Example of a bad logical mistake

12 November, 2020 14:04

~~Assume $P(n)$ holds true for all n~~
~~let us prove $P(n+1)$~~

very bad

$\boxed{\text{Assume } P(n) \text{ is true for a certain } n \text{ let us prove } P(n+1)}$

good

This would prove that
$P(n) \Rightarrow P(n+1)$ is true for all n

Another bad example:
$\{\emptyset\}$ is not the empty set!!
$\emptyset$ is the empty set.

103, 105

12 November, 2020 14:22

104 106

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103 calculating limits by reducing to a derivative

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

$$\lim_{x \to 0} \frac{\sin(x^2)}{x}$$

Define $f(x) = \sin(x^2)$

$f(0) = 0$ $\quad f'(x) = 2x\cos(x^2)$ $\quad 0$

$$\lim_{x \to 0} \frac{\sin(x^2)}{x} = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} =$$

$$= f'(0) = 2 \cdot 0 \cdot \cos(0^2) = 0$$

$$\lim_{x \to a} \frac{\cdots}{x-a}$$

To use this method, we should express the numerator as $f(x)-f(a)$ for some function f.

94 6

12 November, 2020 14:32

$$\lim_{x\to 1} \frac{x-1}{x^n-1}$$

Calculate instead

$$\lim_{x\to 1} \frac{x^n-1}{x-1}$$

Define $f(x) = x^n \qquad f(1)=1$

$$\lim_{x\to 1} \frac{x^n-1}{x-1} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = f'(1)$$

We have $f'(x) = n x^{n-1}$

$$\lim_{x\to 1} \frac{x^n-1}{x-1} = n \cdot 1^{n-1} = n$$

Now we have $\displaystyle\lim_{x\to 1} \frac{x-1}{x^n-1} = \frac{1}{n}$

Advantage of this approach is that it with any exponent n

Ex 103

12 November, 2020 14:41

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

$$\boxed{\begin{array}{l} f(x) = \sqrt{x+1} \qquad f(0) = 1 \\ f'(x) = \dfrac{1}{2\sqrt{x+1}} \end{array}}$$

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

$$= \frac{1}{2\sqrt{0+1}} = \frac{1}{2}$$

This method would not work for

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 2}{x}$$

103

12 November, 2020 14:48

$$\lim_{x \to 1} \frac{e^x - e}{x - 1}$$

$$\boxed{\begin{array}{ll} f(x) = e^x & f(1) = e \\ f'(x) = e^x & \end{array}}$$

$$\lim_{x \to 1} \frac{e^x - e}{x-1} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = f'(1)$$

$$= e^1 = e$$

Ex 105

12 November, 2020 14:50

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^x$$

$$\left(1+\frac{1}{2x}\right)^x = e^{x\ln\left(1+\frac{1}{2x}\right)}$$

Let us calculate

$$\lim_{x\to+\infty} x\ln\left(1+\frac{1}{2x}\right)$$

$$\boxed{\begin{array}{l}\text{Define new variable } u=\frac{1}{2x}\\ \text{when } x\to+\infty,\quad u\to 0^+\\ x=\frac{1}{2u}\end{array}}$$

$$\lim_{x\to+\infty} x\ln\left(1+\frac{1}{2x}\right) = \lim_{u\to0^+}\frac{1}{2u}\ln(1+u)$$

$$=\frac{1}{2}\lim_{u\to0^+}\frac{\ln(1+u)}{u} = \frac{1}{2}\cdot 1=\frac{1}{2}$$

Let us return to the initial limit

$$\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^x = \lim_{x\to+\infty} e^{x\ln\left(1+\frac{1}{2x}\right)}$$

$$= e^{\lim\limits_{x \to +\infty} x \ln\left(1 + \frac{1}{2x}\right)} = e^{\frac{1}{2}} = \sqrt{e}$$

Remarkable limits

12 November, 2020 14:57

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

$$\lim_{x\to 0} \frac{(1+x)^a - 1}{x} = a \qquad \lim_{x\to 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x\to +\infty} \frac{\ln x}{x^a} = 0 \qquad (a>0)$$

$$\lim_{x\to +\infty} \frac{x^a}{e^x} = 0 \qquad (a\in\mathbb{R})$$

105

12 November, 2020 15:01

$$\lim_{x\to 0} \frac{5^x-1}{x}$$

$$\boxed{f(x) = 5^x \quad f(0) = 1 \quad f'(x) = 5^x \ln 5}$$

$$\lim_{x\to 0} \frac{5^x-1}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = f'(0)$$

$$= 5^0 \ln 5 = \ln 5$$

You may take $f(x) = 5^x - 1$ as well. In this case

$$f(0) = 0 \qquad f'(x) = 5^x \ln 5$$

$$\lim_{x\to 0} \frac{5^x-1}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$$

$$= f'(0) = 5^0 \ln 5 = \ln 5$$

Ex 105

12 November, 2020 15:09

$$\lim_{x \to +\infty} x\left(3^{\frac{1}{x}} - 1\right)$$

Define a new variable $u = \frac{1}{x}$ $x = \frac{1}{u}$

When $x \to +\infty$ $u \to 0^+$

$$\lim_{x \to +\infty} x\left(3^{\frac{1}{x}} - 1\right) = \lim_{u \to 0^+} \frac{1}{u}\left(3^u - 1\right)$$

$$= \lim_{u \to 0} \frac{3^u - 1}{u}$$

$f'(u) = 3^u \ln 3$

Take $f(u) = 3^u$ $f(0) = 1$

$$\lim_{u \to 0} \frac{3^u - 1}{u} = \lim_{u \to 0} \frac{f(u) - f(0)}{u - 0} = f'(0)$$

$$= 3^0 \ln 3 = \ln 3$$

Conclusion: $\lim_{x \to +\infty} x\left(3^{\frac{1}{x}} - 1\right) = \ln 3$

Ex 105

12 November, 2020 15:29

$$\lim_{x \to +\infty} \left(1 + \frac{r}{x}\right)^x \qquad r \in \mathbb{R}$$

Did it already with $r = \frac{1}{2}$

$$\lim_{x \to +\infty} \left(1 + \frac{1}{2x}\right)^x$$

The case of general r is similar

let $u = \frac{r}{x}$ be a new variable $x = \frac{r}{u}$

When $x \to +\infty$ $u \to 0^+$ if $r > 0$, and
$u \to 0^-$ if $r < 0$

In any case $u \to 0$

$$\lim_{x \to +\infty} \left(1 + \frac{r}{x}\right)^x = \lim_{x \to +\infty} e^{x \ln\left(1 + \frac{r}{x}\right)}$$

$$= \lim_{u \to 0} e^{\frac{r}{u} \ln(1+u)} = e^{r \lim_{u \to 0} \frac{\ln(1+u)}{u}}$$

$$= e^{r \cdot 1} = e^r$$

Ex 104

12 November, 2020 15:38

$I =]0; \frac{\pi}{2}[\qquad f(x) = \frac{1}{x \tan x} \qquad (x \in I)$

(a) f defines a bijection $I \to J$, where J is an interval to be specified.

$$f'(x) = -\frac{\tan x + \frac{x}{(\cos x)^2}}{(x \tan x)^2}$$

$$\left.\begin{array}{l} \tan x > 0 \text{ for } x \in I \\ \frac{x}{(\cos x)^2} > 0 \text{ for } x \in I \end{array}\right\} \Rightarrow f'(x) < 0 \text{ for } x \in I$$

f is stictly decreasing on I

Hence it defines a bijection $I \to J$, and we only have to specify J.

To do this, let us calculate

$$\lim_{x \to 0^+} f(x) \qquad \lim_{x \to \frac{\pi}{2}^-} f(x)$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x \tan x} = +\infty$$

when $x \to 0^+$ $\tan x \to 0$ and remains positive

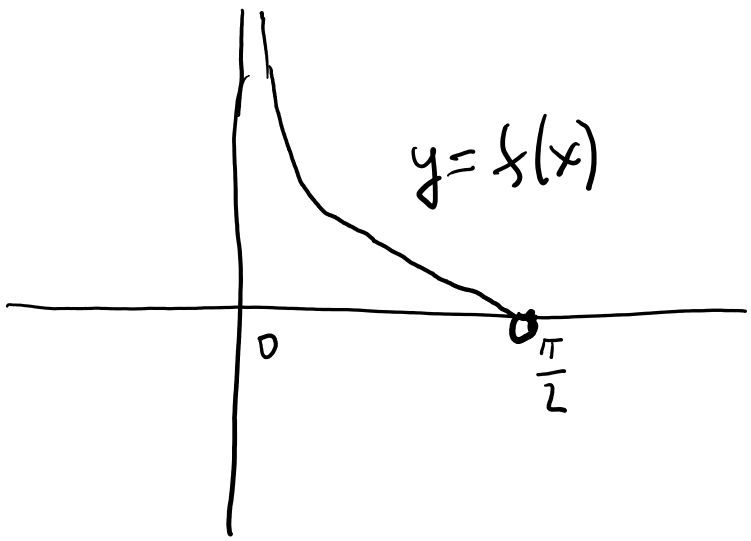
$\frac{1}{x} \to +\infty$, $\frac{1}{\tan x} \to +\infty$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{1}{x \tan x}$$

When $x \to \frac{\pi}{2}^-$ $\tan x \to +\infty$

$$\lim_{x \to \frac{\pi}{2}^-} \frac{1}{x \tan x} = 0$$

Hence $J =]0; +\infty[$



<u>Theorem</u> Let f be a function defined on some interval I

Assume that:

1. f is continuous on I, and
2. f is strictly increasing or strictly decreasing on I

Then f defines a bijection $I \to J$, where J is another interval

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If $I = [a, b]$ then $J = [f(a), f(b)]$ when f increasing

or $[f(b); f(a)]$ when f is decreasing

If $I =]a, b[$ then $f(a), f(b)$ must be replaced by

$\lim\limits_{x \to a^+} f(x)$, $\lim\limits_{x \to b^-} f(x)$

and J is also open

The inverse function $g : J \to I$ continuous when f is continuous

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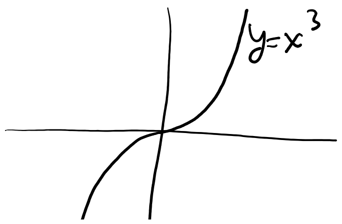
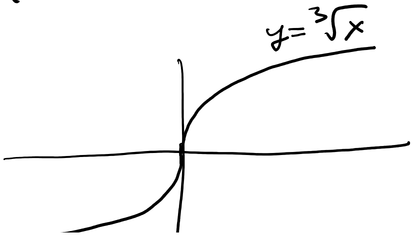
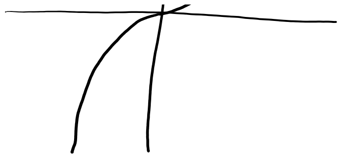
However, it is possible that f is differentiable, but is not

Example: $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$







not differtiable
at 0

The inverse function is differentiable on J if f satisfies

- $f'(x) \neq 0$ for all $x \in I$

12 November, 2020 16:05

Ex. 104 ②

Let $g : J \to I$ be the inverse funtion for f.

- Is g continuous on J?

Yes, because f is continuous on I

- Is g differentiable on J?

Yes, because

$$f'(x) = -\frac{\tan x + \frac{x}{(\cos x)^2}}{(x \tan x)^2} < 0,$$

in particular $f'(x) \neq 0$ for $x \in I$

Ex 106 ①

12 November, 2020 16:20

$$f(x) = x^x \qquad x \in]0; +\infty[$$

$$f'(x) = \left(e^{x\ln x}\right)' = e^{x\ln x} \cdot (x\ln x)' =$$

$$= x^x(\ln x + 1)$$

$$f'(x) = 0 \qquad \ln x + 1 = 0$$

$$\ln x = -1 \qquad x = e^{-1} = \frac{1}{e}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{x\ln x}$$

Let us calculate $\lim_{x\to 0^+} x\ln x$

We introduce the new variable

$u = \dfrac{1}{x}$ When $x \to 0^+$ $u \to +\infty$

$$x = \frac{1}{u}$$

$$\lim_{x\to 0^+} x\ln x = \lim_{u\to+\infty} \frac{1}{u}\ln\frac{1}{u}$$

$$= -\lim_{u\to+\infty} \frac{\ln u}{u} = 0$$

$x\ln x$ $\lim x\ln x$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{x\ln x} = e^{\lim_{x\to 0^+} x\ln x}$$

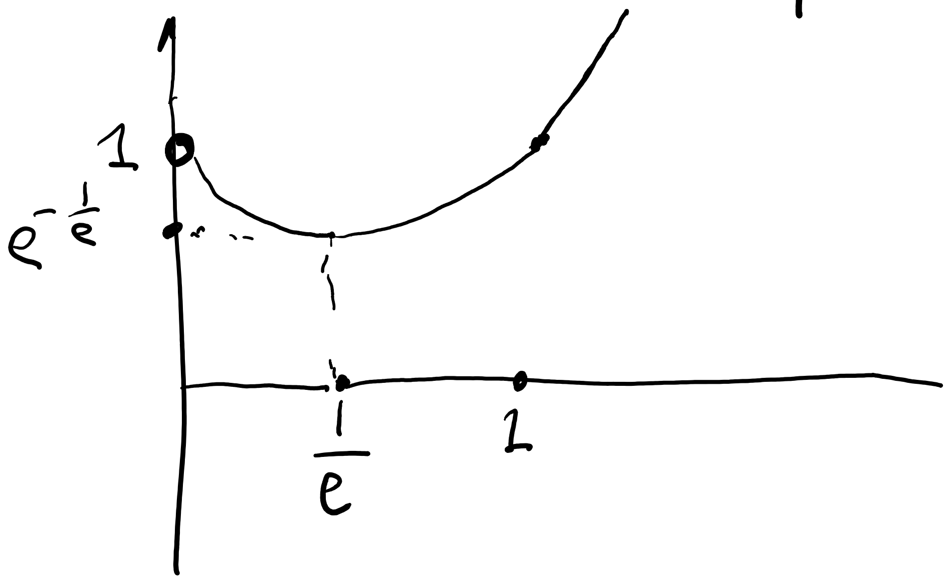
$$= e^0 = 1$$

$$\lim_{x\to +\infty} f(x) = \lim_{x\to +\infty} e^{x\ln x} = +\infty$$

$x \to +\infty \quad \ln x \to +\infty$

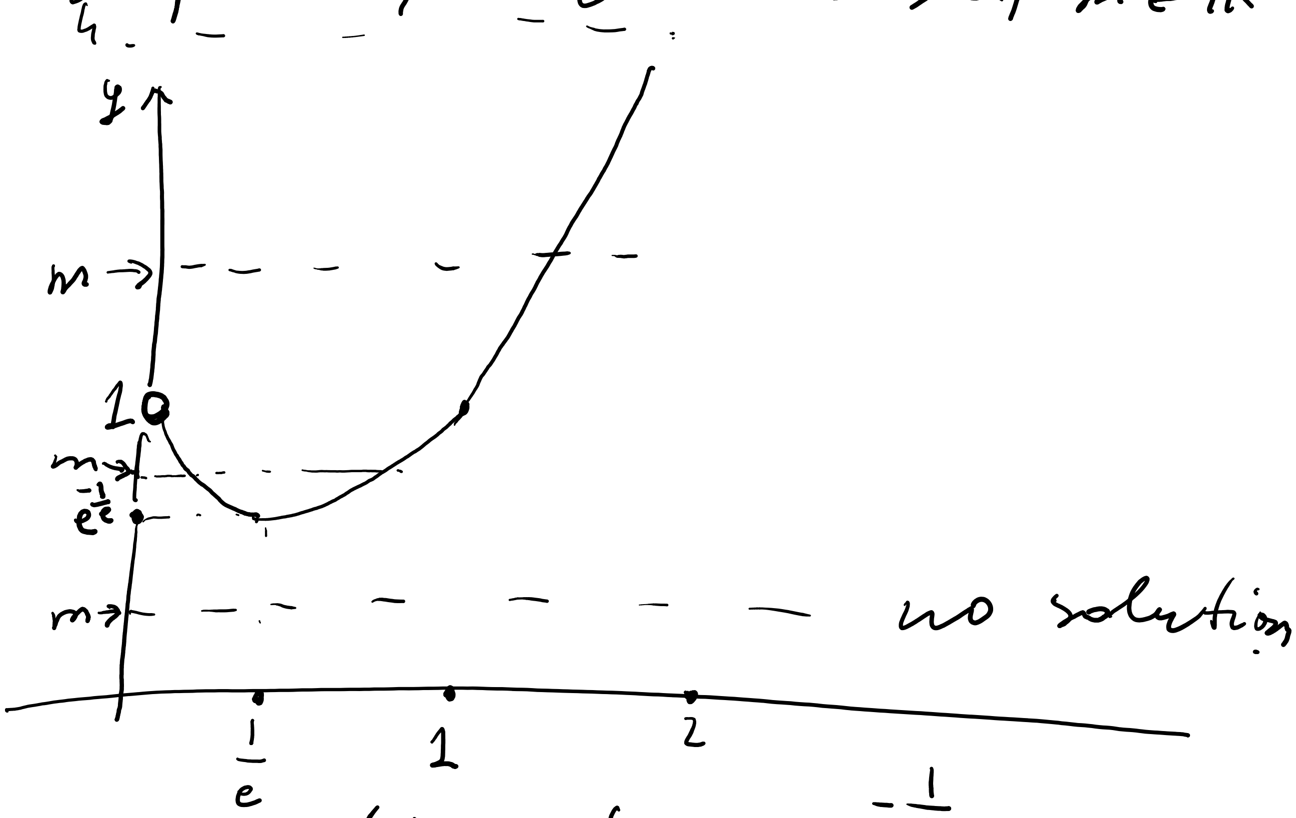
Variation table

x	$\to 0^+$	$]0;\frac{1}{e}[$	$\frac{1}{e}$	$]\frac{1}{e};+\infty[$	$\to +\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$\to 1$	$\searrow$	min $e^{-\frac{1}{e}}$	$\nearrow$	$\to +\infty$



106 (2) Determine the number of solutions of the equation $f(x) = m$ depending on the values of $m \in \mathbb{R}$

12 November, 2020 16:39



No solutions if $m < e^{-\frac{1}{e}}$

If $m = e^{-\frac{1}{e}}$ then there is one solution $x = \frac{1}{e}$

If $e^{-\frac{1}{e}} < m < 1$ then there are 2 solutions

If $m \geq 1$ then there is one solution

For Tuesday Nov. 17

107, 108, 109, 110