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Clifford Algebras and Stein factorization of the Lagrangian Grassmannian of quadrics over local fields of residue characteristic 2

Graduand:
Mattia Zambolin

Supervisor:
prof. Uwe Jannsen

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Introduction

In this thesis we want to build the theoretical foundation necessary to study the computation of the étale cohomology group of an even dimensional quadric over a local field k of characteristic zero, introduced in [1] SGA VII, chapter XV page 21:

> Let Z be the quadratic separable extension of k, which is the center of the even part of the Clifford algebra of a quadratic form Q. Let ϵ be the corresponding character of order 1 or 2of the inertia group I. For $\sigma \in I$, we have
>
> $$\sigma\delta = \epsilon(\sigma)\delta \qquad \text{and} \qquad \sigma\delta' = \epsilon(\sigma)\delta',$$
>
> where δ and δ' are the naturally defined generators of the groups $H^n_{x_0}(X_s, R\Psi_{\bar{\eta}}(\Lambda(m)))$ and $R^n\Psi_{\bar{\eta}}(\Lambda(m))_{x_0}$.

There are no published proofs about this.

The purpose of my thesis is to explore the geometric elements necessary to allow the further study, regarding the vanishing cycles in the étale cohomology theory, for proving this case. In writing my thesis, I have been guided well by my supervisor prof. Jannsen and also by some written hints by Deligne, in correspondence with prof. Jannsen. We are interested in this topic, since it gives a criterion to simplify the computation of the cohomology group of quadrics over local field that have no semistable reduction, because of their ramified character.

This thesis is divided in three chapters.

In the first one, we will firstly introduce the definition and the main properties of the bilinear forms, b, and quadratic forms, Q, defined on a finite dimensional vector space V over a field k of characteristic different from 2 and in the second part we will focus on the quadratic forms defined over even dimensional vector space to highlight the existence of a particular equivalence class of quadratic forms called hyperbolic. For these special quadratic forms we can define the Lagrangian Grassmannian, $Lag(V)$, a projective variety that has deep connections to this theory and in particular to the Clifford algebra, and we prove the following important properties:

> **Proposition** 1.5.6 $Lag(V)$ has two connected components.

> **Proposition** 1.5.8 For any $L, L' \in Lag(V)$, they are in the same connected component if and only if $L \cap L'$ is of even codimension in L (and in L').

In the last part of this chapter, we will analyse two particular cases, the quadrics in $\mathbb{P}^3$ and in $\mathbb{P}^5$, to highlight the results, that we have already proven and to comparing this two different cases.

In the second chapter we focus on the Clifford algebras. They were introduced by William Kingdon Clifford (1845-1879) under the name of geometric algebras. One of his aims was to explain the multiplication rules for Hamilton's quaternions $\mathbb{H}$. The Clifford algebra $C(V, Q)$ is a a k-algebra that one can associate in a functorial way to every quadratic space (V, Q) over a field k, defined as the associative algebra generated by the elements of V, with relations

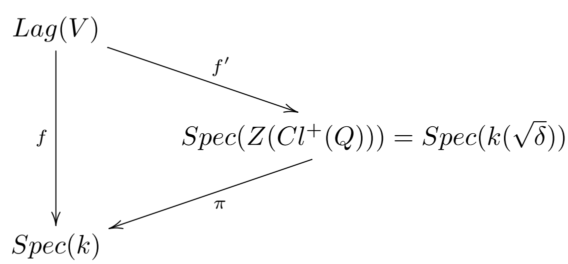
$$v_1v_2 + v_2v_1 = 2b(v_1, v_2) \qquad \text{for every } v_1, v_2 \in V.$$

It is a generalization of exterior algebras, defined in the presence of a symmetric bilinear form. This construction is of fundamental importance in the algebraic theory of quadratic forms. The main goal of this part, as well as to show some properties of this algebraic object, as its universal property, is to show its particular algebraic structure, theorem 2.2.6:

> **Theorem** 2.2.6 Assume $char(k) \neq 2$. Let $\varphi \cong < \alpha_1, \dots, \alpha_n >$ be a non degenerate quadratic space with discriminant δ 1.2.7. Then:
>
> - If $n = 2m$ the Clifford algebra $Cl(\varphi)$ is a central simple algebra over k. It is isomorphic to a tensor product of quaternion algebras. The center of $Cl^+(\varphi)$ is isomorphic to $k[X]/(X^2 - \delta)$. If δ is not a square, $Cl^+(\varphi)$ is a central simple algebra over $K(\sqrt{\delta})$.
> - If $n = 2m+1$ the same statement holds with the roles of $Cl(\varphi)$ and $Cl^+(\varphi)$ reversed: $Cl^+(\varphi)$ is a central simple algebra over k, and a tensor product of quaternion algebras. The center of $Cl(\varphi)$ is isomorphic to $k[X]/(X^2 - \delta)$. If δ is not a square, δ is a central simple algebra over $K(\sqrt{\delta})$.

In the first two chapters we have studied our topic using purely algebraic objects, but in the third chapter we will introduce some elements of algebraic geometry, such proper and projective morphisms in order to explain what the Stein factorization is. My aim in this part is to study the Stein factorization of $f : Lag(V) \to Spec(k)$, in order to prove the relation between the Lagrangian Grassmannian and the center of the even part of the Clifford algebra $Z(Cl^+(Q))$ of the hyperbolic quadratic form Q.

The Stein factorization of $f : Lag(V) \to Spec(k)$ can be explained by the following commutative diagram:



We want to show that the projective and proper morphism f factors through $Z(Cl^+(Q))$.
This result is used in the calculation of vanishing cycles for certain types of quadrics over local fields with residue characteristic 2.

In the last paragraph of this chapter, we show how what we studied in this thesis is related to the case, introduced in [1] SGA VII, chapter XV page 21, which we talked about at the beginning of this thesis. The theory developed in this thesis can be used in the computation of the étale cohomology group of the hyperbolic quadric Q in $\mathbb{P}^{2n-1}$, using the theory of algebraic cycles, of tangent bundles, of the Chern classes.

This is the starting point for a possible further thesis in which we will develop the theory that we briefly introduced in the last chapter in order to complete Deligne's proof of the case in SGA VII, chapter XV page 21.

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Chapter 1

Quadratic Forms and Lagrangian Grassmannians

In this chapter we will firstly introduce the definition and the main properties of quadratic forms defined on a finite dimensional vector space V over a field k of characteristic different from 2 and in the second part we will focus on the Lagrangian Grassmannian, in order to prove the existence of two connected components and we will show a characterization of them. In the last part we will analyse two particular cases, the quadrics in $\mathbb{P}^3$ and in $\mathbb{P}^5$, to highlight the results, that we have already proven and to comparing these two different cases.

1.1 Bilinear Forms and Quadratic Forms

In this paragraph we recall the definition and the main property of the quadratic forms on a finite dimensional vector space V over a field k of characteristic different from 2.

Definition 1.1.1. A *bilinear form* on V is a map $b : V \times V \to k$ such that for every $v_1, v_2, w_1, w_2 \in V$ and for every $\alpha_1, \alpha_2, \beta_1, \beta_2 \in k$ the following holds:

$$b(\alpha_1 v_1 + \beta_1 w_1, \alpha_2 v_2 + \beta_2 w_2) = \alpha_1\alpha_2 b(v_1, v_2) + \alpha_1\beta_2 b(v_1, w_2) + \beta_1\alpha_2 b(v_2, w_1) + \beta_1\beta_2 b(w_1, w_2).$$

Definition 1.1.2. A bilinear form b on V is called:

- *symmetric* if for every $v, w \in V$ we have $b(v, w) = b(w, v)$;
- *skew symmetric* if for every $v, w \in V$ we have $b(v, w) = -b(w, v)$;
- *alternating* if for every $v \in V$ we have $b(v, v) = 0$.

Remark 1.1.3. Every alternating bilinear form is skew symmetric, it can be seen by expanding $b(v + w, v + w) = 0$, for every $v, w \in V$. Moreover, the converse is also true if $char(k) \neq 2$.

The pair (V, b), where V is a vector space and b a symmetric bilinear form, is called a *symmetric bilinear space* (over k) or just *bilinear space* for short.

Definition 1.1.4. Two vectors $w, u \in V$ are b-orthogonal (or b-perpendicular to each other) if $b(v,w) = b(w,v) = 0$. Two subsets W, U of V are called orthogonal if $b(w,u) = 0$ for all $w \in W$ and $u \in U$. Orthogonality will be denoted by $\perp$: $w \perp u, W \perp U$. With each subset W of V is associated its orthogonal space:

$$W^{\perp} = \{v \in V \mid b(v,w) = 0 \text{ for every } w \in W\}$$

Lemma 1.1.5. *$W^{\perp}$ is a subspace of V.*

Proof. If $w, w\prime \in W^{\perp}$, then for all $v \in V$, $b(w,v) = b(w\prime, v) = 0$ holds. Thus $b(w + w\prime, v) = 0$ and hence $(w + w\prime \in W^{\perp})$. If $\alpha \in k$, then for all $v \in W$, $\alpha b(v,w) = 0$ and hence $b(\alpha v, w) = 0$. Thus $\alpha v \in W^{\perp}$. □

It is clear that $W \subset U$ implies $U^{\perp} \subset W^{\perp}$. The inclusion $W \subset (W^{\perp})^{\perp}$ is equally trivial. If W is a subspace of V, then $(W, b_{|W \times W})$ is a symmetric bilinear space, a subspace of (V, b). Instead of $b_{W \times W}$ we will write more simply b_W.

Definition 1.1.6. The kernel of a bilinear form b is the subspace

$$ker(b) = \{v \in V \mid b(v,w) = 0 \text{ for all } w \in W\}.$$

Remark 1.1.7. For any subspace $W \subseteq V$, the restriction of b to W has kernel:

$$ker(b_W) = W \cap W^{\perp}.$$

Definition 1.1.8. A symmetric bilinear space (V, b) is called *non degenerate* if $ker(b) = \{0\}$, i.e. $V \cap V^{\perp} = \{0\} \Rightarrow V^{\perp} = \{0\}$. It means that there doesn't exist any $v \in V$ such that $b(v,w) = 0$ for every $w \in V$.

We can formulate the previous definition, using the following map.

Definition 1.1.9. If (V, b) is a bilinear space, then $\tilde{b} : V \to V^*$ defined by $b(v)(w) = b(v,w)$ is obviously a linear transformation. $\tilde{b}$ is called the *adjoint transformation.*

It's clear from the definition that $ker(b) = ker(\tilde{b})$, so we can say

$$\text{b is non degenerate } \iff \tilde{b} \text{ is an isomorphism.}$$

Lemma 1.1.10. *If W is a subspace of the bilinear space (V, b), then $W^{\perp} = ker(\pi\tilde{b})$, where $\pi : V^* \to W^*$ denotes the canonical projection*

Proof. $v \in W^{\perp}$ means the same thing as $\tilde{b}(v)(w) = b(v,w) = 0$ for all $w \in W$. This means $(\tilde{b})_{|W} = 0$ and hence $v \in ker(\pi\tilde{b})$. □

We introduce now the definition of quadratic forms:

Definition 1.1.11. We call a map $Q : V \to k$, a *quadratic form* on V if:

1. $Q(\alpha v) = \alpha^2 Q(v)$ for every $\alpha \in k$ and $v \in V$;
2. the map $b_Q : V \times V \to k$, $b_Q(v,w) = \frac{1}{2}(Q(v+w) - Q(v) - Q(w))$ is a bilinear form.

This bilinear form b_Q is called the bilinear form associated to Q. The pair (V, Q) is called a quadratic space. We say that Q is non degenerate if and only if b_Q is non degenerate.

It's easy to find that for every $v \in V$, $b_Q(v, v) = Q(v)$. Generalizing this idea to any bilinear form, we can give the following definition.

Definition 1.1.12. If b is a bilinear form on V, then $Q_b : V \to k$ defined by $Q_b(v) := b(v, v)$, for every $v \in V$ is a quadratic form (as is easily checked by a trivial computation). Q_b is called the *quadratic form associated* with b.

Remark 1.1.13. By direct computation we can identify quadratic forms (respectively, quadratic spaces) with symmetric bilinear forms (respectively, symmetric bilinear spaces) by means of the inverse correspondences as defined in the last definitions. By this means all concepts for symmetric bilinear spaces can be carried over to quadratic spaces and conversely. For example, two vectors in a quadratic space (V, Q) are orthogonal when they are orthogonal with respect to b_Q.

For any quadratic form Q on V, with $\mathcal{B} = \{e_1, \ldots, e_n\}$ a basis of V and $a_1, \ldots, a_n \in k$, then:

$$Q\left(\sum_{i=1}^{n} a_i e_i\right) = b_Q\left(\sum_{i=1}^{n} a_i e_i, \sum_{i=1}^{n} a_i e_i\right) = \sum_{i,j=1}^{n} b_Q(e_i, e_j) = \sum_{i=1}^{n} Q(e_i) + 2 \sum_{1 \le i < j}^{n} b_Q(e_i, e_j).$$

Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be a basis of a vector space V, for any bilinear form b on V, then

$$B = B_{\mathcal{B}} = (b(e_i, e_j))_{i,j} = (b_{ij})_{i,j}$$

is called the matrix of b with respect the basis $\mathcal{B}$. In particular, b is symmetric precisely when B is symmetric. For $v = \sum_{i=1}^{n} v_i e_i$, $w = \sum_{i=1}^{n} w_i e_i$ in V we have

$$b(v, w) = (v_1, \ldots, v_n) \begin{pmatrix} b(e_1, e_1) & \cdots & b(e_1, e_n) \\ \vdots & \ddots & \vdots \\ b(e_n, e_1) & \cdots & b(e_n, e_n) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = v^t B w$$

where we identify v and w with the appropriate column vectors.

For $\mathcal{B}' = \{e'_1, \ldots, e'_n\}$ a second basis of V and T the change of basis matrix from $\mathcal{B}$ to $\mathcal{B}'$, we have $e'_j = \sum_{i=1}^{n} t_{ij} e_i$, for $j \in \{1, \ldots, n\}$ and $T = (t_{ij})$. We now compute the matrix of b with respect to $\mathcal{B}'$.

Lemma 1.1.14. $B_{\mathcal{B}'} = T^t B_{\mathcal{B}} T$.

Proof. We compute

$$b(e'_i, e'_j) = b\left(\sum_{k=1}^{n} t_{ki} e_k, \sum_{k=1}^{n} t_{lj} e_l\right) = \sum_{k,l=1}^{n} t_{ki} b(e_k, e_l) t_{lj}$$

yielding the (i, j) entry of $T^t B_{\mathcal{B}} T$.

□

As is well known, two $n \times n$ matrices A, B are called *congruent* if there exists an invertible matrix T such that $B = T^t A T$. The matrix of a bilinear space (V, b) is thus well-defined up to congruence.

Let us see now which is the relation between the matrix of a bilinear form and of its adjoint transormation.

Lemma 1.1.15. *If $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a basis of V and $\mathcal{B}^* = \{e_1^*, \ldots, e_n^*\}$ is the dual basis of V^*, then the matrix of $\tilde{b}$ with respect to the bases $\mathcal{B}, \mathcal{B}'$ is the matrix $B_{\mathcal{B}}$ of b with respect to $\mathcal{B}$.*

Proof. From $\tilde{b}(e_i)(e_j) = b(e_i, e_j)$ follows $\tilde{b}(e_i) = \sum_{j=1}^{n} b(e_i, e_j)e_j^*$ as was to be shown. □

Proposition 1.1.16. *The quadratic form Q or its associated symmetric bilinear form are non degenerate if and only if the determinant of the matrix associated to the bilinear form is invertible.*

Proof. We have already seen that a bilinear form b is non degenerate if and only if $\tilde{b}$ is an isomorphism. From the previous lemma we get the thesis. □

Proposition 1.1.17. *Let (V, b) be non degenerate and W a subspace of V, then*

$$dim_k W + dim_k W^{\perp} = dim_k V \text{ and } (W^{\perp})^{\perp} = W$$

Proof. The adjoint map $\tilde{b} : V \to V^*$ is an isomorphism and the canonical projection $V^* \to W^*$ given by $f \mapsto f\mid_W$ is surjective. As the kernel of the composition $V \to V^* \to W^*$ is $W^{\perp}$, we obtain the first result.

Clearly $W \subset (W^{\perp})^{\perp}$ and by the previous part both spaces have the same dimension, this prove the second result. □

1.2 Singular subspaces

Definition 1.2.1. Let b a symmetric bilinear form on V, W be a subspace of V. We say that:

- $v \in V$ is *isotropic* if $b(v, v) = 0$;
- W is *isotropic* if it has a non zero element in common with $W^{\perp}$, i.e. if $b_{W\times W}$ is degenerate;
- W is *totally isotropic* if $W \subseteq W^{\perp}$, i.e. if $b_{W\times W}$ is the zero form .

We now see a characterization of the non isotropic space.

Proposition 1.2.2. *Let W be a non isotropic subspace of V, then $V = W \oplus W^{\perp}$.*

Proof. We consider $b_{W\times W}$ the restriction to W, and we know that is non degenerate by hypothesis, i.e. $ker(b_{W\times W}) = ker(\tilde{b}_{W\times W})$ then the map $b_{W\times W} : W \to W^*$ is injective and so bijective since W and W^* have same dimension. Thus, for every $w \in V$ do exist one and only one vector $w_0 \in W$ such that $b(v, w) = b(v, w_0)$ for every $v \in V$, so $b(v, w - w_0) = 0$, $w - w_0 \in W^{\perp}$. In this way we have shown that V is direct sum of W and $W^{\perp}$. □

Corollary 1.2.3. *If W is a subspace of V and b is a non degenerate bilinear form. The following conditions are equivalent:*

1. *W is non isotropic;*
2. *$W^{\perp}$ is non isotropic;*
3. *$V = W \oplus W^{\perp}$.*

Proof. The previous proposition shows that *1.* and *2.* imply *3.* The condition *3.* means that $W \cap W^{\perp} = \varnothing$, so implies both *1.* and *2.* □

By the previous corollary, for every totally isotropic subspace W of V, we have:

$$dim_k V = dim_k W + dim_k W^{\perp} \geq 2dim_k W \Rightarrow dim_k W \leq \frac{1}{2} dim_k V$$

Theorem 1.2.4. *Every symmetric bilinear space (V, b) is the orthogonal sum of one-dimensional subspaces. In other words, V has a basis of pairwise orthogonal vectors ("orthogonal basis").*

Proof. By induction on $dim_k V$: If $b = 0$, the assertion is clear as every decomposition of V as a direct sum is also an orthogonal decomposition. If $b \neq 0$, there are vectors $x, y \in V$ with $b(x, y) \neq 0$. Now

$$b(x, y) = \frac{1}{2} \left(b(x + y, x + y) - b(x, x) - b(y, y) \right)$$

and hence there must exist a z with $b(z, z) \neq 0$, namely $z = x, y$, or $x + y$. The one-dimensional subspace $V_1 = zK$ is non degenerate and by the last lemma $V = V_1 \oplus V_1^{\perp}$. A map of the induction hypothesis to $V_1^{\perp}$ completes the proof.

□

Corollary 1.2.5. *Every symmetric matrix B is congruent to a diagonal matrix.*

Proof. By the last theorem choose an orthogonal basis $v_l, \dots, v_n$ for the bilinear space whose symmetric bilinear form, b_B has the matrix B as its associated matrix. Then $(b_B(v_i, v_j))$ is a diagonal matrix congruent to B by 1.1.14.

□

Remark 1.2.6. The previous corollary means that for every field with characteristic different from 2 and for any symmetric matrix A there exist an invertible matrix N such that $N^t A N$ is diagonal. In particular, if k is an algebraically closed field there exist an invertible matrix N such that $N^t A N = \begin{pmatrix} \mathbb{I}_r & 0 \\ 0 & 0 \end{pmatrix}$, where $\mathbb{I}_r$ is the identity matrix of dimension $r = rank(A)$. To any non degenerate bilinear form is associated an invertible matrix, i.e. of maximal rank, it is congruent to the identity matrix, $\mathbb{I}$. This means that a quadratic form defined over an algebraically closed field can be express in a canonical way:

$$Q(x_1, \dots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2.$$

The following definition will be used only in the next chapter to explain the Clifford algebra structure.

Definition 1.2.7. For a quadratic space (V, Q) the *discriminant* $d(V, Q)$ is defined by

$$d(V, Q) = (-1)^n det(V, Q),$$

where $dim(V, Q) = 2n$ or $2n + 1$ and $det(V, Q)$ is the determinant of the associated matrix to Q. Of course, $d(V, Q)$ is considered as an element of the square class group $k^{\times}/(k^{\times})^2$.

In a similar way we can define subspace with the same propertie also for the quadratic form, only replacing isotropic with singular.

Definition 1.2.8. Let Q be a quadratic form on V. A vector $v \in V$ is called *singular* (respect to Q) if $Q(v) = 0$.

A linear subspace W of V is called:

- *singular* if exists a vector $v \neq 0$ in W that is singular and orthogonal to W;
- *totally singular* if the restriction of Q to W is the zero map.

Remark 1.2.9. If Q is a quadratic form on V and b_Q its associated bilinear form then $V^\perp$ is a totally singular subspace of V.

By the definitions of totally isotropic and totally singular subspace we get immediately the following:

Proposition 1.2.10. *Let Q be a quadratic form on V and b_Q its associated bilinear form. Every linear subspace of V is totally singular if and only if it is totally isotropic.*

Definition 1.2.11. Let b a symmetric bilinear form, a totally isotropic subspace W of V is called *maximal totally isotropic* if it is not properly contained in an another totally isotropic subspace. Equivalently, a totally isotropic subspace W is maximal if and only if $W = W^\perp$.
Let Q a quadratic form, a totally singular subspace W of V is called *maximal totally singular* if it is not properly contained in an another totally singular subspace.

Proposition 1.2.12. *Let b_Q a bilinear form associated to a non degenerate quadratic form Q and F a totally singular subspaces of V of dimension r.*

1. *If F' is a totally singular subspace of dimension r such that $F^\perp \cap F' = \{0\}$, then $F + F'$ is non singular, and for every basis $\{f_1, \ldots, f_r\}$ of F, there exist a basis $\{f'_1, \ldots, f'_r\}$ of F' such that $b(f_i, f'_j) = \delta_{ij}$, for $i, j = 1, \ldots, r$.*

2. *If G is a totally singular subspace with $dim_k G \leq r$ such that $G \cap F^\perp = \{0\}$, there exists a totally singular subspace $F' \supseteq G$ of dimension r such that $F^\perp \cap F' = \{0\}$.*

Proof. See [2] § 4, $n°$ 2 Prop. 2. □

Corollary 1.2.13. *If F is totally singular subspace of dimension r, there exists a totally singular subspace F' of dimension r such that $F \cap F' = \{0\}$, and $F \oplus F'$ is not singular.*

Proof. We need only to apply the previous theorem point *2.* with $G = \{0\}$. □

We can show now a property of the maximal totally singular subspaces defined over an algebraically closed field.

Proposition 1.2.14. *If k is an algebraically closed field, any maximal totally singular subspace L in V, $dim_k V = n$, with respect to a non degenerate quadratic form Q, has $dim_k L = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let L a be a maximal totally singular subspace, and let L' be another maximal totally subspace such that $L \cap L' = \{0\}$. Then $L \oplus L'$ is non isotropic, thus even $W = (L \oplus L')^\perp$ is not isotropic. We want to show that $dim_k W \leq 1$.

We first prove that $Q(w) \neq 0$, for every $w \in W \setminus \{0\}$. We suppose for absurd that there exists $\tilde{w} \in W \setminus \{0\}$ such that $Q(\tilde{w}) = 0$, then we have $b_Q(\tilde{w}, v) = 0$ for every $v \in L$, since $W = (L \oplus L')^{\perp}$ Moreover we get

$$Q(v + \tilde{w}) = Q(v) + 2b_Q(v, \tilde{w}) + Q(\tilde{w}) = 0$$

for every $v \in L$. Thus $L \oplus < \tilde{w} >$ is a totally singular subspace of V in which L is properly contained, this is a contradiction, since L is maximal totally isotropic. Therefore, there does not exist there exists $\tilde{w} \in W \setminus \{0\}$ such that $Q(\tilde{w}) = 0$.
Now, we consider $w, w' \in W$, since k is algebraically closed, there exist $a \in k$ such that

$$Q(w - aw') = Q(w) + 2ab_Q(w, w') + a^2 Q(w') = 0.$$

By the previous result it follows that $w - aw' = 0$, which implies $w = aw'$ and so $dim_k W \leq 1$.
Considering the decomposition of V that we applied at the beginning of this proof, we get the following results:

- if n is even the only possibility is $dim_k W = 0$ and $dim_k L = dim_k L' = \frac{n}{2}$;
- if n is odd the only possibility is $dim_k W = 1$ and $dim_k L = dim_k L' = \frac{n-1}{2}$.

This proves the thesis.

□

1.3 The orthogonal group $O(V, b)$

In this paragraph we will explain the concept of map between bilinear symmetric spaces and in paticular we will introduce the definition of orthogonal group that will play a very important role in the paragraph 1.5.

Definition 1.3.1.

- Let (V, b) and (V', b') be symmetric bilinear spaces. An injective linear transformation $\sigma : V \to V'$ is called an *isometry* in case

$$b'(\sigma(x), \sigma(y)) = b(x, y)$$

  for all $x, y \in V$. The composition of isometries is obviously an isometry.

- The spaces (V, b) and (V', b') are called *isometric* or *isomorphic* (notation: $(V, b) \cong (V', b')$), if there exists a bijective isometry $\sigma : V \to V'$. "Isometry" is an equivalence relation.
- With respect to composition the isometries $\sigma : V \to V$ form a group called the *orthogonal group* or the automorphism group of (V, b). This group will be denoted by $O(V, b)$ or $Aut(V, b)$. The elements of the orthogonal group are also called orthogonal transformations. $O(V, b)$ is obviously a subgroup of $GL(V)$, the group of all bijective linear transformations $V \to V$.

Orthogonal vectors are clearly sent to orthogonal vectors by an isometry σ. In particular, for each subset X of V

$$\sigma(X)^{\perp} \supseteq \sigma(X^{\perp})$$

holds. Equality holds if σ is a bijective isometry.

Lemma 1.3.2. *If $(V,b) \cong (V',b')$, then $O(V,b) \cong O(V',b')$.*

Proof. If $\sigma : V \to V'$ is a bijective isometry, then $\tau \to \sigma\tau\sigma^{-1}$ is an isomorphism of the orthogonal groups.

□

We need to see how we can use the concept of isometry between quadratic spaces:

Remark 1.3.3. The concept of isometry also yields nothing new: An isometry between two quadratic spaces (V,Q) and (V',Q') is defined to be an injective linear transformation $\sigma : V \to V'$ satisfying $Q'(\sigma(x)) = Qx)$ for all $x \in V$. A transformation $\sigma : V \to V'$ is an isometry precisely when it is an isometry with respect to b_Q. Conversely, an isometry of symmetric bilinear spaces is also an isometry of their associated quadratic spaces. In spite of this it is often useful to distinguish between the two concepts. Over fields of characteristic 2 quadratic forms and symmetric bilinear forms are essentially different.

If (V,b) and (V',b') are two bilinear spaces with bases $\mathcal{B} = e_1, \dots, e_n$ and $\mathcal{B}' = e'_1, ..., e'_m$, then a linear transformation $\sigma : V \to V'$ is described by an $m \times n$ matrix $S = (s_{ij})$ with $\sigma(e_i) = \sum_{j=1}^{m} e'_j s_{ji}$. The map σ is an isometry precisely when S has rank n and

$$x^t By = b(x,y) = b'(\sigma(x), \sigma(y)) = x^t S^t B' S y$$

holds for all $x, y \in V$, hence $B = S'B'S$. In particular we have thus proven:

Theorem 1.3.4. *Two bilinear spaces are isometric if and only if their associated symmetric matrices (with respect to arbitrary bases) are congruent.*

□

In this part of the section we define the determinant of a bilinear space.

Lemma 1.3.5. *If (V,b) is a non degenerate bilinear space and $\alpha \in O(V,b)$, then $det\ \alpha = \pm 1$.*

Proof. Choose a basis and represent b by the matrix B and α by the matrix A. Then A satisfies the equation $B = A^t BA$. The assertion follows since $\det B \neq 0$.

□

The determinant induces a homomorphism

$$det : O(V,b) \to \{\pm 1\}.$$

The kernel of this homomorphism, the isometries of determinant 1, is called the special orthogonal group and will be denoted by $SO(V,b)$. Let $(k^\times)^2$ denote the subgroup of squares of the multiplicative group $k^\times$ of our ground field k. If b has matrix B with respect to some basis, then b has matrix $B' = A^t BA$ with respect to another basis, where A is the change of basis matrix. As A is invertible and det A^t= det A, det B and det B' differ only by an element of $(k^\times)^2$. From the results of paragraph 1 we have immediately: (V,b) is singular if and only if $\det B = 0$.

We now introduce a particular orthogonal transformation that it will play a very important role in this theory.

Definition 1.3.6. Let $x \in V$ be an anisotropic vector and $W = x^{\perp}$. Then the linear map

$$\tau_x : V \to V \ , \ \tau_x(y) = y - 2\frac{b(x,y)}{b(x,x)}x$$

is called *reflection* in the hyperplane W orthogonal to x.

The name is suggested by statement *1.* of the following lemma.

Lemma 1.3.7. *1. $\tau_x(x) = -x$, $\tau_x|_W = id_W$.*

2. τ_x is an isometry of (V, b).

3. $\tau_x \tau_x = id$.

4. $det\tau_x = -1$.

Proof. The first three statements can be shown by a trivial computation. To see *4.*, choose a basis $e_2, \ldots, e_n$ of W and complete to a basis of V by taking $e_1 = x$. The matrix of τ_x with respect to this basis is

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which has determinant -1. □

We now formulate two basic theorems.

Theorem 1.3.8. *(Cartan-Dieudonné's theorem) Any orthogonal transformation $\phi \in O(V, b)$ can be written as a product of $m \leq dim_k V$ reflections.*

Proof. See [6], chap. 1 §4. □

Theorem 1.3.9. *(Witt's theorem) Let (V, b) be a non degenerate symmetric bilinear space. Let W be a subspace of V and $\sigma : W \to V$ an isometry. Then there exists an isometry $\Sigma : V \to V$ which extends σ, that is $\Sigma|_W = \sigma$.*

Proof. See [8], chap. 1 §5, and [6], chap. 1 §5. □

There is one important consequences of the previous theorem.

Corollary 1.3.10. *All maximal totally isotropic subspaces of (V, b) have the same dimension.*

Proof. Let U and W be such spaces with $dim_k U \leq dim_k W$. Let $\sigma : U \to W$ be an arbitrary injective linear map. Since σ is an isometry, it can be extended to an isometry $\Sigma : V \to V$. Then $\Sigma^{-1}(W) \supset U$ is totally isotropic and hence equals U, thus proving the equality of dimensions. □

Definition 1.3.11. The common dimension of maximal totally isotropic subspaces of a non degenerate space (V, b) is called the *Witt index* of (V, b) and will be denoted by $ind(V, b)$.

1.4 Hyperbolic spaces

Let V be a vector space, $dim_k V = n$, and V^* its dual space. On the vector space $V \oplus V^*$ we consider the following symmetric bilinear form

$$\mathbf{h} = \mathbf{h}_V : (V \oplus V^*) \times (V \oplus V^*) \to k$$

$$\mathbf{h}((x, f), (y, g)) := fy + gx.$$

for every $x, y \in V$ and $f, g \in V^*$. One can easily check that $\mathbf{h}$ is a symmetric bilinear form, by looking at its associated matrix.

Lemma 1.4.1. *1. $(V \oplus V^*, \mathbf{h}_V)$ is non degenerate.*

2. V and V^* are totally isotropic subspaces.

Proof. If $(x, f) \in V \oplus V^*$ and $x \neq 0$, then there is a $g \in V^*$ with $gx \neq 0$, hence

$$\mathbf{h}((x, f), (0, g)) = gx \neq 0.$$

On the other hand if $f \neq 0$, there is a $y \in V$ with $fy \neq 0$ and hence

$$\mathbf{h}((x, f), (y, 0)) = fy \neq 0.$$

This yields *1.* and *2.*, as is obvious from the definition. □

Definition 1.4.2. The non degenerate symmetric bilinear space $(V \oplus V^*, \mathbf{h}_V)$ has $ind(V \oplus V^*, \mathbf{h}_V) = n$, is denoted by $\mathbb{H}(V)$ and called the hyperbolic space on V.

In a canonical way this construction associates a non degenerate bilinear space with each vector space. In an analogous way we can associate an isometry with each vector space isomorphism Let $\alpha : V \to W$ be a vector space isomorphism and $\alpha^* : W^* \to V^*$ the dual isomorphism. Then $\mathbb{H}(\alpha) := \alpha \oplus (\alpha^*)^{-1} : \mathbb{H}(V) \to \mathbb{H}(W)$ is a bijective isometry. This can be easily checked. Let $GL(V) = Aut(V)$ denote the group of invertible endomorphisms of V. Further, one can check that

$$\mathbb{H} : GL(V) \to O(\mathbb{H}(V)) \ , \ \alpha \mapsto \alpha \oplus (\alpha^*)^{-1}$$

is an injective group homomorphism.

Theorem 1.4.3. *Let (V, b) be a non degenerate $2n$-dimensional bilinear space. The following conditions are equivalent:*

1. *$(V, b) \cong \mathbb{H}(W)$ for an n-dimensional vector space W;*
2. *V contains an n-dimensional totally isotropic subspace W;*
3. *$(V, b) \cong (V, b_A)$ where b_A is the symmetric bilinear form that has $A = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}$, where $\mathbb{I}$ denotes the $n \times n$ unit matrix and $\mathbb{O}$ the $n \times n$ zero matrix, as associated matrix.*

Proof. *1.* $\Rightarrow$ *2.* W is a totally isotropic subspace of $\mathbb{H}(W)$.
2. $\Rightarrow$ *3.* Complete an arbitrary basis $e_1, \dots, e_n$ of W to a basis of V. The matrix of b with respect to this basis has the following associated matrix:

$$\begin{pmatrix} \mathbb{O} & C \\ C^t & D \end{pmatrix}$$

where $\mathbb{O}$, C, D are $n \times n$ matrices and $det(C) \neq 0$. This follows by the previous chapter from the following matrix equation:

$$\begin{pmatrix} C & \mathbb{O} \\ E^t & \mathbb{I} \end{pmatrix} \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} \begin{pmatrix} C^t & E \\ \mathbb{O} & \mathbb{I} \end{pmatrix} = \begin{pmatrix} \mathbb{O} & C \\ C^t & D \end{pmatrix} \ , E = \frac{1}{2}D.$$

3. $\Rightarrow$ *1.* Let $e_1, \dots, e_n, e'_1, \dots, e'_n$ be the basis of V with respect to which the matrix of b has the form $\begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix}$. Let $W = e_l k + \dots + e_n k$ and let $e^*_1, \dots, e^*_n$ be the dual basis of W^*. Then $\alpha : V \to W \oplus W^*$ defined by $\alpha(e_i) = e_i$, $\alpha(e'_i) = e^*_i$, for $i \in \{1, \dots, n\}$ is an isometry form (V, b) to $\mathbb{H}(W)$. To prove that α is an isometry, first note that α behaves as an isometry on the basis vectors and the result then follows for arbitrary linear combinations of the basis vectors. This completes the proof of the theorem. □

Definition 1.4.4. Any space which satisfies the conditions of the previous theorem will be called *hyperbolic*. Two vectors x, y with

$$b(x, x) = b(y, y) = 0 \ , \ b(x, y) = 1$$

will be called a *hyperbolic pair*.

Remark 1.4.5. The point *3.* of the previous theorem can be reformulated as follows:

If (V, b) is an hyperbolic space of dimension $2n$, then it admits n hyperbolic pairs that compose a basis of V. In particular, expressing the associated quadratic form Q with respect to this basis we get the following canonical form

$$Q(x_1, \dots, x_{2n}) = \sum_{i=1}^{n} x_i x_{n+i}.$$

Moreover, another consequence is that the discriminant of a hyperbolic space is 1 since

$$det \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathbb{I} & \mathbb{O} \end{pmatrix} = (-1)^n.$$

Remark 1.4.6. As we have already seen in the previous paragraph, if k is an algebraically closed field all the matrices corresponding to the non degenerate quadratic forms, and their associate bilinear forms, are invertible and congruent to each other. It means that, in this particular case, all the non degenerate quadratic forms are hyperbolic, since every non degenerate symmetric bilinear form b defined on a vector space V of dimension $2n$ over k has a maximal totally isotropic subspace of dimension n.

Corollary 1.4.7. *If (V, b) is non degenerate and x is isotropic, then there is a vector y such that x, y is a hyperbolic pair. V then has the decomposition $V = W \oplus W^{\perp}$ where (W, b_W) is a hyperbolic plane, a hyperbolic space of dimension 2.*

Proof. Choose z with $b(x,z) = \alpha \neq 0$. Upon multiplying z by α^{-1} we can assume $b(x,z) = 1$. Then $y = z - \frac{1}{2}b(z,z)x$ is the sought-after vector. Taking $W = kx + ky$ yields the second assertion. □

Definition 1.4.8. A bilinear space (V, b) represents $\alpha \in k$ if there exists an $x \in V$, $x \neq 0$, with $b(x,x) = \alpha$. The space (V, b) is called universal if (V, b) represents all $\alpha \in k^{\times}$.

Lemma 1.4.9. *Every isotropic bilinear space is universal.*

Proof. By the last corollary there exists a hyperbolic pair x, y. For $\alpha \in k$ we then have

$$\alpha = b\left(x + \frac{1}{2}y, x + \frac{1}{2}y\right).$$

□

1.5 Lagrangian Grassmannians

Let (V, b) be an hyperbolic space of dimension $2n$. In this section we will study some properties of the maximal totally isotropic subspaces of an hyperbolic space.

To illustrate their geometric property, we need to introduce this definition:

Definition 1.5.1. In $\mathbb{P}^m$, with $m \in \mathbb{N}$ and $0 \leq r \leq m$, we define the *Grassmannian of the subspaces of dimension r in* $\mathbb{P}^m$ by

$$\mathbb{G}(r,m) = \{L \mid L \subset \mathbb{P}^m \text{linear subspaces with } dim_k L = r\}.$$

The Grassmannian of the subspaces of dimension r in $\mathbb{P}^m$ is a projective variety.

Remark 1.5.2.
- If $r = 0$, then $\mathbb{G}(0,m) = \mathbb{P}^m$.
- If $r = m-1$, then $\mathbb{G}(m-1,m) = (\mathbb{P}^m)^*$.
- If $r = m$, then $\mathbb{G}(m,m) = \{\mathbb{P}^m\}$.

Remark 1.5.3. There is a bijection between the linear subspaces of dimension r in a vector space V of dimension $m+1$ and the linear subspaces of dimension $r-1$ in $\mathbb{P}(V)$, i.e. $\mathbb{G}(r-1,m) \cong \mathbb{G}(r,V)$, where

$$\mathbb{G}(r,V) = \{W \mid W \subset V \text{ linear subspaces with } dim_k L = r\}.$$

The linear subspace W is contained in the quadric, defined by the equation $Q = 0$, if and only if the quadratic form Q vanishes identically on W, i.e., if and only if W is totally singular for Q. The linear subspace of dimension $r \leq n$ contained in Q, then form a closed subvariety of the Grassmannian $\mathbb{G}(r,V)$, a projective variety, thus also the subvariety is projective.

Definition 1.5.4. The family of the maximal totally isotropic subspaces of a hyperbolic space (V, b) is called *Lagrangian Grassmannian* and it is denoted by $Lag(V)$. $Lag(V)$ is a smooth variety.

Obviously, $Lag(V)$ is a closed subvariety of the Grassmannian $\mathbb{G}(n-1, 2n-1)$, thus is projective.

Remark 1.5.5. We choose to use the notation $Lag(V)$ for the Lagrangian Grassmannian of V, instead of using $Lag(V, Q)$, specifying which quadratic form is computed, since all the hyperbolic quadratic forms are equivalent, so the geometric structure and properties depend only on the vector space and not on its hyperbolic quadratic form.

We can define a group action of the orthogonal group, $O(V, b)$, on $Lag(V)$:

$$O(V, b) \times Lag(V) \to Lag(V) \ , \ (\phi, L) \mapsto \phi(L).$$

It's clear from the definition of the orthogonal group that this action is well defined: for every $v, w \in L \in Lag(V)$, $b(\phi(v), \phi(w)) = b(v, w) = 0$, for every $\phi \in O(V, b)$, this means that $\phi(L)$ is an isotropic space of the same dimension as L, so it's maximal.

This group action is not free. For any $L \in Lag(V)$, an orthogonal transformation ϕ such that its restriction to L is the identity belongs to the stabilizer of L.

The group action is transitive by Witt's theorem, paragrapgh 1.3. For every $L, L' \in Lag(V)$, choose a basis for L, $v_1, \dots, v_n$, and a basis for L', $w_1, \dots, w_n$, we can construct an isometry $\sigma : L \to L'$, such that for every $i \in 1, \dots, n$, $\sigma(v_i) = w_i$ and we apply Witt's theorem for extending σ to an element of $O(V, b)$.

Proposition 1.5.6. *$Lag(V)$ has two connected components.*

Proof. To demonstrate this result, we consider the group action of $SO(V, b)$, the special orthogonal group consisting of the elements of $O(V, b)$ with determinant 1, on $Lag(V)$. Since $SO(V, b)$ is a connected subgroup of $O(V, b)$, this action divide $Lag(V)$ in two connected components corresponding to the two orbits defined by this group action. □

Remark 1.5.7. The previous proposition means that for every two elements $L, L' \in Lag(V)$ such that $\phi(L) = L'$, with $\phi \in O(V, b)$:

- L, L' are in the same connected component if and only if $det\phi = 1$;
- L, L' are in different connected components if and only if $det\phi = -1$.

To understand if two elements of the Lagrangian Grassmannian are in the same connected component or not, is of difficult computation using the method just explained. To simplify it, we give another characterization of them.

Proposition 1.5.8. *For any $L, L' \in Lag(V)$, they are in the same connected component if and only if $L \cap L'$ is of even codimension in L (and in L').*

Proof. We prove it by induction on $m = codim_L(L \cap L')$, showing that we can define $\phi \in O(V, b)$ such that $\phi(L) = L'$ as a composition of m reflections.

- for $m = 0$, $L = L'$, so it's clear they are in the same connected component;
- for $m = 1$, we complete an arbitrary basis of $L \cap L'$, $v_1, \dots, v_{n-1}$, to a basis of L and L', respectively with w and w'.
  The vector $w - w'$ is non isotropic, $b(w - w', w - w') = -2b(w, w') \neq 0$, since $w \in L \setminus (L \cap L')$ and $L = L^\perp$, by the definition of maximal totally isotropic subspace. We can use the reflection $\tau_{w-w'} : V \to V$, an orthogonal transformation of V such that $\tau_{w-w'}(w) = w'$, $\tau_{w-w'}(w') = w$ and its restriction to $(w - w')^\perp$ is the identity map. By direct computation, we can show that $\tau_{w-w'}(L) = L'$, and we know that $det(\tau_{w-w'}) = -1$, so they are in different connected components;

- induction step: for $1 < m \leq n$, we complete an arbitrary basis of $L \cap L'$, $v_1, \dots, v_{n-m}$, to a basis of L and L', respectively with $w_1, \dots, w_m$ and $w'_1, \dots, w'_m$ in order to find a basis respectively of L and L'. As in the previous case we use the reflection $\tau_{w_1 - w'_1}$ and we note that replacing L with $L_1 = \tau_{w_1 - w'_1}(L)$ we have $codim_k(L_1 \cap L') = m - 1$. We can apply the induction step and define $\phi \in O(V, b)$ as composition of m reflection.

Therefore,

$$L \text{ and } L' \text{ are in the same connected component} \Leftrightarrow det\phi = (-1)^m = 1 \Leftrightarrow m \text{ is even.}$$

□

Since there exists a bijection between the linear subspaces of dimension m in V and the projective subspaces of dimension $m-1$ in $\mathbb{P}(V)$, the previous proposition implies immediately:

Corollary 1.5.9. *If L and $L' \in Lag(V)$ we define $S = \mathbb{P}(L)$ and $S' = \mathbb{P}(L')$.*
If S and S' are in the same connected component:

- *if n is odd, they can be taken to intersect in one point;*
- *if n is even, they can be taken disjoint.*

If S and S' are in different connected components:

- *if n is even, they can be taken to intersect in one point;*
- *if n is odd, they can be taken disjoint.*

Proof. Follows directly by computation using the previous corollary.
We consider the case of two elements in the same connected component, it means that

$$dim_k S - dim_k(S \cap S') = n - 1 \text{ is even.}$$

If n is odd necessarily $dim_k(S \cap S')$ must be even and so it can be equal to 0, i.e. $S \cap S'$ is a point. If n is even necessarily $dim_k(S \cap S')$ must be odd and so it can be equal to -1 (we denote $dim_k \varnothing = -1$ in the projective space $\mathbb{P}^n$), i.e. S and S' are disjoint. The proof is similar in the other case.

□

Example 1.5.10. In this example, we study a hyperbolic quadric in $\mathbb{P}^1$. It suffices to study a particular one of these quadrics since they are all equivalent. We will focus our study on the quadric $Q(x_1, x_2) = x_1 x_2 = 0$. This is obviously non degenerate, just looking at its associated matrix,

$$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$$

we can see that its determinant it is different from zero. This quadric is composed of two points: $(1:0)$ and $(0:1)$. These are exactly the two connected components. The orthogonal transformation that maps every point in itself is the identity that has determinant 1, instead, to map one of these two point in the other one we can use the orthogonal transformation defined by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ that has determinant -1.

1.6 Geometry of hyperbolic Quadric in $\mathbb{P}^3$

In this section we want to study all the hyperbolic Quadrics in $\mathbb{P}^3 = \mathbb{P}(V)$, where V is a k vector space such that $dim_k V = 4$; we have already seen that all the hyperbolic quadrics are equivalent, so it suffices to study only one of these quadrics.

We will focus our study on the quadric:

$$\Sigma : Q(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3 = 0.$$

This quadric form is non degenerate. We can observe that the matrix associated to this quadric form is invertible, since has determinant equal to $\frac{1}{2^4}$.

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$$

Proposition 1.6.1. *The quadric Σ is smooth.*

Proof. We have to consider the system of equation composed by the equation $Q(x_1, x_2, x_3, x_4) = 0$ and $Q_{x_i}(x_1, x_2, x_3, x_4) = 0$ for every $i \in \{1, 2, 3, 4\}$, the formal partial derivative of the quadric Q.

$$\begin{cases} x_1x_4 - x_2x_3 = 0 \\ x_4 = 0 \\ -x_3 = 0 \\ -x_2 = 0 \\ x_1 = 0 \end{cases}$$

The only possible solution of this system is the point with all zero components, but this one is not beloging to $\mathbb{P}^3$.

□

To highlight all the properties of this specific quadric, we will introduce it as the image of the following map:

Definition 1.6.2. The *Segre map* $\sigma : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ is defined by

$$\sigma((x_1 : x_2), (y_1 : y_2)) = (x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) \in \mathbb{P}^3.$$

Proposition 1.6.3. *σ is well defined.*

Proof. We consider two points of $\mathbb{P}^1$, $(x_1 : x_2)$ and $(y_1 : y_2)$ and we want to show that the image of σ does not depend on the chosen representative of these two points. I have to verify that $\sigma((x_1 : x_2), (y_1 : y_2))$ and $\sigma((\lambda x_1 : \lambda x_2), (\mu y_1 : \mu y_2))$, for every $\lambda, \mu \in k \setminus \{0\}$, are equal.

$$\sigma((\lambda x_1 : \lambda x_2), (\mu y_1 : \mu y_2)) = (\lambda\mu x_1y_1 : \lambda\mu x_1y_2 : \lambda\mu x_2y_1 : \lambda\mu x_2y_2) = \lambda\mu\sigma((x_1 : x_2), (y_1 : y_2))$$

Thus σ is well defined. □

Proposition 1.6.4. *σ is injective.*

Proof. $\sigma((x_1 : x_2), (y_1 : y_2)) = (x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2) \in \mathbb{P}^3$ where at least one of the components of this point is not zero. We can suppose that $x_1y_2 \neq 0$. We suppose, by contradiction, that there exist $(t_1 : t_2) \neq (x_1 : x_2) \vee (z_1 : z_2) \neq (y_1 : y_2)$ such that

$$\sigma((t_1 : t_2), (z_1 : z_2)) = (t_1z_1 : t_1z_2 : t_2z_1 : t_2z_2) = (x_1y_1 : x_1y_2 : t_2y_1 : t_2y_2).$$

Thus, there exists $\lambda \in k$ such that $x_iy_j = \lambda t_iz_j$ for every $i, j \in \{1, 2\}$.

$$\frac{x_2}{x_1} = \frac{x_2y_2}{x_1y_2} = \frac{\lambda t_2z_2}{\lambda t_1z_2} = \frac{t_2}{t_1} \Rightarrow (t_1 : t_2) = (x_1 : x_2)$$

$$\frac{y_1}{y_2} = \frac{x_1y_1}{x_1y_2} = \frac{\lambda t_1z_1}{\lambda t_1z_2} = \frac{z_1}{z_2} \Rightarrow (z_1 : z_2) = (y_1 : y_2)$$

That is a contradiction. Thus σ is injective. □

Proposition 1.6.5. *The image of σ, $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$, is the quadric Σ defined by $Q(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3 = 0$.*

Proof. First of all we have to show that the points of $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset \mathbb{P}^3$ verify the equation of the quadric Σ, $x_1x_4 - x_2x_3 = 0$. Let $(z_1y_1 : z_1y_2 : z_2y_1 : z_2y_2)$ be a points belonging to $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$.

$$x_1x_4 - x_2x_3 = z_1y_1z_2y_2 - z_1y_2z_2y_1 = 0.$$

Thus, $\sigma(\mathbb{P}^1 \times \mathbb{P}^1) \subset \Sigma$.

Let $A(a_1 : a_2 : a_3 : a_4)$ be a point of the quadric Σ such that it satisfies $a_1a_4 = a_2a_3$. We can suppose $a_1 \neq 0$, so we can divide the components of A by a_1 and we got $\left(1 : \frac{a_2}{a_1} : \frac{a_3}{a_1} : \frac{a_4}{a_1}\right)$. Setting $b_i = \frac{a_2}{a_1}$ for $i \in \{2, 3, 4\}$, we want to show that $\sigma((1 : b_3), (1 : b_2))$ is exactly this point A. Using the property of the points of Σ and multiplying by a_1, we have the following result:

$$\sigma((1 : b_3), (1 : b_2)) = (1 : b_2 : b_3 : b_2b_3) = (1 : b_2 : b_3 : b_4) = (a_1 : a_2 : a_3 : a_4).$$

Therefore, $\Sigma \subset \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$. In particular, $\Sigma = \sigma(\mathbb{P}^1 \times \mathbb{P}^1)$. □

Definition 1.6.6. $\sigma(\mathbb{P}^1 \times \mathbb{P}^1)$ is a quadric of equation $\Sigma : x_1x_4 - x_2x_3 = 0$, this is called the *Segre quadric.*

In the previous chapter we proved that, for every hyperbolic quadratic form Q on a vector space V, there exists a maximal totally singular linear subspace L with respect to Q, whose dimension is $\frac{1}{2}dim_kV$ if the dimension of V is even as in the case we are studying. In particular, this means that everyone of these linear subspaces L is contained in the quadric. In the specific case we are examining, $dim_kV = 4$, so every L has $dim_kL = 2$. Every plane L in V corresponds to a projective line $\mathbb{P}(L)$, so the hyperbolic quadric in $\mathbb{P}^3$ contains some projective lines.

With the next proposition we will describe all of these lines.

Proposition 1.6.7. *The quadric Σ contains two families of projective lines: $\sigma(\{A\} \times \mathbb{P}^1)$ and $\sigma(\mathbb{P}^1 \times \{B\})$, for $A, B \in \mathbb{P}^1$. Two line of the same family are skew, while two lines from distinct families intersect each other in a point. This quadric does not contain other lines.*

Proof. We consider $A(a_1 : a_2)$, $\sigma(\{A\} \times \mathbb{P}^1)$, where $(y_1 : y_2) \in \mathbb{P}^1$ is a generic point, given by

$$\begin{cases} x_1 = a_1 y_1 \\ x_2 = a_1 y_2 \\ x_3 = a_2 y_1 \\ x_4 = a_2 y_2 \end{cases}$$

that is the line passing through the points $(a_1 : 0 : a_2 : 0)$ and $(0 : a_1 : 0 : a_2)$ parametrized using y_1, y_2 as parameters.

We consider $B(b_1 : b_2)$, $\sigma(\mathbb{P}^1 \times \{B\})$, where $(y_1 : y_2) \in \mathbb{P}^1$ is a generic point, given by

$$\begin{cases} x_1 = b_1 y_1 \\ x_2 = b_1 y_2 \\ x_3 = b_2 y_1 \\ x_4 = b_2 y_2 \end{cases}$$

that is the line passing through the points $(b_1 : b_2 : 0 : 0)$ and $(0 : 0 : b_1 : b_2)$ parametrized using y_1, y_2 as parameters.

The Segre map σ is a bijection between $\mathbb{P}^1 \times \mathbb{P}^1$ and the quadric Σ, thus the two lines represented by $\sigma(\{A\} \times \mathbb{P}^1)$ and $\sigma(\{\tilde{A}\} \times \mathbb{P}^1)$ if and only if $\{A\} \times \mathbb{P}^1 \cap \{\tilde{A}\} \times \mathbb{P}^1$ but $(A, Y) \neq (\tilde{A}, \tilde{Y})$, for every $Y, \tilde{Y} \in \mathbb{P}^1$. Similarly we get that $\mathbb{P}^1 \times \{B\} \cap \mathbb{P}^1 \times \{\tilde{B}\} =$ if $B \neq \tilde{B}$, then $\sigma(\mathbb{P}^1 \times \{B\}) \cap \sigma(\mathbb{P}^1 \times \{\tilde{B}\}) = \varnothing$. Instead $\sigma(\{A\} \times \mathbb{P}^1) \cap \sigma(\mathbb{P}^1 \times \{B\}) = \sigma(\{A\}, \{B\})$ that is a point. In particular, for every point of the quadric there are two line contained in Σ passing through it, one from each family of lines. We show now that there are no other lines contained in the quadric. Let $P = \sigma(\{A\}, \{B\})$ be a point, we consider $T_P Q$, the tangent plane of the quadric in P, we have that $T_P Q \cap \Sigma$ is a hypersurface of degree 2 in $T_P Q$ unless $T_P Q \subset \Sigma$, but this can not happen since Q is not degenerate and so it is irreducible. If $T_P Q \subset \Sigma$ then the quadric would be a irreducible component of Σ. Two lines $\sigma(\{A\} \times \mathbb{P}^1)$ and $\sigma(\mathbb{P}^1 \times \{B\})$ are both contained in $T_P Q$, since the multiplicity of the intersection with Σ in P is ∞. Then there can not be other lines contained in Σ and passing through P, since if there were, they should be contained also in $T_P Q$, but $T_P Q \cap \Sigma$ has degree 2 and can not contain more than two lines. □

The equation $x_1 x_4 - x_2 x_3 = 0$ can be written as the determinant of the following matrix

$$\begin{vmatrix} x_1 & x_2 \\ x_3 & x_4 \end{vmatrix}$$

this means that the rows and the columns are linearly dependent. If $Y(y_1 : y_2 : y_3 : y_4) \in \Sigma$ then exists $(\lambda_1, \lambda_2) \in k^2 \setminus \{(0,0)\}$ such that $\lambda_1(y_1, y_2) + \lambda_2(y_3, y_4) = 0$ and exist $(\mu_1, \mu_2) \in k^2 \setminus \{(0,0)\}$ such that $\mu_1(y_1, y_3) + \mu_2(y_2, y_4) = 0$.

We get two systems :

$$\begin{cases} \lambda_1 y_1 + \lambda_2 y_3 = 0 \\ \lambda_1 y_2 + \lambda_2 y_4 = 0 \end{cases}$$

and similarly we have the other system consider the rows of the previous matrix. These systems are defining two lines in Σ and each one is described as intersection of two plane in $\mathbb{P}^3$.

Now we check that these two families are the connected components of the Lagrangian grassmanian of the quadric, i.e. two element are in the same connected component if and only if its codimension is even.

Let L and $\tilde{L}$ be two lines contained in the Segre quadric. We want to consider the vector spaces V and W of dimension 2 such that $L = \mathbb{P}(V)$ and $\tilde{L} = \mathbb{P}(W)$. We consider the only case for which $codim_k(L \cap \tilde{L})$ is odd, therefore 1, i.e. the intersection is a line that correspond to a point in the projective space. We have already proved that two lines from different families intersect in one point. Instead, if $codim_k(L \cap \tilde{L})$ is even, we have two possibilities, 0 or 2, both trivial in this situation because the first case corresponds to two lines that coincide and the second to two lines that have no intersection. We have already seen that both of these cases can happen if and only if the two lines are in the same family.

Therefore, we have :

Proposition 1.6.8. *Every quadric defined by a hyperbolic quadric form in $\mathbb{P}^3$ contains two families of lines such that for every point there are exactly two line passing through this point and these two families are the two connected components of the Lagrangian grasmannian of the quadric.*

1.7 Geometry of hyperbolic Quadric forms in $\mathbb{P}^5$

In this section we will study the quadrics defined by a hyperbolic quadratic form $Q = 0$ in $\mathbb{P}^5 = \mathbb{P}(V)$, where V is a vector space with $dim_k = 6$.

We will focus our study on the quadric:

$$\Xi : Q(x_1, x_2, x_3, x_4, x_5, x_6) = x_1x_6 - x_2x_5 + x_3x_4 = 0$$

All the non degnerate quadrics are equivalent, so it suffices to study only one of this quadric. This quadric form is non degenerate. We can observe that the matrix associated to this quadric form is invertible, since has determinant equal to $-\frac{1}{2^6}$.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 1.7.1. *The quadric Ξ is smooth.*

Proof. We have to consider the system of equation composed by the equation $Q(x_1, x_2, x_3, x_4) = 0$ and $Q_{x_i}(x_1, x_2, x_3, x_4, x_5, x_6) = 0$ for every $i \in \{1, 2, 3, 4, 5, 6\}$, the formal partial derivative of the quadric Q.

$$\begin{cases} x_1x_6 - x_2x_5 + x_3x_4 = 0 \\ x_6 = 0 \\ -x_5 = 0 \\ x_4 = 0 \\ x_3 = 0 \\ -x_2 = 0 \\ x_1 = 0 \end{cases}$$

The only possible solution of this system is the point with all zero components, but this does not belong to $\mathbb{P}^5$.

□

As we did it in the previous section, we will define the quadric Ξ as the image of the map.

Definition 1.7.2. We call Plücker map the map $p : \mathbb{G}(1,3) \to \mathbb{P}^5$, such that for every $r \in \mathbb{G}(1,3)$, where $r = \overline{AB}$ is a line passing trough two distinct points $A(a_1 : a_2 : a_3 : a_4)$ and $B(b_1 : b_2 : b_3 : b_4) \in r$, we consider the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{pmatrix}$$

that has rank 2 since at least one of the minor 2×2 is not zero.
We define $p(r) = (p_{12} : p_{13} : p_{14} : p_{23} : p_{24} : p_{34})$, where $p_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} = a_ib_j - a_jb_i$.

Proposition 1.7.3. *p is well defined.*

Proof. If $r = \overline{AB} = \overline{CD}$ with $C(c_1 : c_2 : c_3 : c_4)$ and $D(d_1 : d_2 : d_3 : d_4) \in r$, then the minors 2×2 of

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

Thus, $r = \mathbb{P}(W)$ $A = \mathbb{P}(< v >)$, $B = \mathbb{P}(< w >)$, $C = \mathbb{P}(< v' >)$ and $D = \mathbb{P}(< w' >)$, where W is a linear subspace of dimension 2 in V, $v, w, v', w' \in V$ and $W =< v, w >=< v', w' >$.
Therefore, there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 \in k$ $v' = \lambda_1 v + \mu_1 w$ and $w' = \lambda_2 v + \mu_2 w$, then

$$(c_1 : c_2 : c_3 : c_4) = \lambda_1(a_1 : a_2 : a_3 : a_4) + \mu_1(b_1 : b_2 : b_3 : b_4)$$
$$(d_1 : d_2 : d_3 : d_4) = \lambda_2(a_1 : a_2 : a_3 : a_4) + \mu_2(b_1 : b_2 : b_3 : b_4).$$

$$\begin{aligned} \begin{vmatrix} c_i & c_j \\ d_i & d_j \end{vmatrix} &= c_id_j - c_jd_i = \\ &= (\lambda_1 a_i + \mu_1 b_i)(\lambda_1 a_i + \mu_1 b_i) - (\lambda_1 a_i + \mu_1 b_i)(\lambda_1 a_i + \mu_1 b_i) = \\ &= a_ib_j(\lambda_1\mu_2 - \mu_1\lambda_2) + b_ia_j(\mu_1\lambda_2 - \lambda_1\mu_2) = \\ &= \begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix} (a_ib_j - a_jb_i) \end{aligned}$$

Then the proportionality constant is $\begin{vmatrix} \lambda_1 & \mu_1 \\ \lambda_2 & \mu_2 \end{vmatrix}$, thus p is well defined.

□

Proposition 1.7.4. *p is injective.*

Proof. We consider $r = \overline{AB}$ and $s = \overline{CD}$, we suppose $p(r) = p(s)$, and we want to show that $r = s$. It suffices to show that $C, D \in r$. To demonstrate that $C \in r$, we consider the matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{pmatrix}$$

This matrix has rank less than 3, this means that the vector $(c_1 : c_2 : c_3 : c_4)$ is a linear combination of the vectors $(a_1 : a_2 : a_3 : a_4)$ and $(b_1 : b_2 : b_3 : b_4)$. This implies that in the projective space $C \in r$. By hypothesis $p(r) = p(s)$, therefore, there exists $\lambda \neq 0, \lambda \in k$ such that $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} = \lambda \begin{vmatrix} c_i & c_j \\ d_i & d_j \end{vmatrix}$ holds for every $i < j$. Taking a minor 3×3, with $1 \leq i < j < k \leq 4$, and we consider

$$\begin{vmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{vmatrix} = c_i(a_j b_k - a_k b_j) - c_j(a_i b_k - a_k b_i) + c_k(a_i b_j - a_j b_i) =$$

$$= c_i \lambda(c_j d_k - c_k d_j) - c_j \lambda(c_i d_k - c_k d_i) + c_k \lambda(c_i d_j - c_j d_i) =$$

$$= \lambda \begin{vmatrix} c_i & c_j & c_k \\ d_i & d_j & d_k \\ c_i & c_j & c_k \end{vmatrix} = 0$$

This implies $C \in r$. Similarly, we can prove $D \in r$, so $r = s$. □

Proposition 1.7.5. *Ξ is the image of p, $\Xi = p(\mathbb{G}(1,3))$.*

Proof. We consider the following matrix and developing it respect the first two rows, we get

$$0 = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix} = 2p_{12}p_{34} - 2p_{13}p_{24} + 2p_{14}p_{23}$$

Thus, we prove that the image of p satisfies the equation of the quadric Ξ, $p(\mathbb{G}(1,3)) \subset \Xi$. Conversely, we consider a point $P(p_1 : p_2 : p_3 : p_4 : p_5 : p_6) \in \mathbb{P}^5$ such that $p_1 p_6 - p_2 p_5 + p_3 p_4 = 0$. Looking at the skew symmetric matrix,

$$A = \begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ -p_1 & 0 & p_4 & p_5 \\ -p_2 & -p_4 & 0 & p_6 \\ -p_3 & -p_5 & -p_6 & 0 \end{pmatrix}$$

holds that $detA = (p_1 p_6 - p_2 p_5 + p_3 p_4)^2 = 0$, then rank A is lesser than 4, since every skewsymmetric matrix has even rank, thus rank $A = 2$. This means that there is at least one minor 2×2 that is

not zero. Considering two linear independent rows, they define a line. For example, if we consider the first two rows $\begin{pmatrix} 0 & p_1 & p_2 & p_3 \\ -p_1 & 0 & p_4 & p_5 \end{pmatrix}$, then the minors are identified by
$(p_1^2 : p_1p_2 : p_1p_3 : p_1p_4 : p_1p_5 : p_2p_5 - p_3p_4)$, where $p_2p_5 - p_3p_4 = p_1p_6$.
Then $\Xi \subset p(\mathbb{G}(1,3))$, in particular $\Xi = p(\mathbb{G}(1,3))$. □

We note:

Definition 1.7.6. $p(\mathbb{G}(1,3))$ is a quadric of equation $x_1x_6 - x_2x_5 + x_3x_4 = 0$, that is called *Klein quadric.*

We show that the Klein quadric contains two families of planes of $\mathbb{P}^5$, corresponding to:

1. $\alpha \subset \mathbb{P}^5$ plane, corresponding via p to the plane of $\mathbb{P}^3$; if $\pi \subset \mathbb{P}^3$ is a plane, all the lines contained in π correspond to the points of a plane of Ξ, these planes are called α-planes.

2. $\beta \subset \mathbb{P}^5$ plane, corresponding via p to the family of all the lines of $\mathbb{P}^3$ passing through one point; if $P \in \mathbb{P}^3$ is a point, all the lines passing through P correspond to the points of a plane of Ξ, these planes are called β-planes. To simplify the notation, we will call the family of all the lines of $\mathbb{P}^3$ passing through one point P the star of lines with center P.

These two families of planes are parametrized by $\mathbb{P}^3$.

We will study, now, what corresponds in the quadric Ξ to the condition for two lines of $\mathbb{P}^3$ to intersect each other. We consider $r = \overline{AB}$ and $s = \overline{CD}$, lines of $\mathbb{P}^3$, they intersect each other if and only if they generate a plane contained in $\mathbb{P}^3$. We set $v = (a_1 : a_2 : a_3 : a_4)$, $w = (b_1 : b_2 : b_3 : b_4)$, $u = (c_1 : c_2 : c_3 : c_4)$ and $x = (d_1 : d_2 : d_3 : d_4)$, these are the vectors such that $A = \mathbb{P}(< v >)$, $B = \mathbb{P}(< w >)$, $C = \mathbb{P}(< u >)$ and $D = \mathbb{P}(< x >)$. Then r and s intersect each other if and only if v, w, u, x are linearly dependent if and only if

$$detM = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} = 0$$

Developing the determinant with respect to the first two rows, we get
$detM = r_{12}s_{34} - r_{13}s_{24} + r_{14}s_{23} + s_{12}r_{34} - s_{13}r_{24} + s_{14}r_{23} = 0.$

Recall 1.7.7. Given a quadric Q in $\mathbb{P}^n$ and a point $P(p_1 : p_2 : \cdots : p_{n+1}) \in \mathbb{P}^n$ we define the hyperplane of P with respect to Q by the equation:

$$\Delta_P Q(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} x_i Q_{x_i}(p_1, \ldots, p_{n+1}) =$$
$$= x_1 Q_{x_1}(p_1, \ldots, p_{n+1}) + \cdots + x_{n+1} Q_{x_{n+1}}(p_1, \ldots, p_{n+1}) = 0,$$

where Q_{x_i} is the partial derivative of Q with respect to the $i-th$ variable, x_i, the polar hyperplane of Q at P. By the reciprocity law we have the following property: for every A and $B \in \mathbb{P}^n$ and a quadric Q,

$$A \in (\Delta_B Q = 0) \Leftrightarrow B \in (\Delta_A Q = 0)$$

Moreover if $A \in Q$ and is smooth for Q, the polar hyperplane of A respect Q coincides the tangent hyperplane to Q in A, $T_A Q = \Delta_A Q$.

The last condition about the lines r, s can be reformulated stating that R and S the two points in $\mathbb{P}^5$, corresponding respectively to the previous lines, are conjugated with respect to the Klein quadric, this means that $R \in (\Delta_S Q = 0)$ and $S \in (\Delta_R Q = 0)$. Then $\Xi \cap (\Delta_S Q = 0) = \Xi \cap T_S Q$ represents all the line that have an intersection with the line s. If r and s intersect each other, the line passing through R and S in $\mathbb{P}^5$ is tangent to the Klein quadric in R and in S, thus this line is entirely contained in the quadric.

Proposition 1.7.8. *All the lines contained in the Klein quadric correspond, via p, the Plücker map, to the planar family of lines of $\mathbb{P}^3$ intersecting in a common point.*

Proof. We consider a planar family of lines of $\mathbb{P}^3$ intersecting in a common point, the center, $A(a_1 : a_2 : a_3 : a_4) \in \mathbb{P}^3$ and we prove that this is mapped by the Plücker map onto a line of $\mathbb{P}^5$. Let r be a line of this family and let π be a plane containing A. We consider $B, C \in \pi$ and $B, C \notin r$, such that A, B, C are not aligned. The points $\tilde{B}, \tilde{C}$ in Ξ corresponding respectively to the lines passing through A, B and A, C that are two lines of the family. If $B(b_1 : b_2 : b_3 : b_4)$ and $C(c_1 : c_2 : c_3 : c_4)$, then $\tilde{B}(b_{ij})_{1 \le i<j \le 4}$ and $\tilde{C}(c_{ij})_{1 \le i<j \le 4}$ are the corresponding points in $\mathbb{P}^5$, where $b_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ and $c_{ij} = \begin{vmatrix} a_i & a_j \\ c_i & c_j \end{vmatrix}$. The line $t \subset \pi$ passing through B and C intersect each other in A, in a point with coordinate $(\lambda b_i + \mu c_i)_{1 \le i \le 4}$ with $(\lambda, \mu) \in k/(0,0)$. Consider the minor 2×2 of the matrix having as rows the vectors $(a_1 : a_2 : a_3 : a_4)$ and $(\lambda b_i + \mu c_i)_{1 \le i \le 4}$,

$$\begin{vmatrix} a_i & a_j \\ \lambda b_i + \mu c_i & \lambda b_j + \mu c_j \end{vmatrix} = a_i(\lambda b_j + \mu c_j) - a_j(\lambda b_i + \mu c_i) = \lambda(a_i b_j - a_j b_i) + \mu(a_i c_j - a_j c_i) = \lambda b_{ij} + \mu c_{ij}$$

so the lines of the family define all the points of the line passing through $\tilde{B}$ and $\tilde{C}$ in $\mathbb{P}^5$. This line is contained in the Klein quadric since the corresponding lines to $\tilde{B}$ and $\tilde{C}$ in $\mathbb{P}^3$ intersect each other in A.

Vice versa, we consider $R, S \in \Xi$ two points such that the line t passing through R, S is contained in Ξ. R corresponds to the line $r \subset \mathbb{P}^3$, $r = \mathbb{P}(< v, w >)$, with components $v_i w_j - v_j w_i$. While S corresponds to the line $s \subset \mathbb{P}^3$, $s = \mathbb{P}(< v\prime, w\prime >)$, with components $v\prime_i w\prime_j - v\prime_j w\prime_i$. By hypothesis for every $\lambda, \mu \in k^2 \setminus (0,0)$ the point in $\mathbb{P}^5$ of components $\lambda(v_i w_j - v_j w_i) + \mu(v\prime_i w\prime_j - v\prime_j w\prime_i) \in \Xi$ and so it satisfies the equation of the Klein quadric. From the computation we get

$$\lambda^2(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23}) + \mu^2(s_{12}s_{34} - s_{13}s_{24} + s_{14}s_{23}) +$$
$$\lambda\mu(r_{12}s_{34} - r_{13}s_{24} + r_{14}s_{23} + s_{12}r_{34} - s_{13}r_{24} + s_{14}r_{23}) = 0.$$

Since R, $S \in \Xi$, $r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23} = 0$ and $s_{12}s_{34} - s_{13}s_{24} + s_{14}s_{23} = 0$. The remaining part of the equation represents that r and s are two lines intersecting each other, and using the first part of this proof, we know that t represents all the lines of the planar family of lines of $\mathbb{P}^3$ intersecting in a common point generated by r and S.

□

Now we have all the necessary elements to show which are the two family of plane contained in the Klein quadric.

Proposition 1.7.9. *The Klein quadric, Ξ, contains the following planes:*

1. *α-planes: given a plane of $\mathbb{P}^3$, π, the lines contained in π correspond to all the points of a plane contained in Ξ;*

2. *β-planes: given a point A in $\mathbb{P}^3$, all the lines of the star of lines with center A correspond the points of a plane contained in Ξ;*

Every plane in Ξ can only be one of those in the points 1 or 2.

Proof. 1. We consider a line $a \subset \pi \subset \mathbb{P}^3$, it corresponds to a point $A \in \Xi \subset \mathbb{P}^5$. We consider the two stars of lines with center in $R, S \in a$. The points R, S correspond, by the Plücker map, to the lines r and $s \subset \Xi$. Since these two families contain both the line a, $A \in r$ and $A \in s$. The lines r and s generate a plane $\Pi \subset \mathbb{P}^5$. Now we want to show that $\Pi \subset \Xi$ and that all the points of this plane correspond to the lines contained in π. We consider a line $u \subset \pi$, and we show that the corresponding point $U \in \Pi$. Let T be a point of u, then I have a family of intersecting lines in π with center T that contains u, moreover b, the line passing through T and R, and c, the line passing through T and s, belong to this family. Let t be the line in Ξ corresponding to this family with center T, then this is the line that contains the points corresponding to the lines b and c, but these two belong to r and s, respectively. Therefore t intersect both r and s so $t \subset \Pi$.

Vice versa, we consider a point U of Π and a line $t \subset \Pi$ passing through U, and intersecting r and s in two points which represent two lines, one passing through R and the other one through S, intersecting in T since $t \subset \Xi$. The line u corresponding to U belongs to the family of intersecting lines generated by b and c, for that is entirely contained in π. In this way we are able to define a family of planes contained in the Klein quadric, one for every plane of $\mathbb{P}^3$, so this family is parametrized by $(\mathbb{P}^3)^*$.

2. The proof is similar to the previous one. We consider all the lines passing through A and we take three not coplanar lines, then we got three point in $\mathbb{P}^5$ that determine univocally a plane. Then, one can show that the points of this plane are in bijection with the lines passing through A. This family is parametrized by $\mathbb{P}^3$.

To prove the last claim, let Π be a generic plane contained in Ξ, we consider three not aligned points of Π.These three points correspond to three lines r, s and t of $\mathbb{P}^3$, intersecting each other. The line t can intersect r and s in only one point that is $r \cap s$, this correspond to the star of lines with center $r \cap s$, or in two distinct points, so t is contained in the plane $\pi = < r, s >$. □

We show now that these two family of planes are the two connected component of the Lagrangian grassmanian, i.e. two element are in the same connected component if and only if its codimension is even.

Let Π and $\tilde{\Pi}$ be two planes contained in the Klein quadric, we want to consider the vector spaces V and W of dimension 3 such that $\Pi = \mathbb{P}(V)$ and $\tilde{\Pi} = \mathbb{P}(W)$. All the possibilities for $codim_k(V \cap W)$ are:

- $codim_k(V \cap W) = 0$, i.e. V and V coincide, this means that the two planes coincide.
- $codim_k(V \cap W) = 1$, i.e. $dim_k(V \cap W) = 2$, this means that the two planes have only a line of intersection, and it can happen only when we have one plane from each family, the center P of the star of lines corresponding to the β-plane belongs to the plane corresponding to the α-plane. The intersection between the star and the plane is a planar family of lines intersecting with center P, which correspond to a line contained in the Klein quadric.
- $codim_k(V \cap W) = 2$, i.e. $dim_k(V \cap W) = 1$, this means that the two planes have only one point of intersection, and it can happen only when we have two planes of the same family. If

we have two α-planes this point correspond to the intersection line of the two planes in $\mathbb{P}^3$ corresponding respectively to the two α-planes. If we have two β-planes, this point correspond to the line passing through the two center of the star of lines corresponding respectively to the two β-planes.

- $codim_k(V \cap W) = 3$, i.e. $dim_k(V \cap W) = 0$, this means that the two planes have no intersection. it can happen only when we have one plane from each family and the center of the star of lines corresponding to the β-plane, doesn't belong to plane π, corresponding to the α-plane.

Chapter 2

Clifford Algebras

With every quadratic space (V, Q) over a field k one can associate in a functorial way a k-algebra $C(V, Q)$ called the Clifford algebra of (V, Q), a generalization of exterior algebras, defined in the presence of a symmetric bilinear form. This construction is of fundamental importance in the algebraic theory of quadratic forms.

2.1 Definition and the universal property of the Clifford Algebras

Let k be a field of characteristic different from 2, V be a vector space over k with $dim_k V = n$ and Q be a quadratic form on V with associated non degenerate symmetric bilinear form b.

We recall first the notion of *tensor algebra*. We define

$$T^r(V) = \underbrace{V \otimes \cdots \otimes V}_{n\text{times}} \qquad \text{for } r > 0 \qquad T^0(V) = k$$

$$T^r(V) = 0 \qquad \text{for } r < 0 \qquad T(V) = \coprod_{r \in \mathbb{Z}} T^r(V)$$

Thus $T(V)$ is a $\mathbb{Z}$-graded k-module. The product on $T(V)$ is defined by

$$(x_1 \otimes \cdots \otimes x_m)(x_{m+1} \otimes \cdots \otimes x_r) = x_1 \otimes \cdots \otimes x_m \otimes x_{m+1} \otimes \cdots \otimes x_r$$

Thus $T(V)$ is a $\mathbb{Z}$-graded k-algebra. The inclusion map

$$i : V = T^1(V) \hookrightarrow T(V)$$

has the following universal mapping property: If W is a k vector space and $f : V \to W$ a linear map, then there is a unique algebra homomorphism $g : T(V) \to W$ such that $f = gi$. Of course, the map g is defined

$$g(x_1 \otimes \cdots \otimes x_m) = f(x_1) \ldots f(x_m).$$

We can define now the Clifford algebras of (V, Q).

Definition 2.1.1. Let (V, Q) be a quadratic space over the field k. Let $I(Q)$ be the two-sided ideal of $T(V)$ generated by the set $\{x \otimes x - Q(x) | x \in V\}$. The quotient algebra

$$C(V, Q) = T(V)/I(Q)$$

is called the *Clifford algebra of (V, Q)*.

Proposition 2.1.2. *Let $v, w \in V$, then*

$$v \otimes w + w \otimes v - 2b(v, w) \in I(Q)$$

Proof. Since $v, w, v + w \in V$, we have that

$$v \otimes v - Q(v), \quad w \otimes w - Q(w), \quad (v + w) \otimes (v + w) - Q(v + w)$$

are generators of $I(Q)$, so are elements of $I(Q)$.
Thus, being $I(Q)$ an ideal, we get

$$\begin{aligned}
&(v + w) \otimes (v + w) - Q(v + w) - [v \otimes v - Q(v) + w \otimes w - Q(w)] = \\
&= v \otimes w + w \otimes v - [Q(v + w) - Q(v) - Q(w)] = \\
&= v \otimes w + w \otimes v - 2b(v, w) \in I(Q).
\end{aligned}$$

□

The canonical inclusions $k \hookrightarrow T(V)$ and $V \hookrightarrow T(V)$ descend to two canonical inclusions, by composing them with the canonical projection $\pi : T(V) \to C(V, Q) = T(V)/I(Q)$:

$$\eta : k \to Cl(V, Q) \qquad \rho : V \hookrightarrow Cl(V, Q).$$

It means that k and V are subspace of $Cl(V, Q)$. To simplify the notation we will denote $\rho(x)$ as $\overline{x} \in Cl(V, Q)$, for every $x \in V$.
We may thus characterize $Cl(V, Q)$ as the unitary associative algebra, with generators $v \in V$ and relations

$$\overline{v}^2 = Q(v) \qquad \text{and} \qquad \overline{vw} + \overline{wv} = 2b(v, w) \ \text{ for } v, w \in V$$

From what we know from the first chapter about the quadratic forms, follows directly:

Remark 2.1.3.

- If $x \perp y$ in V, then $\overline{xy} = -\overline{yx}$ in $Cl(V, Q)$. This follows from

$$\overline{x}^2 + \overline{y}^2 = Q(x) + Q(y) = Q(x + y) = (\overline{x} + \overline{y})^2 = \overline{x}^2 + \overline{xy} + \overline{yx} + \overline{y}^2.$$

- If $x \in V$ is non isotropic, $\overline{x}$ is invertible in $Cl(V, Q)$. The inverse is $\overline{x}Q(x)^{-1}$.
- If $x_1, \ldots, x_n$ is an orthogonal basis of (V, Q) the elements $\overline{x_1}, \ldots, \overline{x_n}$ generate $Cl(V, Q)$ as k-algebra. Since $\overline{x_i}^2 = q(x_i)$ and $\overline{x_i x_j} = -\overline{x_j x_i}$ for $i \neq j$, $Cl(V, Q)$ is generated as a k-vector space by

$$\overline{x_1}^{\varepsilon}_1 \ldots \overline{x_n}^{\varepsilon}_n, \qquad \varepsilon_i \in \{0, 1\}.$$

  In particular, $dim_k Cl(V, Q) \leq 2^n$.

- If 2 is invertible ($char(k) \neq 2$) and (V, Q) is a quadratic space of even dimension, $dim_k V = 2n$, we choose an orthogonal basis $e_1, \dots, e_{2n}$ of V. We set

$$z = e_1 \dots e_{2n} \in Cl(V, Q),$$

as $e_i e_j = -e_j e_i$ for $i \neq j$, we have $z e_i = -e_i z$, it means that $z \in Z(Cl^+(V, Q))$, the center of the even algebra of Clifford. Thus,

$$z^2 = (e_1 \dots e_{2n})^2 = (-1)^n Q(e_1) \dots Q(e_{2n}) = (-1)^n det(V, Q),$$

since we have to apply $2n^2 - n$ order change to z^2 in order to get this final result and $2n^2 - n \equiv_2 n$ they are equivalent modulo 2. The element z^2 corresponds to the discriminant of the quadratic space (V, Q).

Example 2.1.4. If the quadratic form Q is the zero form, the Clifford algebra is the exterior algebra $\Lambda(V) = T(V)/I$ where I is the ideal generated by all the squares $x \otimes x$, for every $x \in V$. For $x, y \in V$ the following equations hold in the exterior algebra

$$(\overline{x} + \overline{y})^2 = \overline{x}^2 + \overline{xy} + \overline{yx} + \overline{y}^2 = \overline{x}y + \overline{yx} = 0$$

so that $\overline{xy} = -\overline{yx}$.

From this example, we understand that the exterior algebra is only a particular case of Clifford algebra.

The tensor algebra $T(V) = \coprod_{r \in \mathbb{Z}} T^r(V)$ is naturally a $\mathbb{Z}$-graded algebra, then it inherit a $\mathbb{Z}_2$-graduation dividing the tensor algebra in an even and an odd part:

$$T^+(V) = \coprod_{r \in \mathbb{Z} \text{ even}} T^r(V) \qquad \text{and} \qquad T^-(V) = \coprod_{r \in \mathbb{Z} \text{ odd}} T^r(V).$$

Also the Clifford algebras inherit a $\mathbb{Z}_2$-graduation:

$$Cl^+(V, Q) = T^+(V) / \left(I(Q) \cap T^+(V) \right) \qquad \text{and} \qquad Cl^-(V, Q) = T^-(V) / \left(I(Q) \cap T^-(V) \right),$$

$$Cl(V, Q) = Cl^+(V, Q) \oplus Cl^-(V, Q).$$

The algebra $Cl^+(V, Q)$ is called the *even Clifford algebra*; the subspace $Cl^-(V, Q)$ is called the *odd part of the Clifford algebra.* The following relations hold:

$$Cl^+(V, Q) \cdot Cl^+(V, Q) \subset Cl^+(V, Q) \qquad Cl^-(V, Q) \cdot Cl^-(V, Q) \subset Cl^+(V, Q)$$

$$Cl^+(V, Q) \cdot Cl^-(V, Q) \subset Cl^-(V, Q) \qquad Cl^-(V, Q) \cdot Cl^+(V, Q) \subset Cl^-(V, Q).$$

The Clifford algebra is characterized by the following universal property:

Proposition 2.1.5. *Let A be a filtered super algebra, and $f : V \to A$ a linear map satisfying $f(v)^2 = Q(v)$, for every $v \in V$. Then f extends uniquely to a morphism of filtered super algebras $Cl(V, Q) \to A$.*

Proof. By the universal property of the tensor algebra, f extends to an algebra homomorphism $f_{T(V)} : T(V) \to A$. The property $f(v)^2 = Q(v)$ shows that f vanishes on the ideal $I(Q)$, and hence descends to the Clifford algebra. Uniqueness is clear, since the Clifford algebra is generated by elements of V. □

Suppose b_1, b_2 are symmetric bilinear forms on V_1, V_2 and $\phi : V_1 \to V_2$ is a linear map such that:

$$b_2(\phi(v), \phi(w)) = b_1(v, w), \qquad v, w \in V_1.$$

Viewing ϕ as a map into $Cl(V_2, Q_2)$, the universal property provides a unique extension to a morphism of filtered super algebras $Cl(\phi) : Cl(V_1, Q_1) \to Cl(V_2, Q_2)$.
Clearly, $Cl(\phi_1 \circ \phi_2) = Cl(\phi_1) \circ Cl(\phi_2)$, $Cl(id_V) = id_{Cl(V)}$. The functoriality gives in particular a group homomorphism

$$O(V, Q) \to Aut(Cl(V, Q)), \ \ g \to Cl(g)$$

into algebra automorphisms of $Cl(V, Q)$ (preserving $\mathbb{Z}_2$-grading and filtration). We will usually just write g in place of $Cl(g)$.
For example, the involution $v \mapsto -v$ lies in $O(V, Q)$; hence it defines an involutive algebra automorphism ζ of $Cl(V, Q)$ called the *parity automorphism*. The $+1$ and -1 eigenspaces are the even and odd part of the Clifford algebra, respectively.

2.2 Structure of the Clifford Algebras

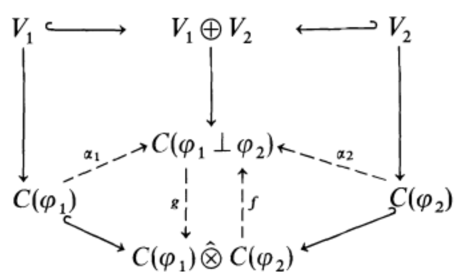
We now introduce a fundamental theorem from which the structure theorems for Clifford algebras will later be derived.

Theorem 2.2.1. *For any two quadratic spaces $\varphi_1 = (V_l, Q_l)$ and $\varphi_2 = (V_2, Q_2)$ there exists a canonical isomorphism*

$$Cl(\varphi_1 \oplus \varphi_2) \cong Cl(\varphi_1) \otimes Cl(\varphi_2)$$

.

Proof. We consider the following diagram of canonical maps; the dotted ones are to be constructed:



The universal property of $Cl(\phi_i)$ gives the algebra homomorphism α_i. It is obvious that the α_i are graded homomorphisms. We claim that the images of α_1 and α_2 anticommute. This has to be checked only for the generators $\overline{x_i}$, for $x_i \in V$, and follows:

$$\alpha_1(\overline{x_1})\alpha_2(\overline{x_2}) + \alpha_2(\overline{x_2})\alpha_1(\overline{x_1}) = \overline{x_1 x_2} + \overline{x_2 x_1} = 0.$$

By the universal property of the graded tensor product we get the graded algebra homomorphism f. This is surjective since the image contains all generators $\overline{x_1} + \overline{x_2}$ of $Cl(\varphi_1 \oplus \varphi_2)$. To construct the inverse, we consider the homomorphism

$$j : V_1 \oplus V_2 \to Cl(\varphi_1) \otimes Cl(\varphi_2), \qquad x_1 + x_2 \to C.$$

This is compatible with the quadratic form:

$$\begin{aligned} j(x_1+x_2)^2 &= (\overline{x_1} \otimes 1 + 1 \otimes \overline{x_2})^2 = \overline{x_1}^2 \otimes 1 + 1 \otimes \overline{x_2}^2 \\ &= Q_1(x_1) + Q_2(x_2) = Q(x_1+x_2). \end{aligned}$$

By the universal property of $Cl(\varphi_1 \oplus \varphi_2)$ we get the algebra homomorphism

$$g : Cl(\varphi_1 \oplus \varphi_2) \to Cl(\varphi_1) \otimes Cl(\varphi_2).$$

Then $gf = id$ since this can be checked trivially for the generators $\overline{x_1} \otimes 1$ and $1 \otimes \overline{x_2}$). □

Using the existence of orthogonal bases and the previous theorem it is easy to determine the structure of Clifford algebras. We begin with the case of 1-dimensional forms.

Lemma 2.2.2. *For the 1-dimensional space $< v >= kv$ with the quadratic form Q such that $Q(v) = \alpha$:*

$$Cl(< v >, Q) \cong k[X]/(X^2 - \alpha) \cong k \oplus kx,$$

with $x^2 = \alpha$.

Proof. This follows at once from the definition. □

Corollary 2.2.3. *If $\varphi = (V, Q)$ is an n-dimensional quadratic space (perhaps singular) and $x_1, \dots, x_n$ an orthogonal basis, then*

1. *$dim_k Cl(\varphi) = 2^n$.*
2. *The elements $\overline{x_1}^{\varepsilon_1}, \dots, \overline{x_n}^{\varepsilon_n}$, $\varepsilon_i = 0, 1$, form a k-basis of $Cl(\varphi)$.*
3. *The canonical map $\rho : V \to Cl(\varphi)$ is injective.*

Proof. *1.* follows from the previous theorem and lemma. The remark 2.1.3. implies *2.*, which in turn implies *3.*. □

From now on we identify V with its image in $C(V, Q)$. For simplicity we denote the 2-dimensional Clifford algebra $Cl(< v >, Q)$ of 2.2.2 by (α).

Lemma 2.2.4. *For $\alpha, \beta \neq 0$ there is a graded isomorphism $(\alpha) \otimes (\beta) = ((\alpha, \beta))$.*

Proof. Choose $e_l \in (\alpha)_1$, $e_2 \in (\beta)_1$ with $e_1^2 = \alpha$, $e_2^2 = \beta$. The basis $1 \otimes 1$, $e_1 \otimes 1$, $1 \otimes e_2$, $e_1 \otimes e_2$ satisfies the defining relations of the quaternion algebra (α, β). Obviously the gradings correspond also. □

Lemma 2.2.5. *For $\alpha, \beta, \gamma \neq 0$, there is a graded isomorphism*

$$(\alpha) \otimes (\beta) \otimes (\gamma) \cong (-\alpha\gamma, -\beta\gamma) \otimes (-\alpha\beta\gamma).$$

Proof. Choose e_l, e_2 as in the previous lemma and $e_3 \in (\gamma)_l$ with $e_3^2 = \gamma$. Let $A = (\alpha) \otimes (\beta) \otimes (\gamma)$ and abbreviate the tensor product in A by juxtaposition. A basis for A_0 is $1, e_l e_2, e_2 e_3, -\gamma e_1 e_3$. These elements satisfy the defining relations of the quaternion algebra $(-\alpha\gamma, -\beta\gamma)$. The element $e_l, e_2, e_3 \in A_l$ commutes elementwise with A_0 and $(e_l e_2 e_3)^2 = -\alpha\beta\gamma$. Using the universal property of the graded tensor product and a dimension argument we get

$$A \cong A_0 \otimes (-\alpha\beta\gamma) \cong (-\alpha\gamma, -\beta\gamma) \otimes (-\alpha\beta\gamma).$$

□

We have already seen in the corollary 1.2.5 that every symmetric matrix is congruent to a diagonal matrix,

$$\begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \alpha_n \end{pmatrix}$$

with $\alpha_1, \dots, \alpha_n \in k$, it means that every bilinear space is isometric to a space, considering it with respect to an orthogonal basis $v_1, \dots, v_n$, in which is defined a quadratic form such that $Q(v_i) = \alpha_i$, for every $i \in \{1, \dots, n\}$. Thus we can always write an arbitrary space in this form, denoting this particular form with $< \alpha_1, \dots, \alpha_n >$, which is often very convenient for computation.

Using these preparations we can prove the main structure theorem:

Theorem 2.2.6. *Assume $char(k) \neq 2$. Let $\varphi \cong < \alpha_1, \dots, \alpha_n >$ be a non degenerate quadratic space with discriminant δ 1.2.7. Then:*

- *If $n = 2m$, then the Clifford algebra $Cl(\varphi)$ is a central simple algebra over k. It is isomorphic to a tensor product of quaternion algebras. The center of $Cl^+(\varphi)$ is isomorphic to $k[X]/(X^2 - \delta)$. If δ is not a square, $Cl^+(\varphi)$ is a central simple algebra over $k(\sqrt{\delta})$.*
- *If $n = 2m+1$, then the same statement holds with the roles of $Cl(\varphi)$ and $Cl^+(\varphi)$ reversed: $Cl^+(\varphi)$ is a central simple algebra over k, and a tensor product of quaternion algebras. The center of $Cl(\varphi)$ is isomorphic to $k[X]/(X^2 - \delta)$. If δ is not a square, δ is a central simple algebra over $k(\sqrt{\delta})$.*

Proof. We compute $Cl(\varphi) \cong (\alpha_l) \otimes \dots \otimes (\alpha_n)$ step by step using the previous lemma. The first step gives

$$(\alpha_l) \otimes \dots \otimes (\alpha_n) = (-\alpha_l\alpha_3, -\alpha_2\alpha_3) \otimes (-\alpha_l\alpha_2\alpha_3) \otimes (\alpha_4) \otimes \dots$$

A trivially graded quaternion algebra occurs on the right side while the number of 2-dimensional factors is reduced by 1. For $n = 2m + l$, we get after m steps a product of m trivially graded quaternion algebras and one last factor $((-1)^m \alpha_1 \dots \alpha_m) = (\delta)$. The product of the quaternion algebras is $Cl^+(\varphi)$; this is therefore a c.s. algebra. It follows easily that the factor (δ) is the center. For $n = 2m$ one gets in the same way after $m-1$ steps a trivially graded product of $m-1$ quaternion algebras with the quaternion algebra

$$((-1)^{m-1}\alpha_1 \dots \alpha_{2m-1})) \otimes (\alpha_{2m}) \cong (((-1)^m \alpha_1 \dots \alpha_{2m-1}, \alpha_{2m})).$$

Then $k[X]/(X^2 - \delta) \cong Cl^+(\varphi)$. □

Remark 2.2.7. In this thesis we are only interested in the even dimensional case. In the previous theorem, we have already proved that the center of $Cl^+(\varphi)$ is isomorphic to $k[X]/(X^2-\delta)$. If $\sqrt{\delta} \notin k$ the polynomial $X^2 - \delta$ is irreducible, thus the ideal generate by it is maximal, so $k[X]/(X^2 - \delta)$ is a field and necessarily we have

$$k[X]/(X^2 - \delta) \cong k(\delta).$$

We can characterize the orthogonal transformation on V applying the Galois theory on the center of the even part of the Clifford algebra, $Z(Cl^+(V,Q))$. Every $\phi \in O(V,Q)$ induces an automorphism ϕ' of the Clifford algebra $Cl(V,Q)$, thus an automorphism of the even Clifford algebra $Cl^+(V,Q)$, and therefore also of the center $Z(Cl^+(V,Q))$ of the even Clifford algebra. Since $Z(Cl^+(V,Q))/k$ is a separable quadratic extension,

$$Aut(Z(Cl^+(V,Q))) \cong \mathbb{Z}_2.$$

Definition 2.2.8. The *Dickson invariant* is the group homomorphism

$$\Delta : O(V,Q) \to \mathbb{Z}_2,$$

defined by

$$\Delta(\phi) = \begin{cases} 0 & \text{if } \phi'|_{Z(Cl^+(V,Q))} = id_{Z(Cl^+(V,Q))} \\ 1 & \text{if } \phi'|_{Z(Cl^+(V,Q))} \neq id_{Z(Cl^+(V,Q))} \end{cases}$$

The kernel of the homomorphism Δ is the special orthogonal group $SO(V,Q)$.

Lemma 2.2.9. $\Delta(\tau_x) = 1$ *for any reflection* $\tau_x \in O(V,Q)$*, for* $x \in V$ *non isotropic.*

Proof. We extend $x = e_1$ to a symplectic basis. We have seen that $e_1 f_1 + \cdots + e_n f_n$ lies in the center of the even Clifford algebra. Then

$$\tau_x(e_1 f_1 + \cdots + e_n f_n) \neq e_1 f_1 + \cdots + e_n f_n.$$

Thus, τ_x is not the identity on the center of the even Clifford algebra. □

Chapter 3

Stein factorization of $f : Lag(V) \to Spec(k)$

In the first two chapters we have studied our topic using purely algebraic objects, but in this chapter we will introduce some elements of algebraic geometry, such as proper and projective morphisms, in order to explain what the Stein factorization consist of. In the first paragraph we will define only the main elements we need to develop this part, referring to Hartshorne [5], chapter II and III, and Grothendieck [4], EGA III § 4, for having a complete vision of this theory. In the second paragraph we will apply the Stein factorization to the proper morphism of schemes $f : Lag(V) \to Spec(k)$, allowing us to study the relation between the two algebraic objects studied in the previous chapters, the Lagrangian Grassmannian and the Clifford algebra. In the last paragraph we will give some hints about a possible further development of this work.

3.1 Some elements of algebraic geometry theory

In this paragraph we will introduce the needed theory to study the Stein factorization of $Lag(Q) \to Spec(k)$, where Q is a non degenerate hyperbolic even dimensional quadratic form and k is a field. Giving only a brief introduction to this theory we are not able to give formal and complete proofs of the theorems that we will state at the end of this paragrapgh, but they can be found referring to Hartshorne [5], chapter II and III, and Grothendieck [4], EGA III § 4.

We denote with X, X', Y and Y' affine schemes. We start defining the following morphism:

Definition 3.1.1. Let $f : X \to Y$ be a morphism of schemes. The diagonal morphism is the unique morphism $\Delta : X \to X \times_Y X$ whose composition with both projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map of $X \to X$. We say that the morphism f is *separated* if the diagonal morphism Δ is a closed immersion. In that case we also say X is *separated* over Y. A scheme X is *separated* if it is separated over $SpecZ$.

Definition 3.1.2. A morphism $f : X \to Y$ is *proper* if it is separated, of finite type, and universally closed. Here we say that a morphism is closed if the image of any closed subset is closed. A morphism $f : X \to Y$ is universally closed if it is closed, and for any morphism $Y' \to Y$, the corresponding morphism $f' : X' \to Y'$ obtained by base extension is also closed.

Our next objective is to define projective morphisms and to show that any projective morphism is proper. We recall that a projective n-space P_A^n over any ring A to be $ProjA[x_0, \dots, x_n]$. Note that if $A \to B$ is a homomorphism of rings, and $SpecB \to SpecA$ is the corresponding morphism of affine schemes, then $P_B^n \cong P_A^n \times_{SpecA} SpecB$. In particular, for any ring A, we have $P_A^n \cong P_Z^n \times_{SpecZ} SpecA$. This motivates the following definition for any scheme Y

Definition 3.1.3. If Y is any scheme, we define projective n-space over Y, denoted $\mathbb{P}_y$, to be $\mathbb{P}_Z^n \times_{SpecZ} Y$. A morphism $f : X \to Y$ of schemes is projective if it factors into a closed immersion $i : X \to P_Y^n$ for some n, followed by the projection $P_Y^n \to Y$. A morphism $f : X \to Y$ is quasi-projective if it factors into an open immersion $j : X \to X'$ followed by a projective morphism $g : X' \to Y$.

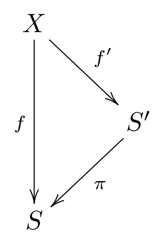
This definition of projective morphisms is slightly different from the one in Grothendieck [EGA II, 5.5]. The two definitions are equivalent in case Y itself is quasi-projective over an affine scheme

We now state the two theorems that we need for proving the results in the next paragraph.

Theorem 3.1.4. *(Zariski's Main theorem) Let $f : X \to Y$ be a birational projective morphism of noetherian integral schemes, and assume that Y is normal. Then for every $y \in Y$, $f^{-1}(y)$ is connected.*

Proof. To see this proof, we refer to Hartshorne [5], chapter III, cor. 11.4. □

Theorem 3.1.5. *Stein factorization theorem Let S be a locally Noetherian scheme. Let $f : X \to S$ be a proper morphism. There exists a factorization $f = \pi \circ f'$,*



with the following properties:

1. *the morphism f' is proper with geometrically connected fibres,*
2. *the morphism $\pi : S' \to S$ is finite,*
3. *we have $f'_*\mathcal{O}_X = \mathcal{O}_{S'}$,*
4. *we have $S' = Spec_S(f_*\mathcal{O}_X)$, and*
5. *S' is the normalization of S in X.*

Proof. To see this proof, we refer to Hartshorne [5], chapter III, cor. 11.5 and Grothendieck [4], EGA III § 4, cor. 4.3.3 .

We give now only a sketch of this proof. Let $S' = Spec(f_*\mathcal{O}_X)$. Then since $f_*\mathcal{O}_X$ is a coherent sheaf of $\mathcal{O}_Y$ algebras, the natural map $\pi : S' \to S$ is finite. On the other hand f clearly factors through π, so we get a morphism $f' : X \to S'$. Since π is separated, we conclude that f' is projective. By construction $f'_*\mathcal{O}_X = \mathcal{O}_S$ so f' has connected fibres by corollary 11.3 in Hartshorne [5]. □

Remark 3.1.6.

- In the proof of the previous theorem it is only necessary to check the first two properties, the last three are direct consequences of the first two.
- Under the hypothesis of Stein factorization, for every $s \in S$, the set of the connected components of the fiber $f^{-1}(s)$ is in one correspondence with the set of the points of the fiber $\pi^{-1}(s)$.

In order to fully understand the previous theorem we need to remind the definition of geometrically connected scheme.

Definition 3.1.7. Let X be a scheme over the field k. We say X is *geometrically connected* over k, if for any field extension k' of k the scheme $X_{k'}$ obtained by base extension to k' is connected as well.

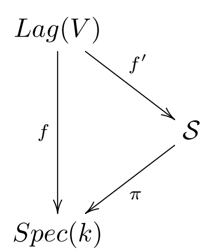
Remark 3.1.8. Let X be a scheme over another scheme S, and let $\overline{k}$ be an extension of the residue field $k(s)$, for $s \in S$. Then we have $f^{-1}(s) = X \times_S s$. We say that $f^{-1}(s)$ is geometrically connected, if for any extension $\overline{k}$ of $k(s)$ the scheme $f^{-1}(s) \times_{k(s)} \overline{k} = X \times_S Spec(\overline{k})$ is connected.

3.2 Computation of the Stein factorization of $f : Lag(V) \to Spec(k)$

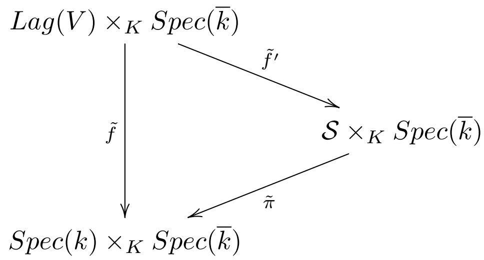
Let V be a vector space of even dimension $2n$ over k a field, and Q a non degenerate hyperbolic form.

In this paragraph, we want to apply the Stein factorization introduced in the previous paragraph to the proper and projective map $f : Lag(V) \to Spec(k)$, where $Lag(V)$ is the scheme over $K = Spec(k)$ of linear subspace S of V, of dimension n on which $Q = 0$.

Applying the Stein factorization to $f : Lag(V) \to Spec(k)$, we get the following diagram:



My purpose in this section is to prove that the scheme $\mathcal{S}$ is exactly the spectrum of the even part of the Clifford algebra defined by Q, $Spec(Z(Cl^+(Q))) = Spec(k(\sqrt{\delta}))$. It suffices to prove that the fibers of f' over $Spec(k(\sqrt{\delta}))$ are geometrically connected, so we need to apply a base change to the previous diagram extending the base field k to $\overline{k}$, its algebraically closed extension, getting the following diagram:



We denote with $\overline{f}, \overline{f'}$ and $\overline{\pi}$ the map induced by the base change of f, f' and π.

We need to prove that the fibers of $\overline{f'}$ are connected and $\mathcal{S} \times_K Spec(\overline{k})$ coincide with $Spec(k(\sqrt{\delta})) \times_K Spec(\overline{k})$.

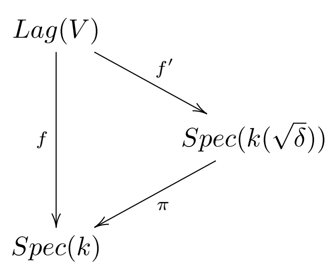
By the remark 3.1.6, we know that for the only point η of $Spec(k)$, the set of the connected components of the fiber $\overline{f}^{-1}(\eta)$ is in bijective correspondence with the points of the fiber $\overline{\pi}^{-1}(\eta)$. In this specific case we have $\overline{f}^{-1}(\eta) = Lag(V) \times_K Spec(\overline{k})$, since the map $\overline{f}$ is surjective because is the map induced by the surjective map f from this base change. In the fisrt chapter, we have proven that the Lagrangian Grassmannian is composed of two connected components, thus $\overline{\pi}^{-1}(\eta) = \mathcal{S} \times_K Spec(\overline{k})$, since $\overline{\pi} \circ \overline{f'} = \overline{f}$ is surjective, is composed of only two points.

We consider now $Spec(k(\sqrt{\delta})) \times_K Spec(\overline{k})$, that is isomorphic to the spectrum of the following tensor product $k(\sqrt{\delta}) \otimes \overline{k}$. Therefore $Spec(k(\sqrt{\delta})) \times_K Spec(\overline{k}) \cong Spec(k(\sqrt{\delta}) \otimes \overline{k}) \cong Spec(\overline{k} \oplus \overline{k})$, that consists of two points as $\mathcal{S} \times_K Spec(\overline{k})$.

The last thing that we need to prove is that the fibers of $\overline{f'}$ over $Spec(k(\sqrt{\delta})) \times_K Spec(\overline{k})$ are connected. Also the map $\overline{f'}$ is surjective since $\overline{\pi}$ and $\overline{\pi} \circ \overline{f'}$ are surjective, thus everyone of the two fibers of $\overline{f'}$ corresponds to one of the two connected component of $Lag(V) \times_K Spec(\overline{k})$, so the fibers of $\overline{f'}$ are connected.

We can finally state the following result:

Proposition 3.2.1. *The Stein factorization of* $f : Lag(V) \to Spec(k)$ *is* $f = \pi \circ f'$, *where* $f : Lag(V) \to Spec(k(\sqrt{\delta}))$ *and* $f : Spec(k(\sqrt{\delta})) \to Spec(k)$, *getting the following diagram:*



This result is used in the calculation of vanishing cycles for certain types of quadrics over local fields with residue characteristic 2.

3.3 Consequence: computation of the étale chomology group $H^{2(n-1)}(X)$

In this last paragraph, we want to show that the theory developed in this thesis can be used in a direct application, the computation of the étale cohomology group of the hyperbolic quadric Q in $\mathbb{P}^{2n-1}$, thus this quadric has dimension $2(n-1)$, using the theory of algebraic cycles, of tangent bundles, of the Chern classes.
We give now a briefly introduction of the different step necessary to show this result.
We consider an m-dimensional smooth quadric hypersurface Q' in P^{m+1}.

1. Using the Lefschetz hyperplane theorem, for $k \neq m$ and $k \in \mathbb{N}$, the restriction maps

$$H^k(\mathbb{P}^n, \mathbb{Z}) \to H^k(Q', \mathbb{Z})$$

   are isomorphisms, and so the cohomology groups of the quadric Q in these degrees certainly come from algebraic cycles. Moreover, by the universal coefficient theorem this also shows that $H^k(Q', \mathbb{Z})$ is torsion-free for all $k \in \mathbb{Z}$.

2. Using the normal bundle sequence

$$T_{Q'} \to T_{\mathbb{P}}^m \to N_{Q'}$$

   one computes the Euler characteristic of Q'; since the Euler characteristic is the same as the degree of the top Chern class of the tangent bundle $T_{Q'}$, if we know the Chern classes of $T_{\mathbb{P}}^m$ and $N_{Q'}$ then we can use the Whitney sum formula to calculate the Chern classes of T_Q.

   Together with the information from the previous step, this shows that

$$H^m(Q', \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

3. Now it just remains to find two non-homologous cycles of middle dimension on an even-dimensional quadric. In the particular case of the hyperbolic quadric Q in $\mathbb{P}^{2n-1}$, it suffice to consider the cycle $cl(S)$ and $cl(S')$, where S and S' are two elements from different connected components of the Lagrangian Grassmanian with respect to Q. Therefore, $cl(S)$ and $cl(S')$ are generators of $H^{2(n-1)}(X)$, $X \subset \mathbb{P}(V)$ (= Grassmannian of the linear subspaces of V of dimension 1) is the quadric $Q = 0$.

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