Università degli Studi di Padova

Université Paris Sud

ERASMUS MUNDUS ALGANT

Mémoire de Master 2 en Mathématiques

On the growth of Cohomology of Arithmetic Quotients of Symmetric Spaces

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Orsay, 2015

To my parents

Acknowledgments

First and foremost I thank my advisor Prof. Laurent Clozel for giving me this topic and for his guidance. Prof. Laurent Clozel helped me a lot throughout my master thesis. He carefully read my thesis and pointed out several errors to me (but I am responsible for any remaining error).

I need to thank the European Commission and the Algant Consortium for supporting me through the Erasmus Mundus Scholarship and for the opportunity to study in the International Masters Programme in Mathematics.

I am grateful to Prof. M. A. Garuti for helping me with many issues during my study at the University of Padova and also providing me the opportunity to study in the University of Paris-Sud for my master second year. A warm thanks also goes to Prof. B. Sury for his support and advice and for providing me some references on the topic discussed in this thesis.

I also have to thank Prof. Michael Harris for answering my questions concerning his theorem in chapter 5 of this thesis and Prof. Guy Henniart for giving me numerous valuable instructions throughout my study at Orsay.

My deepest gratitude goes to my father for his help and constant encouragement for me in every aspect of my life.

Contents

Introduction

Let E be a finite extension of $\mathbb{Q}_p$. Let G_∞ be $\mathbf{G}(\mathbb{R} \otimes_{\mathbb{Q}} F)$ for a connected semisimple linear algebraic group $\mathbf{G}$ over a number field F. Let $\mathfrak{P}$ be an ideal of $\mathcal{O}_F$ lying over p. Let K_∞ be the maximal compact subgroup of G_∞. Let Γ be an arithmetic lattice in G_∞. We define $\Gamma(\mathfrak{P}) := \Gamma \cap G(\mathfrak{P})$ where $G(\mathfrak{P})$ is the intersection of G_∞ with the congruence subgroup of $GL_N(\mathcal{O}_F)$ at level $\mathfrak{P}$. We define $\Gamma(\mathfrak{P}^k)$ similarly. Let Y_k be $\Gamma(\mathfrak{P}^{ek})\backslash G_\infty/K_\infty$, where e is the ramification index of $\mathfrak{P}$ in F. Let $m(\pi, \Gamma(\mathfrak{P}))$ be the multiplicity with which π occurs in the decomposition of the regular representation of G_∞ on $L^2(\Gamma(\mathfrak{P})\backslash G_\infty)$. Let $V(\mathfrak{P})$ be the volume of $\Gamma(\mathfrak{P})\backslash G_\infty$. For nontempered representation π, P. Sarnak and X. X. Xue proved that

$$m(\pi, \Gamma(\mathfrak{P})) \ll V(\mathfrak{P})^{1-\mu} \text{ for some } \mu > 0.$$

In this article we will follow [10] and we are going to find out this value of μ for the cohomological type representations π assuming that G_∞ does not admit a discrete series or if it does then π contributes to the cohomology in degrees other than $\frac{1}{2}\dim(G_\infty/K_\infty)$. We will see that

$$m(\pi, \Gamma(\mathfrak{P}^k)) \ll V(\mathfrak{P}^k)^{1-1/\dim(G_\infty)} \text{ as } k \to \infty.$$

Now let $H^n(Y_k, \mathcal{V}_k)_E$ be $E \otimes_{\mathcal{O}_E} H^n(Y_k, \mathcal{V}_k)$ for the local systems $\mathcal{V}_k$. Let d be the dimension of $G := \varprojlim_k \Gamma/\Gamma(\mathfrak{P}^k)$. We will also prove a theorem concerning the growth of the cohomology spaces of arithmetic quotients of symmetric spaces. We will show that

$$\dim_E H^n(Y_k, \mathbb{C})_E = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k}) \text{ as } k \to \infty$$

for constants r_n and c. The above two theorems are theorem 1.1 and theorem 3.1 of [10]. We will prove the theorem on dimension of the cohomology of locally symmetric spaces using spectral sequences of Matthew Emerton and noncommutative Iwasawa theory.

Chapter 1

Nonarchimedean Functional Analysis

Let K be a nonarchimedean field (equipped with a nonarchimedean absolute value such that K is complete). The ring of integers of K is denoted by $\mathcal{O}$.

Definition 1.1 The field K is called spherically complete if for any decreasing sequence of balls $B_1 \supseteq B_2 \supseteq \cdots$ in K the intersection $\cap_{n\in\mathbb{N}} B_n$ is nonempty.

Examples of spherically complete fields are finite extensions of $\mathbb{Q}_p$ (since any finite extension of $\mathbb{Q}_p$ is locally compact), or any discretely valued field [$|K^\times|$ is a discrete subset of $\mathbb{R}_+^\times$] (cf. Lemma 1.6 [22]).

<u>Seminorms</u>

Let V be a K-vector space. A (nonarchimedean) seminorm q on V is a function $q : V \to \mathbb{R}$ such that

1. $q(av) = |a|q(v)$ for any $a \in K, v \in V$,
2. $q(v + w) \leqslant \max\{q(v), q(w)\}$.

Definition 1.2 : A lattice L in V is a $\mathcal{O}$-submodule which satisfies the condition that for any vector $v \in V$, there exists $0 \neq a \in K^\times$ such that $av \in L$.

For any lattice $L \subseteq V$ we define its gauge p_L by

$$\begin{aligned} p_L : V &\to \mathbb{R} \\ v &\mapsto \inf_{v\in aL} |a|. \end{aligned}$$

One can easily show that p_L defines a seminorm on V. Now let q be any given seminorm on V. Then

$$L(q) := \{v \in V : q(v) \leqslant 1\} \text{ and } L^-(q) := \{v \in V : q(v) < 1\}.$$

Then $L^-(q) \subseteq L(q)$, and $L(q)$ and $L^-(q)$ are lattices in V (cf. the discussion before lemma 2.2 of [22]).

In this way, from a lattice we get a seminorm on V and viceversa.

If a seminorm q also satisfies the property that $q(v) = 0 \Longrightarrow v = 0$, then it is called a norm.

Locally convex vector spaces

Let $(L_j)_{j \in J}$ be a nonempty family of lattices in a K-vector space V such that we have

1. $\cdots$ (lc1) $\cdots$ For any $j \in J$ and any $a \in K^\times$ there exists a $k \in J$ such that $L_k \subseteq aL_j$.
2. $\cdots$ (lc2) $\cdots$ For any two $i, j \in J$ there exists a $k \in J$ with $L_k \subseteq L_i \cap L_j$.

These two conditions ensure that the sets $v + L_j$ for $v \in V, j \in J$ form a basis of a topology on V which is called the locally convex topology on V defined by the family (L_j). For $v \in V$ the sets $v + L_j, j \in J$ form a fundamental system of neighbourhoods of v.

A subset B is called bounded if for any open lattice $L \subseteq V$, there is an $a \in K$ such that $B \subseteq aL$. Here we note from [22] lemma 4.5 that a lattice L is open iff p_L (its gauge) is a continuous map.

Now let V and W be two locally convex K-vector spaces (i.e. equipped with a locally convex topology). We denote by $\mathcal{L}(V, W)$, continuous linear maps from V to W. When $W = K$, $\mathcal{L}(V, K)$ is denoted by $V^{'}$. We describe a general way to construct locally convex topologies on $\mathcal{L}(V, W)$. We choose a nonempty family $\mathcal{B}$ of bounded subsets of V which is closed under finite union. For any $B \in \mathcal{B}$ and any open lattice $M \subseteq W$ the subset

$$\mathcal{L}(B, M) := \{f \in \mathcal{L}(V, W) : f(B) \subseteq M\}$$

is a lattice in $\mathcal{L}(V, W)$: It is clear that $\mathcal{L}(B, M)$ is a $\mathcal{O}$-submodule. If $f \in \mathcal{L}(V, W)$ is any continuous linear map then, since B is bounded there exists $a \in K^\times$ such that $B \subseteq af^{-1}(M)$. This is because f being continuous $f^{-1}(M)$ is a lattice. This gives $f(B) \subseteq aM$ or $a^{-1}f \in \mathcal{L}(B, M)$. Now one can show that $\mathcal{L}(B, M)$ satisfies lc1 and lc2 (cf. Section 3 of [15]).

The corresponding locally convex topology is called the $\mathcal{B}$-topology and we write $\mathcal{L}_{\mathcal{B}}(V, W)$.

When $\mathcal{B}$ is the family of all finite subsets of V then the $\mathcal{B}$-topology is called the weak topology and is denoted by $\mathcal{L}_s(V, W)$.

On the other hand, when $\mathcal{B}$ is the family of all bounded subsets of V then the corresponding topology on $\mathcal{L}(V, W)$ is called the strong topology and is denoted by $\mathcal{L}_b(V, W)$.

Now we give a series of definitions from [22].

1. A locally convex vector space V is called bornological if every lattice L in V which satisfies the following property P is open. Property P is: For any bounded subset $B \subseteq V$, there is an $a \in K$ such that $B \subseteq aL$.

2. A locally convex vector space V is called baralled if every closed lattice in V is open.

3. A net $(v_i)_{i \in I}$ is a family of vectors v_i in V where the index set I is directed (i.e. for $i, j \in I, \exists k \in I$ with $i \leqslant k$ and $j \leqslant k$).

4. A net $(v_i)_{i \in I}$ in V is said to converge to a vector $v \in V$ if for any open lattice $L \subseteq V$ there is an index $i \in I$ such that $v_j - v \in L$ for any $j \geqslant i$; we say that the net is convergent.

5. A net $(v_i)_{i \in I}$ is called a Cauchy net if for any open lattice $L \subseteq V$ there is an index $i \in I$ such that $v_j - v_k \in L$ for any $j, k \geqslant i$.

6. A subset $A \subseteq V$ is called complete if every Cauchy net in A converges to a vector in A.

7. A locally convex vector space V is called quasi-complete if every bounded closed subset of V is complete.

8. A locally convex K-vector space is called a K-Fréchet space if it is metrizable and complete.

9. A subset $H \subseteq \mathcal{L}(V, W)$ is called equicontinuous if for any open lattice $M \subseteq W$ there is an open lattice $L \subseteq V$ such that $f(L) \subseteq M$ for every $f \in H$.

Now we give the main examples of each spaces defined above. We omit the proofs. One can see chapter 1 of [22] for proofs.

Any metrizable vector space V is bornological. Any Banach space is baralled and complete. Any complete space V is quasi-complete. Any Banach space is a Fréchet space. Any Fréchet space is both bornological and baralled.

Now we state analoges in Nonarchimedean Functional Analysis of some important theorems in Archimedean Functional Analysis like the Closed graph theorem, Open mapping theorem e.t.c.

Closed graph Theorem (cf. prop 8.5 of [22])

Let $f : V \to W$ be a linear map from a baralled locally convex K-vector space V into a K-Fréchet space W; if the graph $\Gamma(f) := \{(v, f(v)) : v \in V\}$ is closed in $V \times W$ then f is continuous.

Open mapping Theorem (prop 8.6 of [22])

Let V be a Fréchet space and W be a Hausdorff and baralled space; then every surjective continuous linear map $f : V \to W$ is open.

Hahn-Banach Theorem (prop 9.2 of [22])

Let K be a field which is spherically complete and let U be a K-vector space, q a seminorm on U, and $U_0 \subseteq U$ a vector subspace; for any linear form $l_0 : U_0 \to K$ such that $|l_0(v)| \leq q(v)$ for any $v \in U_0$ there is a linear form $l : U \to K$ such that $l|_{U_0} = l_0$ and $|l(v)| \leq q(v)$ for any $v \in U$.

Banach-Steinhaus Theorem (prop 6.15 of [22])

If V is baralled then any bounded subset $H \subseteq \mathcal{L}_s(V, W)$ is equicontinuous.

A theorem on dual vector space (prop 9.7 of [22])

Let K be spherically complete and V be a locally convex Hausdorff space then the map

$$\begin{aligned} \delta : V &\to (V_s^{'})_s^{'} \\ v &\mapsto \delta_v(l) := l(v) \end{aligned}$$

is a continous bijection. We note that this is not a topological isomorphism in general. Here $V_s^{'} = \mathcal{L}_s(V, K)$.

Now for any topological space M and any locally convex K-vector space W we let $C(M, W)$ denote the space of W-valued continuous functions. If $(V, ||\ ||_v)$ is a K-Banach space and if M is compact then $C(M, V)$ with the sup morm $||f|| = \max_{x \in M} ||f(x)||_v$ is a Banach space. We denote the dual of $C(M, K)$ by $D(M, K)$. (cf. Section 5 of [15]).

Now we introduce the concept of G-equivariant maps between two vector spaces.

Let V be a topological K-vector space and G be a topological group, then we say that a G-action on V is continuous if the action map $G \times V \to V$ is continuous.

If V and W are two K-vector spaces each with a G-action, then the K vector space $Hom_K(V,W)$ of K-linear maps from V to W is also equipped with a G-action, defined by the condition $g(\phi(v)) = (g\phi)(g^{-1}v)$, for $g \in G, \phi \in Hom_K(V,W), v \in V$. An element $\phi \in Hom_K(V,W)$ is called G-equivariant if it is fixed under this action by G. The space of G-equivariant K-linear maps from V to W is denoted by $Hom_G(V,W)$.

Now if V and W are two locally convex K-vector spaces (it will be later abbreviated as convex K-space) equipped with a topological G-action, then we denote the space of continuous G-equivariant K-linear maps from V to W by $\mathcal{L}_G(V,W)$, where $\mathcal{L}(V,W)$ is the space of continuous K-linear maps from V to W, which is obviously a subspace of $Hom(V,W) := Hom_K(V,W)$. $\mathcal{L}_{G,s}(V,W), \mathcal{L}_{G,b}(V,W)$ denote respectively the weak and strong topology.

We end this section by stating a useful lemma which will be used later on.

Let A be any $\mathcal{O}$-submodule in a locally convex K-vector space V. Let V_A be the vector subspace of V generated by A. Now we take any $v \in V_A$, then $v = k_1a_1 + ... + k_na_n$ where $a_i \in A$ and $k_i \in K$. Let $k_i = \frac{m_i}{n_i}$, $m_i, n_i \in \mathcal{O}$. Then $\prod n_i v \in A$. This shows that A is a lattice in V_A. Hence the gauge p_A is a seminorm on V_A. We view V_A as equipped with the locally convex topology defined by p_A which is by definition the coarsest topology such that the map $p_A : V \to \mathbb{R}$ is continuous and all translation map $v+ : V \to V$ are continuous. Now we quote lemma 7.17 of [22].

Lemma 1.1: If B is a bounded $\mathcal{O}$-submodule of V then we have

1. $V_B \subseteq V$ is continuous.
2. If V is Hausdorff then (V_B, p_B) is a normed vector space.
3. If V is Hausdorff and B is complete then (V_B, p_B) is a Banach space.

Chapter 2

Admissible Continuous Representations

2.1 Admissible Banach Space Representations

Throughout this chapter K will be a finite extension of $\mathbb{Q}_p$. We note that K is spherically complete and hence the Hahn-Banach Theorem is true. Now let G be a compact topological group, and W be any Hausdorff locally convex K-vector space then $C(G, W)$ is the set of continuous functions from G to W. We can endow $C(G, W)$ with the locally convex topology. We can construct it in the following way.

Let $q_1, ..., q_r$ be the seminorms which define the topology on W. Then for each q_i we can define a seminorm p_i on $C(G, W)$ as follows. Let $p_i(f) = max_{x \in G} q_i(f(x))$ where $f \in C(G, W)$. Then we endow $C(G, W)$ with the locally convex topology defined by the seminorms p_i's.

Proposition 2.1.1. Let G be a compact topological group and let V be a Hausdorff locally convex K-vector space equipped with a continuous action of G. If W is any Hausdorff locally convex K-vector space then the map $ev_e : C(G, W) \to W$ induces a G - equivariant topological isomorphism of convex spaces

$$\mathcal{L}_{G,b}(V, C(G, W)) \simeq \mathcal{L}_b(V, W).$$

Proof: Prop.5.1.1 of [1], page 86. $\square$

From now on, we assume that G is compact and V is a K- Banach space. So $C(G, K)$ is a Banach space. Now we quote another fact from nonarchimedean functional analysis.

Fact: (cf. Prop 1.1.36 of [1]). Let V and W be Hausdorff locally convex K-vector spaces, and let W is baralled or bornological. Then passing to the transpose induces a topological embedding of locally convex spaces

$$\mathcal{L}_b(V, W) \to \mathcal{L}_b(W_b^{'}, V_b^{'}).$$

By the term 'topological embedding' we mean a homeomorphism onto its image. Now we go back to our case where V is a K-Banach space. Then as $C(G, K)$ is a Banach space, it must be baralled. So we get φ : an embedding

$$\varphi : \mathcal{L}_b(V, C(G, K)) \to \mathcal{L}_b(D(G, K)_b, V_b^{'}).$$

Here we recall that $D(G, K)_b$ is the dual of $C(G, K)$ equipped with the strong topology.

Now from proposition 2.1.1 we get a topological embedding (here we take $W = K$),

$$V_b^{'} \simeq \mathcal{L}_{G,b}(V, C(G, K)) \hookrightarrow \mathcal{L}_{G,b}(D(G, K)_b, V_b^{'})$$

and hence a map $D(G, K)_b \times V_b^{'} \to V_b{'}$ which is G-equivariant in the first variable and separately continuous. This makes $V_b^{'}$ into a left $D(G, K)_b$-module. Now we have an isomorphism of K- vector spaces

$$Hom_{D(G,K)}(D(G, K), V^{'}) \cong \mathcal{L}_G(V, C(G, K)).$$

This is because tautologically

$$Hom_{D(G,K)}(D(G, K), V^{'}) \cong V^{'}$$

and proposition 2.1.1 gives a K-linear isomorphism

$$V^{'} = \mathcal{L}(V, K) \simeq \mathcal{L}_G(V, C(G, K)).$$

Now I state theorem 5.1.15 of [1].

Theorem 2.1.1 : Let G be a compact topological group and let V be a K-Banach space with a continuous G-action. If we equip $V^{'}$ with the action of $D(G, K)$ described above, then any surjection of left $D(G, K)$-modules $D(G, K)^n \to V^{'}$ is obtained by dualizing a closed G - equivariant embedding $V \to C(G, K)^n$. In particular, $V^{'}$ is finitely generated as a left $D(G, K)$-module iff V admits a closed G-equivariant embedding into $C(G, K)^n$ for some natural number n.

Proof : Let us suppose that we have a surjection of left $D(G, K)$-modules

$$\varphi : D(G, K)_b^n \to V_b^{'}.$$

Since the map
$$D(G,K)_b^n \times V_b^{'} \to V_b^{'} \tag{2.1.1}$$
describing $V_b^{'}$ as a left $D(G,K)_b^n$-module is continuous in its first variable, the surjection φ is necessarily continuous.

Now, $V_b^{'}$ is a Banach space. So it is a Fréchet space. Also $D(G,K)_b^n$ is a Fréchet space. So by the open mapping theorem φ is open.

Dualizing the map φ we get a closed G-equivariant embedding

$$(V_b^{'})^{'} \to (D(G,K)_b^n)^{'} \cong ((C(G,K)_b^{'})_b^{'})^n. \tag{2.1.2}$$

The above map is closed because since φ is open we have a topological isomorphism $D(G,K)_b^n/I \cong V_b^{'}$, where $I = \ker\varphi$. So

$$(V_b^{'})^{'} \cong (D(G,K)_b^n/I)^{'} \xrightarrow{closed} (D(G,K)_b^n)^{'}.$$

The map in equation 2.1.2 is an embedding because φ is surjective.

Let $j_i : D(G,K)_b \to V_b^{'}$ denote the i^{th} component of φ, where $1 \leq i \leq n$. By the discussion before the theorem we get that j_i is obtained by dualzing a continuous G-equivariant map $V \to C(G,K)$. Taking the direct sum of these we obtain a continuous G-equivariant map $V \to C(G,K)^n$. We obtain the following diagram.

$$\begin{array}{ccc} V & \xrightarrow{\psi} & C(G,K)^n \\ \downarrow & & \downarrow \\ (V_b^{'})_b^{'} & \longrightarrow & ((C(G,K)_b^{'})_b^{'})^n \end{array}$$

The bottom arrow is a topological embedding and since V and $C(G,K)$ are both Banach spaces, so the vertical arrows are also injective and induces a topological isomorphism onto its image. So ψ is an embedding. It remains to prove that the image of ψ is closed. But V is a Banach space, and hence the image of ψ is complete and hence closed.

Now we note that our field K is a finite extension of $\mathbb{Q}_p$ and hence spherically complete and hence all the nice theorems stated in Chapter 1 including the Hahn-Banach Theorem hold.

Conversely, if we dualize a closed G-equivariant embedding $V \to C(G,K)^n$, then we certainly obtain a surjection of left $D(G,K)$-modules.

$$D(G,K)^n \to V^{'}$$

This proves the theorem. $\square$

Definition 2.1.1: A continuous G-action on a K-Banach space V is said to be admissible continuous representation of G (or an admissible Banach space representation of G) if $V^{'}$ is finitely generated as a left $D(G,K)$-module.

Remark 2.1.1: If V is a K-Banach space equipped with an admissible continuous representation of G, and W is a closed G-invariant K-subspace of V, then the continuous G-representation on W is also admissible. This is because $W^{'}$ is a quotient of $V^{'}$. Here we note that K is spherically complete.

Proposition 2.1.2 : If $0 \to U \to V \to W \to 0$ is a G-equivariant exact sequence of K-Banach spaces equipped with continuous G-action, and if each of U and W is an admissible continuous G-representation, then V is also an admissible continuous G-representation.

Proof : Since

$$0 \to U \to V \to W \to 0$$

is exact, we have the following exact sequence of $D(G,K)$-modules.

$$0 \to W_b^{'} \to V_b^{'} \to U_b{'} \to 0.$$

If each of the end terms is finitely generated over $D(G,K)$, then $V_b{'}$ is also finitely generated $D(G,K)$-module. $\square$

Proposition 2.1.3 : Let K be discrete valued. (So it is spherically complete and the Hahn-Banach Theorem holds). The association of $V^{'}$ to V induces an anti-equivalence between the category of admissible continuous G-representation, (with morphisms being continuous G-equivariant maps) and the category of finitely generated $D(G,K)$-modules.

Proof Sketch: We know that the functor sending V to $V^{'}$ is faithful. This is from the fact that we discussed after Proposition 2.1.1. So the transpose of a continuous linear map between Hausdorff locally convex spaces vanishes iff the map vanishes. We now prove that the functor is full. Let us suppose that V and W are admissible continuous G-representations, and we are given a $D(G,K)$-linear map

$$W^{'} \to V^{'}. \tag{2.1.3}$$

We show that this map arises as the transpose of a continuous G-equivariant map $V \to W$. We choose surjective maps $D(G,K)^m \to V^{'}$ and $D(G,K)^n \to W^{'}$. Now we have a commutative diagram of $D(G,K)$-linear maps.

$$\begin{array}{ccc} D(G,K)^n_b & \longrightarrow & D(G,K)^m_b \\ \downarrow & & \downarrow \\ W^{'}_b & \longrightarrow & V^{'}_b \end{array}$$

Diagram 1

Now the discussion at the beginning of this section shows that if we remove the lower horizontal arrow from diagram 1 then it arises by dualizing a diagram of continuous G-equivariant maps of the form

$$\begin{array}{ccc} V & & W \\ \downarrow & & \downarrow \\ \mathcal{C}(G,K)^m & \longrightarrow & \mathcal{C}(G,K)^n \end{array} \tag{2.1.4}$$

Diagram 2

Now the vertical arrows are closed embeddings. Now the lower horizontal arrow of diagram 1 can be filled in implies that the lower horizontal arrow of diagram 2 restricts to a G-equivariant map $\phi : V \to W$. This is because dualizing the map $W^{'}_b \to V^{'}_b$ gives us a map $(V^{'}_b)^{'} \to (W^{'}_b)^{'}$. Now since K is spherically complete and V is a Banach space we have that the natural map $V \to (V^{'}_b)^{'}$ is a topological isomorphism onto its image and the same is true for W. By construction the transpose of ϕ is equal to the desired morphism $W^{'} \to V^{'}$ in equation 2.1.3. So the functor is full and we proved our claim. Now with some more effort one can prove that this functor turns out to be essentially surjective [cf,[1] Proposition 6.2.10]. Hence we deduce the required anti-equivalence of categories. □

Remark 2.1.2 Let G be a compact p-adic Lie group, and let K be a finite extension of $\mathbb{Q}_p$ and so spherically complete. Let us denote the ring of integers of K by o. Sending an element $g \in G$ to Dirac distribution $\delta_g \in D(G,K)$ extends to an embedding of o- algebras

$$o[[G]] \hookrightarrow D(G,K)$$

whose image can be proved to a lattice. So we have $K \otimes_o o[[G]] \cong D(G,K)$. (cf. [17]).

2.2 Admissible Topological Modules

Let G be a compact locally $\mathbb{Q}_p$ analytic group. Let E be an finite extension of $\mathbb{Q}_p$. Let M be a p-adically separated and complete torsion free $\mathcal{O}_E$-module, then $E \otimes_{\mathcal{O}_E} M$ is an E-Banach space. This is because, M being torsion free, the embedding $M \to E \otimes M$ identifies M with an $\mathcal{O}_E$-lattice of $E \otimes M$ and the gauge of M is the complete norm on $E \otimes M$. This fact becomes clear from lemma 1.1 that we discussed in Chapter 1.

If M is any $\mathcal{O}_E$-submodule, then we denote the torsion subgroup of M by M_{tors}, and M_{tf} is defined as M/M_{tors}. Here 'tf' is the short form of 'torsion free'.

Definition 2.2.1

If M is an $\mathcal{O}[G]$-module, then we say that M is admissible if it satisfies the following two conditions:

1. M_{tors} has bounded exponent i.e there exists a positive integer s such that $p^s M_{tors} = 0$.
2. The G-action on $E \otimes_{\mathcal{O}_E} M$ makes it an admissible continuous representation of G.

Remark 2.2.1: When we write $E \otimes M$ in this article we mean $E \otimes_{\mathcal{O}_E} M$. If $\phi : M \to N$ is an $\mathcal{O}_E$-linear and G-equivariant morphism of admissible $\mathcal{O}_E[G]$-modules, then $\ker \phi$, $coker\phi$, $im\phi$ are all admissible $\mathcal{O}_E[G]$-modules. These facts can be proved easily. Moreover, from proposition 1.2.4 of [14] $\ker \phi \to M$ is a closed embedding, when both have p-adic topologies.

Lemma 2.2.1 Let

$$0 \to M \to N \to P \to 0 \tag{2.2.1}$$

be a short exact sequence of $\mathcal{O}_E[G]$-modules. If M and P are admissible then N is also admissible.

Proof : From the exact sequence below

$$0 \to M_{tors} \to N_{tors} \to P_{tors}$$

we see that if M_{tors} and P_{tors} have bounded exponent then N_{tors} has also bounded exponent. Now we consider the exact sequence of projective systems

$$\{P[p^s]\}_{s \geqslant 1} \xrightarrow{f} \{M/p^s\}_{s \geqslant 1} \xrightarrow{g} \{N/p^s\}_{s \geqslant 1} \to \{P/p^s\}_{s \geqslant 1} \to 0.$$

Here $P[p^s]$ is the p^s-torsion elements of P. We note that we get the above exact sequence by snake lemma.

Now since P_{tors} has bounded exponent we see that the transition maps of the projective system $\{P[p^s]\}_{s \geqslant 1}$ eventually vanish. Hence, $R^1 \varprojlim P[p^s] = \varprojlim P[p^s] = 0$, where R^1 is the first right derived functor. We note that here we have used the following Mittag-Leffler condition (cf. proposition 3.5.7 of [28]):

If the range of morphisms of an inverse system of abelian groups (A_{ij}, f_{ij}) are stationary, that is for every k there exists $j \geqslant k$ such that for all $i \geqslant j$:

$$f_{kj}(A_j) = f_{ki}(A_i)$$

then $R^1 \varprojlim A_i = 0$.

Because of surjectivity of the transition maps of $\{M/p^s\}_{s\geqslant 1}$ we have $R^1 \varprojlim M/p^s = 0$ (again by the Mittag Leffler condition). So we have the exact sequence

$$0 \to \varprojlim M/p^s \to \varprojlim N/p^s \to \varprojlim P/p^s \to 0$$

because the inverse limit functor is left exact. Now since M and P are p-adically separated and complete we have $\varprojlim M/p^s \cong M$ and $\varprojlim P/p^s \cong P$. So using the five lemma of homological algebra we see that $N = \varprojlim N/p^s$. So N is p-adically separated and complete. Now tensoring the short exact sequence 2.2.1 with E over $\mathcal{O}_E$, since E is flat over $\mathcal{O}_E$, we see that N is an admissible $\mathcal{O}_E[G]]$-module because an extension of an admissible continuous representation of G is again an admissible continuous representation and we proved this fact in proposition 2.1.2. $\square$

Lemma 2.2.2 : Let M be a torsion free admissible $\mathcal{O}[G]$-module, then there exists $\delta : M \hookrightarrow C(G, \mathcal{O}_E)^n$, for some natural number n. Moreover, if $N = Coker\delta$. Then there exists a $\mathcal{O}_E$-linear map $\phi : N \to C(G, \mathcal{O}_E)^n$ with the property that the composite of ϕ with the projection map $C(G, \mathcal{O}_E) \to N$ is equal to multiplication by p^s on N.

Proof: We have $i : M \hookrightarrow E \otimes M$. This map is an embeddding since M is torsion free. Now since M is an admissible module then by definition $E \otimes M$ is an admissible representation of G. Hence we have a closed embedding $\theta : E \otimes M \hookrightarrow C(G, E)^n$. This gives us the map by rescaling scalars

$$M \to C(G, \mathcal{O}_E)^n.$$

We describe the process of rescaling. Let $m^{'} \in M$ Let $i(m^{'}) = m, m \in E \otimes_{\mathcal{O}_E} M$. Let us denote $\theta(m)$ by $f \in C(G, E)^n$. Let $f = (f_1, ..., f_n)$. Now we work with f_1. Consider any open set $a + p^n\mathcal{O}_E$ around a where $a \in E$. Since f_1 is continuous there exists a neighbourhood V of G such that $f_1(V) \subseteq a + p^n\mathcal{O}_E$. Now let $a = b_V/c_V$ where $b_V, c_V \in \mathcal{O}_E$. Then $c_V f_1(v) \in \mathcal{O}_E$ for all $v \in V$ Now we vary a and thus we vary V and cover G with open neighbourhoods. Now we use that G is compact. So there exist finitely many neighbourhoods $V_1, ..., V_r$ which cover G. Now we see that $\prod_{i=1}^{i=r} c_{V_i} f_1$ takes values in $\mathcal{O}_E$. Hence obviously there exists a $c \in \mathcal{O}_E$ such that cf takes values in $\mathcal{O}_E$.

Now we have the following exact sequence

$$0 \to E \otimes M \xrightarrow{closed} C(G, E)^n \to E \otimes N \to 0$$

where N is the cokernel of the map $M \to C(G, \mathcal{O}_E)^n$.

Now we quote another fact from nonarchimedean functional analysis (cf. Proposition 10.5 of [22]).

Fact: If V is an E-Banach space, E is a finite extension of $\mathbb{Q}_p$, then for every closed vector subspace $U \subseteq V$ there exists another closed vector space $U_1 \subseteq V$ such that the linear map

$$\begin{aligned} U \oplus U_1 &\to V \\ (v + v_1) &\mapsto v + v_1 \end{aligned}$$

is a topological isomorphism. When this holds we say that any closed subspace in V is complemented.

By the above fact we have a splitting of the above exact sequence $\sigma : E \otimes N \to C(G, E)^n$. Now in the beginning of the proof we showed the process of rescaling scalars for the map θ. We do the same for σ. So for some y, $p^y\sigma$ takes the image of N in $E \otimes N$ which is N/N_{tors}, into $C(G, \mathcal{O}_E)^n$. Now because N_{tors} has bounded exponent we have a natural projection $p^k(\cdot) : N \to N_{tf}$. Let $s = k + y$, and we define $\phi := p^s\sigma \circ p^k(\cdot)$. This ϕ satisfies the required conditions of the theorem because we have constructed σ as the splitting of the map $C(G, E)^n \to E \otimes_{\mathcal{O}_E} N$. □

Remark 2.2.2 : Let $\mathcal{A}_G^{'}$ be the category of $\mathcal{O}_E[[G]]$-modules, and among them we denote the subcategory of the admissible ones by $\mathcal{A}_G$. Let $\mathcal{B}_G^{''}$ denote the category of projective systems $\{M_s\}_{s \geqslant 1}$ of $\mathcal{O}_E[G]$-modules such that M_s is killed by p^s. We call an object of $\mathcal{B}_G^{''}$ <u>essentially null</u> if the transition maps $M_{s'} \to M_s$ vanish if $s^{'}$ is large. This is a Serre subcategory. We denote by $\mathcal{B}_G^{'}$ the Serre quotient category. The concept of Serre category and Serre quotient is discussed in brief in the following paragraph. For reference one can see exercise 10.3.2 of [28].

A (nonempty) full subcategory T of an abelian category A is a Serre subcategory if for any exact sequence

$$0 \to M \to M^{'} \to M^{''} \to 0$$

$M^{'}$ is in T iff M and $M^{''}$ are in T. Now we discuss the process of obtaining Serre quotient. Let A be an abelian category. Let C be a Serre subcategory of A. There exists an abelian category A/C (called the Serre quotient category) and an exact functor $F : A \to A/C$ which is essentially surjective and whose kernel is C. Moreover for any exact functor $G : A \to B$ such that $C \subset \ker(G)$ there exists an exact functor $H : A/C \to B$ such that $G = H \circ F$.

Now it is clear that the essentially null objects of $\mathcal{B}_G^{''}$ form a Serre subcategory.

We list here some facts about categories and functors and we shall not give any proofs. The reader can consult [14] (lemma 1.2.9, page 16) for proofs.

- The map $S : \mathcal{A}_G^{'} \to \mathcal{B}_G^{'}$ sending $M \to \{M/p^s\}_{s\geqslant 1} \to P$ where P the image of $\{M/p^s\}_{s\geqslant 1}$ under the Serre quotient, is a functor. We denote the image of S restricted to $\mathcal{A}_G$ by $\mathcal{B}_G$.
- The projective limit induces a functor $\mathcal{B}_G^{''} \to \mathcal{A}_G^{'}$ which gives us a functor $T : \mathcal{B}_G^{'} \to \mathcal{A}_G^{'}$.
- S restricted to $\mathcal{A}_G$ is exact with the quasi-inverse T restricted to $\mathcal{B}_G$. The category $\mathcal{A}_G$ is equivalent to the category $\mathcal{B}_G$ by the morphisms S and T.
- The subcategory $\mathcal{B}_G$ of $\mathcal{B}_G^{'}$ is closed under kernels and cokernels. This comes from the fact that S restricted to $\mathcal{A}_G$ is exact.

Proposition 2.2.1 : Let $M^\bullet$ be a cochain complex in $\mathcal{A}_G$. Then for $n \in \mathbb{Z}$, the natural map $\{H^n(M^\bullet)/p^s\}_{s\geqslant 1} \to \{H^n(M^\bullet/p^s)\}_{s\geqslant 1}$ is an isomorphism of objects in the category $\mathcal{B}_G$. Here we get the above natural map in the following way. For fixed s we have a natural projection map from $M^\bullet \to M^\bullet/p^s$. This induces the map $\phi : H^n(M^\bullet) \to H^n(M^\bullet/p^s)$ which is also induced from the projection map and hence all elements of the form $p^s a$ for $a \in H^n(M^\bullet)$ belongs to the kernel of ϕ. Hence we get a map

$$\{H^n(M^\bullet)/p^s\}_{s\geqslant 1} \to \{H^n(M^\bullet/p^s)\}_{s\geqslant 1}$$

Proof : The cohomology $H^n(M^\bullet) \in \mathcal{A}_G$. This is because $\mathcal{A}_G$ is closed under taking kernels and cokernels. Also from the facts just jotted down $\mathcal{B}_G$ is closed under taking kernels and cokernels. We also note that $S|_{\mathcal{A}_G}$ is exact, so it commutes with cohomology and S and T are quasi-inverses between categories $\mathcal{A}_G$ and $\mathcal{B}_G$. These facts give us $H^n(S(M^\bullet)) \in \mathcal{B}_G$, $S(H^n(M^\bullet)) \cong H^n(S(M^\bullet))$. Thus we get our required isomorphism. Also $H^n(M^\bullet) \cong TS(H^n(M^\bullet)) \cong T(H^n(S(M^\bullet)))$. Hence we also get that $H^n(M^\bullet) \to \varprojlim H^n(M^\bullet/p^s)$ is an isomorphism in the category of admissible $\mathcal{O}_E[G]$-modules. $\square$

Now if G is trivial we omit G from our previous notations and denote by $\mathcal{A}, \mathcal{A}^{'}, \mathcal{B}, \mathcal{B}^{'}$ the above categories. We call the objects of $\mathcal{A}$ by admissible $\mathcal{O}_E$-modules.

Lemma 2.2.3 : Let us assume that we are given $\mathcal{O}_E$-modules M and N, and $\mathcal{O}_E$-linear maps $M \to N$ and $N \to M$ such that the composition equals multiplication by p^s. Then M is admissible provided N is given to be admissible.

Proof: Let us note that $\mathcal{A}$ consists of M for which M_{tors} has bounded exponent, and M_{tf} is finitely generated. Now let X be the kernel of the map $M \to N$. Then by hypothesis $X \subseteq M[p^s]$. That is X is contained in the torsion submodule of M. This gives us the exact sequence $0 \to X \to M_{tors} \to N_{tors}$, we see that if N_{tors} has bounded exponent then M_{tors} has also bounded exponent. Now using the natural embedding $M_{tf} \to N_{tf}$ we see that M_{tf} is finitely generated over $\mathcal{O}_E$ (which is a PID) since N_{tf} is finitely generated over $\mathcal{O}_E$.

If M is any topological $\mathcal{O}_E$-module with a continuous G-action, then we denote by $C^{\bullet}_{con}(G, M)$, the complex of continuous cochains on G with values in M, and we denote by $H^{\bullet}_{con}(G, M)$, the cohomology of the complex $C^{\bullet}_{con}(G, M)$. If M is an admissible $\mathcal{O}_E[G]$-module then we regard M with its p-adic topology. Now we admit the following fact without proof (cf. [14] Lemma 1.2.17). $\square$

Fact 2.2.1: $H^0_{con}(G, C(G, \mathcal{O}_E)) \cong \mathcal{O}_E$, $H^i_{con}(G, C(G, \mathcal{O}_E)) = 0$ for $i > 0$

Proposition 2.2.2 : If M is an admissible $\mathcal{O}_E[G]$-module, then for $i \in \mathbb{N}$, $H^i_{con}(G, M)$ is an admissible $\mathcal{O}_E$-module.

Proof: We have the natural map $M \to M_{tf} \to M$. The first map is multiplication by p^s where p^s is the exponent of M_{tors}. So we have maps

$$H^i_{con}(G, M) \to H^i_{con}(G, M_{tf}) \to H^i_{con}(G, M).$$

By lemma 2.2.3, it suffices to show that $H^i_{con}(G, M_{tf})$ is an admissible $\mathcal{O}_E$-module. So we prove our proposition when M is torsion free.

We will use induction on i. As M is torsion free there exists an embedding $M \to C(G, \mathcal{O}_E)^n$. This follows from lemma 2.2.2. This gives us

$$H^0_{con}(G, M) \hookrightarrow \mathcal{O}_E^n$$

as $H^0_{con}(G, C(G, \mathcal{O}_E)) \cong \mathcal{O}_E$, from fact 2.2.1. So $H^0_{con}(G, M)$ is an admissible $\mathcal{O}_E$-module. This is because, since $\mathcal{O}_E$ is a PID we can see that $(H^0_{con}(G, M))_{tors}$ has bounded exponent (as $(\mathcal{O}_E^n)_{tors}$ is finitely generated $\mathcal{O}_E$ -module) and $(H^0_{con}(G, M))_{tf}$ is also finitely generated.

We now assume that the case $i - 1$ is know to us. Let us denote by N the module $M/C(G, \mathcal{O}_E)^n$. Now by lemma 2.2.2 we get a map $\phi : N \to C(G, \mathcal{O}_E)^n$ and $proj : C(G, \mathcal{O}_E)^n \to N$ such that $proj \circ \phi = p^s(\cdot)$ for $s \in \mathbb{N}$. Now $C^{\bullet}$ denotes the cokernel of the natural map $C^{\bullet}_{con}(G, M) \to C^{\bullet}_{con}(G, C(G, \mathcal{O}_E)^n)$, then we get that $C^{\bullet} \hookrightarrow C^{\bullet}_{con}(G, N)$. This embedding is because we have

$$0 \to M \to C(G, \mathcal{O}_E) \to N \to 0$$

an exact sequence which implies

$$0 \to C^{\bullet}_{con}(G, M) \to C^{\bullet}_{con}(G, C(G, \mathcal{O}_E)) \xrightarrow{\theta} C^{\bullet}_{con}(G, N)$$

is exact.

So

$$C^{\bullet}_{con}(G, C(G, \mathcal{O}_E))/C^{\bullet}_{con}(G, M) = C^{\bullet}_{con}(G, C(G, \mathcal{O}_E))/\ker\theta \hookrightarrow C^{\bullet}_{con}(G, N).$$

Hence we get that the composite map $C^\bullet \to C^\bullet_{con}(G,N) \to C^\bullet$ is $p^s(\cdot)$, where the second map is induced by ϕ. This gives us

$$H^{i-1}(C^\bullet) \to H^{i-1}_{con}(G,N) \to H^{i-1}(C^\bullet)$$

such that the composite equals $p^s(\cdot)$. Now by the induction hypothesis $H^{i-1}_{con}(G,N)$ is an admissible module which implies that $H^{i-1}(C^\bullet)$ is admissible by lemma 2.2.3. Now we know that a short exact sequence of complexes gives us a long exact cohomology sequence with a connecting δ map between i^{th} and $(i+1)^{th}$ cohomology. Hence we get an exact sequence

$$H^{i-1}(C^\bullet) \to H^i_{con}(G,M) \to H^i_{con}(G, C(G,\mathcal{O}_E)^n)$$

which is an extract of the long exact sequence obtained by considering the exact sequence (exactness is obvious by definition of $C^\bullet$) of complex:

$$0 \to C^\bullet_{con}(G,M) \to C^\bullet_{con}(G, C(G,\mathcal{O}_E)^n) \to C^\bullet \to 0.$$

Now we use the fact that $H^i_{con}(G, C(G,\mathcal{O}_E)) = 0$ for $i > 0$ and so we get that $H^i_{con}(G,M)$ is admissible because $H^i_{con}(G,M)$ is then the quotient of the admissible module $H^{i-1}(C^\bullet)$ and we know that the category of admissible $\mathcal{O}_E$-modules is a Serre subcategory of the category of all $\mathcal{O}_E$-modules (cf. [14] Lemma 1.2.13). This completes our proof. □

Proposition 2.2.3 : Let M be an admissible $\mathcal{O}_E[G]$ modue, then the kernel and cokernel of the natural map of the projective systems

$$\{H^i_{con}(G,M)/p^s\}_{s\geqslant 1} \to \{H^i_{con}(G,M/p^s)\}_{s\geqslant 1}$$

are essentially null projective systems.

Proof: Let M_{tors} has exponent p^{s_0}. Then naturally we have a map $\phi : M_{tf} \to M$ which is $\phi(a + M_{tors}) = p^{s_0}a + M_{tors}$ such that the composite $M \twoheadrightarrow M_{tf} \to M$ is $p^{s_0}(\cdot)$. Now for $s \geqslant s_0$, we write $\phi_s = p^{s-s_0}\phi$. Now we can easily see that the sequence $0 \to M_{tf} \xrightarrow{j} M \to M/p^s \to 0$ is exact, where the second map j is the map by ϕ_s. Now the topology on M/p^s is discrete. So the maps in $C^\bullet_{con}(G,M/p^s)$ are locally constant and hence they can be lifted to the locally constant elements of $C^\bullet_{con}(G,M)$. Hence we obtain an exact sequence $0 \to C^\bullet(G,M_{tf}) \to C^\bullet(G,M) \to C^\bullet(G,M/p^s) \to 0$. Let $H^{i+1}_{con}(G,M_{tf})[\phi_s]$ be all elements $x \in H^{i+1}_{con}(G,M_{tf})$ such that $\phi_s(x) = 0$. Now a short exact sequence gives us a long exact cohomology sequence. Hence we get that the sequence

$$0 \to H^i_{con}(G,M)/\phi_s(H^i_{con}(G,M_{tf})) \xrightarrow{z} H^i_{con}(G,M/p^s) \twoheadrightarrow H^{i+1}_{con}(G,M_{tf})[\phi_s]$$

is exact. This is because we have

$$0 \to C^\bullet(G,M_{tf}) \to C^\bullet(G,M) \to C^\bullet(G,M/p^s) \to 0$$

an exact sequence.

So we have the following exact sequence

$$H^{i-1}(G, M_{tf}) \xrightarrow{\phi_s} H^i(G, M) \xrightarrow{m} H^i(G, M/p^s) \xrightarrow{g} H^{i+1}(G, M_{tf}) \xrightarrow{\phi_s} H^{i+1}(G, M).$$

This implies that

$$H^i(G, M)/\phi_s(H^{i-1}(G, M_{tf})) \cong H^i(G, M)/\ker m \hookrightarrow H^i(G, M/p^s).$$

So z is injective. Also the image of g is the kernel of ϕ_s which is $H^{i+1}_{con}(G, M_{tf})[\phi_s]$.

Now $H^i_{con}(G, M_{tf})[\phi_s]$ is contained in $H^{i+1}_{con}(G, M_{tf})[p^s]$, because the composite $M_{tf} \xrightarrow{\phi_s} M \to M_{tf}$ is $p^s(\cdot)$ and $\phi_s = p^{s-s_0}\phi$. Now we use our proposition 2.2.2 for the fact that $H^{i+1}_{con}(G, M_{tf})$ is an admissible module. Here we note that M is an admissible $\mathcal{O}_E[G]$-module implies that M_{tf} is also admissible (cf. [14] Lemma 1.2.2).

This gives us that the projective system $\{H^{i+1}_{con}(G, M_{tf})[p^s]\}_{s \geqslant s_0}$ is essentially null because $(H^{i+1}_{con}(G, M_{tf}))_{tors}$ has bounded exponent and hence the projective system $\{H^{i+1}_{con}(G, M_{tf})[\phi_s]\}_{s \geqslant s_0}$ is essentially null. We denote this fact by β for providing future reference.

This gives us that

$$H^i_{con}(G, M)/\phi_s(H^i_{con}(G, M_{tf})) \hookrightarrow H^i_{con}(G, M/p^s)$$

has essentially null cokernel for $s \geqslant s_0$.

For $s \geqslant s_0$ we have $p^s H^i_{con}(G, M) \subseteq \phi_s(H^i_{con}(G, M_{tf})) \subseteq p^{s-s_0} H^i_{con}(G, M)$. The second containment is because $\phi_s = p^{s-s_0}\phi$ and $\phi(H^i_{con}(G, M_{tf})) \subseteq H^i_{con}(G, M)$. So by the same type of arguments as we have done in β the natural map of the projective systems

$$q : \{H^i_{con}(G, M)/p^s\}_{s \geqslant s_0} \twoheadrightarrow H^i_{con}(G, M)/\phi_s(H^i_{con}(G, M_{tf}))_{s \geqslant s_0}$$

has essentially null kernel. In total we get that

$$\{H^i_{con}(G, M)/p^s\}_{s \geqslant 1} \to \{H^i_{con}(G, M/p^s)\}_{s \geqslant 1}$$

has essentially null kernel and cokernel. We recall that we get the above map by $z \circ q$. This concludes the proof of our theorem. $\square$

Chapter 3

Iwasawa Algebra

3.1 Results from Lazard

Let G be any abstract group. A p-valuation ω on G is a real valued function

$$\omega : G\backslash\{1\} \to (0, \infty)$$

satisfying

1. $\omega(1) = \infty$ (convention)
2. $\omega(g) > \frac{1}{p-1}$
3. $\omega(g^{-1}h) \geqslant \min\{\omega(g), \omega(h)\}$
4. $\omega([g, h]) \geqslant \omega(g) + \omega(h)$
5. $\omega(g^p) = \omega(g) + 1$

for any $g, h \in G$. Here $[g, h] = ghg^{-1}h^{-1}$. When we put $h = 1$ in (3) we get that $\omega(g^{-1}) \geqslant \omega(g)$ for any $g \in G$ and by symmetry $\omega(g^{-1}) = \omega(g)$. One can also prove using the five properties stated above that $\omega(ghg^{-1}) = \omega(h)$ for all g, h and $\omega(gh) = \min(\omega(g), \omega(h))$ if $\omega(g) \neq \omega(h)$.

Now for any real number $v > 0$, we put,

$$G_v := \{g \in G : \omega(g) \geqslant v\} \text{ and } G_{v+} := \{g \in G, \omega(G) > v\}.$$

Now G_v's form a decreasing filtration of G and hence there exists a unique topology on G (the topology defined by the filtration) such that G_v form a fundamental system of open neighbourhoods of identity. It is called the topology defined by ω.
Let us define for each $v > 0$,

$$\mathrm{gr}_v G := G_v/G_{v+} \text{ and } \mathrm{gr} G := \oplus_{v>0}\, \mathrm{gr}_v G.$$

An element $\xi \in \mathrm{gr}G$ is called homogeneous of degree v if $\xi \in \mathrm{gr}_v G$.
Now, if $\xi =$g G_{v+}, then $\xi^p =$g$^p G_{v+}$, $\omega(g) \geqslant v$, $\Rightarrow \omega(g^p) = \omega(g) + 1 \geqslant v + 1 > v$ so $\xi^p = 0$ ———————————————————————— (α)

Now we make $\mathrm{gr}_v G$ into an $\mathbb{F}_p$-vector space by defining $\overline{a}\xi{=}\xi^{\overline{a}}$ for $\xi \in G_v/G_{v+}$. Because of (α) this is a well defined definition, and so $\mathrm{gr}_v G$, hence $\mathrm{gr}G$ is an $\mathbb{F}_p$-vector space.

Now for any $v, v' > 0$ the map

$$\mathrm{gr}_v G{\times}\mathrm{gr}_{v'} G \to \mathrm{gr}_{v+v'} G$$

sending $(\zeta, \eta) \mapsto [\zeta, \eta] := [g, h]G_{(v+v')+}$ where $\zeta = gG_{v+}$ and $\eta = hG_{v+}$ is well defined, biadditive map. The above fact gives us a graded $\mathbb{F}_p$-bilinear map

$$[,] : \mathrm{gr}G{\times}\mathrm{gr}G \to \mathrm{gr}G$$

For more details one can look at (cf. Lemma 23.4 of [17]). Now as ω is a p-valuation, $\omega(g^p) = \omega(g) + 1$ and with some more effort one can show that $\forall g, h \in G$

1. $\omega(h^{-p}g^{-p}(gh)^p) > \max(\omega(g), \omega(h)) + 1$,
2. $\omega(g^{-ph}h^{pn}) = \omega(g^{-1}h) + n$ for any integer $n \geqslant 1$.

Now let $v > 0$ and $g, h \in G$ such that $\omega(h) \geqslant v = \omega(g)$. Then $\omega(h^{-p}g^{-p}(gh)^p) > \omega(h) + 1 > v + 1$. This implies $h^{-p}g^{-p}(gh)^p \in G_{(v+1)+}$. This gives us $(gh)^p G_{(v+1)+} = g^p h^p G_{(v+1)+}$.

If $\omega(h) > v$ then $\omega(h^p) = \omega(h) + 1 > v + 1$. So $(gh)^p G_{(v+1)+} = g^p G_{(v+1)+}$. Hence the map

$$\begin{aligned} \mathrm{gr}_v G &\to \mathrm{gr}_{v+1} G \\ gG_{v+} &\to g^p G_{(v+1)+} \end{aligned}$$

is well defined and $\mathbb{F}_p$-linear. This gives us a $\mathbb{F}_p$-linear map of degree one

$$P : \mathrm{gr}G \to \mathrm{gr}G.$$

Therefore we can view $\mathrm{gr}G$ as a graded module over $\mathbb{F}[P]$.

Now we let G to be a profinite group and ω be a p-valuation on G which we assume to define the topology of G. So G_v is open and G/G_v is finite, $G = \varprojlim_v G/G_v$. Axiom 5 of the valuation ω gives us that G/G_v is a p-group. So G is a pro-p group. We also recall that in a pro-p group we can take $\mathbb{Z}_p$ power of an element of the group that is if $g \in G$ and $\alpha \in \mathbb{Z}_p$ then g^α is well defined. (cf. Lemma 1.24 page 29 of [2]).

Definition 3.1.1: The pair (G, ω) is called of finite rank if gr$G$ is finitely generated over $\mathbb{F}_p[P]$.

Now one can show that $\mathrm{gr}(G)$ is torsion free module over $\mathbb{F}_p[P]$ (cf. Remark 25.2 of [17]). But a finitely generated torsion free module over a PID $\mathbb{F}_p[P]$ is free. So we can define

$$\mathrm{rank}(G, \omega) := \mathrm{rank}_{\mathbb{F}_p[P]} \mathrm{gr} G.$$

Let us fix $g_1, ..., g_r \in G$. Now we consider the continuous map

$$\mathbb{Z}_p^r \to G$$
$$(x_1, \cdots x_r) \mapsto g_1^{x_1} \cdots g_r^{x_r}.$$

We note that this map depends on the order of $g_1, ..., g_r$ and hence it is not a group homomorphism.

Definition 3.1.2: The sequence of elements $(g_1, ..., g_r)$ in G is called an ordered basis of (G, ω) if the above map is a homeomorphism and

$$\omega(g_1^{x_1}, \cdots g_r^{x_r}) = \min_{1 \leqslant i \leqslant r}(\omega(g_i) + v(x_i)) \text{ for } x_i \in \mathbb{Z}_p.$$

Here v is the valuation map of $\mathbb{Z}_p$. Now we proved previously that if (G, ω) is of finite rank then the rank of gr$G$ over $\mathbb{F}_p[P]$ is finite. Following [17] (proposition 26.5) we can relate the basis of G to the basis of $\mathrm{gr}(G)$ over $\mathbb{F}_p[P]$. In fact, $(g_1, ..., g_r)$ is an ordered basis of (G, ω) iff $\sigma(g_1), ..., \sigma(g_r)$ is a basis of the $\mathbb{F}_p[P]$-module $\mathrm{gr}(G)$. Here for $g \neq 1$, $\sigma(g) := gG_{\omega(g)+} \in \mathrm{gr}(G)$. In analogy with the theory of vector spaces one can also show that any (G, ω) of finite rank has an ordered basis (of length equal to rank of gr$G$ over $\mathbb{F}_p[P]$).

Now we fix a p-valuable group G with a p-valuation ω on it. We fix a complete discrete valuation ring $\mathcal{O} \supseteq \mathbb{Z}_p$ and denote the valuation by v.

Now we pick an ordered basis $(g_1, ..., g_r)$ of (G, ω). By definition we have

$$c : \mathbb{Z}_p^r \to G$$
$$(x_1, \cdots, x_r) \mapsto g_1^{x_1}, \cdots, g_r^{x_r}$$

where the map c is a homeomorphism. Now let $C(G)$ be the continuous functions from G to $\mathcal{O}$. The map c induces an isomorphism of $\mathcal{O}$-modules

$$c^* : C(G) \cong C(\mathbb{Z}_p^r).$$

Now let $\Lambda(G)$ be the Iwasawa algebra of G over $\mathcal{O}$ that is $\Lambda(G) := \mathcal{O}[[G]]$. Now lemma 22.1 of [17] shows that

$$\Lambda(G) = Hom_{\mathcal{O}}(C(G), \mathcal{O}).$$

So dualizing the map c^* we get an isomorphism of $\mathcal{O}$-modules

$$c_* = \Lambda(\mathbb{Z}_p^r) \cong \Lambda(G).$$

This gives us $\Lambda(G) \cong \mathcal{O}[[X_1, ...X_r]]$ as topological $\mathcal{O}$-modules because $\Lambda(\mathbb{Z}_p^r) \cong \mathcal{O}[[X_1, ..., X_r]]$ (cf. prop 20.1 of [17]).

Now let G be any pro-p analytic group that is G is pro-p which is also provided with a p-adic analytic group structure i.e. it is a p-adic manifold with morphisms

$$G \times G \to G$$
$$(x, y) \mapsto xy$$

and the inverse map $x \mapsto x^{-1}$ being analytic maps. In [16] (chapter 5, section 2.2.4, it is proved that if G is a pro-p analytic group, then it admits an open subgroup U which admits a p-valuation such that with respect to the p-valuation, U is complete and its graduation is of finite rank over $\mathbb{F}_p[P]$ and then the Iwasawa algebra of G is both left and right Noetherian. This is a result that we will need later in Chapter 5.

3.2 Construction of the Skew Field of the Iwasawa Algebra

Let R be a non commutive ring. An element $x \in R$ is called right regular if $xr = 0$ implies $r = 0$ for $r \in R$. Similarly left regular elements are defined, and regular means both right and left regular. We denote the set of all regular elements of R by $C_R(0)$ which is a multiplicative closed set.

Let S be a (nonempty) multiplicative closed set of a ring R, and let $assS = \{r \in R | rs = 0 \text{ for some } s \in S\}$.

Then a right quotient ring of R with respect to S is a ring Q (denoted by R_S) together with a homomorphism $\theta : R \to Q$ such that

1. for all $s \in S, \theta(s)$ is a unit in Q,
2. for all $q \in Q, q = \theta(r)\theta(s)^{-1}$ for some $r \in R, s \in S$; and
3. $\ker \theta = assS$.

A multiplicative subset S of R is said to satisfy the right Ore condition if for each $r \in R$ and $s \in S$, $\exists r' \in R, s' \in S$; such that $rs' = sr'$. Now since S satisfies the right Ore condition then $assS$ is a two-sided ideal of R. (cf. [24] Section 2.1.9, Chapter 2) Now we denote by $\bar{S}$ the projection of S in the ring $\bar{R} = R/assS$.

Theorem 3.2.1 : Let S be a multiplicative closed subset of a ring R. Then R_S exists iff S satisfies the right Ore condition and $\bar{S}$ consists of regular elements.

Proof: These conditions are necessary (from [24] Section 2.1.6 and Section 2.1.11). We now describe how to construct the ring of fractions.

Construction: First we consider the set $\Im$ of those right ideals A of R such that $A \cap S \neq \emptyset$. It can be easily verified (using the right Ore condition) that for $A_1, A_2 \in \Im$, and $\alpha \in Hom_R(A_1, R)$ that

1. $A_1 \cap A_2 \in \Im$,

2. $\alpha^{-1}A_2 \equiv \{a \in A, |\alpha(a) \in A_2\} \in \Im$.

Here $Hom_R(A_1, R)$ denotes R-module homomorphism. A_1 is a right ideal so it is obviously a right R-module.

Secondly, we consider the set $\cup\{Hom_R(A, R)|A \in \Im\}$ together with the equivalence relation, $\alpha_i \in Hom_R(A_i, R)$. We say $\alpha_1 \sim \alpha_2$ if α_1 and α_2 coincide on some $A \in \Im, A \subseteq A_1 \cap A_2$. We define the operations on equivalence classes $[\alpha_i]$ by $[\alpha_1] + [\alpha_2] = [\beta]$, where β is the sum of the restrictions of α_1 and α_2 to $A_1 \cap A_2$; and by $[\alpha_1][\alpha_2] = [\gamma]$ with γ being their composition when restricted to $\alpha_2^{-1}A_1$.

The following facts can be checked easily (cf. [24] Section 2.1.12).

1. These operations are well defined, under them, the equivalence classes form a ring denoted by R_S.
2. That, if $r \in R$ is identified with the equivalence class of the homomorphism $\lambda(r) : R \to R$ given by $x \longmapsto rx$, this embeds R in R_S.
3. Under this embedding, each $s \in S$ has an inverse $s^{-1} = [\alpha]$, where $\alpha : sR \longrightarrow R$ is given by $sx \longmapsto x$, and
4. if $\alpha \in Hom_R(A, R)$ with $A \in \Im$, then $[\alpha] = as^{-1}$ where $s \in A \cap S$ and $a = \alpha(s)$.

Thus we have demonstrated the method of construction of the ring of fractions. □

A multiplicative subset S of a ring R which satisfies the right Ore condition will be called a right Ore set. A right Ore set such that the elements of $\bar{S}$ are regular in $\bar{R} = R/assS$ (and so R_S exists) will be called a right denominator set. If S is a right Ore set in a ring R and if the elements of S are regular in R, then it can be proved that S is a right denominator set (cf. [24] Section 2.1.13). The right quotient ring of a ring R with respect to $C_R(0)$, the set of all regular elements, is simply called the right quotient ring of R and is denoted by $Q(R)$. That is, when we say right quotient ring of R we mean that we are considering the multiplicative set $C_R(0)$ and taking the right quotient ring of R with respect to $C_R(0)$. An integral domain (i.e. does not have zero divisors) R is called a right Ore domain if $C_R(0)$ is a right Ore set.

Corollary 3.2.1 : The previous theorem gives that :

An integral domain has a right quotient ring iff it is a right Ore domain.

If R is an integral domain then it is obvious that its right quotient ring will be a division ring which is also called a skew field or a right quotient division ring of R.

Let E be a finite extension of $\mathbb{Q}_p$. Let G be a pro-p analytic group. G can also be a p-adic analytic group which is a torsion free pro-p group. Let $\mathcal{O}_E$ denote the ring of integers of E.

We recall that we are working with $\Lambda_E = E \otimes \mathcal{O}_E[[G]]$ which is by [16] or by [2] (corallary 7.26 and the discussion after it) Noetherian noncommutative integral domain. So if we prove that any right Noetherian integral domain R is a right Ore domain then this will allow us to construct its skew field K and then we can define rank of any Λ_E-module M as $\dim_K(M \otimes_\Lambda K)$.

Theorem 3.2.2 : Any right Noetherian integral domain R is a right Ore domain.

Proof : We show that $aR \cap bR = 0 \Longrightarrow$ the elements $\{a^i b | i \in \mathbb{N}\}$ are R - linearly independent on the right.
If $\Sigma_{i=0}^n a^i b r_i = 0$, then $-br_0 = \Sigma_{i=1}^n a^i b r_i$, but a factors to the left side of this last expression. So both sides are zero, so $r_0 = 0$ and $\sum_{i=1}^n a^{i-1} b r_i = 0$. We do this repeatedly and thus all the r_i's are zero.

This gives us that $\oplus_{i \in \mathbb{N}} a^i b R$ is a submodule of R. But this submodule contains infinite ascending chain. Since R is right Noetherian, this is a contradiction. $\square$

Now we would like to point out that we can make similar construction for constructing left quotient ring of R. However it is not true that the right quotient ring and the left quotient ring of R are the same. They are same iff the set of regular elements of R is both a left and a right Ore set. For proof one can see proposition 6.5 of [3].

So we can now talk of the skew field K of $\Lambda_E = E[[G]]$.

Chapter 4

Completed Cohomology and Spectral Sequence

4.1 Hochschild-Serre Spectral Sequence

We assume that the reader is familiar with basics of category theory and with the definition of cohomology of complexes. We will not discuss them here.

Let G be a profinite group. We begin by defining the n-dimensional continuous cohomology group of G with coefficients in a G-module A. We assume that A is discrete. Let X^n be the set of all continuous maps $x : G^{n+1} \to A$. We also write it as $X^n(G, A)$. Then X^n is naturally a G-module by

$$(\sigma x)(\sigma_0, ..., \sigma_n) = \sigma x(\sigma^{-1}\sigma_0, ..., \sigma^{-1}\sigma_n).$$

We also consider the maps

$$\begin{aligned} d_i : G^{n+1} &\to G^n \\ (\sigma_0, ..., \sigma_n) &\mapsto (\sigma_0, ..., \hat{\sigma_i}, ..., \sigma_n) \end{aligned}$$

where $\hat{\sigma_i}$ denotes that σ_i is omitted.

Now we construct a map $d_i^* : X^{n-1} \to X^n$

$$\begin{aligned} d_i^* : X^{n-1} &\to X^n \\ g &\mapsto g \circ d_i. \end{aligned}$$

Now for each $x \in X^{n-1}$ we define $\delta^n(x)$ as

$$(\delta^n x)(\sigma_0, ..., \sigma_n) = \textstyle\sum_{i=0}^{n}(-1)^i x(\sigma_0, ..., \hat{\sigma_i}, ..., \sigma_n).$$

That is

$$\delta^n = \textstyle\sum_{i=0}^{n}(-1)^i d_i^* : X^{n-1} \to X^n.$$

Following [23] proposition 1.2.1 one can show that

$$0 \to A \xrightarrow{\delta^0} X^0 \xrightarrow{\delta^1} X^1 \to \cdots$$

is exact, where $\delta^0 : A \to X^0$ associates to $a \in A$ the constant function $x(\sigma_0) = a$. The above exact sequence is also called the standard resolution. Now let $C^n(G, A) = X^n(G, A)^G$ which is by definition all elements of X^n fixed by G. Formally $C^n(G, A)$ consists of continuous functions $x : G^{n+1} \to A$ such that

$$x(\sigma\sigma_0, ...\sigma\sigma_0) = \sigma x(\sigma_0, ..., \sigma_n) \text{ for all } \sigma \in G.$$

Now from the standard resolution we obtain the complex

$$C^0(G, A) \xrightarrow{\delta^1} C^1(G, A) \xrightarrow{\delta^2} C^2(G, A) \to \cdots.$$

Then the n-dimensional cohomology group of G with coefficients in A is denoted by $H^n(G, A)$ and is defined as $H^n(G, A) := \ker \delta^{n+1}/image(\delta^n)$.

We note that $H^0(G, A) = A^G = \ker \delta^1 \hookrightarrow C^0(G, A)$.

Now we quote another equivalent definition of cohomology (cf [23] proposition 1.3.9).

Lemma 4.1.1 : If

$$0 \to A \to X^0 \to X^1 \to \cdots$$

is the standard resolution then $H^n(G, A) \cong H^n(H^0(G, X^\bullet))$, where $X^\bullet$ is the complex

$$X^0 \to X^1 \to \cdots.$$

Let $\mathcal{A}$ be an abelian category. A (decreasing) filtration of an object A is a family $(F^pA)_{p\in\mathbb{Z}}$ of subobjects F^pA of A such that $F^pA \supseteq F^{p+1}A$ for all p. We write

$$gr_pA = F^pA/F^{p+1}A.$$

By convention we put $F^\infty A = 0$ and $F^{-\infty}A = A$. We say that the filtration is finite if there exists $n, m \in \mathbb{Z}$ with $F^mA = 0$ and $F^nA = A$. A morphism $f : A \to B$ is said to be compatible with the filtration if $f(F^pA) \subseteq F^pB$ for all $p \in \mathbb{Z}$. and $A, B \in \mathcal{A}$.

Let m belongs to the set of natural numbers. An E_m-spectral sequence in $\mathcal{A}$ is a system $E = (E_r^{pq}, E^n)$ consisting of

1. objects $E_r^{pq} \in \mathcal{A}$ for all $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ and for any integer $r \geq m$,
2. morphisms $d = d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$ with $d \circ d = 0$ such that for each fixed pair $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ the morphisms d_r^{pq} and $d_r^{p-r,q+r-1}$ vanish for sufficiently larger r,

3. isomorphisms $\alpha_r^{pq} : \ker(d_r^{pq})/im(d_r^{p-r,q+r-1}) \cong E_{r+1}^{pq}$,
4. finitely filtered objects $E^n \in \mathcal{A}$ for all $n \in \mathbb{Z}$,
5. isomorphism $\beta^{pq} : E_\infty^{pq} \cong gr_p E^{p+q}$.

Using 2 and 3, the objects E_r^{pq} are independent of r for r sufficiently large and then it is denoted by E_∞^{pq}.

For an E_m-spectral sequence $E = (E_r^{pq}, E^n)$, we usually write

$$E_m^{pq} \Longrightarrow E^{p+q}.$$

If $E_r^{pq} = 0$ for $p < 0$ or $q < 0$ one speaks of a first quadrant spectral sequence.

Let $A^\bullet = (A^n, d^n)_{n \in \mathbb{Z}}$ be a complex. A filtration by subcomplexes of $A^\bullet$ is a filtration of $F^\bullet A^n$ for all $n \in \mathbb{Z}$ such that for each n, $F^n A^\bullet$ is a subcomplex of $A^\bullet$. We say that the filtration $F^\bullet A^\bullet$ is biregular, if for each $n \in \mathbb{Z}$, the filtration $F^\bullet A^n$ is finite. We quote the following proposition from [23] proposition 2.2.1, page 102.

Proposition 4.1.1 : Let $F^\bullet A^\bullet$ be a biregularly filtered cochain complex. For $(p, q) \in \mathbb{Z} \times \mathbb{Z}$ and $r \in \mathbb{Z} \cup \{\infty\}$, we put

- $Z_r^{pq} = \ker(F^p A^{p+q} \to A^{p+q+1}/F^{p+r} A^{p+q+1})$,
- $B_r^{pq} = d(F^{p-r} A^{p+q-1}) \cap F^p A^{p+q}$,
- $E_r^{pq} = Z_r^{pq}/(B_{r-1}^{pq} + Z_{r-1}^{p+1,q-1})$,
- $F^p H^{p+q}(A^\bullet) = image(H^{p+q}(F^p A^\bullet) \to H^{p+q}(A^\bullet))$.

Here $H^n(A^\bullet)$ denotes the n^{th} cohomology of the complex $A^\bullet$. Then the differential of the complex $A^\bullet$ induces homomorphisms

$$d = d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1} \text{ for all } r \in \mathbb{Z}$$

in a natural way. There are canonical isomorphisms

$$\alpha_r^{pq} : \ker(d_r^{pq})/im(d_r^{p-r,q+r-1}) \cong E_{r+1}^{pq}$$

for all $r \in \mathbb{Z}$. For fixed $(p, q) \in \mathbb{Z} \times \mathbb{Z}$, the morphisms d_r^{pq} and $d_r^{p-r,q+r-1}$ vanish for sufficiently large r and we have the natural isomorphism $E_r^{pq} \cong E_\infty^{pq}, r \gg 0$.

Finally, there exists a natural isomorphism $\beta^{pq} : E_\infty^{pq} \cong gr_p H^{p+q}(A^\bullet)$.

In particular, these data define a spectral sequence

$$E_1^{pq} \Longrightarrow H^{p+q}(A^\bullet).$$

A double complex $A^{\bullet\bullet}$ is a collection of objects A^{pq} in a category $\mathcal{A}$ with $p, q \in \mathbb{Z}$, together with differentials $d^{'pq} : A^{pq} \to A^{(p+1)q}$ and $d^{''pq} : A^{pq} \to A^{p,(q+1)}$ such that $d^{'} \circ d^{''} = 0 = d^{''} \circ d^{''}$ and $d^{'} \circ d^{''} + d^{''} \circ d^{'} = 0$. The associated total complex

$$A^{\bullet} = \mathrm{tot}(A^{\bullet\bullet})$$

is a single complex with $A^n = \oplus_{p+q=n} A^{pq}$ whose differential $d : A^n \to A^{n+1}$ is given by the sum of maps

$$d = d^{'} + d^{''} : A^{pq} \to A^{(p+1)q} \oplus A^{p(q+1)},\ p + q = n.$$

Now let $A^{\bullet\bullet}$ be a double complex. We assume that for each n there are only finitely many nonzero A^{pq} on the line $p + q = n$. Let us define $A^{\bullet}$ as $\mathrm{tot}A^{\bullet\bullet}$. Now we have a natural filtration $F^p A^{\bullet}$ of the total complex defined by

$$F^p A^n = \oplus_{i \geqslant p} A^{i,n-1}.$$

Now the above filtration on $A^{\bullet} = \mathrm{tot}(A^{\bullet\bullet})$ is biregular and hence by proposition 4.1.1 induces a spectral sequence

$$E_2^{pq} \Longrightarrow H^{p+q}(A^{\bullet}).$$

This is called the spectral sequence associated to the double complex $A^{\bullet\bullet}$.

Remark 4.1.1: In the above spectral sequence $E_2^{pq} = H^p(H^q(A^{\bullet\bullet}))$. (cf. [23] page 104 chapter 2).

Hochschild-Serre Spectral Sequence

Theorem 4.1.1: Let G be a profinite group. H is a closed normal subgroup and A is a G-module. Then there exists a first quadrant spectral sequence,

$$E_2^{pq} = H^p(G/H, H^q(H, A)) \Longrightarrow H^{p+q}(G, A).$$

It is called the Hochschild-Serre spectral sequence.

Proof Sketch: We consider the standard resolution $0 \to A \to X^{\bullet}$ of the G-module A, where, $X^n = X^n(G, A) = \mathrm{Map}(G^{n+1}, A)$ of all continuous maps $x : G^{n+1} \to A$, i.e. of all continuous functions $x(\sigma_0, ..., \sigma_n)$ with values in A. X^n is in a natural way a G-Module by

$$(\sigma x)(\sigma_0, ..., \sigma_n) = \sigma x(\sigma^{-1}\sigma_0, ..., \sigma^{-1}\sigma_n).$$

We apply the functor $H^0(H, -)$ and get the complex

$$H^0(H, X^0) \to H^0(H, X^1) \to H^0(H, X^2) \to \cdots$$

of G/H-modules. Now for each $H^0(H, X^q)$, we consider the complex

$$H^0(H, X^q)^{G/H} \to C^{\bullet}(G/H, H^0(H, X^q))$$

where $C^n(G, A) = X^n(G, A)^G$. Here we point out that we get the above complex by the following means:

We define $P := H^0(H, X^q)$ and $G^{'} := G/H$, then we know that we have the complex

$$C^0(G^{'}, P) \xrightarrow{\delta^1} C^1(G^{'}, P) \xrightarrow{\delta^2} C^2(G^{'}, P) \to \cdots$$

and $P^{G^{'}} = H^0(G^{'}, P) = \ker \delta^1 \hookrightarrow C^0(G^{'}, P)$.

Now we put

$$C^{pq} := C^p(G/H, H^0(H, X^q)) = X^p(G/H, (X^q)^H)^{G/H} = X^p(G/H, X^q(G, A)^H)^{G/H}$$

where $p, q \geqslant 0$.

We make $C^{\bullet\bullet}$ into a double complex. The differentials are the following:

$$d^{'}_{pq} : C^{pq} \to C^{p+1,q}$$

is the differential of the complex $X^{\bullet}(G/H, X^q(G, A)^H)^{G/H}$ at the p^{th} place.

$$d^{''}_{pq} : C^{pq} \to C^{p,q+1}$$

is the $(-1)^p$ times the differential of the complex $X^p(G/H, X^{\bullet}(G, A)^H)^{G/H}$ at the q^{th} place. To see that $X^p(G/H, X^{\bullet}(G, A)^H)^{G/H}$ is a complex we note that if we are given a complex

$$A^n \xrightarrow{d^n} A^{n+1} \to A^{n+2}$$

then we obtain the following complex

$$X^p(G, A^n) \xrightarrow{p} X^p(G, A^{n+1}) \to X^p(G, A^{n+2})$$

where $p(x) = d^n \circ x$ for $x \in X^p(G, A^n)$.

It is a tedious calculation to verify that $(C^{\bullet\bullet}, d^{'}, d^{''})$ is a double complex and we don't do it here. Now let

$$E_2^{pq} \Longrightarrow E^n$$

be the associated spectral sequence. We now show that this is the Hochschild-Serre spectral sequence. By Remark 4.1.1 $E_2^{pq} = H^p(H^q(C^{\bullet\bullet}))$ and by lemma 4.1.1

$$H^q(H^0(H, X^{\bullet})) = H^q(H, A).$$

Hence by the definition of C^{pq}

$$H^q(C^{p\bullet}) = H^q(C^p(G/H, H^0(H, X^{\bullet}))).$$

Now as the functor $C^p(G/H,-)$ is exact (cf. [23] chapter 1, section 3, Exercise 1) it commutes with the cohomology functor and so

$$H^q(C^p(G/H, H^0(H, X^\bullet))) = C^p(G/H, H^q(H^0(H, X^\bullet))).$$

Now from lemma 4.1.1 we get that

$$C^p(G/H, H^q(H^0(H, X^\bullet))) = C^p(G/H, H^q(H, A)).$$

So

$$E_2^{pq} = H^p(H^q(C^{\bullet\bullet})) = H^p(C^\bullet(G/H, H^q(H, A))) = H^p(G/H, H^q(H, A)).$$

Now with some more effort one can show that $E^n = H^n(G, A)$. (cf. [23] chapter 2, section 4) $\square$

Remark 4.1.2 : We would like to pint out that one needs the following fact to show that $E^n = H^n(G, A)$ in the proof of the Hochschild-Serre spectral sequence in Theorem 4.1.1.

Fact (cf [23] lemma 2.2.4): Consider any first quadrant double complex $A^{\bullet\bullet}$, Suppose for each $q \geqslant 0$ the complex

$$A^{0q} \to A^{1q} \to ....$$

is exact, then $E^n = H^n(B^\bullet)$, where $B^\bullet$ is the complex $\ker(A^{0\bullet} \to A^{1\bullet})$.

4.2 Completed Cohomology and the Spectral Sequence of Matthew Emerton

First of all we will define Galois Coverings and Local System on a topological space which will be needed later in this section.

We recall that, given a topological space X, a covering Y of X is a topological space Y, with a map $f : Y \to X$, such that for every $p \in X$, there is a neighbourhood U_p of p and a set T (with discrete topology) with $f^{-1}(U_p)$ homeomorphic to $T \times U_p$ such that the natural map from $f^{-1}(U_p)$ to U_p is the projection map from $T \times U_p$ to U_p.

Now for any covering map $f : Y \to X$, the group of deck transformations denoted by $\mathrm{Aut}(f)$ is the group of automorphisms of Y, commuting with the map to X. Locally it means that the various covers are permuted. In particular, the deck transformations induce automorphisms of the fiber $f^{-1}(x)$.

We say that $f : Y \to X$ is a Galois covering, if Y is connected and $G :=\mathrm{Aut}(f)$ acts transitively on $f^{-1}(x)$ for any $x \in X$.

Now we give the definition of a local system on a topological space.

We assume that the reader is familiar with the definition of a sheaf. We fix a commutative ring R and a topological space X. An R-local system on X is a sheaf $\mathcal{F}$ of R-modules such that for all $x \in X$, there exists an open neighbourhood U_x of x and a R-module M such that $\mathcal{F}|_{U_x} = \bar{M}_{U_x}$–the constant sheaf corresponding to M. We note that this condition implies that $\Gamma(U_x, \mathcal{F}) \to \mathcal{F}_y$ is an isomorphism. We can also easily see that the stalk of $\mathcal{F}$ at points of X is invariant (upto R-module isomorphism) on the connected components of X. Now we know that the rank of a finite module over a PID is well defined. So if R is a PID then we can define the rank of $\mathcal{F}$ on any connected component of X. We would like to point out here that for sufficiently nice topological space X (pathconnected, locally path connected and semi-locally simply connected) the category of R-local systems on X is equivalent to the category of representations of $\pi_1(X, x_1)$ in R-modules where $\pi_1(X, x_1)$ is the fundamental group of X at the point x_1. We don't prove the above standard result and we will not need it for our purpose.

Our assumptions:
G is a compact locally $\mathbb{Q}_p$ analytic group. We fix a countable basis of open normal subgroups

$$G \supseteq G_1 \supseteq ... \supseteq G_r \supseteq \cdots .$$

We suppose that we are given a sequence of continuous maps of topological spaces

$$\cdots \to X_r \to \cdots \to X_1 \to X_0$$

where each of these spaces have a given right G-action such that

1. the maps are all G-equivariant.
2. G_r acts trivially on X_r.
3. The map $X_r \to X_{r'}$ is a Galois covering map with deck transformations given by the action of $G_{r'}/G_r$ on X_r.

We also assume that we are given a local system of free finite rank $\mathcal{O}_E$-modules $\mathcal{V}_0$ on X_0 and we construct $\mathcal{V}_r$ the pullback of $\mathcal{V}_0$ to X_r for $r \geqslant 0$. Here E is a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}_E$ denotes its ring of integers.

We define:

1. $H^n(\mathcal{V}) := \varinjlim_r H^n(X_r, \mathcal{V}_r)$,
2. $H^n(\mathcal{V})_E := E \otimes_{\mathcal{O}_E} H^n(\mathcal{V})$,
3. $\tilde{H}^n(\mathcal{V}) := \varprojlim_s \varinjlim_r H^n(X_r, \mathcal{V}_r/p^s)$,

4. $\tilde{H}^n(\mathcal{V})_E := E \otimes_{\mathcal{O}_E} \tilde{H}^n(\mathcal{V})$.

Here we admit a fact without proof:

Fact 4.2.1 : $\tilde{H}^j(\mathcal{V})$ is an admissible $\mathcal{O}_E[G]$-module for $j \geqslant 0$. (see [14] theorem 2.1.5).

We fix values of r and s and let $r^{'} \geqslant r$, then we have the Hochschild-Serre spectral sequence

$$E_2^{i,j} = H^i(G_r/G_{r'}, H^j(X_{r'}, \mathcal{V}/p^s)) \Longrightarrow H^{i+j}(X_r, \mathcal{V}/p^s).$$

Since cohomology commutes with inductive limits, we get that

$$E_2^{i,j} = H^i_{con}(G_r, \varinjlim_{r^{'}} H^j(X_{r'}, \mathcal{V}/p^s)) \Longrightarrow H^{i+j}(X_r, \mathcal{V}/p^s). \tag{4.2.1}$$

Now we prove proposition 2.1.11 of [14].

Theorem 4.2.1 Passing to the projective limit in s of the spectral sequence 4.2.1, followed by tensoring with E gives the following spectral sequence:

$$E_2^{i,j} = H^i_{con}(G_r, \tilde{H}^j(\mathcal{V})_E) \Longrightarrow H^{i+j}(X_r, E \otimes \mathcal{V}). \tag{4.2.2}$$

Proof : Proposition 2.2.1 gives us that for any j the natural map of the projective systems

$$\{\tilde{H}^j(\mathcal{V})/p^s\}_{s\geqslant 1} \to \{\varinjlim_{r^{'}} H^j(X_{r'}, \mathcal{V}/p^s)\}_{s\geqslant 1} \tag{4.2.3}$$

has essentially null kernel and cokernel. This follows from proposition 2.2.1 but we note that in order to apply that proposition we need the fact that $\tilde{H}^j(\mathcal{V})$ is an admissible $\mathcal{O}_E[G]$-module which we accept in fact 4.2.1 above.

So the map of the projective systems

$$\{H^i_{con}(G_r, \tilde{H}^j(\mathcal{V})/p^s))\}_{s\geqslant 1} \to \{H^i_{con}(G_r, \varinjlim_{r'} H^j(X_{r'}, \mathcal{V}/p^s))\}_{s\geqslant 1}$$

has essentially null kernel and cokernel. Now, from proposition 2.2.3 we get that the map

$$\{H^i_{con}(G_r, \tilde{H}^j(\mathcal{V}))/p^s)\}_{s\geqslant 1} \to \{H^i_{con}(G_r, \varinjlim_{r'} H^j(X_{r'}, \mathcal{V}/p^s))\}_{s\geqslant 1}$$

has essentially null kernel and cokernel. As X_r is homotopic to a finite simplical complex the natural map

$$\{H^j(X_r, \mathcal{V})/p^s\}_{s\geqslant 1} \to \{H^j(X_r, \mathcal{V}/p^s)\}_{s\geqslant 1} \tag{4.2.4}$$

has essentially null kernel and cokernel. This is because of the following reason given in page 21 of [14].

We fix s. The exact sequence $0 \to \mathcal{V} \xrightarrow{p^s\cdot} \mathcal{V} \to \mathcal{V}/p^s \to 0$ (we can also take trivial coefficients and so we can take $\mathcal{V}$ to be $\mathbb{Z}_p$) gives us that the sequence

$$0 \to H^n(X_r, \mathcal{V})/p^s \xrightarrow{\phi} H^n(X_r, \mathcal{V}/p^s) \to H^{n+1}(X_r, \mathcal{V})[p^s] \to 0$$

is exact. So we get that ϕ is injective and $cokernel(\phi)$ is contained in $H^{n+1}(X_r, \mathcal{V})[p^s]$. But X_r is a finite simplical complex so $H^{n+1}(X_r, \mathcal{V})$ is finitely generated $\mathcal{O}_E$-module. So the transition maps of $\{H^{n+1}(X_r, \mathcal{V})[p^s]\}_{s\geqslant 1}$ eventually vanish. Hence we get that the natural map in 4.2.4 has essentially null kernel and cokernel.

So it is an isomorphism in the category $\mathcal{B}$ that we introduced before in section 2.2. Hence we obtain the following spectral sequence in the category $\mathcal{B}$

$$\{H^i_{con}(G_r, \tilde{H}^j(\mathcal{V}))/p^s\}_{s\geqslant 1} \Longrightarrow \{H^{i+j}(X_r, \mathcal{V}/p^s)\}_{s\geqslant 1}. \tag{4.2.5}$$

We note that to get the above spectral sequence we use equation 4.2.1 and equation 4.2.3.

Now $H^{i+j}(X_r, \mathcal{V})$ is a finitely generated $\mathcal{O}_E$-module as X_r is homotopic to a finite simplical complex. Now any finitely generated $\mathcal{O}_E$ module is p-adically separated and complete (take for example $\mathbb{Z}_p$ over $\mathcal{O}_E$ where E is $\mathbb{Q}_p$). This gives us $H^{i+j}(X_r, \mathcal{V}) \cong \varprojlim_s H^{i+j}(X_r, \mathcal{V})/p^s$. Also since $\tilde{H}^j(\mathcal{V})$ is admissible proposition 2.2.2 gives us that $H^i_{con}(G_r, \tilde{H}^j(\mathcal{V}))$ is admissible. So we get the isomorphism

$$H^i_{con}(G_r, \tilde{H}^j(\mathcal{V})) \cong \varprojlim_s H^i_{con}(G_r, \tilde{H}^j(\mathcal{V}))/p^s. \tag{4.2.6}$$

Now since the category of the admissible $\mathcal{O}_E$-modules and the category $\mathcal{B}$ are equivalent from remark 2.2.2, we take the projective limit of the spectral sequence 4.2.5 in $\mathcal{B}$

$$\{H^i_{con}(G_r, \tilde{H}^j(\mathcal{V}))/p^s\}_{s\geqslant 1} \Longrightarrow \{H^{i+j}(X_r, \mathcal{V}/p^s)\}_{s\geqslant 1}$$

and we obtain the spectral sequence

$$E_2^{i,j} = H^i_{con}(G_r, \tilde{H}^j(\mathcal{V})) \Longrightarrow H^{i+j}(X_r, \mathcal{V}). \tag{4.2.7}$$

We get the above spectral sequence from 4.2.4 and 4.2.6. Now we have $\tilde{H}^j(\mathcal{V})$ admissible from fact 4.2.1 and G is compact, so we get that $E \otimes_{\mathcal{O}_E} C^\bullet_{con}(G, M) \cong C^\bullet_{con}(G, E \otimes M)$. (cf. Prop 1.2.20 [14]) This gives us the isomorphism when passing to the cohomology. Now we tensor the spectral sequence 4.2.7

$$E_2^{i,j} = H^i_{con}(G_r, \tilde{H}^j(\mathcal{V})) \Longrightarrow H^{i+j}(X_r, \mathcal{V})$$

with E and this gives us the desired spectral sequence 4.2.2 of the theorem. □

Chapter 5

Cohomology of Arithmetic Quotients of Symmetric Spaces

5.1 Theorem of Michael Harris

This theorem is from [12] theorem 1.10.

Let p be a prime greater than 2. Let H be a p-adic analytic group introduced in chapter 3 with the property that its Iwasawa algebra has no zero divisors. We can take H to be any p-adic analytic group which is a torsion free pro-p group (cf. [2] discussion after corallary 7.26). We can also take H to be a closed subgroup of the principal congruence subgroup Γ of $GL(N, \mathbb{Z}_p)$ of level p. Let $H_i := H \cap \Gamma(N)_i$ where $\Gamma(N)_i \subset \Gamma$ is the principal congruence subgroup of level p^{i+1}. Let $\Lambda = \mathbb{Z}_p[[H]]$ be the Iwasawa algebra of H.

Theorem 5.1.1 : Let M be a finitely generated compact $\Lambda_H := \Lambda$-module. Let $n = \dim H$, and for each i, let d_i be the $\mathbb{Z}_p$-rank of the free part of the finitely generated $\mathbb{Z}_p$-module $M_i = M/I_{H_i}M$, where $I_{H_i} = \ker(\Lambda \to \mathbb{Z}_p[H/H_i])$. Let r be the rank of M as a Λ-module introduced in chapter 3, section 2. Then $d_i = r[H : H_i] + O(p^{(n-1)i})$. If M is a torsion Λ-module then $d_i = O(p^{(n-1)i})$.

Proof: We assume that $r = 0$. Since Λ is a Noetherian integral domain from chapter 3, the Λ-torsion elements of M form a submodule denoted by M_{tors}. This follows from theorem 3.2.2 of this article and lemma 4.21 of [3]. Now,

$$0 \to M_{tors} \to M \to M/M_{tors} \to 0$$

is exact. Let $K = Frac(\Lambda)$, the skew field of Λ. As K is flat over Λ (cf. corollary 10.13 of [3]), this implies that

$$0 \to M_{tors} \otimes_\Lambda K \to M \otimes_\Lambda K \to M/M_{tors} \otimes_\Lambda K \to 0$$

is exact. Now $\dim_K(M_{tors} \otimes_\Lambda K) = 0$. So rank of M = rank of M/M_{tors}.

Now, we may assume that M has no p-torsion. This is because the p-torsion elements of M are automatically the Λ-torsion elements.

Moreover, M is Λ-torsion iff M/M_{ptors} is Λ-torsion. Here M_{ptors} is the p-torsion elements of M. M is Λ-torsion implies M/M_{ptors} is Λ-torsion. This is clear. Let M/M_{ptors} be Λ-torsion. Now M is finitely generated. Let $m_1, m_2, ..., m_n$ be generators. There exists λ_i nonzero such that $\lambda_i m_i \in M_{ptors}$. $\lambda_i \in \Lambda$. Now there exists n_i such that $p^{n_i}\lambda_i m_i = 0$. So M is Λ-torsion. This says that $r(M) = 0$ iff $r(M/M_{ptors}) = 0$, where $r(M)$ is the rank of M.

Now, $M_i = M/I_{H_i}M$, M_i is also a finitely generated Λ-module. Also $r(M_i) = r(M_i/N)$, where N is any Λ-torsion submodule of M_i. We can take N to be p-torsion elements of M_i. Also, $d_i = \dim_{\mathbb{Q}_p}(M_i \otimes \mathbb{Q}_p)$,

$$M_i = \mathbb{Z}_p^{d_i} \oplus_j \mathbb{Z}/p^{\alpha_j}\mathbb{Z}.$$

So $d_i = \dim_{\mathbb{Q}_p}(M_i/N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$, where N is the p-torsion elements of M_i. So in fact, we can assume that M_i has no p-torsion because if M is Λ-torsion, then M_i is also Λ-torsion, and $d_i(M_i/N) = d_i(M_i)$. Then

$$d_i = \dim_{\mathbb{F}_p}(\bar{M}/I_{H_i}\bar{M})$$

where $\bar{M} = M/pM$, a finitely generated torsion $\Omega_H := \mathbb{F}_p[[H]]$-module. This is because the $\mathbb{Z}_p$-rank of a free $\mathbb{Z}_p$-module is the same as the dimension of its reduction modulo p and we can assume M_i to be a free $\mathbb{Z}_p$-module. So it suffices to prove the estimate $d_i = O(p^{(n-1)i})$ for finitely generated torsion submodules over Ω_H. Now $\bar{M}$ is finitely generated torsion module over Ω_H. Let $m_1, m_2, ..., m_n$ generate $\bar{M}$. There exists $f_i \in \Omega_H$ nonzero such that $f_i m_i = 0$ in $\bar{M}$. So we have a surjective map $\oplus_{i=1}^n \Omega_H/\Omega_H f_i \twoheadrightarrow M$. This is because if M is any finitely generated torsion R-module, then we have $R^n \twoheadrightarrow M$ sending $e_i \mapsto m_i$ Then $f_i e_i \mapsto f_i m_i = 0$. We have $R \simeq Re_i$, $R \oplus ... \oplus R \to M$ sending $f_i Re_i$ to 0. So $\oplus_{i=1}^n Re_i/f_i Re_i \twoheadrightarrow M$. So $\oplus_{i=1}^n R/Rf_i \twoheadrightarrow M$.

Thus we are reduced to the case when $\bar{M} = \Omega_H/\Omega_H f$ for some nonzero $f \in \Omega_H$. Then $d_i = \dim$ *Coker* $f : \Omega_{H,i} \to \Omega_{H,i}$ where $\Omega_{H,i} = \Omega_H/I_{H_i}$. Let $Gr(\Omega_H)$ be the associated graded algebra for the filtration by the images of I_{H_i}.

$$Gr(\Omega_H) = \oplus_i (I_{H_i}/I_{H_{i+1}}).$$

Now the rank of an endomorphism of a finite dimensional filtered $\mathbb{F}_p$-vector space respecting the filtration goes down upon passing to the associated graded space. This is easy to see using the rank nullity theorem of vector spaces and induction on the grading.

We have $f : \Omega_H \to \Omega_H$ which is just the multiplication by f map. Then $Gr(\Omega_H)$ is a polynomial ring. (cf. article of [16]). Let $f^{'} \neq 0$ be the leading coefficient of f in the grading. We have $f^{'} : Gr(\Omega_H) \to Gr(\Omega_H)$. With abuse of notation we write $f^{'} : I_{H_i}/I_{H_{i+1}} \to I_{H_i}/I_{H_{i+1}}$.

Then we have

$$f^{'} : I_{H_i}/I_{H_{i+1}} \to I_{H_i}/I_{H_{i+1}} \to Q_i$$

where $Q_i = Coker f^{'}$. Now d_i is bounded by the dimension of Q_i.

Now we briefly recall the theory of Hilbert Samuel polynomials from chapter 5 of [9].

Let $R = \oplus_{n \geqslant 0} R_n$ be a Noetherian graded ring where R_0 is an Artinian ring. Let $M = \oplus_{n \geqslant 0} M_n$ be a finitely generated graded R-module. Let $l(M_n)$ be the length of M_n as a R_0-module then we have that $l(M_n) < \infty$ and then we define the Hilbert series $P(M, t)$ of M by $P(M, t) := \sum_{n=0}^{\infty} l(M_n) t^n \in \mathbb{Z}[[t]]$. Now let $R = R_0[x_1, ..., x_r]$ with x_i of degree 1 then $P(M, t)$ has the form $f(t)(1 - t)^{-d}$ with $f(t) \in \mathbb{Z}[t]$ and $d \geqslant 0$ (cf. theorem 13.2 of [9]). Sometimes we also write the d above as $d(M)$. Now we state an important corollary of theorem 13.2 of [9] which we need.

Lemma 5.1.1 : Let $d = d(M)$ as described above. Then there is a polynomail $\phi_M(X)$ with rational coefficients of degree $d - 1$ such that for large n we have $l(M_n) = \phi_M(n)$.

The polynomial ϕ_M is called the Hilbert polynomial of the graded module M.

Now using theorem 13.4 of [9] and lemma 5.1.1 above we see that if $R = \mathbb{F}_p[X_1, ...X_d]$ and $M = \oplus_{i \geqq 0} M_i$ be a graded finitely generated R-module then $\dim M_i = O(i^{\delta - 1})$ where $\delta = \dim M$.

Now we have

$$Gr(\Omega_H) = \oplus_i (I_{H_i}/I_{H_{i+1}})$$

and the fact that $Gr(\Omega_H)$ is a polynomial ring with coefficients in $\mathbb{F}_p$ (cf. article of [16] or the discussion before proposition 1.5 of [11]).

So from the above discussion, we see that Q_i is the quotient of a vector space of dimension $O(i^{(n-1)})$ where $n = \dim Gr(\Omega_H)$. Now we take large i of the form p^j for large j. Then we see Q_j has dimension $O({(p^j)}^{n-1})$. Hence we are done for the torsion case.

Now let M be a free Λ-module of rank m. Let $M = \Lambda^m = \mathbb{Z}_p[[H]]^m$. If $m = 1$, then $M = \mathbb{Z}_p[[H]]$. $I_{H_i} = \ker(\mathbb{Z}_p[[H]] \to \mathbb{Z}_p[H/H_i])$. So $M/I_{H_i}M \cong \mathbb{Z}_p[H/H_i]$, hence $d_i = [H/H_i]$. In this way if $M = \Lambda^m$, then

$$M/I_{H_i}M = \Lambda/I_{H_i}\Lambda \times ... \times \Lambda/I_{H_i}\Lambda = (\mathbb{Z}_p[H/H_i])^m.$$

So the theorem is true for Λ-free modules of finite rank.

Now we consider the general case.

There exists an exact sequence

$$0 \to T \to M \to M^{'} \to 0$$

where T is Λ-torsion submodule, and $M^{'} = M/T(M)$ torsion free. Here $T(M) := T$. So we have the following exact sequence

$$T(M)/I_{H_i}T(M) \to M/I_{H_i}M \to (M/T(M))/I_{H_i}(M/T(M)) \to 0.$$

Here we denote the map $T(M)/I_{H_i}T(M) \to M/I_{H_i}M$ by ϕ. We also have the following exact sequence :

(as $\mathbb{Q}_p$ is flat over $\mathbb{Z}_p$),

$$0 \to \underbrace{\ker(\phi) \otimes \mathbb{Q}_p}_{A} \to \underbrace{T(M)/I_{H_i}T(M) \otimes \mathbb{Q}_p}_{B} \to \underbrace{M/I_{H_i} \otimes \mathbb{Q}_p}_{C} \to \underbrace{M^{'}/I_{H_i}M^{'} \otimes \mathbb{Q}_p}_{D} \to 0$$

Let $d_i(-)$ be the dimension of each of these terms over $\mathbb{Q}_p$. Hence we get that $\dim(A) + \dim(C) = \dim(B) + \dim(D)$. Now $\dim(B) = O(p^{(n-1)i})$, but $d_i(A) \leqq d_i(B)$. So $d_i(A) = O(p^{(n-1)i})$. If we show that $d_i(D) = r[H : H_i] + O(p^{(n-1)i})$, then $d_i(C) = \dim(B) + \dim(D) - \dim(A) = r[H : H_i] + O(p^{(n-1)i})$.

So we now show the theorem for M torsion free. We prove that there are Λ_H-free modules V and $V^{'}$ of rank r, and morphisms $M \to V$ and $V^{'} \to M$ with torsion cokernel although we only need the second one.

Since M is torsion free of rank r, there exists an injection

$$\phi : M \hookrightarrow M \otimes_\Lambda K \cong K^r,$$

where K is the skew field of Λ. M is also finitely generated. Let $m_1, m_2, ...m_n$ generate M. Then $\phi(m_j) = \sum \frac{a_{ij}}{b_{ij}} e_i$, where $\langle e_i \rangle = M \otimes K$.

Now we want to find s_i and $c_{ij} \in \Lambda$ such that $s_i = c_{ij}b_{ij}$. If we find them, then $\frac{1}{b_{ij}} = \frac{1}{s_i}c_{ij}$ and then M can be embedded in a Λ-free module of rank r generated by $\{s_i e_i\}$. We recall that Λ is a Noetherian integral domain. So it satisfies the Ore condition by theorem 3.2.2. So there exists s_1 and c_{11} such that $s_1 = c_{11}b_{11}$. Then we consider b_{12}, s_1. By the Ore condition there exists c_{12}, p_{12} such that $p_{12}s_1 = c_{12}b_{12}$. Now we have, $p_{12}s_1 = p_{12}c_{11}b_{11}$, and $p_{12}s_1 = c_{12}b_{12}$. So our s_2 is $p_{12}s_1$. Now by the Ore condition there exists c_{13} and p_{13} such that $p_{13}s_2 = c_{13}b_{13}$. So our new s_3 is $p_{13}s_2$. We proceed like this. This process terminates because b_{ij}'s are finite in number. So we have $M \hookrightarrow F$, where F is

Λ^r, r is the rank of M. So $M \otimes K \hookrightarrow F \otimes K \cong K^r$. But $M \otimes_\Lambda K \cong K^r$. So F/M has rank 0. This implies that F/M is a torsion Λ-module.

Now we prove that there exists $V^{'} \hookrightarrow M$ such that $M/V^{'}$ is torsion, $V^{'}$ is Λ-free of rank(M). If M has rank 1, then $V = \Lambda m$ will suffice for any $m \in M$ which is nonzero. So we consider rank$(M) > 1$ case. Let $m_1 \neq 0$, $m_1 \in M$. We consider $m_1\Lambda \subseteq M$. If $m_1\Lambda = M$, then rank$(m_1\Lambda) = \dim(m_1\Lambda \otimes_\Lambda K) = 1$. (It cannot be 0 because M is torsion free). This is a contradiction. So there exists m_2 such that $m_2 \notin m_1\Lambda$. Consider $m_1\Lambda + m_2\Lambda \subseteq M$. If $M = m_1\Lambda + m_2\Lambda \cong \Lambda^2$, then $\dim_K(\Lambda^2 \otimes_\Lambda K) = 2$. So this is a contraction. Proceeding in this way we see that we get $m_i \in M$, $i = 1, ..., r$, where m_i's are Λ-linearly independent such that $\Lambda m_1 + ... + \Lambda m_r \subseteq M$. So $\Lambda^r \subseteq M$. Now we have an exact sequence,

$$0 \to \Lambda^r \hookrightarrow M/\Lambda^r \to 0.$$

Tensoring this sequence with K we see that M/Λ^r has rank 0. So it is torsion.

If there exists $V^{'} \hookrightarrow M$ such that $M/V^{'}$ is torsion, then we have an exact sequence

$$0 \to \underbrace{\ker(\phi)}_{A} \to \underbrace{V^{'}/I_{H_i}V^{'}}_{B} \to \underbrace{M/I_{H_i}M}_{C} \to \underbrace{(M/V^{'})/I_{H_i}(M/V^{'})}_{D} \to 0$$

So $\dim(C) \leq \dim(A) + \dim(C) = \dim(B) + \dim(D)$. So

$$\dim(C) \leq m[H : H_i] + O(p^{(n-1)i}).$$

But this means $\dim(C) = m[H : H_i] + O(p^{(n-1)i})$, because both of them essentially means that $\dim(C) - m[H : H_i]$ is asymptotically bounded by $p^{(n-1)i}$. This completes the proof of the theorem. $\square$

5.2 Bounds on Cohomology

Let $\mathbf{G}$ be a connected semisimple linear algebraic group over a number field F. Let $\mathcal{O}_F$ be the ring of integers of F. Let $G_\infty := \mathbf{G}(\mathbb{R} \otimes_\mathbb{Q} F)$. Let K_∞ be a maximal compact subgroup of G_∞. We fix an embedding $G \hookrightarrow GL_N$ for some N. Let $\mathfrak{P}$ denote a prime of $\mathcal{O}_F$ lying over p. Now the congruence subgroup of $GL_N(\mathcal{O}_F)$ at level $\mathfrak{P}$ is defined as the kernel of the reduction modulo $\mathfrak{P}$ map $GL_N(\mathcal{O}_F) \to GL_N(\mathcal{O}_F/\mathfrak{P})$. Let $G(\mathfrak{P})$ be the intersection of G_∞ with the congruence subgroup of $GL_N(\mathcal{O}_F)$ at level $\mathfrak{P}$. We now take $\mathbf{G} = SL_n(\mathbb{Q})$ and then $G_\infty = SL_n(\mathbb{R})$. Now we define an arithmetic subgroup of $\mathbf{G}$.

Subgroups T_1 and T_2 of any group T are called commensurable if $T_1 \cap T_2$ has finite index in both T_1 and T_2. Now let $U = SL_n(\mathbb{Z})$. We call any subgroup Γ of $SL_n(\mathbb{Z})$ commensurable with U an arithmetic subgroup of $\mathbf{G}$.

Now we go back to our general $\mathbf{G}$. We fix an arithmetic lattice Γ of G_∞. If $G_\infty = SL_n(\mathbb{R})$ then $\Gamma = SL_n(\mathbb{Z})$.

Let us denote $\Gamma(\mathfrak{P}) := \Gamma \cap G(\mathfrak{P})$. Let us denote by e and f the ramification and inertia degrees of $\mathfrak{P}$ in F. Hence we get that $[F_{\mathfrak{P}} : \mathbb{Q}_p] = ef$ where $F_{\mathfrak{P}}$ is the completion of F at $\mathfrak{P}$. We have the embeddings

$$\mathbf{G}(F_{\mathfrak{P}}) \hookrightarrow GL_N(F_{\mathfrak{P}}) \hookrightarrow GL_{efN}(\mathbb{Q}_p).$$

We define $G_k := G \cap (1 + p^k M_{efN}(\mathbb{Z}_p))$ where $G := \varprojlim_k \Gamma/\Gamma(\mathfrak{p}^k)$ which is the closure of Γ in $\mathbf{G}(F_{\mathfrak{P}})$. These G_k form a fundamental system of open neighbourhoods of identity of G. We can take $\Gamma = SL_n(\mathbb{Z})$ and see that if we replace Γ by $\Gamma(\mathfrak{P}^k)$ for large k then G becomes the inverse limit of p-groups and so G is a pro-p group. Thus G is an analytic pro-p group. For $k \geqslant 0$, we define $Y_k := \Gamma(\mathfrak{P}^{ek}) \backslash G_\infty / K_\infty$. Then we have a natural right action of $\Gamma/\Gamma(\mathfrak{P}^{ek})$ on Y_k. So we have a natural action of $G/G_k \cong \Gamma/\Gamma(\mathfrak{P}^{ek})$ on Y_k. Hence we get an action of G on Y_k which is compatible with the projections $Y_k \to Y_l$ for $0 \leq l \leq k$. Now we fix a finite dimension representation W of $\mathbf{G}$ over F, and let W_0 be a G-invariant $\mathcal{O}_E$-lattice in W. We denote by $\mathcal{V}_k$ the local system of finite free rank $\mathcal{O}_E$-modules on Y_k associated to W_0. Now we are on the usual setting and notation that we introduced while discussing the spectral sequence of M. Emerton. Moreover, Y_k constructed above is homotopic to a finite simplical complex. The proof uses the Borel-Serre compactification and will not be discussed in this article. For reference one can see [5] and [4].

In this setting Emerton's spectral sequence 4.2.2 is

$$E_2^{i,j} = H^i_{con}(G_k, \tilde{H}^j(\mathcal{V})_E) \Longrightarrow H^{i+j}(Y_k, \mathcal{V}_k)_E. \tag{5.2.1}$$

Here the subscript '*con*' denotes that we are working with continuous cohomology. Let E be a finite extension of $\mathbb{Q}_p$. Let V be a E-Banach space equipped with a continuous representation of G.

The following lemma is due to Frank Calegari and Matthew Emerton, cf. lemma 2.2 in [10].

Lemma 5.2.1 : Let V be an admissible continuous G-representation. Then

$$\dim_E H^i(G_k, V) \ll p^{(d-1)k} \text{ for large } k \text{ where } d = \dim(G),\ i \geqslant 1.$$

Proof: Let $C := C(G, E)$ denote the Banach space of continuous functions on G, equipped with the right regular G-action. Its dual, denoted by $D(G, E)$ is a free module over $D(G, E)$ of rank 1. This proves that C is an injective object in the category of all admissible representations. This is because if $f : X \hookrightarrow Y$ be any injective G-equivariant map between admissible representations, and let $g : X \to C$ be any map, then we need to show that there exists a map $q : Y \to C$ with $q \circ f = g$. But if we dualize all maps and denote the dual of f by $f^{'}$ then we easily see that since $f^{'}$ is surjective we get a map from $D(G, E)$ to $Y^{'}$ which we denote by $l^{'}$. Now dualizing the map $l^{'}$ gives us the required map q. The commutative relation follows easily because of the commutative relation of the dual maps. So C is injective. Hence it is acyclic. Now we have an exact sequence

$$0 \to V \hookrightarrow C^n \to W \to 0.$$

In fact here V is a closed subspace of C^n from theorem 2.1.1. As the quotient of an admissible representation by a closed invariant subspace is admissible (cf. remark 2.1.1) so W is also an admissible continuous representation of G. Now we know that a short exact sequence gives a long exact cohomology sequence and $H^0(G_k, V) = V^{G_k}$. Hence we obtain the following exact sequence of cohomology

$$0 \to V^{G_k} \to \left(C^{G_k}\right)^n \to W^{G_k} \to H^1(G_k, V) \to 0.$$

We get the last 0 term of the exact sequence because C is acyclic, so its cohomology is 0 in case of H^1. We also get that

$$H^i(G_k, V) \cong H^{i-1}(G_k, W) \ i \geqslant 2.$$

Now we use induction to prove our lemma. Now we observe that because $H^i(G_k, V) \cong H^{i-1}(G_k, W)$, $i \geqslant 2$ we only need to prove the base case $i = 1$. If the result is true for $i = m$ then it is obviously true for $i = m + 1$. So we prove the lemma for $i = 1$. Now from the above exact sequence we get that

$$\dim_E(V^{G_k}) + \dim_E(W^{G_k}) = \dim_E((C^{G_k})^n) + \dim H^1(G_k, V).$$

Now from the theorem of Harris that is theorem 5.1.1 we get that

$$\dim_E V^{G_k} = r[G : G_k] + O(p^{(d-1)k})$$

where r is the rank of the dual of V (called the corank of V). Now from the exact sequence $0 \to V \hookrightarrow C^n \to W \to 0$ we see that the corank of V and the corank of W equals n. Also $\dim_E((C^{G_k})^n) = n[G : G_k]$. Hence we see that $\dim_E H^1(G_k, V) = O(p^{(d-1)k})$. Hence our lemma is proved. □

Now we are ready to prove our main result of this section.

Theorem 5.2.1 : Let $n \geqslant 0$. Let r_n be the corank of $\tilde{H}^n(\mathcal{V})$, then

$$\dim_E H^n(Y_k, \mathcal{V}_k)_E = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k}) \text{ for large } k.$$

Here the constant c only depends on G.

Proof. Let $i, j \geqslant 0$ and $l \geqslant 2$. Let $E_l^{i,j}(Y_k)$ be the terms of the spectral sequence 5.2.1

$$E_2^{i,j} = H^i(G_k, \tilde{H}^j(\mathcal{V})_E) \Longrightarrow H^{i+j}(Y_k, \mathcal{V}_k)_E.$$

Now $\tilde{H}^j(\mathcal{V})$ is admissible from fact 4.2.1 so lemma 5.2.1 implies that

$$\dim_E H^i(G_k, \tilde{H}^j(\mathcal{V})_E) \ll p^{(d-1)k} \text{ for } i \geqslant 1$$

which implies that $\dim_E E_l^{i,j}(Y_k) \ll p^{(d-1)k}$ for $i \geqslant 1, l \geqslant 2$ because by definition of a spectral sequence in section 4.1, we get that

$$E_{r+1}^{pq} \cong \ker(d_r^{pq})/image(d_r^{p-r,q+r-1}).$$

So $\dim_E E_l^{i,j}(Y_k) \leqslant \dim_E E_{l-1}^{i,j}(Y_k)$. Proceeding like this we get that

$$\dim_E E_l^{i,j}(Y_k) \leqslant \dim_E E_2^{i,j}(Y_k) := H^i(G_k, \tilde{H}^j(\mathcal{V})_E). \tag{5.2.2}$$

Now from the theorem of Harris that is theorem 5.1.1 we get that

$$\dim_E \tilde{H}^n(\mathcal{V})_E^{G_k} = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k}).$$

Now we recall that the spectral sequence 5.2.1

$$E_2^{i,j} = H^i(G_k, \tilde{H}^j(\mathcal{V})_E) \Longrightarrow H^{i+j}(Y_k, \mathcal{V}_k)_E$$

comes from a Hochschild-Serre spectral sequence (see Section 4.2 before theorem 4.2.1) which is a first quadrant spectral sequence. So in fact, the spectral sequence 5.2.1 is a first quadrant spectral sequence. Now we consider

$$E_2^{0,n} = H^0(G_k, \tilde{H}^j(\mathcal{V})_E) = \left(\tilde{H}^j(\mathcal{V})_E\right)^{G_k}.$$

So $\dim_E E_2^{0,n} = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k})$.

Now $\dim_E E_3^{0,n}$ = dimension of the kernel of the map $E_2^{0,n} \xrightarrow{d_2^{0n}} E_2^{2,n-1}$. Now by the rank nullity theorem

$$\dim E_2^{0,n} = \dim \ker d_2^{0n} + \dim_E(image(d_2^{0n})).$$

Now $\dim_E(image(d_2^{0n})) \leqslant \dim E_2^{2,n-1} \ll p^{(d-1)k}$ as we know that

$$\dim_E E_l^{i,j}(Y_k) \ll p^{(d-1)k} \text{ for } i \geqslant 1, l \geqslant 2.$$

So $\dim \ker d_2^{0n} = \dim E_3^{0n} = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k})$.

Also by the definition of a spectral sequence (section 4.1) for any pair $(p,q) \in \mathbb{Z} \times \mathbb{Z}$ the morphisms d_r^{pq} and $d_r^{p-r,q+r-1}$ vanish for sufficiently large r. Proceeding like this we see that $\dim_E E_\infty^{0,n}$ which is obtained by taking successive kernels of $d_l^{0,n}$ maps has the following dimension:

$$\dim_E E_\infty^{0,n} = r_n \cdot c \cdot p^{dk} + O(p^{d-1)k}.$$

Now since 5.2.1 is a Hochschild-Serre spectral sequence, we see from point d of the definition of a spectral sequence in section 4.1 after lemma 4.1 that $H^n(Y_k, \mathcal{V}_k)_E$ has finite length filtration and the associated graded pieces are isomorphic to $E_\infty^{i,j}$ for $i+j=n$. Hence we conclude that

$$\dim_E H^n(Y_k, \mathcal{V}_k)_E = r_n \cdot c \cdot p^{dk} + O(p^{(d-1)k}) \text{ for large } k.$$

This concludes the proof of the theorem. $\square$

5.3 Bounds on Multiplicities of Unitary Representations

Let G be a locally compact unimodular topological group. So we choose a Haar measure on G. Let Γ be a discrete subgroup of G. For all our purpose we will take $G = G_\infty$ and we will take Γ to be an arithmetic lattice in G_∞ as introduced in section 5.2 of chapter 5 in this article. Let K_∞ be a maximal compact subgroup of G_∞. Let F be a number field and let $\mathfrak{P}$ be an ideal of $\mathcal{O}_F$. Let $\Gamma(\mathfrak{P}) := \Gamma \cap G(\mathfrak{P})$ which is defined in section 5.2 of this article. Unless otherwise stated we will take $\Gamma\backslash G$ to be compact. Let X be the quotient $\Gamma\backslash G$ and let dx be the induced Haar measure on X.

We want to define a representation of G on the space of functions $f(x)$ on X with square integrable modulus taking values in $\mathbb{C}$. We denote this space by $L^2(\Gamma\backslash G)$.

$$||f||^2 = \int_X |f(x)|^2 dx < \infty.$$

We associate with each element $g \in G$ an operator $T(g)$ satisfying $T(g)f(x) = f(xg)$ for $x \in X$ and $g \in G$. Here xg denotes the point of X into which x is carried by g.

Now we give the general construction of a representation of G induced by Γ. Let $\gamma \in \Gamma$ and let $\chi(\gamma)$ be a finite-dimensional unitary representation of Γ acting over a Hilbert space V. By 'unitary representation of Γ' we mean that $\chi(\gamma)$ is a unitary operator on the Hilbert space V for every $\gamma \in \Gamma$. We consider the Hilbert Space $H(\chi)$ of all measurable functions $f(g)$ on G with values in V satisfying

1. $f(\gamma g) = \chi(\gamma) f(g)$ for $\gamma \in \Gamma$,
2. $(f, f) = \int_X [f, f] dx < \infty$,

where $[f_1, f_2]$ is the inner product in the finite dimensional space V. The representation associates with every element $g_0 \in G$ an operator $T(g_0)$ of the following form:

$$T(g_0)f(g) = f(gg_0).$$

We notice that our previous construction of the representation of G on $L^2(\Gamma\backslash G)$ is a special case of the above construction. If we take V to be an one dimensional space and $\chi(\gamma)$ to be the unit representation, then $f(g)$ is a scalar function and condition 1 above reduces to $f(\gamma g) = f(g)$. So f can be realised as a function on $\Gamma\backslash G$.

Now we state a theorem from section 2, chapter 1 of [26].

Theorem 5.3.1: If $X = \Gamma\backslash G$ is compact, then the representation $T(g)$ of G induced by Γ splits into a discrete sum of countable number of irreducible unitary representations, each of finite multiplicity.

Let $\hat{G}$ be the unitary dual of G which is by definition the set of equivalence classes of irreducible unitary representations of G. So from the above theorem we get that the representation of G on $L^2(\Gamma\backslash G)$ can be decomposed in

$$L^2(\Gamma\backslash G) = \hat{\oplus}_{\pi\in\hat{G}} m(\pi,\Gamma)\mathcal{H}_\pi$$

completed discrete sum of G-invariant irreducible subspaces indexed by the unitary dual $\hat{G}$ of G, where each $m(\pi,\Gamma)$ is finite.

Now we briefly recall the definition of the injectivity radius of a Riemannian manifold. Let M be a Riemannian manifold. Let $p \in M$. The injectivity radius of p is

$$i(p) := \sup\{\rho > 0 : exp_p \text{ defined on } B_\rho(0) \subset T_p(M) \text{ is a diffeomorphism}\}$$

where $B_\rho(0)$ is an open ball of radius ρ around 0 inside the tangent space and exp_p is the exponential map at point p defined for any Riemannian manifold which maps the tangent space T_pM to M. On the open ball having the injectivity radius, the exponential map at p is a diffeomorphism from the tangent space to the manifold and the injectivity radius is the largest such radius. The injectivity radius of M is defined as follows.

$$i(M) := \inf_{p\in M} i(p).$$

Now we come back to our original situation. So let G be a connected linear semisimple real Lie group. Let $K \subset G$ be a maximal compact subgroup of G. We can take G to be G_∞ and K to be K_∞ be to compatible with our notations in section 5.2 of this article. Let $M = G/K$. Let $\mathfrak{g}_0$ and $\mathfrak{k}_0$ be the respective Lie algebras of G and K. Let B denote the Killing form of $\mathfrak{g}_0$. Let $\mathfrak{p}_0 = \{X \in \mathfrak{g}_0 | B(X,Y) = 0, Y \in \mathfrak{k}_0\}$. Then $B|_{\mathfrak{p}_0\times\mathfrak{p}_0}$ is positive definite. Let $p : G \to G/K = M$ be the natural projection. We put the G-invariant Riemannian structure $\langle,\rangle$ on M that makes $p_{*1}|_{\mathfrak{p}_0} : \mathfrak{p}_0 \to T(M)_{1\cdot K}$ into an isometry. Here if f is a C^∞ map, f_{*g} is the differential of f at g, 1 is the identity of G, and $T(M)_x$ is the tangent space of M at x. Let $d(x,y)$ be the Riemannian distance on M corresponding to $\langle,\rangle$. If $x \in G$, we define $\sigma(x) = d(1\cdot K, x\cdot K)$. Now let $\Gamma \subset G$ be a discrete, torsion free, cocompact subgroup i.e $\Gamma\backslash G$ is compact. Then $\Gamma\backslash M$ is a C^∞ manifold and $\langle,\rangle$ pushes down to a Riemannian structure on $\Gamma\backslash M$. Now for $r > 0$, let $B_r = \{x \in G | \sigma(x) \leqslant r\}$.

Let us consider a tower of subgroups inside Γ:

$$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots$$

such that $\Gamma_i\backslash G$ is compact, Γ_i is normal in Γ and $\cap_{i=1}^{\infty}\Gamma_i = \{1\}$. In our case we are interested where Γ is an arithmetic subgroup and Γ_i's are deeper and deeper congruence subgroups. Now we quote theorem 2.1 of [21].

Theorem 5.3.2: For $x \in \Gamma\backslash G$, let $\psi_x : G \to \Gamma\backslash G$ be defined by $\psi_x(g) = xg$. Then

1. there exists a positive real number r such that if $x \in G$ then $\psi_x|B_r$ is a homeomorphism which is a diffeomorphism on the interior of B_r.
2. If we set $r(\Gamma)$ equal to the supremum of r's in part 1 which is by definition the injectivity radius of $\Gamma\backslash G$, then $\lim_{j\to\infty} r(\Gamma_j) = \infty$.

We know that the exponential map of $T(\Gamma\backslash M)_x$ is injective on a ball in $T(\Gamma\backslash M)_x$ of positive radius $r(x)$ for $x \in \Gamma\backslash M$. Using the compactness argument we get that $r(x)$ is independent of x. [cf. proof of theorem 2.1 of [21]].

After all the above discussions we see that the injective radius of $\Gamma(\mathfrak{P})\backslash G_\infty$ goes to infinity as we go to deeper and deeper congruence subgroups, i.e as $N_{F/\mathbb{Q}}(\mathfrak{P}) \to \infty$ where N is the norm map of the number field F.

Now we recall that we have in the cocompact case

$$L^2(\Gamma\backslash G) = \hat{\oplus}_{\pi\in\hat{G}} m(\pi,\Gamma)\mathcal{H}_\pi$$

where each $m(\pi,\Gamma)$ is finite. Corollary 3.2 of [21] gives a bound on $m(\pi,\Gamma)$ which is the following:

$$m(\pi,\Gamma) \leqslant (\textstyle\int_{B_{r(\Gamma)}} |\phi(g)|^2 dg)^{-1} vol(\Gamma\backslash G_\infty)$$

where *vol* denotes the volume and $\phi(g) = \langle \pi(g)v, v\rangle$ with $v \in \mathcal{H}_\pi$.

The matrix coefficients of $\pi \in \hat{G}$ is defined as functions $c_{v,w} : g \mapsto \langle \pi(g)v, w\rangle$. By definition the representation π is a discrete series (or sometimes called square integrable) if at least one matrix coefficient of π with $v, w \neq 0$ is square integrable with respect to the Haar measure on G.

If π is not a discrete series then corollary 3.3 on [21] states that

$$\lim_{N_{F/\mathbb{Q}}(\mathfrak{P}^k)\to\infty} vol(\Gamma(\mathfrak{P}^k)\backslash G_\infty)^{-1} m(\pi, \Gamma(\mathfrak{P}^k)) = 0. \tag{5.3.1}$$

The proof of the above fact is simple. If π is not square integrable then we have

$$\lim_{j\to\infty} \textstyle\int_{B_{r(\Gamma_j)}} |\phi(g)|^2 dg = \infty$$

by theorem 5.3.2 part 2. But we know that

$$m(\pi,\Gamma) \leqslant (\textstyle\int_{B_{r(\Gamma)}} |\phi(g)|^2 dg)^{-1} vol(\Gamma\backslash G_\infty).$$

Hence we have the desired limit in equation 5.3.1.

Now let $G^{'}$ be any Lie group acting continuously on a topological $G^{'}$-module V, then a vector $v \in V$ is called $G^{'}$-finite if it is contained in a finite dimensional subspace of V which is stable under $G^{'}$.

Now let $p(\pi)$ be the infimum over $p \geqslant 2$ such that the K-finite matrix coefficients of π are in $L^p(G)$ where K is the maximal compact subgroup of G. Then lemma 1 of [27] gives us that for a nontempered $\pi \in \hat{G}$ (i.e. $p(\pi) > 2$),

$$m(\pi, \Gamma(\mathfrak{P})) \ll vol(\Gamma(\mathfrak{P})\backslash G)^{1-\mu}$$

for some $\mu > 0$.

Now we briefly recall the definition of Relative Lie algebra cohomology from [7].

Let $\mathfrak{g}$ be a real Lie algebra acting on a $\mathbb{C}$-vector space V. More precisely, let V be a $\mathfrak{g}$-module i.e. a $\mathbb{C}$-vector space together with a morphism of Lie algebras $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. Then we define

$$C^q := C^q(\mathfrak{g}; V) := Hom_{\mathbb{C}}(\wedge^q \mathfrak{g}, V)$$

where q is a positive integer and $d : C^q \to C^{q+1}$ is defined as

$$\begin{aligned} df(Y_0 \wedge \cdots \wedge Y_q) = &\sum_{j=0}^{q} (-1)^j Y_j \cdot f(Y_0 \wedge \cdots \wedge \hat{Y_j} \wedge \cdots \wedge Y_q) \\ &+ \sum_{r<s} (-1)^{r+s} f([Y_r, Y_s] \wedge Y_0 \wedge \cdots \hat{Y_r} \cdots \wedge \hat{Y_s} \wedge \cdots \wedge Y_q) \end{aligned}$$

where, $\hat{}$ over an argument means that the argument is omitted. Let $x \in \mathfrak{g}$. We also define $\mathcal{L}_x : C^q \to C^q$ and $\mathcal{J}_x : C^q \to C^{q-1}$ by

$$\begin{aligned} (\mathcal{L}_x f)(x_1 \wedge \cdots \wedge x_q) = &\sum_{j=1}^{q} f(x_1 \wedge \cdots \wedge [x_1, x] \wedge \cdots \wedge x_q) \\ &+ x \cdot f(x_1 \wedge \cdots \wedge x_q), \end{aligned}$$

and

$$(\mathcal{J}_x f)(x_1 \wedge \cdots \wedge x_{q-1}) = f(x \wedge x_1 \wedge \cdots \wedge x_{q-1}).$$

Now let $G(\mathbb{R})$ be the connected semisimple Lie group with Lie algebra $\mathfrak{g}$, K be the maximal compact subgroup of $G(\mathbb{R})$. Let $\mathfrak{k}$ be the Lie algebra of K. Then we define $C^q(\mathfrak{g}, \mathfrak{k}; V)$ to be

$$C^q(\mathfrak{g}, \mathfrak{k}; V) := C^q(\mathfrak{g}, K; V) := \{f \in C^q(\mathfrak{g}; V) | \mathcal{J}_x f = 0 \text{ and } \mathcal{L}_x f = 0 \text{ for all } x \in \mathfrak{k}\}$$

We also note that

$$C^q(\mathfrak{g},\mathfrak{k};V) = Hom_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}),V)$$

where the action of $\mathfrak{k}$ on $\wedge^q(\mathfrak{g}/\mathfrak{k})$ is induced by the adjoint representation, i.e, $C^q(\mathfrak{g},\mathfrak{k};V)$ may be identified with the subspace of elements $f \in Hom_{F'}(\wedge^q(\mathfrak{g}/\mathfrak{k}),V)$ which satisfies the relation

$$\sum_i f(x_1,...,[x,x_i],...,x_q) = x \cdot f(x_1,...,x_q)$$

where $x \in \mathfrak{k}, x_i \in \mathfrak{g}/\mathfrak{k}, i = 1,...,q$. Then one can show that $C^i(\mathfrak{g},\mathfrak{k};V)$ is stable under d [cf. chapter 1, section 1, [7]] and it forms a complex denoted by $C^\bullet(\mathfrak{g},\mathfrak{k};V)$ which is

$$0 \to C^0(\mathfrak{g},\mathfrak{k};V) \xrightarrow{d} C^1(\mathfrak{g},\mathfrak{k};V) \xrightarrow{d} \cdots.$$

Now $H^q(\mathfrak{g},K;V)$ is defined as $H^q(C^\bullet(\mathfrak{g},\mathfrak{k};V))$.

Now we have a natural identification

$$\Omega^p(G,\mathbb{C}) = Hom(\wedge^p(\mathfrak{g}), C^\infty(G,\mathbb{C}))$$

where $\Omega^p(G,\mathbb{C})$ is the differential p-forms on G with coefficients in $\mathbb{C}$. Later on we will omit the word 'differential' and will simply call p-forms. The left multiplication map with g^{-1} gives a canoical isomorphism between $T_g(G)$ and $T_e(G)$. Then let $\omega \in \Omega^p(G,\mathbb{C})$, we identify ω with $h \in Hom(\wedge^p(\mathfrak{g}), C^\infty(G,\mathbb{C}))$ where h is given by

$$h(\lambda)(g) = \omega(g;\lambda_1,...,\lambda_p) \text{ for } \lambda = (\lambda_1 \wedge \cdots \wedge \lambda_p) \in \wedge^p(\mathfrak{g}), g \in G.$$

Now let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition where $\mathfrak{k}$ is the Lie algebra of K, the maximal compact subgroup of a connected semisimple real Lie group G with Lie algebra $\mathfrak{g}$. From definition

$$C^p(\mathfrak{g},K;C^\infty(\Gamma\backslash G;\mathbb{C})) = Hom_K(\wedge^p(\mathfrak{p}), C^\infty(\Gamma\backslash G;\mathbb{C})).$$

where K acts on $\mathfrak{p}$ via the adjoint representation and on $C^\infty(\Gamma\backslash G;\mathbb{C})$ via the right regular representation. With the above identification one can easily see that $C^p(\mathfrak{g},K;C^\infty(\Gamma\backslash G,\mathbb{C}))$corresponds to a p-form on G coming from a p-form on G/K (via the pullback of the map $\pi : G \to G/K$) which is invariant under the action of Γ. Now let ω be a p-form on $X = G/K$ invariant by Γ i.e.

$$\omega(x;v_1,...,v_p) = \omega(\gamma x;\gamma v_1,...,\gamma v_p) \text{ for all } \gamma \in \Gamma,$$

then the application $\omega \to \omega \circ \pi$ induces an isomorphism of graded complexes $\Omega^*(\Gamma\backslash X,\mathbb{C})$ and $C^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G;\mathbb{C})$ which in turn gives a canonical isomorphism between $H^*(\Gamma\backslash X,\mathbb{C})$ and $H^*(\mathfrak{g},K;C^\infty(\Gamma\backslash G))$ (cf. proposition 2.5, chapter 7 of [7]). Now we know that

$$L^2(\Gamma\backslash G) = \hat{\oplus}_{\pi\in\hat{G}} m(\pi,\Gamma)\mathcal{H}_\pi$$

which gives us the following Matsushima's formula. For proof one can see theorem 3.2 of [7].

Theorem 5.3.3: $H^*(\Gamma, \mathbb{C}) = \oplus_{\pi \in \hat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; \mathcal{H}_\pi)$ where the sum is finite and is restricted to the irreducible unitary representation π on $\mathcal{H}_\pi$ such that $H^*(\mathfrak{g}, K; \mathcal{H}_\pi)$ is not zero.

From lemma 1.1.1 and theorem 1.0.2 of [8] we note that we can restrict the above decomposition to those unitary irreducible representation π such that $\pi(C) = 0$ where C is the Casimir element of $\mathfrak{g}$ lying in the centre of the universal enveloping algebra $U(\mathfrak{g})$. This is because a differential form ω contributes to the cohomology of $\Gamma\backslash X$ in degree i iff $\Delta\omega = 0$, where Δ is the Hodge laplacian on the differential forms on X (for its Riemannian structure). For more vanishing results in this direction one can see chapter 2, proposition 3.1 of [7] or [29].

Here we like to state that if G admits a discrete series representation then this representation contributes to the cohomology only in dimension $\frac{1}{2}\dim(G/K)$ and all other relative Lie algebra cohomologies in dimension $\neq \frac{1}{2}\dim(G/K)$ become 0. For a proof one can see chapter 2, theorem 5.4 of [7]. Other finer results about the concentration of cohomology of π can be found in the book by Laurent Clozel and N. Bergeron which is [8]. For example one can see theorem 2.2.4 in chapter 2.

Now let T be a unitary representation of G on a Hilbert space $\mathcal{H}$. A vector $\phi \in \mathcal{H}$ is called *smooth* or C^∞ if the function $g \to T(g)\phi$ is C^∞. Let $\mathcal{H}^\infty$ be the subspace of smooth vectors in $\mathcal{H}$. If $X \in \mathfrak{g}$ and $\phi \in \mathcal{H}^\infty$ we pose:

$$T(X)\phi = [\tfrac{d}{dt} T(exp(tX))\phi]_{|t=0}$$

Now let $\mathcal{H}_\pi^\infty$ be the subspace of smooth vectors of $\mathcal{H}_\pi$. We define

$$\mathcal{H}_\pi^K = \{v \in \mathcal{H}_\pi^\infty : \dim\langle \pi(K)v\rangle < \infty\}$$

where $\langle \pi(K)v\rangle$ is the subspace generated by the vectors of the form $\pi(k)v$ with $k \in K$.

One says that $\pi \in \hat{G}$ is cohomological of degree i if it satisfies the following two properties:

1. $\pi(C) = 0$ where C is the Casimir element of $\mathfrak{g}$.
2. $\mathrm{Hom}_K(\wedge^i \mathfrak{p}, \mathcal{H}_\pi^K) \neq 0$

We note that point 2 above is equivalent to $H^i(\mathfrak{g}, K; \mathcal{H}_\pi^k) \neq 0$ (cf. theorem 1.0.2 and section 1.3 of [8].)

Now we state the bound on $m(\pi, \Gamma)$ due to F. Calegari and M. Emerton. We quote theorem 1.1 and theorem 1.2 from [10]. We follow our notations as in section 5.2 in this article.

Theorem 5.3.4: Let $\pi \in \hat{G}_\infty$ be of cohomological type. Suppose either that G_∞ does not admit discrete series, or, if it admits discrete series, that π contributes to cohomology in degrees other than $\frac{1}{2}\dim(G_\infty/K_\infty)$. Then

$$m(\pi, \Gamma(\mathfrak{P}^k)) \ll V(\mathfrak{P}^k)^{1-1/\dim(G_\infty)} \text{ as } k \to \infty$$

where $V(\mathfrak{P}^k)$ is the volume of $\Gamma(\mathfrak{P}^k)\backslash G_\infty$.

Now we state theorem 1.2 of [10] which gives the bound on cohomology.

Theorem 5.3.5: Let $n \geq 0$, and suppose that G_∞ admits discrete series and let $n \neq \frac{1}{2}\dim(G_\infty/K_\infty)$. Then

$$\dim H^n(\Gamma(\mathfrak{P}^k)\backslash G_\infty/K_\infty, \mathbb{C}) \ll V(\mathfrak{P}^k)^{1-1/\dim G_\infty} \text{ as } k \to \infty.$$

We prove theorem 5.3.5 and theorem 5.3.4 in the cocompact case. For the proofs of the above two theorems in the non cocompact case one can see lemma 3.3 and theorem 3.4 of [10].

We assume that G_∞ admits discrete series and $n \neq \frac{1}{2}\dim(G_\infty/K_\infty)$. We know that the discrete series representation contributes to the cohomology only in dimension $\frac{1}{2}\dim(G_\infty/K_\infty)$. Then equation 5.3.1 and theorem 5.3.3 gives that

$$\lim_{k\to\infty} V(\mathfrak{P}^k)^{-1} \dim H^n(\Gamma(\mathfrak{P}^k)\backslash G_\infty/K_\infty, \mathbb{C}) \to 0. \tag{5.3.2}$$

But $V(\mathfrak{P}^k)$ is like $c \cdot p^{dk}$ (cf. discussion before inequality 3.1 of [10]), where c is a constant and $d = \dim G$. Now from theorem 5.2.1 and equation 5.3.2 we get that $r_n = 0$ where r_n is the corank of $\tilde{H}^n(\mathcal{V})$ as defined in theorem 5.2.1. In the above two theorems we take trivial local system $\mathbb{C}$.

Now again applying theorem 5.2.1 we get that

$$\dim H^n(\Gamma(\mathfrak{P}^k)\backslash G_\infty/K_\infty, \mathbb{C}) \ll p^{(d-1)k} \text{ as } k \to \infty.$$

This gives us theorem 5.3.5 as $\dim(G) \leqslant \dim(G_\infty)$ (cf. inequality 3.1 of [10]).

Now theorem 5.3.4 is an easy consequence of theorem 5.3.5 because in theorem 5.3.4 we have asssumed that π contributes to the cohomology in degrees other than $\frac{1}{2}\dim(G_\infty/K_\infty)$. So we can apply theorem 5.3.5 and theorem 5.3.3 giving us the required bound

$$m(\pi, \Gamma(\mathfrak{P}^k)) \ll V(\mathfrak{P}^k)^{1-1/\dim(G_\infty)} \text{ as } k \to \infty.$$

This completes the proof of theorem 5.3.4. $\square$

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