that ρ is crystalline.

(2) Assume that K'/K is totally ramified. Then $K_0' = K_0$ so we have

$$\dim_{K_0}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_K} \leq \dim_{K_0(=K_0')}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_{K'}} \leq d.$$

We have equality on the right if and only if ρ' is crystalline, equality on the left if and only if H acts trivially, and equality everywhere if and only if ρ is crystalline.

(3) If ρ is crystalline then ρ is trivially potentially crystalline and the I_K-action on $\mathbb{D}_{\mathrm{pst}}(V)$ is trivial. Indeed, if ρ is crystalline then inertia acts trivially on $\mathbb{D}_{st}(V) = \mathbb{D}_{\mathrm{cris}}(V)$ by [Fra94, exp.III 5.1.7].

So assume ρ is potentially crystalline with trivial I_K-action on $\mathbb{D}_{\mathrm{pst}}(V)$. Choose a finite Galois extension K'/K such that $\rho' := \rho|_{G'_K}$ is crystalline. By (1), ρ is crystalline if and only if $\rho|_{G_L}$ is crystalline, where L is the maximal unramified extension of K inside K'. Since $\rho|_{G'_K} = (\rho|_{G_L})|_{G_{K'}}$, we may replace K by L and thereby assume that K'/K is totally ramified.

So to prove that ρ is crystalline it suffices, by (2), to show that the induced H-action is trivial.

One can combine (1) and (2) to show that

$$\mathbb{D}_{\mathrm{pst}}(V) \cong (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_{K'}} \otimes_{K_0} K_0^{\mathrm{nr}}$$

(where the tensor is over K_0 since $K_0 = K_0'$ in this setting) and the I_K-action on $\mathbb{D}_{\mathrm{pst}}(V)$ is the extension of scalars of the natural action of I_K on $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{G_{K'}}$ via the surjection $I_K \twoheadrightarrow H$. Since the action of I_K is trivial so is the action of H, as needed.

□

Secondly, we need an extension of the C_{st}-conjecture to the scenario in which the model is an algebraic space:

Theorem 2.4 ([CLL17], Theorem 2.4)**.** *Let X be a smooth and proper variety over K and assume that there exists a smooth and proper algebraic space*

$$\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$$

whose generic fiber is X and whose special fiber $\mathcal{X}_k$ is a scheme. Then

(1) the G_K-representation on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline for all n;

(2) for all n, there exist isomorphisms

$$\mathbb{D}_{\mathrm{cris}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{cris}}(\mathcal{X}_k, K_0)$$

of F-isocrystals over K_0.

This theorem was essentially proven by Colmez-Nizioł in [CN17] and Bhatt-Morrow-Scholze in [BMS16].

With these in hand, we may prove Theorem 2.1.

Proof of Theorem 2.1. For every prime l, including p, write $V_l := H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ and let $\rho_l : G_K \to \mathrm{GL}(V_l)$ be the associated Galois representation. The $\mathbb{Q}_l$-dimension of V_l is independent of l and is equal to the second Betti number $b_2 := b_2(X)$.

Suppose first that X has good reduction after a finite unramified extension L/K. Then the G_L-representation $H^2_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_l) = H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for all $l \neq p$. But since L/K is unramified we have $L^{\mathrm{nr}} \subseteq K^{\mathrm{nr}}$, so

$$\begin{aligned} I_L &= \mathrm{Gal}(\overline{L}/L^{\mathrm{nr}}) \\ &= \mathrm{Gal}(\overline{K}/L^{\mathrm{nr}}) \\ &\supseteq \mathrm{Gal}(\overline{K}/K^{\mathrm{nr}}) \\ &= I_K \end{aligned}$$

Thus if the inertia group I_L acts trivially on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ then so does I_K, that is, the G_K-representation is unramified as well. This proves (1) ⇒ (3).

Clearly (3) ⇒ (2), and (2) ⇒ (1) is the substance of Theorem 2.3. Thus (1), (2), and (3) are equivalent.

Suppose again that X has good reduction after a finite unramified extension L/K. By Theorem 2.4 the representation ρ_L is a crystalline representation. But $\rho_L = \rho_K|_{G_L}$ so by Lemma 2.1 (1) the representation ρ_K is also crystalline, which establishes (1) $\Rightarrow$ (4).

Now assume that ρ_p is crystalline. We may assume again that K is complete, since the invariants are the same. By Lemma 2.1 (3), the induced I_K-action on $\mathbb{D}_{\mathrm{pst}}(V_p)$ is trivial, so in particular for every $g \in I_K$ the trace of the map g^* on $\mathbb{D}_{\mathrm{pst}}(V_p)$ is equal to b_2. It then follows from [Och99, Theorem 3.1] that, for K3 surfaces, we have

$$\mathrm{Tr}(g^*, H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)) = \mathrm{Tr}(g^*, \mathbb{D}_{\mathrm{pst}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$$

for all $l \neq p$. In particular, the trace of the map g^* on $V_l = H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is equal to b_2 for all $l \neq p$.

On the other hand, since ρ_p is crystalline, Theorem 2.2 tells us that there exists a finite, possibly ramified extension L/K such that X_L has good reduction over $\mathcal{O}_L$. It follows that for every $l \neq p$ the image of I_K in $GL(V_l)$ is finite; since I_L acts trivially, the action of I_K factors through the finite quotient I_K/I_L. By a general linear algebra argument, since $\mathbb{Q}_l$ has characteristic zero, a linear automorphism of finite order on V_l is trivial if and only if its trace is equal to b_2.

Putting these together, it follows that g acts trivially on V_l for all $g \in I_K$, i.e., the G_K-representation on V_l is unramified. This proves (4) $\Rightarrow$ (3) and completes the proof of Theorem 2.1. $\square$

These results on potential good reduction of K3 surfaces cleared the necessary ground for Chiarellotto, Lazda, and Liedtke's to establish their criterion for good reduction of K3 surfaces.

Chapter 3

Good Reduction

The previous chapter established a fairly comprehensive criterion for potential good reduction of K3 surfaces. We recall it here:

Theorem 2.1 ([CLL17], Theorem 1.1)**.** *Suppose that $p > 0$, and let X be a K3 surface over K that satisfies $(*)$. Then, the following are equivalent:*

(1) X has good reduction after a finite and unramified extension of K.

(2) The G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for one $l \neq p$.

(3) The G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified for all $l \neq p$.

(4) The G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline.

The goal now is to add extra conditions to criteria (2)-(4) which ensure that X has good reduction, not only good reduction after a finite unramified extension. To phrase this criterion we need to have at hand more refined models of K3 surfaces, constructed by Liedtke and Matsumoto.

Suppose X is a K3 surface over K that satisfies $(*)$ and the equivalent conditions of Theorem 2.1. Let Y be the canonical reduction of X, as defined in Section 2.5. Chiarellotto, Lazda, and Liedtke's criterion for good reduction states that we can detect K3 surfaces with good reduction among those with potential good reduction by comparing its cohomology with the cohomology of its canonical reduction:

Theorem 3.1 ([CLL17], Theorem 1.6)**.** *Let X be a K3 surface satisfying $(*)$ and satisfying the equivalent conditions of Theorem 2.1. Then the following are equivalent:*

(1) X has good reduction over K,

(2) There exists a prime $l \neq p$ (equivalently, for all $l \neq p$) such that the G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified and there exists an isomorphism

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \cong H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l).$$

of G_k-modules,

(3) If $p > 0$, $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, and there exists an isomorphism

$$\mathbb{D}_{\text{cris}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^2_{\text{cris}}(Y/K_0)$$

of F-isocrystals over K_0.

The key data that we exploit when our K3 surface has potential good reduction is the Galois action which accompanies the base change to a finite unramified extension. When a K3 surface X over K has good reduction after a Galois extension L/K, the surface X_L inherits a $G = \text{Gal}(L/K)$-structure, and we can understand the behavior of the model by understanding this action. This will be made more precise in the following sections, in which we describe a criteria for good reduction in terms of the Galois action.

3.1 Galois actions on models

Let X be a K3 surface over K satisfying $(*)$ and the equivalent conditions of Theorem 2.1. So X has potential good reduction after a finite unramified extension X/K, say $\mathcal{Y} \to \mathcal{O}_L$. Since the generic fiber X_L has a natural semi-linear $G := \text{Gal}(L/K)$-action, the space $\mathcal{Y}$ has a rational G-action. The proposition which enables us to move between potential good reduction and good reduction is the following small proposition, which allows us to phrase good reduction in terms of the behavior of the G-action on $\mathcal{Y}$:

Proposition 3.1 ([CLL17], Proposition 7.1)**.** *Let X be a K3 surface over K satisfying $(*)$ and the equivalent conditions of Theorem 2.1. Then in the above notation, the quotient $\mathcal{X} := \mathcal{Y}/G$ exists as a smooth and proper algebraic space over K, and we have a G-equivariant isomorphism $\mathcal{Y} \cong \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L$.*

Proof. Since L/K is unramified the map $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$ is finite étale. The G-action on $\mathcal{Y}$ is equivalent to descent data for $\mathcal{Y}$ with respect to the morphism $\operatorname{Spec} \mathcal{O}_L \to \operatorname{Spec} \mathcal{O}_K$, completely analogously to Section 1.5.4.

Following the procedure in [Ols10, Example 5.3.1], we may take the quotient of $\mathcal{Y}$ by the action of G. Since the equivalence relation defined by G is étale and hence flat, $\mathcal{X} := \mathcal{Y}/G$ exists as an algebraic space by [Art74b, Corollary 6.3]. Since taking quotients is the inverse to base change by the group action, we have that $\mathcal{Y} \cong \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathcal{O}_L$.

This shows that the descent data given by the G-action is effective. Since Lemma [Stacks, Tag 0422] and Lemma [Stacks, Tag 0429] testify that the properties of being smooth and proper are fpqc local on the base for algebraic spaces, respectively, the smoothness and properness of $\mathcal{Y}$ descend to $\mathcal{X}$. □

In particular, X has good reduction over K if and only if such a $\mathcal{Y}$ can be found for which the G-action is regular. This translates the problem of adding extra criteria to ensure good reduction to adding criteria on the G-action which guarantees that it is regular.

To this end, we will take a detailed look at the behavior of birational maps on models of K3 surfaces. First we need to introduce some vocabulary and machinery to grapple with the singular behavior of the special fiber.

3.1.1 Weyl groups and K3 surfaces

Recall that the canonical reduction Y of X was defined to be the minimal resolution of singularities of $P(X, \mathcal{L})_k$, a surface with canonical singularities. Minimal resolutions of canonical singularities on surfaces over general fields can be comprehensively described in the language of root lattices and Weyl groups.

As geometric motivation, let V be a finite-dimensional Euclidean space with bilinear form $(-,-) : V \times V \to \mathbb{R}$. If $v \in V$ is nonzero, let $H_v := \{w \in V \mid (w, v) = 0\}$ denote the hyperplane perpendicular to v. The *reflection* about H_v is the linear map $s_v : V \to V$ which maps v to $-v$ and fixes every $w \in H_v$. Since $V = H_v \oplus \mathrm{Span}\{v\}$, an arbitrary $w \in V$ can be written in the form $w = av + z$ for $a \in \mathbb{R}$ and $z \in H_v$. Then $(v, w) = a(v, v)$ and $s_v(w) = -av + z = w - 2av$, so

$$\begin{aligned} s_v &: V \to V \\ w &\mapsto w - 2\frac{(v,w)}{(v,v)}v \end{aligned}$$

For $v, w \in V$, write

$$\langle w, v\rangle = 2\frac{(v,w)}{(v,v)}$$

It is easy to check that the reflection s_u for any $u \in V$ is an orthogonal transformation, i.e., it preserves the linear product.

Definition 3.1. Let V be as above. A *root system* for V is a finite subset $\Phi \subseteq V$ such that

1. $0 \notin \Phi$ and $\mathrm{Span}(\Phi) = V$;
2. if $\alpha, \lambda\alpha \in \Phi$ with $\lambda \in \mathbb{R}$ then $\lambda = \pm 1$;
3. $s_\alpha(\beta) \in \Phi$ for all $\alpha, \beta \in \Phi$;
4. $\langle \alpha, \beta\rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

Definition 3.2. Let Φ be a root system. The *Weyl group* of Φ is the subgroup of linear transformations of V generated by all reflections s_α with $\alpha \in \Phi$, i.e.,

$$W = W(\Phi) := \langle s_\alpha \mid \alpha \in \Phi\rangle.$$

Intuitively, the Weyl group is the group of automorphisms generated by reflections of the root vectors in a root system, which are highly symmetric configurations of vectors. For our exposition this baseline geometric intuition is all we will need. There is much more to the theory that we omit, for example the theory of Cartan matrices and Dynkin diagrams. For these concepts, examples, and further results, see [Die].

The above constructions can be imitated to describe the exceptional divisors of a minimal resolution of singularities of a surface, in which the inner product is substituted by the intersection form on divisors. Let F be an arbitrary field, S/F a surface with canonical singularities, and $\tilde{S} \to S$ its minimal resolution of singularities. Let $E_P \subseteq \tilde{S}$ be the exceptional divisor over a singular point $P \in S$ with irreducible components $E_{P,i} \subseteq E_P$. If we let

$$n_{P,i} = h^0(E_{P,i}) := \dim_{F(P)} H^0(E_{P,i}, \mathcal{O}_{E_{P,i}})$$

then the intersection matrix is

$$E_{P,i} \cdot E_{P,j} = \begin{cases} -2n_{P,i} & \text{if } i = j \\ \max(n_{P,i}, n_{P,j}) & \text{if } i \neq j \text{ and } E_{P,i} \cap E_{P,j} \neq \varnothing \\ 0 & \text{if } E_{P,i} \cap E_{P,j} = \varnothing. \end{cases}$$

Definition 3.3. In the above notation, the subgroup $\Lambda_S \subseteq \mathrm{Pic}(\tilde{S})$ freely generated by the integral components $E_{P,i}$ for all P and i, equipped with its intersection pairing, is called the *root lattice* of S.

This is analogous to the $\mathbb{Z}$-lattice generated by a root system over a vector space. The reflection with respect to an irreducible component $E_{P,i}$ is defined to be

$$\begin{aligned} s_{P,i} : \Lambda_S &\to \Lambda_S \\ D &\mapsto D + \frac{1}{n_{P,i}}(D \cdot E_{P,i})E_{P,i} \end{aligned}$$

which we see is analogous to the classical definition of a reflection along a vector in light of the above explicit description of the intersection pairing. Thus we make the following definition:

Definition 3.4. The *Weyl group* $\mathcal{W}_S \subseteq \mathrm{Aut}_{\mathbb{Z}}(\Lambda_S)$ of S is the group generated by the reflections $s_{P,i}$ for all P and i. Using the same formula, we view $\mathcal{W}_S$ as a subgroup of $\mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(\tilde{S}))$.

This analogue to the classical theory of root systems extends to their Dynkin diagrams, and the compatibility of these constructions with base change by a Galois extension, is explored in [CLL17, Section 3, 4].

Let us return to our setting. Let X be a K3 surface over K with potential good reduction, and let $Y \to P(X, \mathcal{L})_k$ be the minimal resolution of singularities of $P(X, \mathcal{L})$. We denote by $E_{X,\mathcal{L}}$ the exceptional locus of $Y \to P(X, \mathcal{L})$, by $\Lambda_{X,\mathcal{L}} \subseteq \mathrm{Pic}(Y)$ the corresponding root lattice, and by $\mathcal{W}_{X,\mathcal{L}} \subseteq \mathrm{Aut}_{\mathbb{Z}}(\Lambda_{X,\mathcal{L}})$ the Weyl group of $(X, \mathcal{L})$. The following is an exposition on the results in [CLL17, Section 7], which connects these geometric objects to birational maps of models.

Definition 3.5. We say that a smooth model $\mathcal{X}$ of X is $\mathcal{L}$-terminal, that $(\mathcal{X}, \mathcal{L})$ is terminal, or even that $\mathcal{X}$ is a terminal model of $(X, \mathcal{L})$, if the specialization $\mathcal{L}_k$ on the special fiber $\mathcal{X}_k$ is big and nef.

In addition to specializing to the generic fiber, we may push $\mathcal{L}$ along a birational map. More precisely, if we have a birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ of smooth models of K3 surfaces with $(\mathcal{X}, \mathcal{L})$ terminal, we may push forward the polarization $\mathcal{L}$ to a line bundle $\mathcal{L}^+$ on the generic fiber of $\mathcal{X}^+$.

Definition 3.6. We say that f is a *terminal* birational map if $(\mathcal{X}^+, \mathcal{L}^+)$ is also terminal.

Remark. By [LM18, Proposition 4.5], if X is any smooth and proper surface with numerically trivial $\omega_{X/K}$ with some smooth model we can, up to birational equivalence, find an $\mathcal{L}$-terminal model.

A birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ of K3 surfaces restricts to isomorphisms on both its generic and special fibers. Indeed, such a birational map is an isomorphism outside of a collection of curves on its special fiber. Thus it induces an isomorphism on the generic fiber by restriction. The restriction to the special fiber, on the other hand, is only an isomorphism outside of these curves, but by minimality of K3 surfaces such a map must be an isomorphism.

That being said, this does not mean that f is itself an isomorphism, and when we consider the restriction of f to its special fiber we lose information about the exceptional curves. This motivates the following workaround for pulling back objects along birational maps.

First we describe the classical case, which is interesting in its own right. We refer to [Voi03] for details. Recall that a *correspondence* between two smooth varieties X and Y is a cycle $\Gamma \in \mathrm{CH}(X \times Y)$, the Chow ring of $X \times Y$. If Y is projective, a correspondence $\Gamma \in \mathrm{CH}_k(X \times Y)$ induces a morphism

$$\begin{aligned} \Gamma^* : \mathrm{CH}(Y) &\to \mathrm{CH}(X) \\ Z &\mapsto p_{1*}(p_2^*(Z) \cdot \Gamma) \end{aligned}$$

where the $p_1 : X \times Y \to X$ and $p_2 : X \times Y \to Y$ are the projection maps, $\cdot$ is the intersection form, and p_{1*} and p_2^* are defined as in [Voi10, Lemma 9.7] and [Voi10, Section 9.2.1]. In particular, the *graph* of a rational map $f : X \dashrightarrow Y$ with indeterminacy set I_f, defined to be

$$\Gamma_f := \overline{\{(x, y) \in X \times Y \mid x \notin I_f \text{ and } y = f(x)\}},$$

is a cycle in $\mathrm{CH}_{\dim X}(X \times Y)$. Since $\mathrm{CH}_1(X) = \mathrm{Pic}(X)$ and the intersection form is an intersection for divisors which properly intersect, Γ_f induces a map

$$\begin{aligned} (\Gamma_f)^* : \mathrm{Pic}(Y) &\to \mathrm{Pic}(X) \\ D &\mapsto p_{1*}(p_2^* D \cap \Gamma_f). \end{aligned}$$

When f is regular, this is the typical pullback of a line bundle.

On the cohomological side, if X and Y are complex projective algebraic manifolds, the graph Γ_f also defines an action $f^* : H^*(Y) \to H^*(X)$ on cohomology, defined by

$$f^*\alpha := p_{1^*}(p_2^*\alpha \smile [\Gamma_f])$$

where $[\Gamma_f]$ is the fundamental cohomology class of the closed submanifold Γ_f, p_2^* is the classical pullback, and $p_1* := \mathrm{PD}_X^{-1} \circ p_{1\#} \circ \mathrm{PD}_{X \times Y}$ where PD is the Poincaré duality isomorphism and $p_{1\#}$ is the usual (covariant) pushforward on homology. This coincides with the usual definition of pullback when f is regular (for more, see [Roe15]).

These are connected via the class map

$$\mathrm{cl} : \mathrm{CH}_l(X) \to H^{2n-2l}(X, \mathbb{Z})$$

which takes a cycle Z to its fundamental class $[Z]$, which is well-defined by [Voi03, Lemma 9.18]. By [Voi03, Proposition 9.20] the class map is compatible with the structures of intersection product and cup product, in the sense that

$$\mathrm{cl}(Z \cdot Z') = \mathrm{cl}(Z) \smile \mathrm{cl}(Z') \in H^{2k+2l}(X, \mathbb{Z})$$

for $Z \in \mathrm{CH}^l(X)$ and $Z' \in \mathrm{CH}^k(X)$. To summarize, we can emulate the pullback of a map on divisors and cohomology when the morphism is only rational by recourse to its graph, which encodes the locus of indeterminacy.

Returning to our arithmetic situation, let $\mathcal{X}$ be a model of X and let $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ be a birational map to some other smooth model of a K3 surface X^+ over K. Let

$$\Gamma_f := \overline{\Gamma_{f_K}} \subseteq \mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}^+$$

be the graph of f, where f_K is restriction of f to its generic fiber. The generic fiber Γ_{f_K} is simply the graph of the morphism f_K, since f is an isomorphism on the generic fiber. The same is not necessarily true for the special fiber $\Gamma_{f,k}$ since the isomorphism $f_k : Y \to \mathcal{X}_k^+$ is *not* the restriction of f to the special fiber due to the exceptional curves on the special fiber. The special fiber induces a homomorphism

$$\begin{aligned} \tilde{s}_f : \mathrm{Pic}(\mathcal{X}_k^+) &\to \mathrm{Pic}(Y) \\ D &\mapsto p_{1*}(p_2^* D \cap \Gamma_{f,k}) \end{aligned}$$

where $p_i : Y \times_k \mathcal{X}_k^+ \rightrightarrows Y, \mathcal{X}_k^+$ are the respective projections. This is the arithmetic analogue of the pullback on Chow groups describe above. Composing with the push-forward via f_k, we obtain a morphism

$$s_f : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y)$$

which is also the map induced by the pull-back cycle $\Gamma \subseteq Y \times_k Y$ by the same formula. That is to say, the map $\tilde{s}_f$ with $D \in \mathrm{Pic}(Y)$ instead coincides with the pullback of $\tilde{s}_f$ along the map π in the below commutative diagram

$$\begin{tikzcd} Y \times_k Y \arrow[r, "\pi"] \arrow[d] & Y \times_k \mathcal{X}_k^+ \arrow[r, "p_1"] \arrow[d, "p_2"] & Y \arrow[d] \\ Y \arrow[r, "f_k"] & \mathcal{X}_k^+ \arrow[r] & k \end{tikzcd}$$

If s_f is to behave like its regular counterpart, it ought to preserve intersection pairing and work nicely with composition. And it does:

Proposition 3.2 ([CLL17], Proposition 7.4)**.** *1. The map s_f preserves the intersection pairing, that is,*

$$s_f(D_1) \cdot s_f(D_2) = D_1 \cdot D_2$$

for $D_i \in \mathrm{Pic}(Y)$.

(2) If f, g are composable birational maps, then we have

$$s_{g \circ f} = s_f \circ f_k^* \circ s_g \circ (f_k^*)^{-1}$$

(3) The map s_f is invertible, and we have

$$(s_f)^{-1} = f_k^* \circ s_{f^{-1}} \circ (f_k^*)^{-1}.$$

Proof. Since $s_{\mathrm{id}} = \mathrm{id}$ we immediately have that (2) $\Rightarrow$ (3). And by the above remark, it suffices to prove (1) and (2) by working with $\tilde{s}_f$, which means that we prove compatibility with the intersection pairing on $\mathrm{Pic}(Y)$ and $\mathrm{Pic}(\mathcal{X}_k^+)$ and that $\tilde{s}_{g\circ f} = \tilde{s}_f \circ \tilde{s}_g$.

To this end, we generalize and interpret $\tilde{s}_f$ cohomologically by using the class map to switch from divisors (Chow groups) to cohomology. Fix a prime $l \neq p$. Via the class map for étale cohomology, the function $\tilde{s}_f$ extends to the map

$$\begin{aligned}\tilde{s}_{f,l} : H^2_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Q}_l) &\to H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)\\ \alpha &\mapsto p_{1*}([\Gamma_{f,k}] \smile p_2^*\alpha).\end{aligned}$$

This map extends $\tilde{s}_f$ because the map $\mathrm{Pic}(Y) \to H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)$ is injective. To see this, note that the map decomposes as the composition

$$\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_{\overline{k}}) \to H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l).$$

The first morphism arises from the Grothendieck spectral sequence with the functors $(-)^{\mathrm{Gal}(\overline{K}/K)}$ and $\Gamma(-, \mathbb{G}_m)$, which states in our case, since Y_k is projective, that we have a spectral sequence

$$E_2^{p,q} = H^p(\mathrm{Gal}(\overline{k}/k), H^q(Y_{\overline{k}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(Y_K, \mathbb{G}_m).$$

This has a first terms exact sequence

$$0 \to H^1(\mathrm{Gal}(\overline{k}/k, \overline{k}^*)) \to \mathrm{Pic}(Y_k) \to \mathrm{Pic}(Y_{\overline{k}})$$

since $H^1(Y, \mathbb{G}_m) \cong \mathrm{Pic}(Y)$ and similarly for $Y_{\overline{k}}$. But $H^1(\mathrm{Gal}(\overline{k}/k), \overline{k}^*) = 0$ by Hilbert's Theorem 90, so the map $\mathrm{Pic}(Y) \to \mathrm{Pic}(Y_{\overline{k}})$ is injective. The second map is injective by [Lui07, Proposition 6.2] and the fact that for K3 surfaces the Néron-Severi group coincides with the Picard group (Proposition 1.1).

Thus it suffices to prove that $\tilde{s}_{f,l}$ is compatible with composition and, by the compatibility of the class map with the intersection pairing, that $\tilde{s}_{f,l}$ preserves the Poincaré pairing, and this was proved in [LM18, Lemma 5.6]. $\square$

Since s_f preserves the intersection pairing, we can consider it as an element $s_f \in \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y))$. Chiarellotto, Lazda, and Liedtke showed that the elements of $\mathcal{W}_{X,\mathcal{L}} \leq \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y))$ are precisely those of the form s_f for f a terminal map, in the sense of the following results:

Proposition 3.3 ([CLL17], Proposition 7.5)**.** *Let $(\mathcal{X}, \mathcal{L})$ be terminal and assume that all irreducible components of $E_{X,\mathcal{L}}$ are geometrically irreducible. Then, for any element $w \in \mathcal{W}_{\mathcal{X},\mathcal{L}}$, there exists a terminal birational map*

$$f : \mathcal{X} \dashrightarrow \mathcal{X}^+$$

such that $s_f = w$ as automorphisms of $\mathrm{Pic}(Y)$*.*

Theorem 3.2 ([CLL17], Theorem 7.6)**.** *Let $\mathcal{X}$ be a terminal model of a polarized K3 surface $(X, \mathcal{L})$ over K and $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ a terminal birational map.*

(1) $s_f \in \mathcal{W}_{X,\mathcal{L}} \leq \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y))$.

(2) f is an isomorphism if and only if $s_f = \mathrm{id}$.

We omit their proofs.

In particular, Theorem 3.2 states that to detect whether a terminal birational map is regular, i.e., an isomorphism, we can simply look at whether the corresponding automorphism $s_f \in \mathcal{W}_{X,\mathcal{L}}$ acts as the identity. We use this to state the first form of the obstruction to good reduction.

3.1.2 Group Cohomology and Good Reduction

Now we apply this to the case where a K3 surface X over K has potential good reduction over an unramified extension. Suppose X has good reduction after a finite unramified extension L/K, say with Galois group $G = \mathrm{Gal}(L/K)$ and residue field extension k_L/k. For any object over L or k_L and $\sigma \in G$, let $(-)^\sigma$ denote the base change by σ for any $\sigma \in G$.

Fix a polarization $\mathcal{L}$ of X and as above let $P(X, \mathcal{L})$ be the RDP model over $\mathcal{O}_K$ and let $Y \to P(X, \mathcal{L})_k \to k$ be the canonical reduction of X. Choose an $\mathcal{L}_L$-terminal model $\mathcal{Y}$ of X_L over $\mathcal{O}_L$. We use the above results to define a map

$$\alpha_{\mathcal{Y}} : G \to \mathcal{W}_{X_L, \mathcal{L}_L}$$

as follows.

For any $\sigma \in G$, the G-action on the generic fiber X_L induces an $\mathcal{O}_L$-linear birational map

$$f_\sigma : \mathcal{Y} \dashrightarrow \mathcal{Y}^\sigma.$$

which is terminal with respect to $\mathcal{L}_L$ on the generic fiber. By Theorem 3.2 we may associate to this an element $s_{f_\sigma} \in \mathcal{W}_{X_L, \mathcal{L}_L}$ of the Weyl group. This defines a map

$$\begin{aligned} \alpha_{\mathcal{Y}} : G &\to \mathcal{W}_{X_L, \mathcal{L}_L} \\ \sigma &\mapsto s_{f_\sigma}. \end{aligned}$$

Note also that this birational map induces an isomorphism

$$f_{\sigma, k_L} : Y_{k_L} \to Y_{k_L}^\sigma$$

on the special fibers, which coincides with the base change map. As such, the G-action on $\mathcal{W}_{X_L, \mathcal{L}_L} \leq \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y_{k_L}))$ can be explicitly described as

$$\sigma(s) = f_{\sigma, k_L}^* \circ s^\sigma \circ (f_{\sigma, k_L}^*)^{-1}.$$

This allows us to show the following proposition, which shows that the regularity of the G-action on $\mathcal{Y}$ can be interpreted via group cohomology.

Proposition 3.4 ([CLL17], Proposition 7.10)**.** *The map*

$$\alpha_{\mathcal{Y}} : G \to \mathcal{W}_{X_L, \mathcal{L}_L}$$

is a 1-cocycle for the G-action on $\mathcal{W}_{X_L, \mathcal{L}_L}$. Moreover, the G-action on $\mathcal{Y}$ is regular if and only if this cocycle is trivial, that is, satisfies $\alpha_{\mathcal{Y}}(\sigma) = \mathrm{id}$ for all $\sigma \in G$.

Proof. Note that $f_{\sigma\tau} = f_\tau^\sigma \circ f_\sigma$. Thus

$$\begin{aligned}\alpha_{\mathcal{Y}}(\sigma\tau) = s_{f_{\sigma\tau}} &= s_{f_\tau^\sigma \circ f_\sigma} \\ &= s_{f_\sigma} \circ f^*_{\sigma,k_L} \circ s_{f_\tau^\sigma} \circ (f^*_{\sigma,k_L})^{-1} \\ &= s_{f_\sigma} \circ f^*_{\sigma,k_L} \circ s^\sigma_{f_\tau} \circ (f^*_{\sigma,k_L})^{-1} \\ &= s_{f_\sigma} \circ \sigma(s_{f_\tau}) \\ &= \alpha_{\mathcal{Y}}(\sigma) \circ \sigma(\alpha_{\mathcal{Y}}(\tau))\end{aligned}$$

as desired; the second equality follows from Proposition 3.2 and the fourth from the above remarks. The G-action being regular means that f_σ is a regular map for all σ. By Theorem 3.2 this happens if and only if $s_{f_\sigma} = \mathrm{id}$ for all σ, that is, if $\alpha_{\mathcal{Y}}(\sigma) = 1$ for all $\sigma \in G$. $\square$

As a 1-cocycle, we may interpret this map as an element $[\alpha_{\mathcal{Y}}] \in H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L})$ in non-abelian cohomology. The following shows that while $\alpha_{\mathcal{Y}}$ depends on the terminal model $\mathcal{Y}$, the corresponding 1-cocycle does not.

Proposition 3.5 ([CLL17] Proposition 7.11)**.** *Let $\mathcal{Y}^+$ be another $\mathcal{L}_L$-terminal model for X_L, and let $f : \mathcal{Y} \dashrightarrow \mathcal{Y}^+$ be the rational map given by identity on the generic fibers. Let $s_f \in \mathcal{W}_{X_L,\mathcal{L}_L}$ be the element of the Weyl group associated to f, as in Theorem 3.2. Then for all $\sigma \in G$ we have*

$$\alpha_{\mathcal{Y}^+}(\sigma) = s_f^{-1} \circ \alpha_{\mathcal{Y}}(\sigma) \circ \sigma(s_f).$$

Proof. As automorphisms, $\alpha_{\mathcal{Y}}$ and $\alpha_{\mathcal{Y}^+}$ are functions

$$\begin{aligned}\alpha_{\mathcal{Y}} &: G \to \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(\mathcal{Y}_{k_L})) \\ \alpha_{\mathcal{Y}^+} &: G \to \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(\mathcal{Y}^+_{k_L}))\end{aligned}$$

respectively. Thus we need to show that

$$f^*_{k_L} \circ \alpha_{\mathcal{Y}^+} \circ (f^*_{k_L})^{-1} = s_f^{-1} \circ \alpha_{\mathcal{Y}}(\sigma) \circ \sigma(s_f)$$

The diagram

$$\begin{tikzcd} \mathcal{Y} \arrow[r, dashed, "f_\sigma"] \arrow[d, dashed, "f"'] & \mathcal{Y}^\sigma \arrow[d, dashed, "f^\sigma"] \\ \mathcal{Y}^+ \arrow[r, dashed, "f_\sigma^+"] & \mathcal{Y}^{+,\sigma} \end{tikzcd}$$

since both morphisms consist of the composition of the Galois action on the identity on the generic fibers. Thus

$$\begin{aligned}\alpha_{\mathcal{Y}} \circ \sigma(s_f) &= s_{f_\sigma} \circ f^*_{\sigma,k_L} \circ s_f^\sigma \circ (f^*_{\sigma,k_L})^{-1} \\ &= s_{f_\sigma} \circ f^*_{\sigma,k_L} \circ s_{f^\sigma} \circ (f^*_{\sigma,k_L})^{-1} \\ &= s_{f^\sigma \circ f_\sigma} \\ &= s_{f_\sigma^+ \circ f} \\ &= s_f \circ f^*_{k_L} \circ \alpha_{\mathcal{Y}^+}(\sigma) \circ (f^*_{k_L})^{-1}\end{aligned}$$

which was what we wanted to show. $\square$

Given this, if X has good reduction over L, we associate a cohomology class

$$\alpha^L_{X,\mathcal{L}} := [\alpha_{\mathcal{Y}}] \in H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L})$$

which is independent of our choice of terminal model.

With this in hand we can prove the main results of this section, which is a criterion for good reduction in terms of the above nonabelian cohomology group. The first of them essentially says that if the cohomology class $\alpha^L_{X,\mathcal{L}}$ is trivial, then in fact it can be represented by a trivial cocycle.

Proposition 3.6 ([CLL17], Proposition 7.12)**.** *There exists an $\mathcal{L}_L$-terminal model of X_L over $\mathcal{O}_L$ for which the G-action is regular if and only if the cohomology class*

$$\alpha^L_{X,\mathcal{L}} \in H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L})$$

is trivial. In particular, X has good reduction over K if and only if $\alpha^L_{X,\mathcal{L}}$ is trivial.

Proof. If the G-action is regular then s_{f_σ} is an isomorphism for all σ, so certainly $\alpha^L_{X,\mathcal{L}}$ is trivial. So suppose $\alpha := \alpha^L_{X,\mathcal{L}}$ is trivial for some $\mathcal{L}_L$-terminal model $\mathcal{Y}$. By assumption and definition, there exists $w \in \mathcal{W}_{X_L,\mathcal{L}_L}$ such that $\alpha(\sigma) = w^{-1}\sigma(w)$ for all $\sigma \in G$. Since elements of the Weyl group can always be constructed out of birational maps by Proposition 3.3, there exists some other terminal model $f : \mathcal{Y} \dashrightarrow \mathcal{Y}^+$ such that $s_f = w^{-1}$. Let

$$\alpha^+ : G \to \mathcal{W}_{X_L,\mathcal{L}_L}$$

be the cocycle associated to the model $\mathcal{Y}^+$. Then by Proposition 3.5 we have, for any $\sigma \in G$,

$$\begin{aligned}\alpha_{\mathcal{Y}^+}(\sigma) &= s_f^{-1} \circ \alpha(\sigma) \circ \sigma(s_f) \\ &= w \circ (w^{-1}\sigma(w)) \circ \sigma(w^{-1}) \\ &= 1.\end{aligned}$$

Hence by Proposition 3.4 the rational G-action on $\mathcal{Y}^+$ is in fact regular.

For the second statement, if $\alpha^L_{X,\mathcal{L}}$ is trivial then by the first statement there is a model of X_L over $\mathcal{O}_L$ over which the G-action is regular, from which Proposition 3.1 guarantees that X has good reduction over K. Conversely, suppose X has good reduction over K, say with model $\mathcal{X}$. Then we may assume without loss of generality that $\mathcal{X}$ is $\mathcal{L}$-terminal. Then $\mathcal{X} \times_{\mathcal{O}_K} \mathcal{O}_L$ is an $\mathcal{L}_L$-terminal model of X_L to which the G-action extends, that is, for which the G-action is regular. Then the first statement guarantees that $\alpha^L_{X,\mathcal{L}}$ is trivial. □

Because the construction of RDP models and canonical reductions is compatible with base change we may define

$$\mathcal{W}^{\mathrm{nr}}_{X,\mathcal{L}} := \varinjlim_{K \subseteq L \subseteq K^{\mathrm{nr}}} \mathcal{W}_{X_L,\mathcal{L}_L} \cong \mathcal{W}_{X_{K^{\mathrm{nr}}},\mathcal{L}_{K^{\mathrm{nr}}}}$$

where the limit is taken over all finite and unramified extension of K inside our fixed algebraic closure $\overline{K}$. There is a natural $\mathrm{Gal}(K^{\mathrm{nr}}/K) \cong G_k$-action on $\mathcal{W}^{\mathrm{nr}}_{X,\mathcal{L}}$. The above cohomology classes

$$\alpha^L_{X,\mathcal{L}} \in H^1(\mathrm{Gal}(L/K), \mathcal{W}_{X_L,\mathcal{L}_L})$$

for those finite and unramified extensions which are Galois and admit good reduction are compatible, and give rise to a well-defined cohomology class

$$\alpha_{X,\mathcal{L}}^{\mathrm{nr}} \in H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\mathrm{nr}})$$

in continuous cohomology. This construction allows us to phrase the above criterion in a way independent of our choice of unramified extension L/K:

Corollary 3.1 ([CLL17], Corollarly 7.13)**.** *Let X be a K3 surface over K satisfying $(*)$ and the equivalent conditions of Theorem 2.1. Then the following are equivalent:*

1. *X has good reduction over K;*
2. *The cohomology class $\alpha_{X,\mathcal{L}}^{\mathrm{nr}} \in H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\mathrm{nr}})$ is trivial for* all *ample line bundles $\mathcal{L}$ on X;*
3. *The cohomology class $\alpha_{X,\mathcal{L}}^{\mathrm{nr}} \in H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\mathrm{nr}})$ is trivial for* some *ample line bundle $\mathcal{L}$ on X.*

Proof. Given that

$$H^1(G_k, \mathcal{W}_{X,\mathcal{L}}^{\mathrm{nr}}) = \varinjlim_{K \subseteq L \subseteq K^{\mathrm{nr}}} H^1(\mathrm{Gal}(L/K), \mathcal{W}_{X_L,\mathcal{L}_L}).$$

this follows immediately from Proposition 3.6. □

With this result, we've obtained a cohomological criterion for good reduction. Our next task is to convert this group-theoretic criterion into cohomological criteria on the special fiber.

3.2 Compatibilities in Étale and Crystalline Cohomology

We first need a general result on the compatibility of cycle class maps and comparison isomorphisms. It is similar in flavor to the constructions in Section 3.1.1, and we point the reader there for the motivation for the following constructions.

Suppose that we have a birational map $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ between smooth models of K3 surfaces X and X^+, respectively. We have the graph of f

$$\Gamma_f \subseteq \mathcal{X} \times_{\mathcal{O}_K} \mathcal{X}^+$$

along with its generic and special fibers

$$\Gamma_{f_K} \subseteq X \times_K X^+$$
$$\Gamma_{f,k} \subseteq \mathcal{X}_k \times_k \mathcal{X}_k^+.$$

As we described in Section 3.1.1, f is defined outside of a finite collection of curves on the special fibers, the generic fiber Γ_{f_K} coincides with the graph of the isomorphism f_K induced by f on the generic fibers, but not so with f_k. On the special fibers we have associated maps on cohomology

$$\Gamma_{f,k}^* : H_{\text{ét}}^n(X_{\overline{k}}^+, \mathbb{Q}_l)) \to H_{\text{ét}}^n(\mathcal{X}_{\overline{k}}, \mathbb{Q}_l)$$
$$\alpha \mapsto p_{1*}([\Gamma_{f,k}] \smile p_2^*(\alpha)).$$

for $l \neq p$ and

$$\begin{aligned}\Gamma_{f,k}^* : H^n_{\mathrm{cris}}(\mathcal{X}_k^+/K_0) &\to H^n_{\mathrm{cris}}(\mathcal{X}_k/K_0)\\ \alpha &\mapsto p_{1*}([\Gamma_{f,k}] \smile p_2^*(\alpha))\end{aligned}$$

for $p > 0$. The comparison theorem we need is the following:

Lemma 3.1 ([CLL17], Lemma 8.1)**.** *In the above situation, the diagrams*

$$\begin{tikzcd} H^n_{\text{ét}}(X^+_{\overline{K}}, \mathbb{Q}_l) \arrow[r] \arrow[d, "f_K^*"'] & H^n_{\text{ét}}(\mathcal{X}^+_{\overline{k}}, \mathbb{Q}_l) \arrow[d, "\Gamma^*_{f,k}"] \\ H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \arrow[r] & H^n_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Q}_l) \end{tikzcd}$$

for $l \neq p$, and

$$\begin{tikzcd} \mathbb{D}_{\mathrm{cris}}\left(H^n_{\text{ét}}(X^+_{\overline{K}}, \mathbb{Q}_p)\right) \arrow[r] \arrow[d, "f_K^*"'] & H^n_{\mathrm{cris}}(\mathcal{X}^+_k/K_0) \arrow[d, "\Gamma^*_{f,k}"] \\ \mathbb{D}_{\mathrm{cris}}\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\right) \arrow[r] & H^n_{\mathrm{cris}}(\mathcal{X}_k/K_0) \end{tikzcd}$$

for $p > 0$ commute, where the horizonal arrows are the isomorphisms provided by smooth and proper base change and the crystalline comparison theorem, respectively.

Proof. As remarked in the proof of Proposition 3.2, the result for $l \neq p$ was proved in [LM18, Lemma 5.6]. As usual, we may assume that $K = \hat{K}$ is complete. We do not have a Berthelot-Ogus comparison theorem for algebraic spaces, which bars us from passing between the generic and special fibers directly. But in the proof of Theorem 2.4 in [CLL17] it was shown that, since $\mathcal{X}$ and $\mathcal{X}^+$ has schematic fibers, the completions $\mathfrak{X}$ and $\mathfrak{X}^+$ along their special fibers are smooth and proper formal schemes. It thus follows from the Berthelot-Ogus comparison isomorphism that we have

$$H^n_{\mathrm{cris}}(\mathfrak{X}_k/W) \otimes_W K \cong H^n_{\mathrm{dR}}(\mathfrak{X}_K)$$

where $\mathfrak{X}_K$ is the generic fiber of $\mathfrak{X}$ considered as a rigid space over K. Since $\mathfrak{X}_k$ is proper and smooth, we have isomorphisms

$$H^n_{\mathrm{dR}}(\mathfrak{X}_K) = H^n_{\mathrm{dR}}(\mathfrak{X}_K^{\mathrm{rig}}) = H^n_{\mathrm{rig}}(\mathfrak{X}_k)$$

and by the proof of Theorem 2.4, we additionally have canonical isomorphisms

$$H^n_{\mathrm{dR}}(\mathfrak{X}_K) \cong H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \cong H^n_{\mathrm{dR}}(X)$$

where the isomorphisms $H^n_{\mathrm{dR}}(\mathfrak{X}_K) \cong H^n_{\mathrm{dR}}(\mathfrak{X}_K^{\mathrm{rig}})$ and $H^n_{\mathrm{dR}}(X^{\mathrm{an}}) \cong H^n_{\mathrm{dR}}(X)$ arise from rigid-analytic GAGA theorems. In addition, since $\mathfrak{X}_k = \mathcal{X}_k$ by construction, the above Berthelot-Ogus comparison isomorphism provides an isomorphism

$$H^n_{\mathrm{cris}}(\mathcal{X}_k/W) \otimes_W K \cong H_{\mathrm{dR}}(X/K)$$

and similarly for $\mathcal{X}^+$. By [CN17, Corollally 5.26], this isomorphism fits into the commutative diagram

$$\begin{tikzcd} \mathbb{D}_{\mathrm{cris}}\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\right) \otimes_{K_0} K \arrow[r, "\cong"] \arrow[d, "\cong"'] & H^n_{\mathrm{cris}}(\mathcal{X}_k/W) \otimes_W K \arrow[d, "\cong"] \\ \mathbb{D}_{\mathrm{dR}}\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\right) \arrow[r, "\cong"] & H^n_{\mathrm{dR}}(X/K). \end{tikzcd}$$

Since the map $f_K : X \to X^+$ is regular, it follows that the diagram

$$\begin{tikzcd} \mathbb{D}_{\mathrm{dR}}\left(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)\right) \arrow[r, "\cong"] \arrow[d, "\mathbb{D}_{\mathrm{dR}}(f_K^*)"'] & H^n_{\mathrm{dR}}(X/K) \arrow[d, "f_K^*"] \\ \mathbb{D}_{\mathrm{dR}}\left(H^n_{\text{ét}}(X^+_{\overline{K}}, \mathbb{Q}_p)\right) \arrow[r, "\cong"] & H^n_{\mathrm{dR}}(X^+/K) \end{tikzcd}$$

commutes. Thus to prove our result it suffices to prove that the Berthelot-Ogus comparison isomorphisms above are compatible with $\Gamma^*_{f,k}$, i.e., that the diagram

$$\begin{tikzcd} H^n_{\mathrm{cris}}(\mathcal{X}_k/W) \otimes_W K) \arrow[r, "\cong"] \arrow[d, "\Gamma^*_{f,k}"'] & H_{\mathrm{dR}}(X/K) \arrow[d, "f_K^*"] \\ H^n_{\mathrm{cris}}(\mathcal{X}^+_k/W) \otimes_W K \arrow[r, "\cong"] & H^n_{\mathrm{dR}}(X^+/K) \end{tikzcd}$$

commutes. Since $\Gamma^*_{f,k}$ is defined by taking cup products and cycle classes, this amounts to showing that the horizonal isomorphisms are compatible with cycle classes. More precisely, we are given a smooth and proper algebraic space $\mathcal{Z}$ over $\mathcal{O}_K$, with schematic fibers Z and $\mathcal{Z}_k$, and a closed, integral subspace $\mathcal{T} \subseteq \mathcal{Z}$ taht is flat and of relative codimension c over $\mathcal{O}_K$ with generic and special fibers T and $\mathcal{T}_k$, respectively. We have to show that the isomorphism

$$H^{2c}_{\mathrm{cris}}(\mathcal{Z}_k/W) \otimes_W K \cong H^{2c}_{\mathrm{dR}}(Z/K)$$

identifies $\mathrm{cl}(\mathcal{T}_k) \otimes 1$ with $\mathrm{cl}(T)$.

In the situation where $\mathcal{Z}$ is a smooth $\mathcal{O}_K$-scheme, [CCM13, Corollary 1.5.1] tells us that the diagram

$$\begin{tikzcd} \mathrm{CH}^c(Z/K) \otimes \mathbb{Q} \arrow[r, "\eta_{\mathrm{dR}}"] \arrow[d, "\mathrm{sp}_{\mathrm{CH}}"'] & H^{2c}_{\mathrm{dR}}(Z) \arrow[d, "\mathrm{sp}"] \\ \mathrm{CH}^c(\mathcal{Z}_k/k) \otimes \mathbb{Q} \arrow[r, "\eta_{\mathrm{rig}}"] & H^{2c}_{\mathrm{rig}}(\mathcal{Z}_k/K) \end{tikzcd}$$

commutes, where $\mathrm{sp}_{\mathrm{CH}}$ maps $\mathrm{cl}(T) \otimes 1$ to $\mathrm{cl}(\mathcal{T}_k) \otimes 1$, the horizonal maps are the cycle class maps, and sp is the specialization map introduced in Section 1.4.5. This is essentially the compatibility we are looking for - to push this compatibility to algebraic spaces, we approximate $\mathcal{Z}$ by smooth schemes using étale descent. Let $\mathcal{Z}_\bullet \to \mathcal{Z}$ be an étale hyperrcover of $\mathcal{Z}$ by smooth $\mathcal{O}_K$-schemes. The subspace $\mathcal{T} \subseteq \mathcal{Z}$ induces a closed subscheme $\mathcal{T}_i \subseteq \mathcal{Z}_i$ for all i in a way compatible with taking generic and special fibers. Thus at every level i of the hypercovering we obtain an equality

$$\mathrm{cl}((\mathcal{T}_i)_k) \otimes 1 = \mathrm{sp}(\mathrm{cl}((\mathcal{T}_i)_K)).$$

Since rational crystalline cohomology and algebraic de Rham cohomology both satisfy étale descent, this equality descends and we obtain an equality

$$\mathrm{cl}(\mathcal{T}_k) \otimes 1 = \mathrm{sp}(\mathrm{cl}(T)).$$

Since rigid cohomology coincides with crystalline cohomology for smooth and proper schemes, it follows that the isomorphism

$$H^{2c}_{\mathrm{cris}}(\mathcal{Z}_k/W) \otimes_W K \cong H^{2c}_{\mathrm{dR}}(Z/K)$$

sends $\mathrm{cl}(\mathcal{T}_k) \otimes 1$ to $\mathrm{cl}(T)$, as claimed. □

3.2.1 Actions of the Weyl group on cohomology

As we saw in Proposition 3.2, the elements of the Weyl group, *a priori* automorphisms of $\mathrm{Pic}(Y)$, can be extended to automorphisms on cohomology. Here we develop this theme further. Let $(X, \mathcal{L})$ be a polarized K3 surface over K and $\mathcal{X}$ an $\mathcal{L}$-terminal model for X. We have the associated Weyl group

$$\mathcal{W}_{X,\mathcal{L}} \leq \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y))$$

and base changing to K^{nr} provides the G_k-equivariant version

$$\mathcal{W}^{\mathrm{nr}}_{X,\mathcal{L}} \leq \mathrm{Aut}_{\mathbb{Z}}(\mathrm{Pic}(Y_{\overline{k}})).$$

The latter extends to an automorphism on étale and crystalline cohomology, as follows:

Lemma 3.2 ([CLL17], Lemma 8.3)**.** *There are G_k-equivariant and injective homomorphisms*

$$\begin{aligned} i_l &: \mathcal{W}^{\mathrm{nr}}_{X,\mathcal{L}} \to \mathrm{Aut}_{\mathbb{Q}_l}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))) \\ i_p &: \mathcal{W}^{\mathrm{nr}}_{X,\mathcal{L}} \to \mathrm{Aut}_{K_0^{\mathrm{nr}},F}(H^2_{\mathrm{cris}}(Y/K_0)(1) \otimes_{K_0} K_0^{\mathrm{nr}}) \end{aligned}$$

for $l \neq p$ and $p > 0$, respectively.

Proof. By linearity, it suffices to define the morphism for reflections s_E for $E \subseteq E_{X,\mathcal{L},\overline{k}}$ an irreducible component of the geometric exceptional locus. For $l \neq p$, we define it as

$$\begin{aligned} H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)) &\to H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)) \\ \alpha &\mapsto \alpha + ([E] \smile \alpha)[E] \end{aligned}$$

using the canonical isomorphism $H^4_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(2)) \cong \mathbb{Q}_l$. This morphism is injective by the nondegeneracy of the cup product. Indeed, if s_E is not the trivial reflection then $[E] \neq 0$, and by nondegeneracy there exists α such that $[E] \smile \alpha) \neq 0$. This means that the above morphism is not the identity, which is what we wanted.

For $p > 0$ we do an analogous construction. Let L/K be a finite unramified extension with residue field k_L over which E is defined. We then send s_E to the automorphism

$$\begin{aligned} H^2_{\mathrm{cris}}(Y_{k_L}/L_0)(1) &\to H^2_{\mathrm{cris}}(Y_{k_L}/L_0)(1) \\ \alpha &\mapsto \alpha + ([E] \smile \alpha)[E] \end{aligned}$$

where $L_0 = W(k_L)[1/p]$ using the identification $H^4_{\mathrm{cris}}(Y_{k_L}/L_0)(2) \cong L_0$. Passing to the limit, we have our desired action. □

In Theorem 3.2 we showed that if $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ is a terminal birational map between terminal models of K3 surfaces then $s_f \in \mathcal{W}_{X,\mathcal{L}}$. Given the above lemma, we can ask in particular about the action of s_f on cohomology. Lemma 3.1 allows us to provide an elegant description in terms of pullback maps.

Let $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ be a terminal birational map of K3 surfaces. As we described above, this induces isomorphisms $f_K : X \to X^+$ and $f_k : Y \to \mathcal{X}^+$ on the special and generic fibers, respectively. We have the natural pullback maps on the special fibers

$$f_{k,l}^* : H^2_{\text{ét}}(\mathcal{X}^+_{\overline{k}}, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))$$
$$f_{k,p}^* : H^2_{\text{cris}}(\mathcal{X}^+_k/K_0)(1) \to H^2_{\text{cris}}(Y/K_0)(1)$$

for $l \neq p$ and $p > 0$, respectively. We also have a variation derived from the action of the generic fibers

$$f_{K,l}^* : H^2_{\text{ét}}(\mathcal{X}^+_{\overline{k}}, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))$$
$$f_{K,p}^* : H^2_{\text{cris}}(\mathcal{X}^+_k/K_0)(1) \to H^2_{\text{cris}}(Y/K_0)(1)$$

defined by the commutative diagrams

$$\begin{tikzcd}
H^2_{\text{ét}}(X^+_{\overline{K}}, \mathbb{Q}_l(1)) \arrow[r, "\cong"] \arrow[d] & H^2_{\text{ét}}(\mathcal{X}^+_{\overline{k}}, \mathbb{Q}_l(1)) \arrow[d, "f_{K,l}^*"] \\
H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l(1)) \arrow[r, "\cong"] & H^2_{\text{ét}}(\mathcal{X}_{\overline{k}}, \mathbb{Q}_l(1))
\end{tikzcd}$$

for $l \neq p$ and

$$\begin{tikzcd}
\mathbb{D}_{\text{cris}}\left(H^2_{\text{ét}}(X^+_{\overline{K}}, \mathbb{Q}_p(1))\right) \arrow[r, "\cong"] \arrow[d] & H^2_{\text{cris}}(\mathcal{X}^+_k/K_0)(1) \arrow[d, "f_{K,p}^*"] \\
\mathbb{D}_{\text{cris}}\left(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(1))\right) \arrow[r, "\cong"] & H^2_{\text{cris}}(\mathcal{X}_k/K_0)(1)
\end{tikzcd}$$

for $p > 0$, where the horizontal arrows are the appropriate comparison theorems and the left hand maps are the usual pullbacks on the generic fiber. These maps suffice to describe the action of i_l and i_p on s_f, as the following theorem shows:

Theorem 3.3 ([CLL17], Theorem 8.4)**.** *Let $f : \mathcal{X} \dashrightarrow \mathcal{X}^+$ be a birational map between terminal models of K3 surfaces over K. Then we have*

$$i_l(s_f) = f_{K,l}^* \circ (f_{k,l}^*)^{-1} : \qquad H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)) \to H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))$$
$$i_p(s_f) = f_{K,p}^* \circ (f_{k,p}^*)^{-1} : \qquad H^2_{\text{cris}}(Y/K_0)(1) \to H^2_{\text{cris}}(Y/K_0)(1)$$

for $l \neq p$ and $p > 0$, respectively.

Proof. Since the proof for the p- and l-adic cases are virtually identical, we prove the l-adic equality. Let $l \neq p$. The definition of i_l is such that if E is an irreducible component of the geometric exceptional locus and $s_E \in W^{\text{nr}}_{X,\mathcal{L}} \leq \operatorname{Aut}_{\mathbb{Z}}(\operatorname{Pic}(Y_{\overline{k}}))$ is regarded as an automorphism of $\operatorname{Pic}(Y_{\overline{k}})$ then the following diagram

$$\begin{tikzcd}
\mathrm{Pic}(Y_{\overline{k}}) \arrow[r, "\mathrm{cl}"] \arrow[d, "s_E"'] & H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)) \arrow[d, "i_l(s_E)"] \\
\mathrm{Pic}(Y_{\overline{k}}) \arrow[r, "\mathrm{cl}"] & H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))
\end{tikzcd}$$

commutes, where $\mathrm{cl} : D \mapsto [D]$ is the class map. In other words, $i_l(s_E)$ extends the action of s_E on the Picard group to cohomology.

In the l-adic setting we have by definition that $\Gamma^*_{f,k} = \tilde{s}_{f,l}$, where $\Gamma^*_{f,k}$ was defined in the beginning of Section 3.2 and $\tilde{s}_{f,l}$ was defined in the proof of Proposition 3.2. By Lemma 3.1, we additionally have $\Gamma^*_{f,k} = f^*_{K,l}$. Since $s_f = \tilde{s}_f \circ (f^*_k)^{-1}$ and $\tilde{s}_{f,l}$ is the extension of $\tilde{s}_f$ to cohomology by definition, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Pic}(Y_{\overline{k}}) \arrow[r, "\mathrm{cl}"] \arrow[d, "(f^*_k)^{-1}"] \arrow[dd, bend right=60, "s_f"'] & H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)) \arrow[d, "(f^*_{k,l})^{-1}"] \\
\mathrm{Pic}(\mathcal{X}^+_{\overline{k}}) \arrow[r, "\mathrm{cl}"] \arrow[d, "\tilde{s}_f"] & H^2_{\text{ét}}(\mathcal{X}^+_{\overline{k}}, \mathbb{Q}_l(1)) \arrow[d, "\tilde{s}_{f,l} = f^*_{K,l}"] \\
\mathrm{Pic}(Y_{\overline{k}}) \arrow[r, "\mathrm{cl}"] & H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))
\end{tikzcd}$$

Thus $f^*_{K,l} \circ (f^*_{k,l})^{-1}$ also extends s_f to cohomology. One can check that the construction of $\Gamma^*_{f,k} = \tilde{s}_{f,l}$ coincides with that of i_l, and it thus follows that $i_l(s_f) = f^*_{K,l} \circ (f^*_{k,l})^{-1}$, as desired. $\square$

The results in this section equip the cohomology groups of the special fiber with an action of the Weyl group. But the Weyl group, as we saw in Section 3.1.2, can itself be endowed with a Galois action, and in this way the cohomology groups can be given an action of the Galois group.

In more detail, let L/K be a finite unramified Galois extension over which X has good reduction. Let $G := \mathrm{Gal}(L/K)$, write k_L/k denote the residual extension, and $L_0 := W(k_L)[1/p] \subseteq \hat{L}$. We have the exact sequence

$$1 \to G_{k_L} \to G_k \to G \to 1.$$

Taking G_{k_L}-invariants in Lemma 3.2 and twisting, we obtain $G_k/G_{k_L} \cong G$-equivariant homomorphisms

$$\mathcal{W}_{X_L,\mathcal{L}_L} \to \mathrm{Aut}_{G_{k_L}}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l))$$
$$\mathcal{W}_{X_L,\mathcal{L}_L} \to \mathrm{Aut}_{L_0,F}(H^2_{\mathrm{cris}}(Y_{k_L}/L_0))$$

for $l \neq p$ and $p > 0$, respectively. Combined with the automorphisms

$$\alpha^L_{X,\mathcal{L}} \in H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L})$$

we are lead to the following

Definition 3.7. We define the *l-adic and p-adic realizations* of $\alpha^L_{X,\mathcal{L}}$ to be

$$\beta^L_{X,\mathcal{L},l} := i_l(\alpha^L_{X,\mathcal{L}}) \in H^1(G, \mathrm{Aut}_{G_{k_L}}\left(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)\right))$$
$$\beta^L_{X,\mathcal{L},p} := i_l(\alpha^L_{X,\mathcal{L}}) \in H^1(G, \mathrm{Aut}_{L_0,F}(H^2_{\mathrm{cris}}(Y_{k_L}/L_0)))$$

for $l \neq p$ and $p > 0$, respectively.

We now twist the natural G_k-module and F-isocrystal structures on $H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l))$ and $H^2_{\text{cris}}(Y_{k_L}/L_0)$, respectively, as in Sections 1.5.5 and 1.5.6. We obtain new G_k-modules and F-isocrystals

$$H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)^{\beta^L_{X,\mathcal{L},l}} \qquad \text{and} \qquad H^2_{\text{cris}}(Y/K_0)^{\beta^L_{X,\mathcal{L},p}}.$$

These are intimately related to the cohomology of the generic fiber. In fact, the below theorem states that these twists are precisely what is necessary to make the comparison isomorphisms between the geometric generic and special fibers of a K3 surface X over K, which are only well-behaved over an extension L/K on which X has good reduction, descend to K.

Theorem 3.4 ([CLL17], Theorem 8.6). *Let $(X, \mathcal{L})$ be a polarized K3 surface over K satisfying (*) and the equivalent conditions of Theorem 2.1. Then there are natural isomorphisms*

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \cong H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)^{\beta^L_{X,\mathcal{L},l}}$$
$$\mathbb{D}_{\text{cris}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^2_{\text{cris}}(Y/K_0)^{\beta^L_{X,\mathcal{L},p}}$$

for $l \neq p$ and $p > 0$, respectively.

Proof. Let L/K be a finite unramified extension for which there exists a smooth, $\mathcal{L}_L$-terminal $\mathcal{Y}$ for X_L over $\mathcal{O}_L$. We thus have a comparison isomorphism

$$\text{comp}_{\mathcal{Y},l} : H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \xrightarrow{\sim} H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)$$

which is G_{k_L}-equivariant but not G_k-equivariant in general. In other words, $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is an L-form of $H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)$ equipped with its standard G_k-representation via $\text{comp}_{\mathcal{Y},l}$ (more precisely, its inverse $\text{comp}^{-1}_{\mathcal{Y},l}$). As an L-form, it has an associated 1-cocycle

$$\beta_{\mathcal{Y},l} : G \to \text{Aut}_{\mathbb{Q}_l}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l))$$

which in this case is explicitly described as

$$\beta_{\mathcal{Y},l}(\sigma) = \text{comp}_{\mathcal{Y},l} \circ \sigma^*_g \circ \text{comp}^{-1}_{\mathcal{Y},l} \circ (\sigma^*_s)^{-1}$$

where σ^*_g is the action on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ and σ^*_s that on $H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)$; see Section 1.5.6.

On the other hand, Theorem 3.3 provides another description of this automorphism. Fix an arbitrary $\sigma \in G$ and let $f_\sigma : \mathcal{Y} \dashrightarrow \mathcal{Y}^\sigma$ be the birational map defined by the action of σ on the generic fiber, as in Section 3.1.2. Then in the notation of Theorem 3.3 we have a commutative diagram

$$\begin{tikzcd}
H^2_{\text{ét}}(X^\sigma_{\overline{K}}, \mathbb{Q}_l) \arrow[r, "\text{comp}_{\mathcal{Y},l}"] \arrow[d, "\sigma^*_g"] & H^2_{\text{ét}}(Y^\sigma_{\overline{k}}, \mathbb{Q}_l) \arrow[d, "f^*_{K,l}"', shift right] \\
H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \arrow[r, "\text{comp}_{\mathcal{Y},l}"] & H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l) \arrow[u, bend right, "(\sigma^*_s)^{-1} = (f^*_{k,l})^{-1}"']
\end{tikzcd}$$

Theorem 3.3 then states that

$$\begin{aligned}(i_l \circ \alpha_{\mathcal{Y}})(\sigma) :&= i_l(s_{f_\sigma}) \\ &= f_{K,l}^* \circ (f_{k,l}^*)^{-1} \\ &= f_{K,l}^* \circ (\sigma_s^*)^{-1} \\ &= \left(\mathrm{comp}_{\mathcal{Y},l} \circ \sigma_g^* \circ \mathrm{comp}_{\mathcal{Y},l}^{-1}\right) \circ (\sigma_s^*)^{-1} \\ &= \beta_{\mathcal{Y},l}(\sigma).\end{aligned}$$

Thus $\beta_{X,\mathcal{L},l}^L = [\beta_{\mathcal{Y},l}]$. The bijection between 1-cocycles and L-forms then states that the G_{k_L}-equivariant isomorphism $\mathrm{comp}_{\mathcal{Y},l}$ extends to a G_k-equivariant isomorphism

$$\begin{aligned}H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) &\cong H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)^{\beta_{\mathcal{Y},l}} \\ &= H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)^{\beta^L_{X,\mathcal{L},l}}\end{aligned}$$

as desired.

The p-adic case is similar, using descent for F-isocrystals instead. Since L/K is unramified and the representation ρ_L on $H^2_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_p)$ is crystalline by assumption, we have that

$$\left(H^2_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}}\right)^{G_L} \cong \left(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}}\right)^{G_K} \otimes_K L$$

(see, for instance, the proof of Lemma 2.1). Similarly we have

$$H^2_{\text{cris}}(Y_{k_L}/L_0) \cong H^2_{\text{cris}}(Y_k/K_0) \otimes_{K_0} L_0.$$

Thus the comparison isomorphism

$$\left(H^2_{\text{ét}}(X_{\overline{L}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}}\right)^{G_L} \xrightarrow{\sim} H^2_{\text{cris}}(Y_{k_L}/L_0)$$

over L can be read as saying that $\mathbb{D}_{\text{cris}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ is an L-form for the F-isocrystal $H^2_{\text{cris}}(Y/K_0)$. Running a nearly identical argument to the l-adic case with the cocycle induced by this isomorphism, we find that the comparison isomorphism descends to an isomorphism of F-isocrystals over K after we twist by $i_p \circ \alpha_{\mathcal{Y}}$. □

This can be lifted to $\overline{k}$ as well. Using the G_k-equivariant homomorphisms

$$\begin{aligned}i_l &: \mathcal{W}^{\text{nr}}_{X,\mathcal{L}} \to \mathrm{Aut}_{\mathbb{Q}_l}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))) \\ i_p &: \mathcal{W}^{\text{nr}}_{X,\mathcal{L}} \to \mathrm{Aut}_{K_0^{\text{nr}},F}(H^2_{\text{cris}}(Y/K_0)(1) \otimes_{K_0} K_0^{\text{nr}})\end{aligned}$$

from Lemma 3.2 and twisting, we obtain cohomology classes

$$\begin{aligned}\beta^{\text{nr}}_{X,\mathcal{L},l} &:= i_l(\alpha^{\text{nr}}_{X,\mathcal{L}}) \in H^1(G_k, \mathrm{Aut}_{\mathbb{Q}_l}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l))) \\ \beta^{\text{nr}}_{X,\mathcal{L},p} &:= i_p(\alpha^{\text{nr}}_{X,\mathcal{L}}) \in H^1(G_k, \mathrm{Aut}_{K_0^{\text{nr}},F}(H^2_{\text{cris}}(Y/K_0) \otimes_{K_0} K_0^{\text{nr}}))\end{aligned}$$

for $l \neq p$ and $p > 0$, respectively. We have the following analogue over $\overline{k}$:

Corollary 3.2. *Let $(X, \mathcal{L})$ be a polarized K3 surface over K satisfying (*) and the equivalent conditions of Theorem 2.1. Then there are natural isomorphisms*

$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \cong H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l)^{\beta^{\text{nr}}_{X,\mathcal{L},l}}$$
$$\mathbb{D}_{\text{cris}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^2_{\text{cris}}(Y/K_0)^{\beta^{\text{nr}}_{X,\mathcal{L},p}}$$

for $l \neq p$ and $p > 0$, respectively.

Synthesizing this comparison between the cohomology of the generic and special fibers with the connection between $\alpha^L_{X,\mathcal{L}}$ and good reduction established in Section 3.1.2, we can finally present the proof of Chiarellotto, Lazda, and Liedtke's criterion for good reduction of K3 surfaces.

3.3 A Néron-Ogg-Shafarevich Criterion for K3 Surfaces

For convenience, we restate the result:

Theorem 3.5 ([CLL17], Theorem 1.6). *Let X be a K3 surface satisfying $(*)$. Then the following are equivalent:*

(1) X has good reduction over K,

(2) There exists a prime $l \neq p$ (equivalently, for all $l \neq p$) such that the G_K-representation on $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l)$ is unramified and there exists an isomorphism
$$H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_l) \cong H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l).$$
of G_k-modules,

(3) If $p > 0$, $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline, and there exists an isomorphism
$$\mathbb{D}_{\text{cris}}(H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^2_{\text{cris}}(Y/K_0)$$
of F-isocrystals over K_0.

The implications (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) are the smooth and proper base change and crystalline comparison theorems, respectively. It remains to show that (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). This amounts to showing that, given some ample line bundle $\mathcal{L}$ on X, the induced maps

$$i_l : H^1(G_k, \mathcal{W}^{\text{nr}}_{X,\mathcal{L}}) \to H^1(G_k, \text{Aut}_{\mathbb{Q}_l}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)))$$

for $l \neq p$ and

$$i_p : H^1(G_k, \mathcal{W}^{\text{nr}}_{X,\mathcal{L}}) \to H^1(G_k, \text{Aut}_{K_0^{\text{nr}},F}(H^2_{\text{cris}}(Y/K_0) \otimes_{K_0} K_0^{\text{nr}}))$$

for $p > 0$ have trivial kernel. Indeed, by Corollary 3.2 the assumption in (2) means that $\beta^{\text{nr}}_{X,\mathcal{L},l} = \text{id}$. If i_l has trivial kernel then this forces $\alpha^{\text{nr}}_{X,\mathcal{L}}$ to be trivial as well, and we know by Corollary 3.1 that this happens only when X has good reduction. An identical argument works for the p-adic case.

We may equivalently show it for one extension L/K, since the direct limit is over a directed set. That is, if we have some finite and unramified Galois extension L/K with Galois group G and residue field extension k_L/k over which X has good reduction, it suffices to prove that the maps

$$i_l : H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L}) \to H^1(G, \mathrm{Aut}_{G_{k_L}}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)))$$

for $l \neq p$ and

$$i_p : H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L}) \to H^1(G, \mathrm{Aut}_{L_0,F}(H^2_{\text{cris}}(Y_{k_L}/L_0)(1)))$$

for $p > 0$ have trivial kernel. We proceed by making reductions to a known case.

Let

$$V_l \subseteq H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1))$$
$$V_p \subseteq H^2_{\text{cris}}(Y_{k_L}/L_0)(1)$$

denote the subspaces fixed by G_{k_L} and Frobenius, respectively. By construction these are $G_k/G_{k_L} \cong G$-representations within finite-dimensional $\mathbb{Q}_l$ (resp. $\mathbb{Q}_p$) vector spaces. The restriction map $\varphi \mapsto \varphi|_{V_l}$ defines G-equivariant maps

$$\mathrm{Aut}_{G_{k_L}}(H^2_{\text{ét}}(Y_{\overline{k}}, \mathbb{Q}_l(1)) \to \mathrm{GL}(V_l)$$
$$\mathrm{Aut}_{L_0,F}(H^2_{\text{cris}}(Y_{k_L}/L_0)(1)) \to \mathrm{GL}(V_p)$$

for $l \neq p$ and $p > 0$, respectively. This induces a morphism in group cohomology, and it suffices to show that the composition

$$i_l : H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L}) \to H^1(G, \mathrm{GL}(V_l))$$

has trivial kernel for all l, including $l = p$ when $p > 0$.

Proposition 3.7 ([CLL17], Proposition 9.1)**.** *For all primes l, including $l = p$ when $p > 0$, if $\alpha \in H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L})$ maps to the trivial class in*

$$H^1(G, \mathrm{GL}(V_l))$$

then it maps to the trivial class in

$$H^1(G, \mathrm{GL}(\Lambda_{X_L,\mathcal{L}_L,\mathbb{Q}_l})).$$

Proof. Consider the exact sequence

$$0 \to \Lambda_{X_L,\mathcal{L},\mathbb{Q}_l} \to V_l \to T_l \to 0$$

of G-representations. This is compatible with the action of $\mathcal{W}_{X_L,\mathcal{L}_L}$ and the induced action of the Weyl group on T_l is trivial. Since G is finite the category of G-representations is semi-simple the above exact sequence splits. Hence $V_l^\alpha \cong V_l$ if and only if $\Lambda^\alpha_{X_L,\mathcal{L}_L,\mathbb{Q}_l} \cong \Lambda_{X_L,\mathcal{L}_L,\mathbb{Q}_l}$.

But by the equivalence from Section 1.5 between elements in non-abelian cohomology and forms, α being trivially in $H^1(G, \mathrm{GL}(V_l))$ is equivalent to an isomorphism $V_l^\alpha \cong V_l$ as G-representations, and similarly for $H^1(G, \mathrm{GL}(\Lambda_{X_L,\mathcal{L}_L,\mathbb{Q}_l}))$. Through this the equivalence of the above isomorphisms implies the claim. □

This reduces the problem, finally, to showing that

$$H^1(G, \mathcal{W}_{X_L,\mathcal{L}_L}) \to H^1(G, \mathrm{GL}(\Lambda_{X_L,\mathcal{L}_L,\mathbb{Q}_l}))$$

has trivial kernel. This is precisely the content of [CLL17, Theorem 4.1], which completes the proof.

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