I am solving 83. Those not interested may try 89.

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82 : expressed $(\cos x)^5$ as sum of first powers of cosines of multiple arguments

83 : want to express $\cos(5\theta)$ in terms of $\cos\theta$ and $\sin\theta$

Again use Euler's formula & binomial formula.

$$e^{\theta i} = \cos\theta + i\sin\theta$$

$$(e^{\theta i})^5 = e^{5\theta i} = \cos(5\theta) + i\sin(5\theta) \quad \checkmark$$

$$(e^{\theta i})^5 = (\cos\theta + i\sin\theta)^5 =$$

$$\boxed{(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5}$$

$$(\cos\theta)^5 + 5(\cos\theta)^4 \cdot i\sin\theta + 10(\cos\theta)^3(i\sin\theta)^2$$
$$+ 10(\cos\theta)^2(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 =$$
$$= (\cos\theta)^5 + 5i(\cos\theta)^4\sin\theta - 10(\cos\theta)^3(\sin\theta)^2$$
$$- 10i(\cos\theta)^2(\sin\theta)^3 + 5\cos\theta(\sin\theta)^4 + i(\sin\theta)^5$$
$$= (\cos\theta)^5 - 10(\cos\theta)^3(\sin\theta)^2 + 5\cos\theta(\sin\theta)^4$$
$$+ i\left(5(\cos\theta)^4\sin\theta - 10(\cos\theta)^2(\sin\theta)^3 + (\sin\theta)^5\right)$$

$$\cos(5\theta) = (\cos\theta)^5 - 10(\cos\theta)^3(\sin\theta)^2 + 5\cos\theta(\sin\theta)^4$$
$$\sin(5\theta) = 5(\cos\theta)^4\sin\theta - 10(\cos\theta)^2(\sin\theta)^3 + (\sin\theta)^5$$

Ex 89 1

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$z^3 - 2z^2 - iz + 3 - i = 0$

Equation of degree 3

We know that this equation has a real root !!

Let us find this real root !!

Denote it $x \in \mathbb{R}$

$x^3 - 2x^2 - ix + 3 - i = 0$ — real, ~ imaginary

$(x^3 - 2x^2 + 3) + i(-x-1) = 0$

$\begin{cases} x^3 - 2x^2 + 3 = 0 & (1) \\ -x-1 = 0 & (2) \end{cases}$

(2) $\Rightarrow \boxed{x = -1}$ And this is a solution of (1) as well: $(-1)^3 - 2(-1)^2 + 3 = 0$

$z^3 - 2z^2 - iz + 3 - i = 0$ has a root $z_0 = -1$

A general theorem: If $f(z) = 0$ of degree n has a root z_0 then

$f(z) = (z - z_0)\, g(z)$, where $g(z)$ is of degree $n-1$

In our case $f(z) = z^3 - 2z^2 - iz + 3 - i$

$z_0 = -1$

So there must be $f(z) = (z+1)\, g(z)$

where $g(z)$ is of degree 2

$g(z) = az^2 + bz + c$. Let us determine a, b, c

$z^3 - 2z^2 - iz + 3 - i = (z+1)(az^2 + bz + c)$

$(z+1)(az^2 + bz + c) = az^3 + bz^2 + cz + az^2 + bz + c = az^3 + (a+b)z^2 + (b+c)z + c$

$\quad 1z^3 \quad - 2z^2 \quad - iz \quad + 3 - i$

$\begin{cases} a = 1 & (1) \\ a + b = -2 & (2) \end{cases}$ (1) $\Rightarrow a = 1$; (4) $\Rightarrow c = 3 - i$

$$\begin{cases} a = 1 & (1) \\ a + b = -2 & (2) \\ b + c = -i & (3) \\ c = 3 - i & (4) \end{cases}$$

(1) $\Rightarrow a = 1$

(4) $\Rightarrow c = 3 - i$

(2) $\Rightarrow b = -3$

(3) $\Rightarrow b = -3$

$$g(z) = z^2 - 3z + 3 - i$$

$$(z^3 - 2z^2 - iz + 3 - i) = (z+1)(z^2 - 3z + 3 - i) = 0$$

Let us solve $z^2 - 3z + 3 - i = 0$

$$\Delta = (-3)^2 - 4(3-i) = -3 + 4i$$

We must find δ such that $\delta^2 = -3 + 4i$

$\delta = u + vi \qquad \delta^2 = u^2 - v^2 + 2uvi$

$$\begin{cases} u^2 - v^2 = -3 & (1) \\ 2uv = 4 & (2) \\ u^2 + v^2 = 5 & (3) \end{cases}$$

$|\delta^2| = |\delta|^2 = u^2 + v^2$

$|\delta^2| = |-3 + 4i| = 5$

(1) + (3) $\quad 2u^2 = 2 \quad u^2 = 1 \quad u = \pm 1$

(3) − (1) $\quad 2v^2 = 8 \quad v^2 = 4 \quad v = \pm 2$

(2) $\Rightarrow$ $\quad u = 1 \quad v = 2$

$\quad u = -1 \quad v = -2$

$\boxed{\delta = 1 + 2i}$ (or $-1 - 2i$)

$$z_1 = \frac{3 - (1+2i)}{2} = 1 - i$$

$$z_2 = \frac{3 + (1+2i)}{2} = 2 + i$$

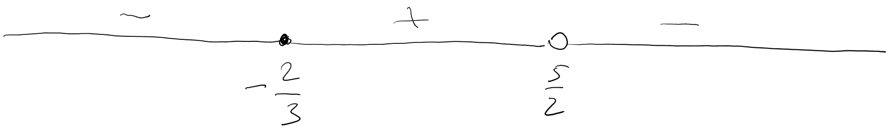
Recall that we had the root $z_0 = -1$. The set of solutions is $\{-1, 1-i, 2+i\}$

Ex 91 5 November, 2020 16:26 $f(x) = \sqrt{\dfrac{2+3x}{5-2x}}$

f is defined when $\dfrac{2+3x}{5-2x} \geq 0$

$2+3x = 0$ when $x = -\dfrac{2}{3}$

$5-2x = 0$ when $x = \dfrac{5}{2}$



The domain of definition of f is $[-\frac{2}{3}; \frac{5}{2}[$

$g(x) = \sqrt{x^2 - 2x - 5}$

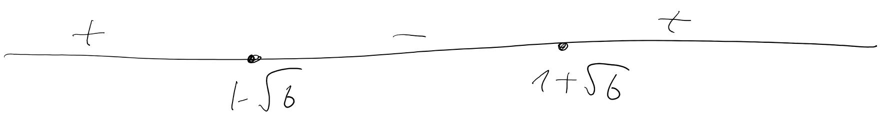
g is defined when $x^2 - 2x - 5 \geq 0$

$x^2 - 2x - 5 = 0 \qquad \Delta = 24$

$x_1 = \dfrac{2 - \sqrt{24}}{2} = 1 - \sqrt{6}$

$x_2 = \dfrac{2 + \sqrt{24}}{2} = 1 + \sqrt{6}$

$x^2 - 2x - 5 = (x - (1-\sqrt{6}))(x - (1+\sqrt{6}))$



The domain of definition of g is

$]-\infty; 1-\sqrt{6}] \cup [1+\sqrt{6}, +\infty[$

$h(x) = \ln(4x+3)$

h is defined at x when $4x+3 > 0$

$x > -\dfrac{3}{4}$

The domain of definition is

$]-\frac{3}{4}; +\infty[$

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$$\lim_{x\to a} \frac{3x^3+7x^2+2x}{x^3-4x}$$

a can be $+\infty$, $-\infty$, 0, 2, -2

Let us simplify the function by factorizing the numerator & the denominator.

$$3x^3+7x^2+2x = x\left(3x^2+7x+2\right)$$

Let us solve $3x^2+7x+2=0$

$\Delta = 25$ $\quad x_1 = \frac{-7-\sqrt{25}}{6} = -2$

$x_2 = \frac{-7+\sqrt{25}}{6} = -\frac{1}{3}$

$$3x^2+7x+2 = 3(x+2)\left(x+\frac{1}{3}\right) = (x+2)(3x+1)$$

$$3x^3+7x^2+2x = x(x+2)(3x+1)$$

$$x^3-4x = x\left(x^2-4\right) = x(x-2)(x+2)$$

$$\lim_{x\to a} \frac{3x^3+7x^2+2x}{x^3-4x} = \lim_{x\to a} \frac{x(x+2)(3x+1)}{x(x-2)(x+2)}$$

$$= \lim_{x\to a} \frac{3x+1}{x-2}$$

$$\lim_{x\to+\infty} \frac{3x+1}{x-2} = \lim_{x\to+\infty} \frac{3+\frac{1}{x}}{1-\frac{2}{x}} = \frac{3+0}{1-0} = 3$$

Same when $x \to -\infty$

General advice: when you have $x \to -\infty$ make variable change $x=-u$ so that $u \to +\infty$

Not so important for Q4 (1)
But very important Q4 (2) if we want do lim at $-\infty$ therein

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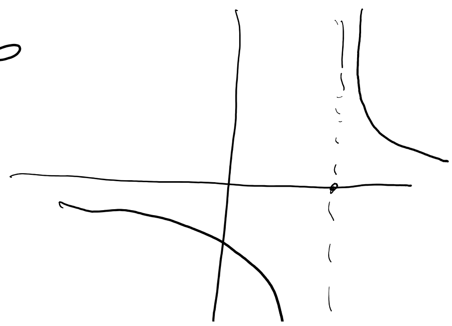
Q4 (1) lim at 0

$$\lim_{x\to 0} \frac{3x+1}{x-2} = -\frac{1}{2}$$

$$\lim_{x\to -2} \frac{3x+1}{x-2} = \frac{-5}{-4} = \frac{5}{4}$$

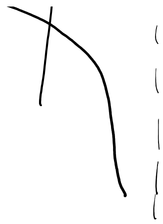
$$\lim_{x\to 2} \ ? \ ?$$

$$\lim_{x\to 2^-} \frac{3x+1}{x-2} = -\infty$$



$$\lim \frac{3x+1}{} = +\infty$$

$$\lim_{x \to 2^+} \frac{3x+1}{x-2} = +\infty$$



$$\lim_{x \to 2} \frac{3x+1}{x-2}$$ does not excist

94 (2) $\lim_{x\to+\infty}$ solved

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Let us try

$$\lim_{x\to-\infty} \frac{x-\sqrt{x^2+1}}{2x-\sqrt{4x^2+1}} = \lim_{u\to+\infty} \frac{-u-\sqrt{u^2+1}}{-2u-\sqrt{4u^2+1}}$$

$$\boxed{x=-u \quad \begin{matrix} x\to-\infty \\ u\to+\infty \end{matrix}}$$

$$= \lim_{u\to+\infty} \frac{u+\sqrt{u^2+1}}{2u+\sqrt{4u^2+1}} = \lim_{u\to+\infty} \frac{1+\frac{\sqrt{u^2+1}}{u}}{2+\frac{\sqrt{4u^2+1}}{u}}$$

$$\boxed{u=\sqrt{u^2} \quad (u>0)} \qquad \boxed{\frac{\sqrt{u^2+1}}{u} = \frac{\sqrt{u^2+1}}{\sqrt{u^2}} = \sqrt{\frac{u^2+1}{u^2}}}$$

$$= \lim_{u\to+\infty} \frac{1+\sqrt{\frac{u^2+1}{u^2}}}{2+\sqrt{\frac{4u^2+1}{u^2}}} = \lim_{u\to+\infty} \frac{1+\sqrt{1+\frac{1}{u^2}}}{2+\sqrt{4+\frac{1}{u^2}}}$$

$$= \frac{1+\sqrt{1+0}}{2+\sqrt{4+0}} = \frac{2}{4} = \frac{1}{2}$$

$$\lim_{x\to-\infty} = \frac{1}{2} \qquad \lim_{x\to+\infty} = 2$$

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$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} =$$

$$= \lim_{x\to 0} \frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})} =$$

$$= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})} = \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}$$

$$= \lim_{x\to 0} \frac{2}{\sqrt{1+x}+\sqrt{1-x}} = \frac{2}{\sqrt{1-0}+\sqrt{1+0}} = 1$$

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$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x}$$

$$= \lim_{x\to 0} \frac{(\sqrt{1+x}-\sqrt{1+x^2})(\sqrt{1+x}+\sqrt{1+x^2})}{x(\sqrt{1+x}+\sqrt{1+x^2})} =$$

$$= \lim_{x\to 0} \frac{(1+x)-(1+x^2)}{x(\sqrt{1+x}+\sqrt{1+x^2})} =$$

$$\lim_{x\to 0} \frac{x-x^2}{x(\sqrt{1+x}+\sqrt{1+x^2})} = \lim_{x\to 0} \frac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}}$$

$$x \to 0 \qquad x\left(\sqrt{1+x}+\sqrt{1+x^2}\right) \quad = \quad x \to 0 \quad \sqrt{1+x}+\sqrt{1+x^2}$$

$$= \frac{1-0}{\sqrt{1+0}+\sqrt{1+0^2}} = \frac{1}{2}$$

94 6 $\lim_{x \to 1} \frac{x-1}{x^n-1}$

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$n=1$ $\lim_{x \to 1} \frac{x-1}{x-1} = \boxed{1}$

$n=2$ $\lim_{x \to 1} \frac{x-1}{x^2-1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x+1)} =$

$= \lim_{x \to 1} \frac{1}{x+1} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$

$n=3$ $\lim_{x \to 1} \frac{x-1}{x^3-1}$??

$x^3-1 = (x-1)(x^2+x+1)$

$\lim_{x \to 1} \frac{x-1}{x^3-1} = \lim \frac{x-1}{(x-1)(x^2+x+1)} =$

$= \lim_{x \to 1} \frac{1}{x^2+x+1} = \frac{1}{1^2+1+1} = \boxed{\frac{1}{3}}$

$\lim_{x \to} \frac{x-1}{x^n-1} =$?? $\frac{1}{n}$

Sum of geome~~tric~~ progression

$1+x+\dots+x^{n-1} = \frac{1-x^n}{1-x}$ $(x \neq 1)$

$1 - x$

$$\boxed{\begin{array}{ll} n=2 & 1+x = \dfrac{1-x^2}{1-x} \\ n=3 & 1+x+x^2 = \dfrac{1-x^3}{1-x} \end{array}}$$

Let us prove this formula.

Denote $S = 1 + x + x^2 + \dots + x^{n-2} + x^{n-1}$

Then $xS = x + x^2 + x^3 + \dots + x^{n-1} + x^n$

$$S - xS = 1 - x^n$$

$$S(1-x) = 1 - x^n \qquad S = \frac{1-x^n}{1-x}$$

Return to the limit

$$\lim_{x\to 1} \frac{x-1}{x^n - 1} = \lim_{x \to 1} \frac{x-1}{(x-1)(x^{n-1} + x^{n-2} + \dots + x + 1)}$$

$$= \lim_{x\to 1} \frac{1}{x^{n-1} + x^{n-2} + \dots + x + 1} =$$

$$= \frac{1}{1^{n-1} + 1^{n-2} + \dots + 1 + 1} =$$

$$= \frac{1}{1+1 \dots} = \frac{1}{\dots}$$

$$= \frac{}{\underbrace{1+1+\ldots+1+1}_{n}} = \frac{1}{n}$$

At home: Ex 95, 96