1|2 |3|4|5
4|2.5|4|4|4

18.5/20

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EXERCICE 5. Déterminer l'ensemble des solutions des systèmes suivants :

$$(1)\ \begin{cases} x + 2y - 3z = -7 \\ 4x\ \ 2y\ \ 2z = 14 \\ -3x + y - z = -9 \end{cases} \qquad (2)\ \begin{cases} x + 2y - 3z = -7 \\ 4x\ \ 2y\ \ 2z = 14 \\ -5x + z = -5 \end{cases}$$

1) $\begin{cases} x + 2y - 3z = -7 \\ 4x - 2y + 2z = 14 \\ -3x + y - z = -9 \end{cases} \overset{L_1 \leftrightarrow L_2}{\Longleftrightarrow} \begin{cases} 4x - 2y + 2z = 14 \\ x + 2y - 3z = -7 \\ -3x + y - z = -9 \end{cases} \overset{L_3 \to L_3 + 3L_2}{\Longleftrightarrow}$

$\begin{cases} 4x - 2y + 2z = 14 \\ x + 2y - 3z = -7 \\ 7y - 10z = -30 \end{cases} \underset{L_2 \to 2L_2}{\overset{L_2 \to L_2 - \frac{L_1}{4}}{\Longleftrightarrow}} \begin{cases} 4x - 2y + 2x = 14 \\ 5y - 7z = -21 \\ 7y - 10z = -30 \end{cases} \overset{L_3 \to L_3 - \frac{7}{5}L_2}{\Longleftrightarrow}$

$\begin{cases} 4x - 2y + 2x = 14 \\ 5y - 7z = -21 \\ -\frac{1}{5}z = -\frac{3}{5} \end{cases}$

Now: $-\frac{1}{5}z = -\frac{3}{5}$

$\boxed{z = 3}$

$5y - 7 \times 3 = -21$

$5y = 0$

$\boxed{y = 0}$

The set of solutions is $\{(2; 0; 3)\}$

$4x - 2 \times 0 + 2 \times 3 = 14$

$4x + 6 = 14$

$4x = 8$

$\boxed{x = 2}$

2) $\begin{cases} x + 2y - 3z = -7 \\ 4x - 2y + 2z = 14 \\ -5x + z = -5 \end{cases} \overset{L_2 \leftrightarrow L_1}{\Longleftrightarrow} \begin{cases} 4x - 2x + 2z = 14 \\ x + 2y - 3x = -7 \\ -5x + z = -5 \end{cases} \overset{L_3 \to L_3 + 5L_2}{\Longleftrightarrow}$

$\begin{cases} 4x - 2x + 2z = 14 \\ x + 2y - 3x = -7 \\ 10y - 14z = -40 \end{cases} \underset{L_2 \to 2L_2}{\overset{L_2 \to L_2 - \frac{L_1}{4}}{\Longleftrightarrow}} \begin{cases} 4x - 2y + 2z = 14 \\ 5y - 7z = -21 \\ 10y - 14z = -40 \end{cases} \overset{L_3 \to L_3 - 2L_2}{\Longleftrightarrow} \begin{cases} 4x - 2y + 2z = 14 \\ 5y - 7y = -21 \\ 0z = 2 \end{cases}$

/!\

That is impossible, so the set of solutions is

That is impossible, so the set of solutions is empty.

The set of solutions is $\emptyset$.

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Exercice 4. Résoudre dans $\mathbb{R}$ l'inéquation suivante

$$x - 1 + |x - 2| \geq |2x + 4|.$$

$x-1+|x-2| \geqslant |2x+4|$

$|x-2| \Rightarrow \begin{cases} x-2, & x \geqslant 2 \\ -x+2, & x \leq 2 \end{cases}$

$x-2=0$

$\underline{x=2}$

$|2x+4| \Rightarrow \begin{cases} 2x+4, & x \geqslant -2 \\ -2x-4, & x \leq -2 \end{cases}$

$2x+4=0$

$2x=-4$

$\underline{x=-2}$

So if $x \in]-\infty; -2]$

$x-1+(-x+2) \geqslant -2x-4$

$x-1-x+2 \geqslant -2x-4$

$1 \geqslant -2x-4$

$5 \geqslant -2x$

$-\frac{5}{2} \leq x$

$S = [-\frac{5}{2}; -2]$

If $x \in [-2; 2]$

$x-1+(-x+2) \geqslant 2x+4$

$x-1-x+2 \geqslant 2x+4$

$1 \geqslant 2x+4$

$-3 \geqslant 2x$

$$-3 \geqslant 2x$$

$$\boxed{-\frac{3}{2} \geqslant x} \qquad S = \left[-2 ; -\frac{3}{2}\right]$$

If $x \in [2 ; +\infty[$

$$x - 1 + (x-2) \geqslant 2x + 4$$

$$x - 1 + x - 2 \geqslant 2x + 4$$

$$\underbrace{2x - 3 \geqslant 2x + 4}$$ (!) impossible!

$S = \varnothing$ for $[2 ; +\infty[$

The set of solutions for $x - 1 + |x-2| \geqslant |2x+4|$

is $\left[-\frac{5}{2} ; -2\right] \cup \left[-2 ; -\frac{3}{2}\right] \cup \{\varnothing\} \Longleftrightarrow \left[-\frac{5}{2} ; -\frac{3}{2}\right]$ **4/4**

Exercice 1. Soient P et Q deux propositions logiques :

(1) Établir la table de vérité de la proposition

$$(P \vee Q) \Rightarrow (P \wedge Q).$$

(2) Établir la table de vérité de la proposition

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P).$$

(3) En déduire que la proposition $(P \vee Q) \Rightarrow (P \wedge Q)$ est équivalente à $P \iff Q$.

1) To obtain the truth table of $(P \cup Q) \Rightarrow (P \wedge Q)$ let's observe the truth tables of $(P \cup Q)$ and $(P \wedge Q)$.

P	Q	$P \cup Q$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	F	F	F
F	T	T	F

F	T	T	F

Let's now look at the truth table of $(P \vee Q) \Rightarrow (P \wedge Q)$.

$P \vee Q$	$P \wedge Q$	$(P \vee Q) \Rightarrow (P \wedge Q)$
T	T	T
T	F	F
F	F	T
T	F	F

2) To obtain the truth table of $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ let's observe the truth tables of $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	F	T	T
F	T	T	F

Thus $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is:

$(P \Rightarrow Q)$	$(Q \Rightarrow P)$	$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$
T	T	T
F	T	F
T	T	T
T	F	F

T	F	F

3) Let's look at the truth table of $P \Leftrightarrow Q$.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	F	T
F	T	F

We can see that $(P \Leftrightarrow Q)$'s truth table is the same as $(P \vee Q) \Rightarrow (P \wedge Q)$, so the two are equivalent.

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EXERCICE 2. Soit f une fonction de $\mathbb{R}$ dans $\mathbb{R}$, on dit que f admet un minimum si la proposition logique suivante est vraie :

$$\exists x \in \mathbb{R},\ \forall y \in \mathbb{R},\ f(x) \leq f(y).$$

(1) A quelle proposition logique associerez-vous la définition de "f admet un maximum" ?

(2) Quelle serait la définition de "f n'admet pas de minimum" ?

(3) Pour une fonction f de $\mathbb{R}$ dans $\mathbb{R}$, on considère la proposition :

$$\forall y \in \mathbb{R},\ \exists x \in \mathbb{R},\ f(x) \leq f(y).$$

Montrer que quelle que soit f, cette dernière proposition est toujours vraie.

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, f(x) \leq f(y)$

1) The proposition for "f admits a maximum" would be:

$$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, f(x) \geq f(y)$$

2) The definition of "f doesn't admit a minimum" would be:

$$\neg(\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, f(x) \leq f(y))$$

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, f(x) > f(y)$$

3) We want to show that $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) \leq f(y)$ is always true.

We can proceed by the absurd.

We can proceed by the absurd.
So let's assume that non the proposition is true
so:

$$\neg(\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) \leq f(y))$$

$$\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) > f(y)$$

But if non the proposition is true then the proposition itself is false. Thus the proposition is always true.

???

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EXERCICE 3.

(1) Montrer par récurrence que pour tout $n \in \mathbb{N}^*$:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

(2) Montrer par récurrence que pour tout $n \in \mathbb{N}^*$:

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2.$$

1) Let's demonstrate by induction that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}^*$$

Initialization: for $n=1$

$$\sum_{k=1}^{1} k = 1 \quad || \quad \frac{1(1+1)}{2} = \frac{1 \times 2}{2} = 1$$

So the proposition is true at rank 1, so the induction is initialized.

Induction step:

Let's assume that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ is true for rank n.

Let's check for the rank $(n+1)$.

Induction hypothesis (IH): $\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}$, $\forall n \in \mathbb{N}^*$

$$\sum^{n+1} k \quad \sum^{n} k \quad \sum^{n+1} k$$

$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + \sum_{k=n+1}^{n+1} k$$

$$= \sum_{k=1}^{n} k + n+1$$

(due to the IH)

$$= \frac{n(n+1)}{2} + n+1$$

$$= \frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$= \frac{n(n+1) + 2(n+1)}{2}$$

$$= \frac{n^2+n+2n+2}{2}$$

$$= \frac{n^2+3n+2}{2}$$

$$= \frac{(n+1)(n+2)}{2}$$

However $\frac{(n+1)((n+1)+1)}{2} = \frac{(n+1)(n+2)}{2}$; thus the proposition is hereditary.

We can then conclude that $\forall n \in \mathbb{N}^*$ $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$.

② Let's demonstrate by induction that

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2, \forall n \in \mathbb{N}^*$$

<u>Initialization:</u> for $n=1$

$$\sum_{k=1}^{1} k^3 = 1^3 = 1 \qquad || \left(\sum_{k=1}^{1} k\right)^2 = 1^2 = 1$$

The proposition is true at rang 1. The induction is initiazed.

Induction step:

Let's assume the $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$ is true for rank n.

Let's check for rank $(n+1)$.

I.H: $\sum_{k}^{n+1} k^3 = \left(\sum_{k=1}^{n+1} k\right)^2 = \left(\sum_{k=1}^{n+1} k\right)^2$

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + \sum_{k=n+1}^{n+1} k^3$$

$\leftarrow$ due to IR.

$$= \left(\sum_{k=1}^{n} k\right)^2 + (n+1)^3$$

based on ① $\rightarrow$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4}$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

$$= \frac{(n+1)^2(n^2+4(n+1))}{4}$$

$$= \frac{(n+1)^2(n^2+4n+4)}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4}$$

$$= \left(\frac{(n+1)(n+2)}{2}\right)^2$$

Based on the previous question we know that $\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$. Thus $\left(\sum_{k=1}^{n+1} k\right)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2$, which is in turn equal to $\sum_{k=1}^{n+1} k^3$.

In conclusion $\forall n \in \mathbb{N}^*$, $\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.