Ex 95

$$f(x) = \begin{cases} -x^2, & x < 0 \\ c, & x = 0 \\ ax+b, & x > 0 \end{cases}$$

f is continuous on $]-\infty; 0[$ because on this interval $f(x) = x^2$, which is a continuous function

f is continuous on $]0; +\infty[$, because on this interval $f(x) = ax+b$, again a continuous function.

We are left with $x = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x^2) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (ax+b) = b$$

$$f(0) = c$$

f is continuous at 0 if & only if the 3 numbers above are equal

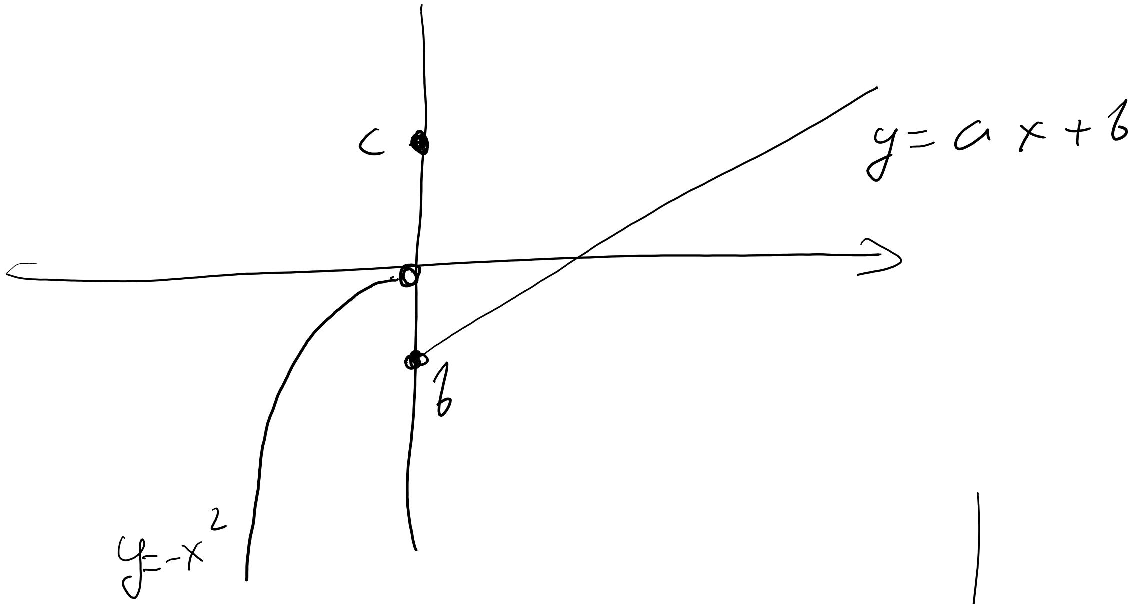
$$0 = b = c$$

The set of parameters $(a, b, c)$ such that f is continuous is

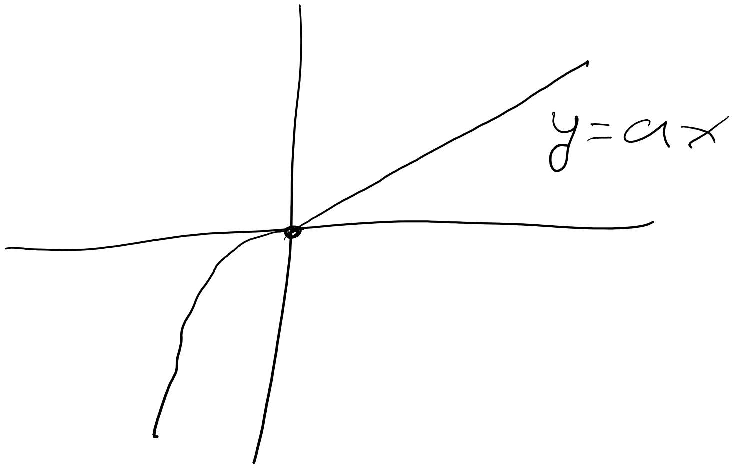
$$\{(a, 0, 0) : a \in \mathbb{R}\}$$

---

arbitrary $a$, $b$, $c$



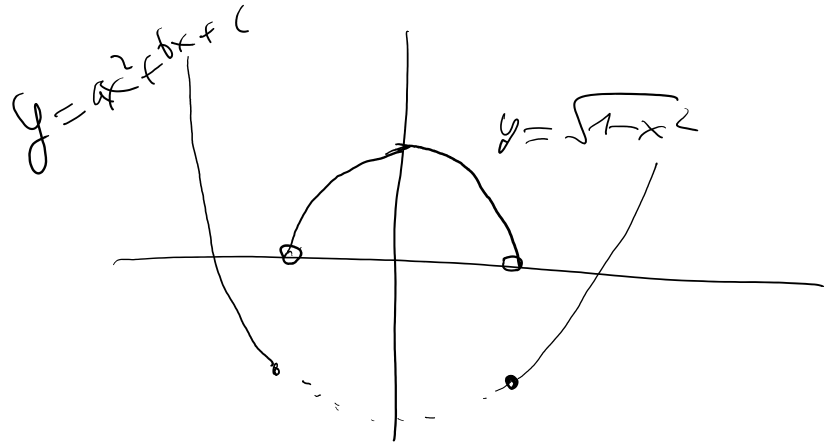

$b = c = 0$



Ex 96

10 November, 2020 08:16

$$f(x) = \begin{cases} \sqrt{1-x^2}, & x \in ]-1;1[ \\ ax^2+bx+c, & x \in ]-\infty;-1] \cup [1;+\infty[ \end{cases}$$




$f$ is continuous on each of the intervals $]-\infty;-1[$, $]-1;1[$ and $]1;+\infty[$.

At $-1$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (ax^2+bx+c) = a-b+c$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{1-x^2} = 0$$

$$f(x) = a-b+c$$

$f$ is continuous at $-1$ if & only if

$$a-b+c=0$$

---

At 1 $\quad$ $f$ is continuous if & only if

$$a + b + c = 0$$

$f$ is continuous on $\mathbb{R}$

$$\Updownarrow$$

$$\begin{cases} a - b + c = 0 \\ a + b + c = 0 \end{cases}$$

Let us solve the system

$$L_2 \to L_2 - L_1$$

$$\begin{cases} a - b + c = 0 \\ 2b = 0 \end{cases}$$

$b = 0$

$a + c = 0$

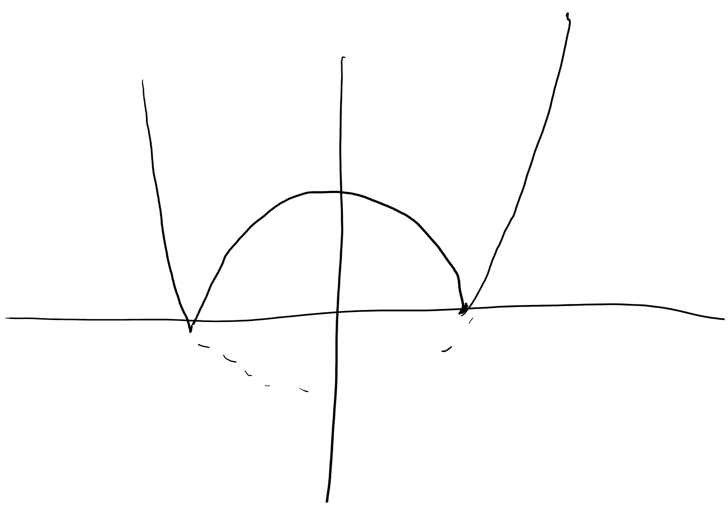
$a = -c$

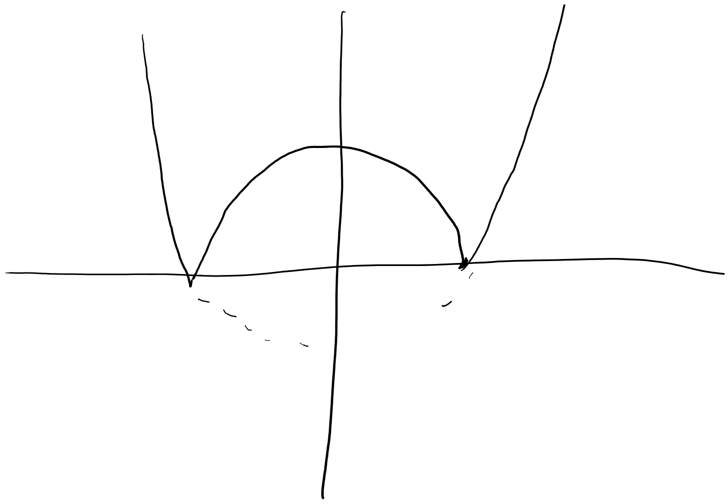
The set of solutions is

$$\{ (-c, 0, c) : c \in \mathbb{R} \}$$

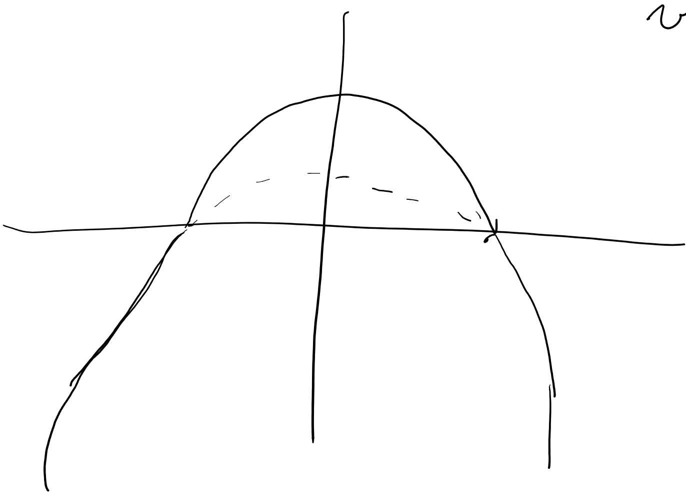
This is the set of parameters $(a, b, c)$ such that $f$ is continuous

$a > 0$





when $a < 0$



Ex97

10 November, 2020 09:07

$f(x) = \frac{1}{1+x^2}$

f defines a bijection of $[0;+\infty[$ to some interval J, to be determined.

$f'(x) = \frac{-2x}{(1+x^2)^2}$

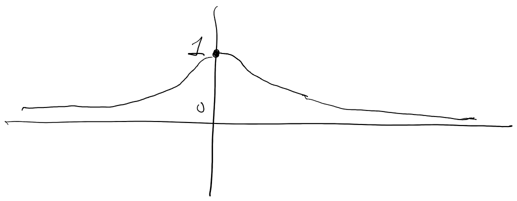
$f'(x) < 0$ on $]0;+\infty[$

f is strictly decreasing on $[0;+\infty[$

$f(0) = 1$ $\quad \lim_{x \to +\infty} f(x) = 0$

f defines a bijection

$[0;+\infty[ \to ]0;1]$




I do not like notation $f^{-1}$ for the inverse function, because it can be confused with $\frac{1}{f}$

$g: ]0;1] \to [0,+\infty[$ be the invers function for f

both f and g are strictly decreasing.

Let us determine g explicitly

$\frac{1}{1+x^2} = y$

The method:

1. Express x in terms of y
2. Swap x & y

$1 = y(1+x^2)$

$1+x^2 = \frac{1}{y}$ $\quad y \in ]0,1]$, $y \neq 0$

$x^2 = \frac{1}{y} - 1$ $\quad x = \pm\left(\frac{1}{y} - 1\right)^{\frac{1}{2}}$

Since ... $x = \left(\frac{1}{...} ...\right)^{\frac{1}{2}}$

Break until 9:40

$y \; - 1 \qquad x = \pm\left(\frac{1}{y} - 1\right)$

Since $x \geq 0 \qquad x = \left(\frac{1}{y} - 1\right)^{\frac{1}{2}}$

$g(x) = \left(\frac{1}{x} - 1\right)^{\frac{1}{2}}$

# Ex98

10 November, 2020 09:36

$f(x) = |x-1|$

continuity & derivability at

$x = 0, 1, 2$

$$f(x) = \begin{cases} -x+1, & x \in ]-\infty; 1[ \\ 0, & x = 1 \\ x-1, & x \in ]1; +\infty[ \end{cases}$$

f is continuous and differentiable on $]-\infty; -1[$ and $]1; +\infty[$

In particular, it is continuous & differentiable at $x=0$ and at $x=2$

on $]-\infty; 1[$ $f(x) = -x+1$

$f'(x) = -1$ In particular

$\boxed{f'(0) = -1}$

On $]1; +\infty[$ $f(x) = x-1$

$f'(x) = 1$ in particular
$f'(2) = 1$

---

we are left with $x = 1$

Continuity:

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-x+1) = 0$$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x-1) = 0$$

$$f(1) = 0$$

$f$ is continuous at $x = 1$

---

Derivability:

$$f'_-(1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x-1} =$$

$$\lim_{x \to 1^-} \frac{-x+1-0}{x-1} = -1$$

$$f'_+(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} =$$

$$\lim \frac{x-1-0}{} \quad 1$$

$$= \lim_{x \to 1^+} \frac{x-1-0}{x-1} = 1$$

$$f'_-(1) \neq f'_+(1)$$

f is not differentiable at 1

Ex 100

10 November, 2020 09:54

$$f(x) = \begin{cases} 0, & x = 0 \\ x^2 \sin\frac{1}{x}, & x \neq 0 \end{cases}$$

$f(x) = x^2 \sin\frac{1}{x}$ on $]-\infty;0[$

and on $]0;+\infty[$

f is continuous and differentiable

on $]-\infty;0[$ and on $]0,+\infty[$

$$f'(x) = 2x \sin\frac{1}{x} + x^2\left(\cos\frac{1}{x}\right) \cdot \frac{-1}{x^2}$$

$$= 2x \sin\frac{1}{x} - \cos\frac{1}{x}$$

on $]-\infty;0[$ and on $]0;+\infty[$

---

Let us study $x=0$

Continuity:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^2 \sin\frac{1}{x} \quad ??$$

We have to use the "policemen rule"

$$\left|\sin\frac{1}{x}\right| \leq 1 \qquad |f(x)| \leq x^2$$

$$-x^2 \leq f(x) \leq x^2$$

$$\lim_{x\to 0^-}(-x^2) = \lim_{x\to 0^-} x^2 = 0$$

$$\lim_{x\to 0^-} f(x) = 0$$

Totally similarly

$$\lim_{x\to 0^+} f(x) = 0$$

$$f(0) = 0$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$f$ is continuous at $0$.

---

Derivability at $0$

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} =$$

$$\lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x} = \lim_{x\to 0} x \sin\frac{1}{x}$$

$$\left|x \sin\frac{1}{x}\right| \le |x| \quad \text{because } \left|\sin\frac{1}{x}\right| \le 1$$

$$-|x| \le x \sin\frac{1}{x} \le |x|$$

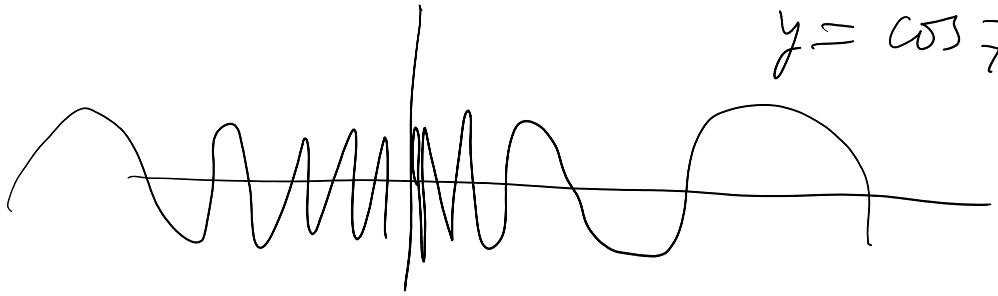
$$\lim_{x\to 0} |x| = 0 \qquad \lim_{x\to 0} (-|x|) = 0$$

$$\lim_{x\to 0} x \sin\frac{1}{x} = 0 \qquad f'(0) = 0$$

$$f'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

$\lim_{x \to 0} f(x)$ does not exist

$y = \cos \frac{1}{x}$



Ex 103

10 November, 2020 10:08

$$\lim_{x \to 1} \frac{\sin(\pi x)}{x-1} = \lim_{x \to 1} \frac{f(x)-f(1)}{x-1} =$$

$$\boxed{f(x) = \sin(\pi x) \qquad f(1) = 0}$$

$$= f'(1) = \pi \cos \pi = -\pi$$

$$f'(x) = \pi \cos(\pi x)$$

$$\lim_{x \to 0} \frac{e^{2x}-1}{x} = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$$

$$\boxed{f(x) = e^{2x} \qquad f(0) = 1}$$

$$= f'(0) = 2e^{2 \cdot 0} = 2$$

$$\boxed{f'(x) = 2e^{2x}}$$

Ex 105

10 November, 2020 10:33

$$\lim_{x \to +\infty} \left(1+\frac{1}{2x}\right)^x$$

$$\ln\left(\left(1+\frac{1}{2x}\right)^x\right) = x \ln\left(1+\frac{1}{2x}\right)$$

$$\lim_{x \to +\infty} x \ln\left(1+\frac{1}{2x}\right)$$

$$\boxed{\frac{1}{2x} = u \qquad x \to +\infty \quad u \to 0^+ \qquad x = \frac{1}{2u}}$$

$$= \lim_{u \to 0^+} \frac{\ln(1+u)}{2u} = \frac{1}{2} \lim_{u \to 0^+} \frac{\ln(1+u)}{u} = \frac{1}{2}$$

$$\lim_{x \to +\infty} \left(1+\frac{1}{2x}\right)^x = \lim_{x \to +\infty} e^{x \ln\left(1+\frac{1}{2x}\right)}$$

$$\boxed{\left(1+\frac{1}{2x}\right)^x = e^{x \ln\left(1+\frac{1}{2x}\right)}}$$

$$= e^{\lim_{x \to +\infty} x \ln\left(1+\frac{1}{2x}\right)} = e^{1/2} = \sqrt{e}$$

---

finish 103, 105

106, 104