140,143,145,146

14 December, 2020 00:53

140

$$\int_2^4 \frac{1}{(x-1)^2}\,dx$$

$x = u+1$ $x=2$ $u=1$

$x-1=u$ $x=4$ $u=3$

$dx = du$

$$= \int_1^3 u^{-2}\,du = \left[\frac{u^{-1}}{-1}\right]_1^3 = -\left[\frac{1}{u}\right]_1^3 =$$

$$= -\left(\frac{1}{3}-1\right) = \frac{2}{3}$$

$$\int_0^1 \frac{x^2}{1+x^2}\,dx = \int_0^1 \frac{x^2+1-1}{1+x^2}\,dx =$$

$$\int_0^1 \left(1 - \frac{1}{1+x^2}\right)\,dx = \left[x - \arctan x\right]_0^1$$

$$= 1 - \frac{\pi}{4}$$

$$\int_0^1 \frac{2x}{x^2+4}\,dx$$

$x^2+4 = u$ $x=0$ $u=4$

$x=1$ $u=5$

$du = 2x\,dx$

$$= \int_4^5 \frac{du}{u} = \left[\ln u\right]_4^5 = \ln 5 - \ln 4$$

$- \quad {}_4 \quad u \quad -4$

$$\int_0^1 \frac{2x+2}{x^2+2x+5}\,dx$$

$x^2+2x+5=u$

$x=0 \quad u=5$

$x=1 \quad u=8$

$du=(2x+2)\,dx$

$$=\int_5^8 \frac{du}{u} = \Big[\ln u\Big]_5^8 = \ln 8-\ln 5$$

$$F(x) = \int_0^x \frac{du}{u^2+3u+2}$$

u^2+3u+2 has 2 roots $u_1 = -1$
$u_2 = -2$

$$u^2+3u+2 = (u+1)(u+2)$$

$$g(u) = \frac{1}{(u+1)(u+2)} = \frac{a}{u+1} + \frac{b}{u+2}$$

$$g(u)(u+1) = a + \frac{b}{u+2}(u+1)$$

$$\frac{1}{u+2} \qquad u = -1$$

$$\boxed{1 = a}$$

$$g(u)(u+2) = \frac{a}{u+1}(u+2) + b$$

$$\frac{1}{u+1} \qquad u = -2$$

$$\frac{1}{-2+1} = \frac{a}{-2+1}(-2+2) + b \qquad \boxed{-1 = b}$$

$$g(u) = \frac{1}{u+1} - \frac{1}{u+2}$$

Another method

$$\frac{1}{(u+1)(u+2)} = \frac{a}{u+1} + \frac{b}{u+2}$$

$$\frac{a}{u+1} + \frac{b}{u+2} = \frac{a(u+2) + b(u+1)}{(u+1)(u+2)} =$$

$$= \frac{(a+b)u + 2a + b}{(u+1)(u+2)} = \frac{1}{(u+1)(u+2)}$$

$$\begin{cases} a+b=0 \\ 2a+b=1 \end{cases} \qquad L_2 \to L_2 - 2L_1$$

$$\begin{cases} a+b=0 \\ -b=1 \end{cases} \qquad \begin{aligned} a &= 1 \\ b &= -1 \end{aligned}$$

$$F(x) = \int_0^x \frac{du}{u^2+3u+2} = \int_0^x \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du$$

$$= \Big[\ln(u+1) - \ln(u+2)\Big]_0^x =$$

$$\ln(x+1) - \ln(x+2) - (\ln 1 - \ln 2)$$

$$= \ln(x+1) - \ln(x+2) + \ln 2$$

$$= \ln \frac{2(x+1)}{x+2}$$

$$\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \ln \frac{2(x+1)}{x+2} =$$

$$= \lim_{x\to+\infty} \ln \frac{2\left(1+\frac{1}{x}\right)}{1+\frac{2}{x}} = \ln \frac{2(1+0)}{1+0} =$$

$$= \boxed{\ln 2}$$

$$\int_0^{\frac{\pi}{2}} \frac{\cos t}{(\sin t)^2 + 3\sin t + 2} \, dt$$

$u = \sin t$

$t = 0 \quad u = 0$

$t = \frac{\pi}{2} \quad u = 1$

$du = \cos t \, dt$

$$= \int_0^1 \frac{du}{u^2+3u+2} = F(1) = \ln \frac{2(1+1)}{1+2} = \ln \frac{4}{3}$$

$$F(x) = \ln \frac{2(x+1)}{x+2}$$

$$F(x) = \int_0^x \frac{du}{u^2+3u+2}$$

145 (3)

$xy' - y = x^3$

Solving the homogeneous equation

$xy' - y = 0$

$xy' = y \qquad \frac{y'}{y} = \frac{1}{x}$

$(\ln y)' = \frac{1}{x}$

$\ln y = \int \frac{1}{x} dx = \ln x + \ln C$

$y = e^{\ln x + \ln C} = Cx$

$y = Cx$ solution of the homogeneous equation.

$e^{a+b} = e^a \cdot e^b \qquad e^{\ln x} = x$

$xy' - y = x^3$ use "variation of the constant"

$y = zx$ z new uknown

$$y' = z'x + z$$

$$x(z'x + z) - zx = x^3$$

$$z'x^2 + zx - zx = x^3$$

$$z'x^2 = x^3 \qquad z' = x$$

$$z = \int x\,dx = \frac{x^2}{2} + C$$

$$y = zx = \frac{x^3}{2} + Cx$$

solution of the non-homogeneous equation

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$y' + a(x)y = 0$

the solution is $Ce^{-\int a(x)dx}$

$xy' - y = 0$

$y' - \frac{1}{x}y = 0$

$y = Ce^{-\int -\frac{1}{x}dx} = Ce^{-(-\ln x)} =$

$Ce^{\ln x} = Cx$

$y' + a(x)y = 0$

$y' = -a(x)y$

$\frac{y'}{y} = -a(x)$

$$(\ln y)' = -a(x) \qquad A'(x) = a(x)$$

$$\ln y = -A(x) + \ln C$$

$$y = e^{-A(x) + \ln C} = Ce^{-A(x)}$$

146

15 December, 2020 08:45

(1) $y' + y = 0 \qquad y(0) = 1$

$y' = -y \qquad \frac{y'}{y} = -1$

$(\ln y)' = -1 \qquad \ln y = -x + \ln c$

$y = e^{-x + \ln c} = ce^{-x}$ general solution

$y(0) = 1 \qquad ce^{-0} = 1 \quad c = 1$

$y = e^{-x}$ solution of the Cauchy problem

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(5) $y' + 3y = 4e^{x}$

solve the homogeneous equation

$y' + 3y = 0$

$y' = -3y \qquad \frac{y'}{y} = -3$

$(\ln y)' = -3 \qquad \ln y = -3x + \ln c$

$y_h = c\, e^{-3x}$ solution of the homogeneous equation solve the non-homogeneous equation

$$y' + 3y = 4e^x$$

2 methods

- variation of constant
- "guessing" a particular solution

variation of constant

$$y = z\, e^{-3x} \qquad z \text{ new unknown}$$

$$y' = z' e^{-3x} - 3 z e^{-3x}$$

$$z' e^{-3x} \underbrace{- 3 z e^{-3x}} \underbrace{+ 3 z e^{-3x}} = 4e^x$$

$$z' e^{-3x} = 4e^x$$

$$z' = 4e^{4x} \qquad z = \int 4e^{4x}\, dx =$$

$$= e^{4x} + C$$

$$\boxed{y_{nh} = z e^{-3x} = e^x + Ce^{-3x}}$$

general solution of the non-homogeneous equation.

Second method: look for a

particular solution of special form

$y' + 3y = 4e^x$

look for a particular solution

$y_p = a e^x$ a some constant

$(ae^x)' + 3ae^x = 4e^x$

$a e^x + 3a e^x = 4 e^x$

$4ae^x = 4e^x$ $4a = 4$ $\boxed{a=1}$

$y_p = e^x$ particular solution

$$\boxed{y_{nh} = y_p + y_h}$$

$y_{nh} = e^x + C e^{-3x}$ general solution of the non homogeneous equation

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Cauchy problem

$y' + 3x = 4e^x$ $y(0) = -2$

$y = e^x + c e^{-3x}$ have to determine

C from the initial condition

$$y(0) = e^{0} + C e^{-3\cdot 0} = 1 + C$$

$$1 + C = -2 \qquad C = -3$$

$$\boxed{y = e^{x} - 3e^{-3x}}$$ Solution of the Cauchy problem

Cauchy problem: 3 steps

1) solve the homogeneous equation

2) solve (find the general solution) of the non homogeneous equation (using variation of constant or "guessing" method)

3) Determine the value of the constant using the initial condition.

146 ④

$y' + 5y = 3 \quad y(0) = 0$

1. Solving homogeneous equation

$y' + 5y = 0 \qquad \frac{y'}{y} = -5$

$(\ln y)' = -5 \qquad \ln y = -5x + \ln c$

$\boxed{y_h = c e^{-5x}}$

2. Solving non-homogeneous equation. Use the "guessing" method

$y' + 5y = 3$ ← constant

look for particular solution as $y = a \qquad y' = 0$

$5a = 3 \qquad a = \frac{3}{5}$

$y_p = \frac{3}{5}$ particular solution

$\boxed{y_{nh} = \frac{3}{5} + c e^{-5x}}$ general solution of the nonhomogeneous equation

3. Solving Cauchy

$y' + 5y = 3 \qquad y(0) = 0$

$y = \frac{3}{5} + C e^{-5x}$

$y(0) = \frac{3}{5} + C e^{-5 \cdot 0} = \frac{3}{5} + C$

$\frac{3}{5} + C = 0 \qquad C = -\frac{3}{5}$

$\boxed{y = \frac{3}{5} - \frac{3}{5} e^{-5x}}$ solution of the Cauchy problem

147 (2) $3y' + 2y = 2x^3 + 9x^2 + 14$

solving the homogeneous equation

$3y' + 2y = 0$ $\quad \frac{y'}{y} = -\frac{2}{3}$

$(\ln y)' = -\frac{2}{3}$ $\quad \ln y = -\frac{2}{3}x + \ln C$

$y_h = Ce^{-\frac{2}{3}x}$ general solution of the homogeneous equation.

$3y' + 2y = 2x^3 + 9x^2 + 14$

variation of constant leads complicated partial integration (try it!!)

Insted, look for solution of the form $y_p = ax^3 + bx^2 + cx + d$

$y_p' = 3ax^2 + 2bx + c$

$$3y_p' + 2y_p = 3(3ax^2 + 2bx + c) +$$
$$+ \ 2(ax^3 + bx^2 + cx + d) =$$
$$= 2ax^3 + (9a + 2b)x^2 + (6b + 2c)x + (3c + 2d)$$
$$= 2x^3 + \quad 9x^2 + \quad 0x \ + 14$$

$$\begin{cases} 2a = 2 & a = 1 \\ 9a + 2b = 9 & b = 0 \\ 6b + 2c = 0 & c = 0 \\ 3c + 2d = 14 & d = 7 \end{cases}$$

$y_p = x^3 + 7$ particular solution of the nh equation

Control verification

$$3(x^3+7)' + 2(x^3+7) = 3 \cdot 3x^2 + 2x^3 + 14$$
$$= 2x^3 + 9x^2 + 14 \quad \text{OK}$$

general solution of the non homogeneous equation is homogeneous equation is

$y_{nh} = y_p + y_h = \boxed{x^3 + 7 + Ce^{-\frac{2}{3}x}}$

149

15 December, 2020 10:13

$$y' + y = e^{-x}(\cos x + x^2) + x^2$$

homogeneous $y' + y = 0$

$y = Ce^{-x}$

$$y' + y = \underbrace{e^{-x}\cos x} + \underbrace{x^2 e^{-x}} + \underbrace{x^2}$$

y_{p1} solution of $y' + y = e^{-x}\cos x$

y_{p2} - - - - $y' + y = x^2 e^{-x}$

y_{p3} - - - - $y' + y = x^2$

y_{p3} look in the form $ax^2 + bx + c$

$y_{p3}' = 2ax + b$

$y_{p3}' + y_{p3} = 2ax + b + ax^2 + bx + c =$

$= ax^2 + (2a + b)x + b + c = x^2$

$$\begin{cases} a = 1 \\ 2a + b = 0 \\ b + c = 0 \end{cases} \qquad \begin{aligned} a &= 1 \\ b &= -2 \\ c &= 2 \end{aligned}$$

$y_{p3} = x^2 - 2x + 2$

y_{p1} solution of

$y' + y = e^{-x}\cos x$

look for $y_{p1} = e^{-x}(a\cos x + b\sin x)$

$$y_{p1}' = -e^{-x}(a\cos x + b\sin x) + e^{-x}(-a\sin x + b\cos x) =$$

$$= e^{-x}((-a-b)\sin x + (-a+b)\cos x) = e^{-x}\cos x$$

$$\begin{cases} -a-b=0 \\ -a+b=1 \end{cases} \qquad L_2 \to L_2 - L_1$$

$$\begin{cases} -a-b=0 \\ 2b=1 \end{cases} \qquad a=-\frac{1}{2} \quad b=\frac{1}{2}$$

$$y_{p1} = \frac{1}{2}e^{-x}(-\cos x + \sin x)$$

---

y_{p2} solution of $y' + y = e^{-x}x^2$

want to look for y_{p2} in the form $e^{-x}(ax^2+bx+c)$ would not work

look for $e^{-x}(ax^3+bx^2+cx)$

$$y_{p2}' = -e^{-x}(ax^3+bx^2+cx) + e^{-x}(3ax^2+2bx+c)$$

$$= e^{-x}(-ax^3+(3a-b)x^2+(2b-c)x+c)$$

$$y_{p2}' + y_{p2} = e^{-x}\left(-ax^3 + (3a-b)x^2 + (2b-c)x + c\right) +$$
$$e^{-x}\left(ax^3 + bx^2 + cx\right) =$$
$$= e^{-x}\left(3a x^2 + 2bx + c\right)$$
$$= e^{-x} x^2$$

$$\begin{cases} 3a = 1 & a = \frac{1}{3} \\ 2b = 0 & b = 0 \\ c = 0 & c = 0 \end{cases}$$

$$\boxed{y_{p2} = \frac{1}{3} x^3 e^{-x}}$$

$$y_{p1} = \frac{1}{2} e^{-x}\left(-\cos x + \sin x\right)$$

$$y_{p3} = x^2 - 2x + 2$$

$$y_p = \frac{1}{2} e^{-x}\left(-\cos x + \sin x\right) + \frac{1}{3} e^{-x} x^3 + x^2 - 2x + 2$$

$$\boxed{y_{nh} = \frac{1}{2} e^{-x}\left(-\cos x + \sin x\right) + \frac{1}{3} e^{-x} x^3 + x^2 - 2x + 2 + Ce^{-x}}$$

general solution.

HW 147 4, 150 (use variation of constant)

151 (use variation of constant)

The "guessing" method only works for equations with constant coefficients

$y' + ay = b(x)$ $\quad$ a constant

does not work for

$y' + a(x)y = b(x)$ $\quad$ $a(x)$ non constant

How to "guess" the particular solution?

- if rhs (right-hand side) is polynomial of degree d, look for a polynomial of gegree d.
- if rhs is $\text{const} \cdot e^{\lambda x}$, look for $a e^{\lambda x}$ with $a \in \mathbb{R}$. (like in 146 5)
- if rhs $f(x) e^{\lambda x}$, f polynomial of degree d, look for (polynomial of degree d)$\cdot e^{\lambda x}$
- If rhs $A \sin \lambda x + B \cos \lambda x$, look for $a \sin \lambda x + b \cos \lambda x$

$$a, b \in \mathbb{R}$$

- If the things go wrong,

multiply by x !!!

- If rhs is $e^{\lambda x}(A\cos\alpha x + B\sin\alpha x)$
  look for $e^{\lambda x}(a\cos\alpha x + b\sin\alpha x)$

  $a, b \in \mathbb{R}$

147 (3)

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$$y' - y = \sin x + 2\cos x$$

homogeneous: $y' - y = 0$

$$\frac{y'}{y} = 1 \qquad (\ln y)' = 1$$

$$\ln y = x + \ln C \qquad y = C e^{x}$$

look for particular solution of non-homogeneous of the form $a \sin x + b \cos x = y_p$

$$y_p' = a\cos x - b \sin x$$

$$y_p' - y_p = \sin x + 2\cos x$$

$$a\cos x - b\sin x - (a\sin x + b\cos x)$$

$$= \sin x + 2\cos x$$

$$(-a-b)\sin x + (a-b)\cos x = \sin x + 2\cos x$$

$$\begin{cases} -a - b = 1 \end{cases}$$

$$\begin{cases} -a-b=1 \\ a-b=2 \end{cases} \qquad L_2 \to L_2 + L_1$$

$$\begin{cases} -a-b=1 \\ -2b=3 \end{cases} \qquad \begin{aligned} a &= \frac{1}{2} \\ b &= -\frac{3}{2} \end{aligned}$$

$$y_p = \frac{1}{2}\sin x - \frac{3}{2}\cos x$$

$$\boxed{y_{nh} = \frac{1}{2}\sin x - \frac{3}{2}\cos x + Ce^{x}}$$

general solution of the non-homogeneous equation.

147 (1) $y' + y = xe^{-x} + 1$

---

homogeneous $y' + y = 0$ $\frac{y'}{y} = -1$

$(\ln y)' = -1$ $\ln y = -x + \ln c$

$y = ce^{-x}$

---

non homogeneous

$y' + y = xe^{-x} + 1$

$y_p = y_{p1} + y_{p2}$

y_{p1} solution of $y' + y = xe^{-x}$

y_{p2} " " $y' + y = 1$

Start from y_{p2} $y' + y = 1$

look for $y_{p2} = a$ $a \in \mathbb{R}$

$a' = 0$ $a' + a = 1$ $\boxed{a = 1}$

$\boxed{y_{p2} = 1}$

---

y_{p1} Solution of $y' + y = x e^{-x}$

look for $y_{p1} = (ax+b)\, e^{-x}$

$$y_{p1}' = a e^{-x} - (ax+b)\, e^{-x}$$

$$= \left(-ax + (a-b)\right) e^{-x}$$

$$y_{p1}' + y_{p1} = \left(-ax + (a-b)\right) e^{-x} + (ax+b) e^{-x}$$

$$= a e^{-x} \qquad = x e^{-x}$$

does not work !!

Look for $y_{p1} = (ax+b)\, x e^{-x}$

$$= (ax^2 + bx)\, e^{-x}$$

$$y_{p1}' = (2ax+b)\, e^{-x} - (ax^2+bx)\, e^{-x}$$

$$= \left(-ax^2 + (2a-b)x + b\right) e^{-x}$$

$$y_{p1}' + y_{p1} = \left(-ax^2 + (2a-b)x + b\right) e^{-x} + (ax^2 + bx)\, e^{-x}$$

$$= (2ax+b)\, e^{-x} \qquad = x e^{-x}$$

$\{ 2a = 1$ ... $= 1$

$$\begin{cases} 2a = 1 \\ b = 0 \end{cases} \qquad \begin{aligned} a &= \tfrac{1}{2} \\ b &= 0 \end{aligned}$$

$$y_{p1} = \frac{1}{2} x^2 e^{-x}$$

$$y_p = y_{p1} + y_{p2} = \frac{1}{2} x^2 e^{-x} + 1$$

$$y_{nh} = y_p + y_n = \boxed{\frac{1}{2} x^2 e^{-x} + 1 + Ce^{-x}}$$