131 132 135 136(3) 137 139

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(131) $\int_0^{\frac{\pi}{6}} (\cos x)^2 (\sin x)^5 \, dx = \int_0^{\frac{\pi}{2}} (\cos x)^2 (\sin x)^4 \cdot \sin x \, dx$

$-\int_0^{\frac{\pi}{6}} (\cos x)^2 (\sin x)^4 (\cos x)' \, dx =$

$= -\int_0^{\frac{\pi}{6}} (\cos x)^2 \left(1-(\cos x)^2\right)^2 (\cos x)' \, dx$

$u = \cos x \qquad x = 0 \quad u = 1 \qquad du = (\cos x)' dx$

$\qquad\qquad x = \frac{\pi}{6} \quad u = \frac{\sqrt{3}}{2}$

$= -\int_1^{\frac{\sqrt{3}}{2}} u^2 (1-u^2)^2 \, du = \int_{\frac{\sqrt{3}}{2}}^1 (u^2 - 2u^4 + u^6) \, du$

$= \left[\frac{u^3}{3} - \frac{2}{5}u^5 + \frac{u^7}{7}\right]_{\frac{\sqrt{3}}{2}}^1 =$

$\frac{1}{3} - \frac{2}{5} + \frac{1}{7} - \left(\frac{\sqrt{3}}{8} - \frac{18}{5}\sqrt{3} + \frac{27}{7}\sqrt{3}\right)$

$= \dots$

$$(\sin x)^4 = ((\sin x)^2)^2 = (1-(\cos x)^2)^2$$

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132 $\int_0^\pi (\cos x)^4 \, dx$

Using complex numbers

$$2\cos x = e^{xi} + e^{-xi}$$

$$\cos x = \frac{1}{2}(e^{xi} + e^{-xi})$$

$$(\cos x)^4 = \frac{1}{16}(e^{xi} + e^{-xi})^4$$

$$= \frac{1}{16}(e^{4xi} + 4e^{2xi} + 6 + 4e^{-2xi} + e^{-4xi})$$

$$= \frac{1}{16}(e^{4xi} + e^{-4xi} + 4e^{2xi} + 4e^{-2xi} + 6)$$

$$= \frac{1}{16}(2\cos 4x + 8\cos 2x + 6) =$$

$$= \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$$

$$\int_0^\pi (\cos x)^4 \, dx = \int_0^\pi \left(\frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}\right) dx$$

$$= \left[\frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3}{8}x\right]_0^\pi = \frac{3}{8}\pi$$

Pascal triangle

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0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$e^{xi} = \cos x + i \sin x$$

$$e^{-xi} = \cos x - i \sin x$$

$$2\cos x = e^{xi} + e^{-xi}$$

$$2i \sin x = e^{xi} - e^{-xi}$$

$$e^{3xi} + e^{-3xi} = 2\cos 3x$$

etc....

(135)

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$$\int_0^1 \frac{e^t}{e^{2t}+1}\,dt \qquad u=e^t \quad du=e^t dt$$

$t=0 \quad u=1$

$t=1 \quad u=e$

$$= \int_1^e \frac{1}{u^2+1}\,du = \left[\arctan u\right]_1^e =$$

$$= \arctan e - \arctan 1 = \arctan e - \frac{\pi}{4}$$

---

136 (3), 137

Let us do 137

$$\int_{-1}^1 \sqrt{4-x^2}\,dx$$

If it were $\sqrt{1-x^2}$ then $x=\sin u$ would do

For $\sqrt{4-x^2}$ take $x=2\sin u$

In general for $\sqrt{a^2-x^2}$ take $x=a\sin u$

$x=2\sin u \qquad dx = 2\cos u\,du$

$u=\arcsin\frac{x}{2} \qquad x=-1 \quad u=\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$x=1 \quad u=\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}$

$$\int_{-1}^1 \sqrt{4-x^2}\,dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{4-(2\sin u)^2}\;2\cos u\,du$$

$$= 2\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{4-4(\sin u)^2}\;\cos u\,du =$$

$$= 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{1-(\sin u)^2}\;\cos u\,du =$$

$$= 4\int^{\frac{\pi}{6}} \sqrt{(\cos u)^2}\;\cos u\,du =$$

$$= 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \sqrt{(\cos u)^2} \cos u \, du =$$

$\cos u \geq 0$ for $u \in [-\frac{\pi}{6}, \frac{\pi}{6}]$

$$= 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} |\cos u| \cos u \, du$$

$$= 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos u)^2 \, du = 4 \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{1}{2}\cos 2u + \frac{1}{2}\right) du$$

$$\cos u = \frac{1}{2}\left(e^{ui} + e^{-ui}\right)$$

$$(\cos u)^2 = \frac{1}{4}\left(e^{ui} + e^{-ui}\right)^2 =$$

$$\frac{1}{4}\left(e^{2ui} + 2 + e^{-2ui}\right) =$$

$$= \frac{1}{4}\left(e^{2ui} + e^{-2ui} + 2\right) =$$

$$= \frac{1}{4}\left(2\cos 2u + 2\right) = \frac{1}{2}\cos 2u + \frac{1}{2}$$

$$\to 2 \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} (\cos 2u + 1) \, du$$

$$= \left[\sin 2u + 2x\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{\sqrt{3}}{2} + \frac{\pi}{3} - \left(-\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right) =$$

$$= \sqrt{3} + \frac{2}{3}\pi$$

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$$\int \frac{1}{(x-1)^2(x+1)}\,dx$$

$$\frac{1}{(x-1)^2(x+1)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1}$$

various methods for calculating a, b, c

1. by identification

$$\frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1} = \frac{a(x-1)(x+1) + b(x+1) + c(x-1)^2}{(x-1)^2(x+1)} =$$

$$\frac{a(x^2-1) + b(x+1) + c(x^2-2x+1)}{(x-1)^2(x+1)} =$$

$$\frac{(a+c)x^2 + (b-2c)x + (-a+b+c)}{(x-1)^2(x+1)}$$

$$= \frac{1}{(x-1)^2(x+1)}$$

$$\begin{cases} a+c=0 \\ b-2c=0 \\ -a+b+c=1 \end{cases}$$ Solve, obtain a, b, c

(2) trick allowing to find b and c quickly

$$g(x) = \frac{1}{(x-1)^2(x+1)} = \frac{a}{x-1} + \frac{b}{(x-1)^2} + \frac{c}{x+1}$$

$$g(x)(x+1) = \frac{a}{x-1}(x+1) + \frac{b}{(x-1)^2}(x+1) + c$$

$$\shortparallel$$

$$\frac{1}{(x-1)^2} \qquad x = -1$$

$$c = \frac{1}{(-1-1)^2} = \frac{1}{4}$$

Let us find b

$$g(x)(x-1)^2 = a(x-1) + b + \frac{c}{x+1}(x-1)^2$$

$$\shortparallel$$

$$\frac{1}{x+1} \qquad x = 1$$

$$b = \frac{1}{1+1} = \frac{1}{2}$$

Cannot find a this way.
But knowing b and c, easy to

find a

$$\frac{1}{(x-1)^2(x+1)} = \frac{a}{x-1} + \frac{1}{2}\frac{1}{(x-1)^2} + \frac{1}{4}\frac{1}{x+1}$$

To determine a, take some value of x say $x = 0$

$$1 = \frac{a}{-1} + \frac{1}{2} + \frac{1}{4} \qquad a = -\frac{1}{4}$$

$$\frac{1}{(x-1)^2(x+1)} = -\frac{1}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{(x-1)^2} + \frac{1}{4}\frac{1}{x+1}$$

simple fractions

$$\int \frac{1}{(x-1)^2(x+1)}\,dx = \int\left(-\frac{1}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{(x-1)^2} + \frac{1}{4}\frac{1}{x+1}\right)dx$$

$$= -\frac{1}{4}\ln(x-1) - \frac{1}{2}\frac{1}{x-1} + \frac{1}{4}\ln(x+1) + C$$

Differential equations

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$$y' = f(x, y) \qquad (*)$$

solving $(*)$ means finding <u>all</u> functions y satisfying the equation

$$\begin{cases} y' = f(x,y) & \leftarrow \text{equation} \\ y(a) = b & \leftarrow \text{initial condition} \end{cases} \quad \text{Cauchy problem}$$

We will solve only very simple equations: linear differential equations

$$y' = g(x)\, y + h(x)$$

$y' = g(x)y$ homogeneous linear equation

$y' = g(x)\, y + \underline{\underline{h(x)}}$ non homogeneous linear equation.

145 (1)

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$$y' = xy$$

$$\frac{y'}{y} = x$$

$$(\ln y)' = x$$

$$\ln y = \int x\,dx = \frac{x^2}{2} + \ln C$$

$$y = e^{\frac{x^2}{2} + \ln C} = C e^{\frac{x^2}{2}}$$

$$\boxed{y = C e^{\frac{x^2}{2}}}$$ where $C \in \mathbb{R}$

general solution

$$y = e^{\frac{x^2}{2}} \quad (C=1)$$

$$y = -3 e^{\frac{x^2}{2}} \quad (C=-3)$$

$$y = 0 \quad (C=0)$$

particular solutions

145 (2)

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$y' + xy = x$ non homogeneous equation.

First of all, let us solve the homogeneous equation

$y' + xy = 0$

$y' = -xy$

$\frac{y'}{y} = -x$

$(\ln y)' = -x$

$\ln y = \int -x\,dx = -\frac{x^2}{2} + \ln c$

$y = Ce^{-\frac{x^2}{2}}$ general solution of the homogeneous equation

Solving non-homogeneous equation: variation of the constant

$y = z\, e^{-\frac{x^2}{2}}$ z new unknown function

$$y' + xy = x$$

$$\left(z e^{-\frac{x^2}{2}}\right)' + x z e^{-\frac{x^2}{2}} = x$$

$$z' e^{-\frac{x^2}{2}} \underbrace{- z x e^{-\frac{x^2}{2}} + x z e^{-\frac{x^2}{2}}}_{\text{miracle !!}} = x$$

$$z' e^{-\frac{x^2}{2}} = x$$

$$z' = x e^{\frac{x^2}{2}}$$

$$z = \int x e^{\frac{x^2}{2}} dx = e^{\frac{x^2}{2}} + C$$

$$y = z e^{-\frac{x^2}{2}} = \left(e^{\frac{x^2}{2}} + C\right) e^{-\frac{x^2}{2}}$$

$$\underline{y = 1 + C e^{-\frac{x^2}{2}}}$$

general solution of the non-homogeneous equation.

Remark

$$y = \underbrace{1}_{\substack{\text{p.s.} \\ \text{of n.h.e.}}} + \underbrace{C e^{-\frac{x^2}{2}}}_{\text{g.s. of h.e.}}$$

of n.h.e.

$$\boxed{\begin{array}{c}\text{general solution of nh equation} = \\ \text{particular solution of nh equation} \\ + \\ \text{general solution of homogeneous e.}\end{array}}$$

In certain cases, instead of doing variation of constant, it is easier to guess a particular solution of the non homogeneous equation.

Example

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$y' + y = x^2$

homogeneous equation

$y' + y = 0 \qquad y' = -y$

$\frac{y'}{y} = -1 \qquad (\ln y)' = -1$

$\ln y = -x + \ln C$

$\boxed{y = Ce^{-x}}$ general solution of homogeneous equation.

Now: ① either use variation of constan
② or try to guess a particular solution.

Let us explore option ②

Look for a particular solution in the form $ax^2 + bx + c$ (polynomial of degree ≤ 2)

$y' + y = x^2$

$(ax^2 + bx + c)' + ax^2 + bx + c = x^2$

$$2ax + b + ax^2 + bx + c = x^2$$

$$ax^2 + (2a+b)x + b + c = x^2$$

$$\begin{cases} a = 1 \\ 2a + b = 0 \\ b + c = 0 \end{cases} \qquad \begin{aligned} a &= 1 \\ b &= -2 \\ c &= 2 \end{aligned}$$

$x^2 - 2x + 2$ is a particular solution of the non-homog.. equation.

$$\boxed{y = x^2 - 2x + 2 + Ce^{-x}}$$

general solution of the nh equation.

you may try variation of constant you will obtain the same answer.

146 (4)

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$$\begin{cases} y' + 5y = 3 \\ y(0) = 0 \end{cases}$$

(1) Solving the homogeneous equation

$y' + 5y = 0 \qquad y' = -5y \qquad \frac{y'}{y} = -5$

$(\ln y)' = -5 \qquad \ln y = -5x + \ln C$

$\boxed{y = Ce^{-5x}}$ general solution of homogeneous equation.

(2) Solving the non-homogeneous equations

first method: variation of constan:

$y = z e^{-5x}$

$y' + 5y = 3$

$(z e^{-5x})' + 5 z e^{-5x} = 3$

$z' e^{-5x} \underbrace{- 5 z e^{-5x}} \underbrace{+ 5 z e^{-5x}} = 3$

$z' e^{-5x} = 3$

$z' = 3e^{5x} \qquad z = 3\int e^{5x} dx =$

$$= \frac{3}{5} e^{5x} + C$$

$$y = z e^{-5x} = \frac{3}{5} + C e^{-5x}$$

---

second method: try to guess a particular solution.

Look for a constant particular solution $y = a$

$$y' + 5y = 3$$

$$\underset{0}{\underset{\shortparallel}{a'}} + 5a = 3 \qquad 5a = 3 \qquad a = \frac{3}{5}$$

$y = \frac{3}{5}$ particular solution

$y = \frac{3}{5} + Ce^{-5x}$ general solution

3 use the initial condition

$$y(0) = 0$$

$$y = \frac{3}{5} + C e^{-5x}$$

$$y(0) = \frac{3}{5} + C$$

$$y(0) = 0 \qquad C = -\frac{3}{5}$$

Conclusion: solution of the Cauchy

problem

$$\begin{cases} y' + 5y = 3 \\ y(0) = 0 \end{cases}$$

is $\boxed{y = \frac{3}{5} - \frac{3}{5} e^{-5x}}$

---

3 steps
1. homogeneous
2. non homogen
3. initial condition

HW 140, 143, 145 (to finish), 146,