Ex 85 (6) $\quad z^2 = 3-4i$

$z = x + yi \qquad z^2 = x^2 - y^2 + 2xyi$

$$\begin{cases} x^2 - y^2 = 3 & (1) \\ 2xy = -4 & (2) \\ x^2 + y^2 = 5 & (3) \end{cases}$$

$|z|^2 = x^2 + y^2$

$|z|^2 = |z^2| = |3-4i| = \sqrt{3^2 + (-4)^2} = 5$

$(1) + (3) \quad 2x^2 = 8 \quad x^2 = 4 \quad x = \pm 2$

$(3) - (1) \quad 2y^2 = 2 \quad y^2 = 1 \quad y = \pm 1$

$(2) \Rightarrow \quad x = 2,\ y = -1 \qquad z = 2 - i$

or $\qquad\qquad\qquad\qquad$ or

$x = -2 \quad y = 1 \qquad z = -2 + i$

The set of solutions $\{2-i, -2+i\}$

Equations of degree 2

Real theory	Complex theory
$ax^2 + bx + c = 0$ $a, b, c \in \mathbb{R},\ a \neq 0$ solve in $x \in \mathbb{R}$ $\Delta = b^2 - 4ac$ $\boxed{\text{3 cases}}$	$az^2 + bz + c = 0$ $a, b, c \in \mathbb{C},\ a \neq 0$ solve in $z \in \mathbb{C}$ $\Delta = b^2 - 4ac$ $\boxed{\text{2 cases}}$
$\Delta > 0$ 2 roots $x_1 = \frac{-b - \sqrt{\Delta}}{2a} \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}$ $\Delta = 0$ 1 double root $x = \frac{-b}{2a}$	$\Delta \neq 0$ look for $\delta \in \mathbb{C}$ such that $\delta^2 = \Delta$ 2 roots: $z_1 = \frac{-b - \delta}{2a} \quad z_2 = \frac{-b + \delta}{2a}$
$\Delta < 0$ no real roots	$\Delta = 0$ 1 double root $z = \frac{-b}{2a}$

we

<u>Remark</u> Even in the complex case it may happen that $\Delta \in \mathbb{R}$

If $\Delta \in \mathbb{R}$, $\Delta > 0$ then $\delta = \sqrt{\Delta}$

If $\Delta \in \mathbb{R}$, $\Delta < 0$ then $\delta = i\sqrt{|\Delta|}$

Ex 86 (1) 3 November, 2020 08:29 $z^2 - 5iz - 7 + i = 0$

$$\Delta = (-5i)^2 - 4(-7+i) = -25 + 28 - 4i$$
$$= 3 - 4i$$

Find δ such that $\delta^2 = 3 - 4i$

Did it in ex 85 (6). Using the result of ex 85 (6) find $\delta = 2 - i$ (or $\delta = -2 + i$)

2 roots

$$z_1 = \frac{5i - (2-i)}{2} = -1 + 3i \qquad z_2 = \frac{5i + 2 - i}{2} = 1 + 2i$$

The set of solutions is $\{-1+3i, 1+2i\}$

Please work on Ex 86 (2), 87, 88

Plan: before the break, and right after
Speak on 86 (2), 87, 88

Afterwards 80, 81, 82, 83

Ex 86 2 $z^2 + 2i\sqrt{2}\, z - 2(1+i)$

3 November, 2020 08:56

$\Delta = (2i\sqrt{2})^2 - 4 \cdot (-2(1+i)) =$

$= -8 + 8 + 8i = 8i$

Find δ such that $\boxed{\delta^2 = 8i}$

let us use ~~exponential~~ exponential form

$8i = 8e^{\frac{\pi}{2}i}$ $\quad \delta = \rho e^{\theta i}$ $\quad \delta^2 = \rho^2 e^{2\theta i}$

$\rho^2 = 8$ $\quad \rho = 2\sqrt{2}$

$2\theta = \frac{\pi}{2} + 2\pi k$ $\quad k \in \mathbb{Z}$

$\theta = \frac{\pi}{4} + \pi k$ $\quad k=0$ $\quad \theta_0 = \frac{\pi}{4}$ $\quad \delta_0 = 2\sqrt{2}e^{\frac{\pi}{4}i} =$

$= 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= 2 + 2i$

$k=1$ $\quad \theta_1 = \frac{\pi}{4} + \pi$

$\delta_1 = -2 - 2i$

May use δ_0 or δ_1 as δ

For instance take $\delta = 2+2i$

$$z_1 = \frac{-2\sqrt{2}i - (2+2i)}{2} = -1 - (1+\sqrt{2})i$$

$$z_2 = \frac{-2\sqrt{2}i + 2 + 2i}{2} = 1 + (1-\sqrt{2})i$$

The set of solutions is

$$\{-1-(1+\sqrt{2})i,\ 1+(1-\sqrt{2})i\}$$

87 2 $z^2+2z+5=0$

3 November, 2020 09:12

$\Delta = 2^2 - 4\cdot 5 = -16$

$\delta^2 = -16 \qquad \delta = 4i \quad (\text{or } -4i)$

$z_1 = \frac{-2-4i}{2} = -1-2i$

$z_2 = \frac{-2+4i}{2} = -1+2i$

The set of solutions is

$\{-1-2i, -1+2i\}$

---

88 1 $z^2+z+1=0$

$\Delta = 1^2 - 4\cdot 1 = -3 \qquad \delta^2 = -3$

$\delta = i\sqrt{3} \quad (\text{or } -i\sqrt{3})$

$z_1 = \frac{-1-i\sqrt{3}}{2} \qquad z_2 = \frac{-1+i\sqrt{3}}{2}$

---

88 2 $|z_1| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$

$z_1 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{-\frac{2\pi}{3}i}$

Similarly $z_2 = e^{\frac{2\pi}{3}i}$

88 3 $z^4+z^2+1=1$

new variable $w = z^2$

$w^2+w+1=1 \qquad w_1 = e^{-\frac{2\pi}{3}i}$

$w_2 = e^{\frac{2\pi}{3}i}$

$\boxed{z^2 = w_1} \qquad z^2 = 1e^{-\frac{2\pi}{3}i}$

$z = \rho e^{\theta i} \qquad z^2 = \rho^2 e^{2\theta i}$

$\rho^2 = 1 \qquad \rho = 1$

$2\theta = -\frac{2\pi}{3} + 2\pi k \qquad k \in \mathbb{Z}$

$\theta = -\frac{\pi}{3} + \pi k$

$k=0 \qquad \theta_0 = -\frac{\pi}{3} \qquad z_0 = 1\cdot e^{-\frac{\pi}{3}i} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$k=1 \qquad \theta_1 = -\frac{\pi}{3}+\pi = \frac{2\pi}{3} \qquad z_1 = 1\, e^{\frac{2\pi}{3}i} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$z^2 = w_2 = e^{\frac{2\pi}{3}i}$ solved similarly

$z_2 = e^{\frac{\pi}{3}i} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

$\frac{4\pi}{}i$

$$z_3 = e^{\frac{4\pi}{3}i} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

The set of solutions of $z^4 + z^2 + 1 = 0$

$$\left\{ \frac{1}{2} - \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ \frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \right\}$$

80 $z = 1 - i\sqrt{3}$ 5 November, 2020 09:43

Determine $n \in \mathbb{Z}$ such that

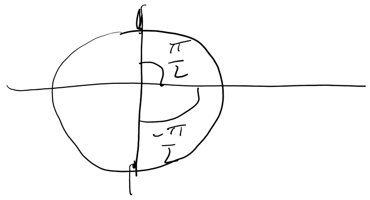
① z^n pure imaginary ② z^n real negative

---

① w is pure imaginary if $\underline{\operatorname{Re} w = 0}$

not practical for $w = z^n$

w is pure imaginary if $\boxed{\arg w = \frac{\pi}{2}\ [2\pi] \text{ or } \arg w = -\frac{\pi}{2}\ [2\pi]}$



$\Updownarrow$ for smart people

$\arg w = \frac{\pi}{2}\ [\pi]$

We have to determine $\arg(z^n)$

Need to write z in exponential form

$z = 1 - i\sqrt{3}$ $|z| = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$z = 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 2e^{-\frac{\pi}{3}i}$

$z^n = 2^n e^{-\frac{\pi}{3}ni}$ $\boxed{\arg(z^n) = -\frac{\pi}{3}n\ [2\pi]}$

The problem is reduced to:

determine $n \in \mathbb{Z}$ such that

$\boxed{-\frac{\pi}{3}n = \frac{\pi}{2}\ [2\pi]}$ A or $\boxed{-\frac{\pi}{3}n = -\frac{\pi}{2}\ [2\pi]}$ B

Solving A $-\frac{\pi}{3}n = \frac{\pi}{2} + 2\pi k$ $k \in \mathbb{Z}$

$-\frac{1}{3}n = \frac{1}{2} + 2k$ (we cancelled π out)

Let us clear the denominators

Multiply both sides by 6

$$-\underbrace{2n}_{\text{even}} = \underset{\text{odd}}{\underset{\uparrow}{3}} + \underbrace{12k}_{\text{even}} \quad \text{impossible}$$

There is no $n \in \mathbb{Z}$ satisfying (A)

Similarly one shows that (B) is impossible as well

The set of $n \in \mathbb{Z}$ such that z^n is pure imaginary is $\emptyset$.

w is real negative if $\arg w = \pi \; [2\pi]$

Let us determine $n \in \mathbb{Z}$ such that

$\arg(z^n) = \pi \; [2\pi]$

$-\frac{\pi}{3} n = \pi \; [2\pi]$

$-\frac{\pi}{3} n = \pi + 2\pi k \quad k \in \mathbb{Z}$

$-\frac{1}{3} n = 1 + 2k$

$-n = 3 + 6k \quad n = -3 - 6k$

The set of $n \in \mathbb{Z}$ such that

z^n is real negative is

$\{-3 - 6k : k \in \mathbb{Z}\} =$

$\{\dots, -15, -9, -3, 3, 9, 15, 21, \dots\}$

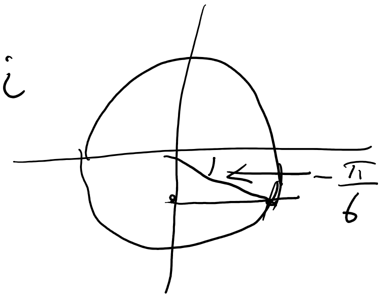
Ex 8.1 $z = \dfrac{\sqrt{3}-i}{1-i}$ write cartesian & exponential form of z

8 November, 2020 10:05

Exponential form: $z = \dfrac{z_1}{z_2}$

$z_1 = \sqrt{3} - i \qquad |z_1| = 2$

$$z_1 = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) = 2e^{-\frac{\pi}{6}i}$$



$z_2 = 1 - i \qquad |z_2| = \sqrt{2}$

$$z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \sqrt{2}e^{-\frac{\pi}{4}i}$$

$$z = \frac{2e^{-\frac{\pi}{6}i}}{\sqrt{2}e^{-\frac{\pi}{4}i}} = \sqrt{2}e^{\frac{\pi}{12}i} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$

Cartesian form

$$z = \frac{\sqrt{3}-i}{1-i} = \frac{(\sqrt{3}-i)(1+i)}{(1-i)(1+i)} = \frac{\sqrt{3}+\sqrt{3}i-i+1}{2} =$$

$$= \frac{\sqrt{3}+1}{2} + i\,\frac{\sqrt{3}-1}{2}$$

$$\sqrt{2}\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2} \qquad \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

Similarly $\quad \sin\frac{\pi}{2} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

Exercise 82. Motivation:

3 November, 2020 10:16

$$(\cos x)^2 = \frac{1+\cos(2x)}{2}$$ is a useful formula

(we will see why when we study integration)

Good to have similar formulas for

$(\cos x)^3$, $(\cos x)^4$, ...

In exercise 82 obtain a formula for $(\cos x)^5$. We use 2 fundamental facts:

- Euler's formula
- Newton's binomial formula.

Euler's formula:

$$\left.\begin{aligned} e^{\theta i} &= \cos\theta + i\sin\theta \\ e^{-\theta i} &= \cos\theta - i\sin\theta \end{aligned}\right\}$$ sum them up!!

$$e^{\theta i} + e^{-\theta i} = 2\cos\theta$$

$$\cos\theta = \frac{1}{2}\left(e^{\theta i} + e^{-\theta i}\right)$$

$$(\cos\theta)^5 = \frac{1}{2^5}\left(e^{\theta i} + e^{-\theta i}\right)^5$$ ← need Newton's

Binomial Formula

Pascal triangle

3 November, 2020 10:25

0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

$$\left(e^{\theta i}+e^{-\theta i}\right)^5 = e^{5\theta i}+5e^{3\theta i}+10e^{\theta i}+10e^{-\theta i}+5e^{-3\theta i}+e^{-5\theta i}$$

$$e^{5\theta i}+e^{-5\theta i}+5\left(e^{3\theta i}+e^{-3\theta i}\right)+10\left(e^{\theta i}+e^{-\theta i}\right)$$

$$= 2\cos(5\theta)+10\cos(3\theta)+20\cos\theta$$

$$(\cos\theta)^5 = \frac{1}{2^5}\left(e^{\theta i}+e^{-\theta i}\right)^5 =$$

$$= \frac{1}{32}\left(2\cos(5\theta)+10\cos(3\theta)+20\cos\theta\right) =$$

$$= \frac{1}{16}\cos(5\theta)+\frac{5}{16}\cos(3\theta)+\frac{5}{8}\cos\theta$$