120,125,126

3 December, 2020 14:02

(120)

$$f(x)=\begin{cases} \arctan x, & x\in]-\infty;0[\\ 0, & x=0 \\ e^{-\frac{1}{x}}, & x\in]0;+\infty[\end{cases}$$

f continuous, derivable on $]-\infty;0[$ and on $]0,+\infty[$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\arctan x=0$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} e^{\overbrace{-\frac{1}{x}}^{\to -\infty}}=0$$

$$0=f(0)=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x)$$

f is continuous at 0

f is continuous on $\mathbb{R}$.

$$f'_-(0)=\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^-}\frac{\arctan x}{x}$$

$$=\arctan'_-(0)=\arctan'(x)=1$$

$$f'_+(0)=\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0^+}\frac{e^{-\frac{1}{x}}}{x}$$

$$f_+(0) = \lim_{x \to 0^+} \frac{\quad}{x-0} = \lim_{x \to 0^+} \frac{\quad}{x}$$

$$\boxed{\begin{array}{ll} x = \frac{1}{u} & \frac{1}{x} = u \\ x \to 0^+ & u \to +\infty \end{array}}$$

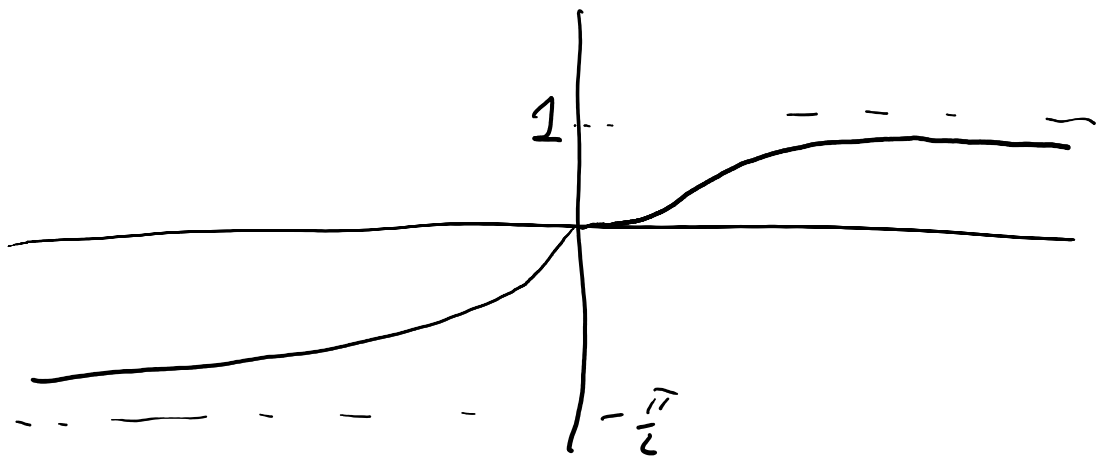
$$= \lim_{u \to +\infty} \frac{e^{-u}}{1/u} =$$

$$= \lim_{u \to +\infty} \frac{u}{e^u} = 0$$

$$f'_-(0) = 1 \qquad f'_+(0) = 0$$

f is not derivable at 0

The domain of derivability of f is $\mathbb{R} - \{0\}$



125 2

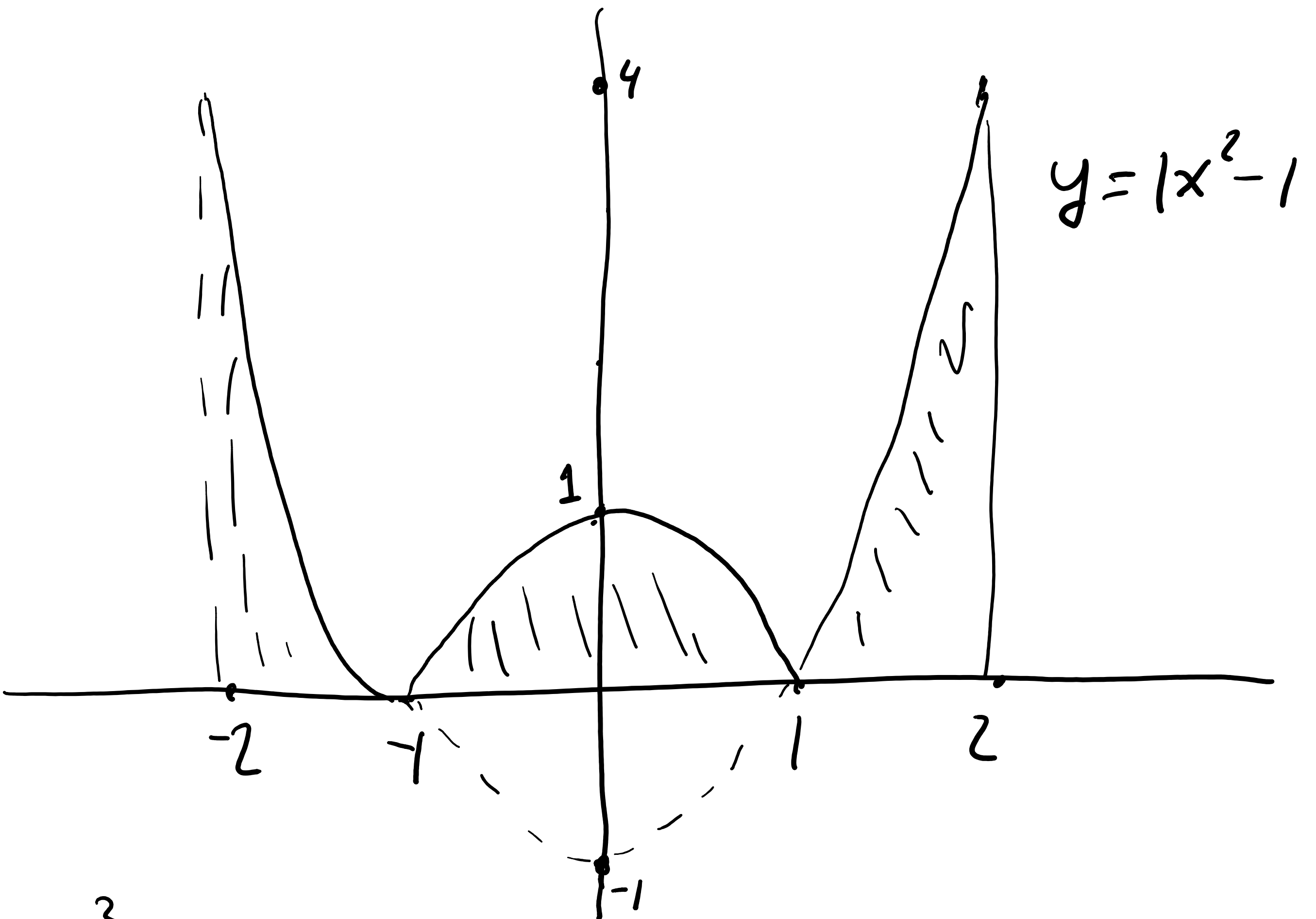
3 December, 2020 14:22

$$\int_{-2}^{2} |x^2-1| dx$$

$$|x^2-1| = \begin{cases} x^2-1, & x^2-1 \geq 0 \\ 1-x^2, & x^2-1 \leq 0 \end{cases}$$

$$x^2-1 \leq 0 \Leftrightarrow x \in [-1, 1]$$

$$|x^2-1| = \begin{cases} x^2-1, & x \in]-\infty; -1] \\ 1-x^2, & x \in [-1, 1] \\ x^2-1, & x \in [1; +\infty[\end{cases}$$



$$\int_{-2}^{2}|x^2-1|\,dx = \int_{-2}^{-1}(x^2-1)\,dx+\int_{-1}^{1}(1-x^2)\,dx+$$

$$+\int_{1}^{2}(x^2-1)\,dx = \left[\frac{x^3}{3}-x\right]_{-2}^{-1}+\left[x-\frac{x^3}{3}\right]_{-1}^{1}+$$

$$\left[\frac{x^3}{3}-x\right]_{1}^{2}=\left(\frac{2}{3}-\left(-\frac{2}{3}\right)\right)+\left(\frac{2}{3}-\left(-\frac{2}{3}\right)\right)$$

$$+\left(\frac{2}{3}-\left(-\frac{2}{3}\right)\right) = 4$$

Since the function is even,

$$\int_{-2}^{2} |x^2-1|\,dx = 2\int_{0}^{2} |x^2-1|\,dx =$$

$$= 2\left(\int_{0}^{1} (1-x^2)\,dx + \int_{1}^{2} (x^2-1)\,dx\right) =$$

$$= 2\left(\left[x-\frac{x^3}{3}\right]_0^1 + \left[\frac{x^3}{3}-x\right]_0^1\right) = 2\left(\frac{2}{3}+\frac{2}{3}\right)$$

$$= 4$$

125 3a

find primitive for $\frac{1}{(\cos x)^2}$

specifying the possible intervals

$\cos x = 0 \iff x = \frac{\pi}{2} + \pi k$

one must take intervals not containing points of the form $\frac{\pi}{2} + \pi k$

$\dots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}; \frac{5\pi}{2}, \dots$

possible intervals are $]-\frac{\pi}{2}; \frac{\pi}{2}[$,

$]\frac{\pi}{2}; \frac{3\pi}{2}[$, $]\frac{3\pi}{2}; \frac{5\pi}{2}[$, $]-\frac{3\pi}{2}; -\frac{\pi}{2}[, \dots$

etc...

on each of these intervals

$$\int \frac{1}{(\cos x)^2}\, dx = \tan x + C$$

12.5 3 6

$$\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{3/2} + C =$$

$$= \frac{2}{3}\,x\sqrt{x} + C \quad \text{on } [0; +\infty[$$

$$\int 5^x\,dx = \int e^{x\ln 5}\,dx = \frac{e^{x\ln 5}}{\ln 5} + C$$

$$\boxed{\int e^{ax}\,dx = \frac{e^{ax}}{a} + C} \quad = \frac{5^x}{\ln 5} + C \quad \text{on } \mathbb{R}$$

3 December, 2020 14:46

12.6 1

$$\int x^{\alpha}dx = \begin{cases} \frac{x^{\alpha+1}}{\alpha+1} + C, & \alpha \neq -1 \\ \ln x + C, & \alpha = -1 \end{cases}$$

$$\int u(x)^{\alpha} u'(x)\,dx = \begin{cases} \frac{u(x)^{\alpha+1}}{\alpha+1} + C, & \alpha \neq -1 \\ \ln u(x) + C, & \alpha = -1 \end{cases}$$

12.6 1b

$$\int x^4 (1+x^5)^5 dx = \frac{1}{5}\int u(x)^5 u'(x)\,dx =$$

$$\boxed{u(x) = 1 + x^5 \qquad u'(x) = 5x^4}$$

$$\frac{1}{5}\frac{u(x)^6}{6} + C = \frac{(1+x^5)^6}{30} + C$$

12.6 1c

3 December, 2020 14:54

$$\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx =$$

$u(x) = \sin\frac{x}{2}$ $\quad u'(x) = \frac{1}{2}\cos\frac{x}{2}$

$$= 2\int u(x)\,u'(x)\,dx = 2\cdot\frac{u(x)^2}{2} + C$$

$$= \boxed{\left(\sin\frac{x}{2}\right)^2 + C}$$

Second method:

$v(x) = \cos\frac{x}{2}$ $\quad v'(x) = -\frac{1}{2}\sin\frac{x}{2}$

$$\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx = -2\int v(x)\,v'(x)\,dx =$$

$$= -2\cdot\frac{v(x)^2}{2} + C = \boxed{-\left(\cos\frac{x}{2}\right)^2 + C}$$

Third method:

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx = \frac{1}{2}\int \sin x\,dx =$$

$$\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx = \frac{1}{2}\int \sin x\,dx =$$

$$= \boxed{-\frac{1}{2}\cos x + C}$$

$$\cos x = \left(\cos\frac{x}{2}\right)^2 - \left(\sin\frac{x}{2}\right)^2$$

$$= 2\left(\cos\frac{x}{2}\right)^2 - 1 = 1 - 2\left(\sin\frac{x}{2}\right)^2$$

$$-\frac{1}{2}\cos x + C = -\frac{1}{2}\left(2\left(\cos\frac{x}{2}\right)^2 - 1\right) + C$$

$$= -\left(\cos\frac{x}{2}\right)^2 + \frac{1}{2} + C$$

12.6 1d on $]-\frac{\pi}{2};\frac{\pi}{2}[$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx =$$

$u(x) = \cos x \quad u'(x) = -\sin x$

$$= -\int \frac{u'(x)}{u(x)} dx = -\ln(\cos x) + C$$

$\ln(\cos x)$ is well defined because

$\cos x > 0$ on $]-\frac{\pi}{2};\frac{\pi}{2}[$

126 1e belongs to 126 2

126 1f $\int (\sin t)^3 \cos t \, dt$

$u(t) = \sin t \qquad u'(t) = \cos t$

$$= \int u(t)^3 \, u'(t) \, dx = \frac{u(t)^4}{4} + C$$

$$= \frac{(\sin t)^4}{4} + C$$

12.6 1g 3 December, 2020 15:11 on $]-\frac{\pi}{2};\frac{\pi}{2}[$

$$\int \frac{\sin x}{(\cos x)^{4/5}}\,dx = -\int u(x)^{-\frac{4}{5}} u'(x)\,dx$$

$u(x) = \cos x \qquad u'(x) = -\sin x$

$$= -\frac{u(x)^{\frac{1}{5}}}{1/5} + C = -5(\cos x)^{\frac{1}{5}} + C$$

1h

$$\int \frac{x^3}{\sqrt{1-x^4}}\,dx = \frac{-1}{4}\int u(x)^{-\frac{1}{2}} u'(x)\,dx$$

$u(x) = 1 - x^4 \qquad u'(x) = -4x^3$

$$= -\frac{1}{4}\,\frac{u(x)^{\frac{1}{2}}}{1/2} + C = -\frac{1}{2}\sqrt{1-x^4} + C$$

126 2 3 December, 2020 15:16

$$\int e^x dx = e^x + C$$

$$\left(e^{u(x)}\right)' = e^{u(x)} u'(x)$$

$$\int e^{u(x)} u'(x)\, dx = e^{u(x)} + C$$

Here belongs 126 1 e

$$\int \frac{1}{x^2} e^{-\frac{1}{x}} dx = \int e^{u(x)} u'(x)\, dx$$

$u(x) = -\frac{1}{x}$ $\quad u'(x) = \frac{1}{x^2}$

$$= e^{u(x)} + C = e^{-\frac{1}{x}} + C$$

---

126 2a $\quad \int e^{1-3x} dx = -\frac{1}{3} \int e^{u(x)} u'(x)\, dx$

$\boxed{u(x) = 1-3x \quad u'(x) = -3}$

$$= -\frac{1}{3} e^{u(x)} + C$$

$$= -\frac{1}{3} e^{1-3x} + C$$

126 (2b) $\int x e^{x^2} dx = \frac{1}{2}\int e^{u(x)} u'(x) dx$

3 December, 2020 15:21

$$\boxed{u(x) = x^2 \quad u'(x) = 2x}$$

$= \frac{1}{2} e^{u(x)} + C =$

$= \frac{1}{2} e^{x^2} + C$

126 (2c) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = 2\int e^{u(x)} u'(x) dx$

$$\boxed{u(x) = \sqrt{x} \quad u'(x) = \frac{1}{2\sqrt{x}}}$$

$= 2 e^{u(x)} + C$

$= 2 e^{\sqrt{x}} + C$

Integration by parts

Integration is harder that derivation!! Why??

$$\left.\begin{aligned}(u+v)' &= u'+v' \\ (uv)' &= u'v+uv'\end{aligned}\right\}\ \text{formulas for sum \& prod}$$

Integration: there is formula for sum

$$\int (u+v)\,dx = \int u\,dx + \int v\,dx$$

There is no formula for product:

$$\int uv\,dx - ???$$

Partial integration is kind of analogue for the formula for product.

$$(uv)' = u'v + uv'$$

$$\underbrace{\int (uv)'dx}_{uv} = \int u'v\,dx + \int uv'dx$$

uv

$$\boxed{\int uv'\,dx = uv - \int u'v\,dx}$$

$$\boxed{\int_a^b uv'\,dx = [uv]_a^b - \int_a^b u'v\,dx}$$

It is not always obvious what should be taken as u and what as v'

Some indications:

$\int x^a e^x\,dx$ then $x^a = u$, $e^x = v'$

$\int x^a (\cos x)\,dx$ $x^a = u$ $\cos x = v'$

HW for Dec 8:

127, 128, try 129