127, 128, 129

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int\left(1-\frac{1}{1+x^2}\right)dx = x - \arctan x + C$$

$$\boxed{\int \frac{1}{1+x^2}\,dx = \arctan x + C}$$

$$\int \frac{(\cos x)^3}{(\sin x)^5}\,dx = \int \frac{(\cos x)^2}{(\sin x)^5}(\sin x)'\,dx$$

$$\boxed{(\sin x)^2 + (\cos x)^2 = 1}$$

$$= \int \frac{1-(\sin x)^2}{(\sin x)^5}(\sin x)'\,dx =$$

$$= \int\left((\sin x)^{-5} - (\sin x)^{-3}\right)(\sin x)'\,dx$$

$$= \frac{\sin x^{-4}}{-4} - \frac{(\sin x)^{-2}}{-2} + C =$$

$$= -\frac{1}{4(\sin x)^4} + \frac{1}{2(\sin x)^2} + C$$

$$\int (\cos x)^2\,dx$$

$$(\cos x)^2 = \frac{1+\cos(2x)}{}$$

$$2$$

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$2\cos x = e^{ix} + e^{-ix}$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$(\cos x)^2 = \frac{1}{4}\left(e^{ix} + e^{-ix}\right)^2 = \frac{1}{4}\left(e^{2ix} + 2 + e^{-2ix}\right)$$

$$= \frac{1}{4}\left(e^{2ix} + e^{-2ix} + 2\right) = \frac{1}{4}\left(2\cos(2x) + 2\right)$$

$$= \frac{\cos(2x) + 1}{2}$$

$$\int (\cos x)^2\,dx = \frac{1}{2}\int \left(\cos(2x) + 1\right)dx =$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2x + x\right) + C$$

$$= \frac{1}{4}\sin 2x + \frac{1}{2}x + C$$

12.8 Integration by parts

$$\int u v' dx = uv - \int u' v \, dx$$

$$\int_a^b u v' dx = \left[uv\right]_a^b - \int_a^b u' v \, dx$$

$$\int_0^2 \underbrace{(3x+2)}_{u} \underbrace{e^{-x}}_{v'} dx = \int_0^2 (3x+2)\left(-e^{-x}\right)' dx$$

$$e^{-x} = (\underbrace{-e^{-x}}_{v})'$$

$$= -\left[(3x+2)e^{-x}\right]_0^2 - \int_0^2 (3x+2)'\left(-e^{-x}\right) dx =$$

$$= -\left(8e^{-2} - 2\right) + 3\int_0^2 e^{-x} dx =$$

$$= 2 - 8e^{-2} - 3\left[e^{-x}\right]_0^2 = 2 - 8e^{-2} - 3\left(e^{-2} - 1\right)$$

$$= 5 - 11e^{-2}$$

$$\int_0^1 \frac{t}{\sqrt{t+1}} dt = \int_0^1 \underset{u}{t} \cdot \underbrace{\frac{1}{\sqrt{t+1}}}_{v'} dt =$$

$$= 2\int_0^1 t \left(\sqrt{t+1}\right)' dt = 2\left[t\sqrt{t+1}\right]_0^1 - 2\int_0^1 t' \sqrt{t+1} \, dt$$

$$= 2\sqrt{2} - 2\int_0^1 (t+1)^{\frac{1}{2}} dt = 2\sqrt{2} - 2\left[\frac{(t+1)^{3/2}}{\ldots}\right]^1$$

$$= 2\sqrt{2} - 2\int_0^1 (t+1)^{\frac{1}{2}}\,dt = 2\sqrt{2} - 2\left[\frac{(t+1)^{3/2}}{3/2}\right]_0^1$$

$$= 2\sqrt{2} - \frac{4}{3}\left(2\sqrt{2} - 1\right) = \frac{4}{3} - \frac{2}{3}\sqrt{2}$$

Another method

$$\int_0^1 \frac{t}{\sqrt{t+1}}\,dt = \int_0^1 \frac{t+1-1}{\sqrt{t+1}}\,dt =$$

$$\int_0^1 \left(\sqrt{t+1} - \frac{1}{\sqrt{t+1}}\right) dt = \int_0^1 \left((t+1)^{\frac{1}{2}} - (t+1)^{-\frac{1}{2}}\right) dt$$

$$\left[\frac{(t+1)^{\frac{3}{2}}}{3/2} - \frac{(t+1)^{1/2}}{1/2}\right]_0^1 =$$

$$= \left[\frac{2}{3}(t+1)^{\frac{3}{2}} - 2(t+1)^{\frac{1}{2}}\right]_0^1 = \frac{4}{3}\sqrt{2} - 2\sqrt{2} - \left(\frac{2}{3} - 2\right) =$$

$$= \frac{4}{3} - \frac{2}{3}\sqrt{2}$$

$$\int_0^1 x^2 e^x\,dx = \underbrace{\int_0^1 x^2 (e^x)'\,dx}_{\text{good}} =$$

$$\cancel{\underbrace{\int_0^1 \left(\frac{x^3}{3}\right)' e^x\,dx}_{\text{bad}}}$$

u

$$\left[x^2 e^x\right]_0^1 - \int_0^1 (x^2)' e^x\,dx$$

$$= e - 2\int_0^1 x e^x\,dx =$$

$$= e - 2\int_0^1 x (e^x)'\,dx =$$

$$e - 2\left[x e^x\right]_0^1 + 2\int_0^1 x' e^x\,dx$$

$$= e - 2e + 2\int_0^1 e^x\,dx = -e + 2\left[e^x\right]_0^1 =$$

$$= e - 2$$

Similar:

$$\int_0^\pi \underset{u}{t^2}\, \underbrace{\cos t}_{v'}\,dt = \int_0^\pi t^2 (\sin t)'\,dt =$$

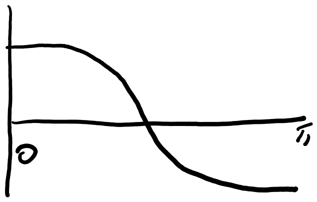
$$\left[t^2 \sin t\right]_0^\pi - \int_0^\pi (t^2)' \sin t\,dt =$$

$$-2\int_0^\pi t\sin t\,dt = 2\int_0^\pi t(\cos t)'\,dt$$

$$= \left[2t\cos t\right]_0^\pi - 2\int_0^\pi t'\cos t\,dt =$$

$$= -2\pi - 2\int_0^\pi \cos t\,dt = -2\pi - 2\left[\sin t\right]_0^\pi$$

$$= -2\pi$$



$y = \cos x$

$y = x^2\cos x$

Should expect negative number

8 December, 2020 08:57

Morale: $\int p(x)\, e^{ax}\, dx$

$p(x)$ polynomial $\int p(x) \cos(ax)\, dx$

$\int p(x) \sin(ax)\, dx$

Always take $p(x) = u$

$e^{ax}, \cos(ax), \sin(ax) = v'$

$$y = \boxed{\int_0^x \underbrace{e^t}_{u} \underbrace{\cos t}_{v'}\, dt} = \int_0^x e^t (\sin t)'\, dt$$

$$= \left[e^t \sin t\right]_0^x - \int_0^x (e^t)' \sin t\, dt =$$

$$= e^x \sin x - \int_0^x e^t \sin t\, dt$$

$$= e^x \sin x + \int_0^x e^t (\cos t)'\, dt =$$

$$= e^x \sin x + \left[e^t \cos t\right]_0^x - \int_0^x (e^t)' \cos t\, dt$$

$$= e^x \sin x + e^x \cos x - 1 - \boxed{\int_0^x e^t \cos t\, dt}$$

$$y = e^x \sin x + e^x \cos x - 1 - y$$

$$2y = e^x \sin x + e^x \cos x - 1$$

$$y = \frac{1}{2}\left(e^x \sin x + e^x \cos x - 1\right)$$

I suggest: calculate the same integral $\int_0^x e^t \cos t \, dt$

taking $\cos t$ as u and e^t as v'

8 December, 2020 09:09

$$\int_1^2 x^2 \ln x \, dx = \int_1^2 \left(\frac{x^3}{3}\right)' \ln x \, dx =$$

$\boxed{\ln x \text{ always } u}$

$$\left[\frac{x^3}{3} \ln x\right]_1^2 - \frac{1}{3}\int_1^2 x^3 (\ln x)' \, dx$$

$$= \frac{8}{3}\ln 2 - \frac{1}{3}\int_1^2 x^2 \, dx = \frac{8}{3}\ln 2 - \frac{1}{9}\left[x^3\right]_1^2 =$$

$$= \frac{8}{3}\ln 2 - \frac{7}{9}$$

$$\int_1^2 \ln t \, dt = \int_1^2 \underbrace{1}_{s'} \cdot \underbrace{\ln t}_{u} \, dt =$$

$$\int_1^2 t' \ln t \, dt = \left[t \ln t\right]_1^2 - \int_1^2 t (\ln t)' \, dt$$

$$= 2\ln 2 - \int_1^2 dt = 2 \ln t - \left[t\right]_1^2 =$$

$$= 2\ln 2 - 1$$

8 December, 2020 09:37

$$\int_0^{\frac{\pi}{4}} \frac{x}{(\cos x)^2}\,dx = \int_0^{\frac{\pi}{4}} x(\tan x)'\,dx =$$

$$\boxed{\frac{1}{(\cos x)^2} = \tan' x}$$

$$= \left[x\tan x\right]_0^{\frac{\pi}{4}} -$$

$$- \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos x}\,dx =$$

$$= \frac{\pi}{4} + \int_0^{\frac{\pi}{4}} \frac{(\cos x)'}{\cos x}\,dx = \frac{\pi}{4} + \left[\ln\cos x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}} = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

129 $I_n = \int_1^e x^2 (\ln x)^n dx$

8 December, 2020 09:43

$$I_0 = \int_1^e x^2 dx = \left[\frac{x^3}{3}\right]_1^e = \frac{e^3 - 1}{3}$$

$$I_1 = \int_1^e x^2 \ln x \, dx = \int_1^e \left(\frac{x^3}{3}\right)' \ln x \, dx$$

$$= \left[\frac{x^3}{3} \ln x\right]_1^e - \frac{1}{3} \int_1^e x^2 dx$$

$$= \frac{e^3}{3} - \frac{1}{3} I_0 = \frac{e^3}{3} - \frac{1}{3} \frac{e^3 - 1}{3} =$$

$$= \frac{2}{9} e^3 + \frac{1}{9}$$

We expressed I_1 in terms of I_0

$$I_1 = \frac{e^3}{3} - \frac{1}{3} I_0$$

In a similar fashion one can express I_{n+1} in terms of I_n

$$I_{n+1} = \int_1^e x^2 (\ln x)^{n+1} dx =$$

...

$$\int_1^e \left(\frac{x^3}{3}\right)' (\ln x)^{n+1}\, dx = \left[\frac{x^3}{3}(\ln x)^{n+1}\right]_1^e -$$

$$-\frac{1}{3}\int_1^e x^3 \left((\ln x)^{n+1}\right)' dx =$$

$$= \frac{e^3}{3} - \frac{n+1}{3}\int_1^e x^3 \cdot (\ln x)^n \cdot \frac{1}{x}\, dx$$

$$= \frac{e^3}{3} - \frac{n+1}{3}\underbrace{\int_1^e x^2 (\ln x)^n\, dx}_{I_n}$$

$$I_{n+1} = \frac{e^3}{3} - \frac{n+1}{3} I_n$$

holds true for $n \geq 0$

$$I_0 = \frac{e^3 - 1}{3} \qquad I_1 = \frac{2}{9}e^3 + \frac{1}{9}$$

$$I_2 = \frac{e^3}{3} - \frac{2}{3} I_1 = \frac{e^3}{3} - \frac{2}{3}\left(\frac{2}{9}e^3 + \frac{1}{9}\right)$$

$$= \frac{5}{27}e^3 - \frac{2}{27}$$

$I_3, I_4, \ldots$

Variable change

8 December, 2020 09:56

did alredy (ex 126 !!)

derivation	integration
$(u+v)' = u'+v'$	$\int (u+v)\,dx = \int u\,dx + \int v\,dx$
$(uv)' = uv' + u'v$	$\int uv'\,dx = uv - \int u'v\,dx$
$(u \circ v(x))' = u'(v(x)) \cdot v'(x)$	$\int u'(v(x))\, v'(x)\,dx = u \circ v(x) + C$

Variable change with limits

$$\int_a^b u(v(x))\, v'(x)\,dx = \int_{v(a)}^{v(b)} u(x)\,dx$$

To remember, it is useful to write

$$\boxed{v'(x)\,dx = dv}$$

$$\int_a^b u(v(x))\, \underbrace{v'(x)\,dx} = \int_{v(a)}^{v(b)} u(v)\,dv$$

Examples:

134 $\int^{\pi} (\cos x)^3\,dx = \int^{\pi} (\cos x)^2 (\sin x)'\,dx$

$$\int_0 (\cos x) \, dx = \int_0 (\cos x) \, (\sin x)' \, dx$$

$$= \int_0^{\pi} \left(1 - (\sin x)^2\right) (\sin x)' \, dx$$

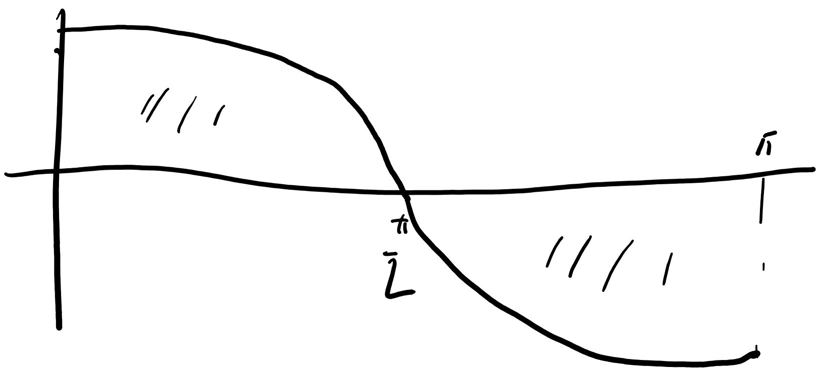
$\sin x = v \qquad x = 0 \quad v = 0$

$\qquad\qquad\qquad x = \pi \quad v = 0$

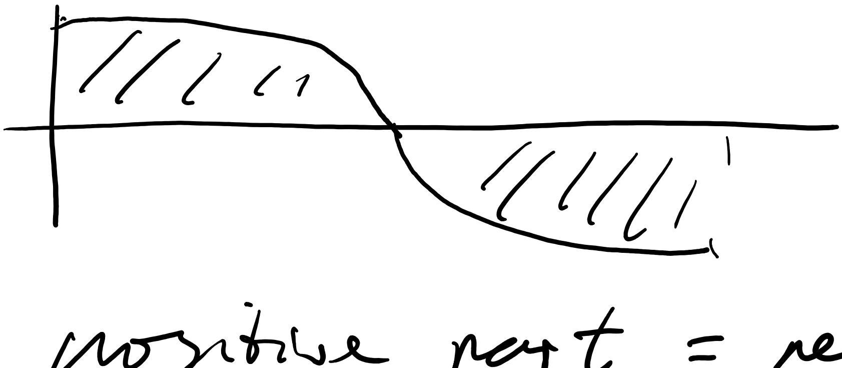
$dv = (\sin x)' \, dx$

$$= \int_0^0 (1 - v^2) \, dv = 0$$

Geometric explanation why it is 0



$y = \cos x$



$y = (\cos x)^3$

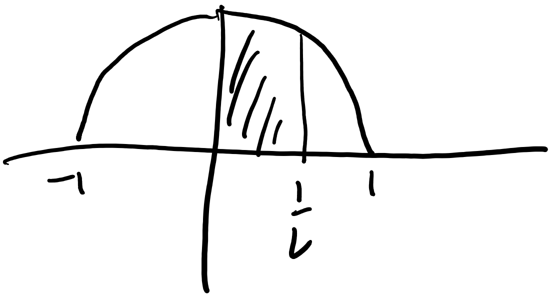
positive part = negative part.

136 (2)

8 December, 2020 10:12

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$$

what is $y=\sqrt{1-x^2}$



$x^2+y^2=1 \quad y\geq 0$

$x=\cos u \qquad u=\arccos x$

$x=0 \qquad u=\frac{\pi}{2}$

$x=\frac{1}{2} \qquad u=\frac{\pi}{3}$

$dx=(\cos u)'\,du=-\sin u\,du$

$$\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx=\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{1-(\cos u)^2}\,(-\sin u)\,du$$

$$\boxed{\begin{array}{l}\sqrt{1-(\cos u)^2}=\sqrt{(\sin u)^2}=|\sin u|=\sin u \\ \text{because} \\ \text{for } u\in\left[\frac{\pi}{3};\frac{\pi}{2}\right] \quad \sin u\geq 0\end{array}}$$

π

$$= -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} (\sin u)^2 \, du = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (\sin u)^2 \, du =$$

$$\boxed{(\sin u)^2 = \frac{1-\cos(2u)}{2}}$$

$$\frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1-\cos 2u)\, du = \frac{1}{2}\left[u - \frac{1}{2}\sin 2u\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} =$$

$$= \frac{1}{2}\left(\frac{\pi}{2} - \left(\frac{\pi}{3} - \frac{1}{2}\,\frac{\sqrt{3}}{2}\right)\right) =$$

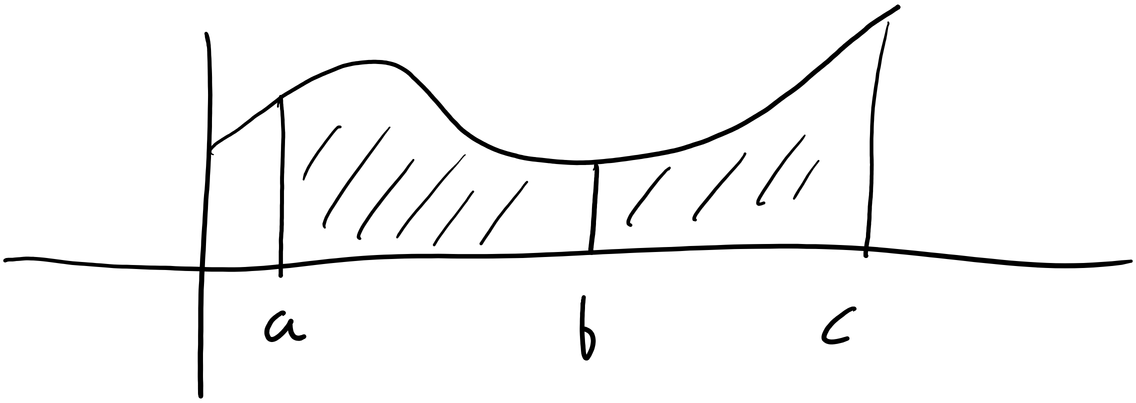
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{8}$$

Do this yourselves with $x = \sin u$

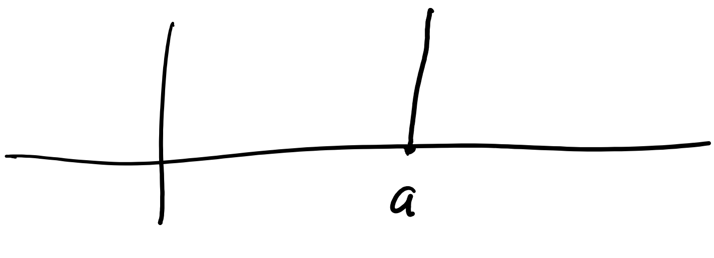
Remark $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

Why ??

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = \int_a^a f(x)\,dx$$
$$= 0$$



Integration of rational functions

rational function $\frac{f(x)}{g(x)}$

$f(x)$, $g(x)$ polynomials

Ex 138

$$\int_{-1}^{1} \frac{1}{x^2+2x+5}\, dx = \frac{\pi}{8}$$

$$\int_{-1}^{1} \frac{x}{x^2+2x+5}\, dx$$

x^2+2x+5 $\qquad \Delta = -16$

does not factor

Can express it as sum of squares

$$x^2+2x+5 = x^2+2x+1+4 = (x+1)^2+2^2$$

$$\int_{-1}^{1} \frac{dx}{(x+1)^2+2^2} =$$

$$\boxed{\begin{array}{ll} x+1=u & \\ x=-1 & u=0 \\ x=1 & u=2 \\ dx=du & \end{array}}$$

$$= \int_0^2 \frac{du}{u^2+2^2} =$$

If it were $\dfrac{1}{u^2+1}$, we get arctan

One more variable change

$u = 2v$ $\quad u=0 \; v=0$

$u=2 \; v=1$

$du = 2\,dv$

$$= \int_0^1 \frac{2\,dv}{(2v)^2+2^2} = \frac{2}{2^2}\int_0^1 \frac{dv}{v^2+1} =$$

$$= \frac{1}{2}\left[\arctan v\right]_0^1 = \frac{\pi}{8}$$

$$\int_{-1}^{1} \frac{dx}{x^2+2x+5} = \frac{\pi}{8}$$

8 December, 2020 10:38

$$\int_{-1}^{1} \frac{x}{x^2+2x+5}\,dx =$$

$$(x^2+2x+5)' = 2x+2 = 2(x+1)$$

$$\int_{-1}^{1} \frac{x+1}{x^2+2x+5}\,dx = \frac{1}{2}\int_{-1}^{1} \frac{(x^2+2x+5)'}{x^2+2x+5}\,dx$$

$$= \frac{1}{2}\Big[\ln(x^2+2x+5)\Big]_{-1}^{1} = \frac{1}{2}(\ln 8 - \ln 4)$$

$$= \frac{1}{2}\ln 2$$

$$\int_{-1}^{1} \frac{x}{x^2+2x+5}\,dx = \int_{-1}^{1} \frac{x+1-1}{x^2+2x+5}\,dx$$

$$= \int_{-1}^{1} \frac{x+1}{x^2+2x+5}\,dx - \int_{-1}^{1} \frac{1}{x^2+2x+5}\,dx =$$

$$= \frac{1}{2}\ln 2 - \frac{\pi}{8}$$

HW 131, 132, 135, 136 3

137, 139