107,108,109,110

17 November, 2020 07:47

107 $f(x) = e^{\cos(\sin x)}$

defined on $\mathbb{R}$

$f'(x) = e^{\cos(\sin x)} \cdot (\cos(\sin x))' =$

$= e^{\cos(\sin x)} \cdot (-\sin(\sin x)) \cdot \cos x$

$= -e^{\cos(\sin x)} \sin(\sin x) \cos x$

$f(x) = \ln(\sin x)$

$f'(x) = \frac{1}{\sin x} \cdot \cos x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$

Defined where $\sin x > 0$

$]-2\pi, -\pi[\cup]0; \pi[\cup]2\pi, 3\pi[\cup]4\pi, 5\pi[\ldots$

$\bigcup_{k \in \mathbb{Z}}]2\pi k ; 2\pi k + \pi[$

$f(x) = \ln(\ln x)$

Defined when $x > 0$

$\ln x > 0 \Leftrightarrow x > 1$

Defined on $]1 ; +\infty[$

$$f'(x) = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x \ln x}$$

$u(x) = (x^3+1)^{3/2}$

Defined when $x^3+1 \geq 0$

$x^3 \geq -1 \qquad x \geq -1$

Defined on $[-1; +\infty[$

$u'(x) = \frac{3}{2}(x^3+1)^{\frac{1}{2}} \cdot 3x^2 = \frac{9}{2}x^2\sqrt{x^3+1}$

$f(x) = \frac{\sqrt{x}}{e^x+1}$

Defined when $x \geq 0$

on $[0, +\infty[$

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}(e^x+1) - \sqrt{x}\,e^x}{(e^x+1)^2} =$$

$$= \frac{e^x+1-2xe^x}{2\sqrt{x}(e^x+1)^2}$$

$f(x) = \exp(x^3-2x+1)$

Defined on $\mathbb{R}$

$f'(x) = e^{x^3-2x+1}$

$$f'(x) = e^{x^3-2x+1} \cdot (3x^2-2)$$

$f(x) = \big(a\cos(\omega x) + b\sin(\omega x)\big)e^{cx}$

Defined on $\mathbb{R}$

$(uv)' = u'v + uv'$

$u = a\cos(\omega x) + b\sin(\omega x)$

$v = e^{cx}$

$f'(x) = \big(-a\omega\sin(\omega x) + b\omega\cos(\omega x)\big)e^{cx}$

$+ \big(a\cos(\omega x) + b\sin(\omega x)\big)ce^{cx} =$

$= \Big(\big(b\omega + ac\big)\cos(\omega x) + \big(-a\omega + bc\big)\sin(\omega x)\Big)e^{cx}$

108 $\lim\limits_{x\to+\infty}(\sqrt{x}-x)$

$\sqrt{x}\to+\infty$ $(+\infty)-(+\infty)$??

$x\to+\infty$

$\lim\limits_{x\to+\infty}(\sqrt{x}-x)=\lim\limits_{x\to+\infty}\sqrt{x}(1-\sqrt{x})=-\infty$

$\sqrt{x}\to+\infty$

$1-\sqrt{x}\to-\infty$ $+\infty\cdot(-\infty)=-\infty$

$\lim\limits_{x\to+\infty}(\sqrt{x}-x)=\lim\limits_{x\to+\infty}x\left(\frac{1}{\sqrt{x}}-1\right)=-\infty$

$x\to+\infty$ $\frac{1}{\sqrt{x}}-1\to-1$

$\lim\limits_{x\to+\infty}e^{\sqrt{x}-x}=0$, because

$\lim\limits_{x\to+\infty}(\sqrt{x}-x)=-\infty$

$\lim\limits_{t\to-\infty}e^{t}=0$

$$\lim_{x\to+\infty} \frac{\ln(1+x)}{\ln x} = \cdot$$

$\ln(ab) = \ln a + \ln b$

$$\boxed{\ln(1+x) = \ln x + \ln\left(\frac{1}{x}+1\right)}$$

$$= \lim_{x\to+\infty} \frac{\ln x + \ln\left(\frac{1}{x}+1\right)}{\ln x} =$$

$$= \lim_{x\to+\infty} \left(1 + \frac{\overbrace{\ln\left(\frac{1}{x}+1\right)}^{\to 0}}{\underbrace{\ln x}_{\to+\infty}}\right) = 1$$

Reminder $\quad \lim_{x\to+\infty} \frac{x^a}{e^x} = 0$

$$\lim_{x\to+\infty} \frac{\ln x}{x^a} = 0 \quad (a>0)$$

$$\lim_{x\to+\infty} \frac{\ln x}{e^x} = 0$$

$$\lim_{x\to+\infty} \frac{x-2\ln x}{e^x-1} = \lim_{x\to+\infty} \frac{\frac{x}{e^x} - 2\frac{\ln x}{e^x}}{1-\frac{1}{e^x}} =$$

$0 - 2\cdot 0$

$$\frac{\qquad}{1-0} = 0$$

$$\lim_{x\to+\infty} \frac{e^x}{e^{e^x}} = \lim_{u\to+\infty} \frac{u}{e^u} = 0$$

$u = e^x \qquad x \to +\infty$

$u \to +\infty$

$$\lim_{x\to+\infty} e^{x-e^x} = \lim_{x\to+\infty} e^{\boxed{e^x\left(\frac{x}{e^x}-1\right)}}$$

$\to -\infty$

$$\frac{x}{e^x}-1 \Rightarrow -1 \qquad e^x\left(\frac{x}{e^x}-1\right) \to -\infty$$

109 1

$$\lim_{x \to \frac{\pi}{4}} \frac{\sin x - \frac{1}{\sqrt{2}}}{x - \frac{\pi}{4}} =$$

$$\boxed{\begin{array}{ll} f(x) = \sin x & \\ & f'(x) = \cos x \\ f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} & \end{array}}$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{f(x) - f\left(\frac{\pi}{4}\right)}{x - \frac{\pi}{4}} = f'\left(\frac{\pi}{4}\right) =$$

$$= \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

2

$$\lim_{x \to \frac{1}{2}} \frac{e^{2x} - e}{2x - 1} = \frac{1}{2} \lim_{x \to \frac{1}{2}} \frac{e^{2x} - e}{x - \frac{1}{2}}$$

$$\boxed{\begin{array}{ll} f(x) = e^{2x} & f\left(\frac{1}{2}\right) = e \\ f'(x) = 2e^{2x} & \end{array}}$$

$$= \frac{1}{2} \lim_{x \to \frac{1}{2}} \frac{f(x) - f\left(\frac{1}{2}\right)}{x - \frac{1}{2}} = f'\left(\frac{1}{2}\right) = 2e$$

109 (3)

$$\lim_{x\to-\infty} \frac{\sqrt{x^2+3}-x+2}{x} =$$

$$\boxed{x=-u \qquad x\to-\infty \qquad u\to+\infty}$$

$$= \lim_{u\to+\infty} \frac{\sqrt{u^2+3}+u+2}{-u} =$$

$$= -\lim_{u\to+\infty} \frac{\frac{\sqrt{u^2+3}}{u}+1+\frac{2}{u}}{1} =$$

let us find $\lim_{u\to+\infty} \frac{\sqrt{u^2+3}}{u}$

$$\lim_{u\to+\infty} \frac{\sqrt{u^2+3}}{u} = \lim_{u\to+\infty} \sqrt{\frac{u^2+3}{u^2}} =$$

$$= \lim_{u\to+\infty} \sqrt{1+\frac{3}{u^2}} = \sqrt{1+0} = 1$$

$$= \lim_{u \to +\infty} \sqrt{1+\frac{1}{u^2}} - \sqrt{1+0} - 1$$

$$- \quad \frac{1+1+0}{1} = -2$$

$$\lim_{x \to +\infty} \frac{\sqrt{x^2+3} - x + 2}{x}$$

Claim: $\lim\limits_{x \to +\infty} \left(\sqrt{x^2+3} - x\right) = 0$

$$\lim_{x \to +\infty} \left(\sqrt{x^2+3} - x\right) =$$

$$= \lim_{x \to +\infty} \frac{\left(\sqrt{x^2+3} - x\right)\left(\sqrt{x^2+3} + x\right)}{\sqrt{x^2+3} + x} =$$

$$= \lim_{x \to +\infty} \frac{x^2 + 3 - x^2}{\sqrt{x^2+3} + x} =$$

$$= \lim_{x \to +\infty} \frac{3}{\sqrt{x^2+3} + x} = 0$$

$$\lim_{x \to +\infty} \frac{\overbrace{\sqrt{x^2+3} - x}^{\to 0} + 2}{x} = 0$$

$\longrightarrow +\infty$

110 ① $\lim_{x\to 1} \frac{\ln x}{x-1} =$

$= \lim_{x\to 1} \frac{\ln x - \ln 1}{x-1} = \ln'(1) = \frac{1}{1} = 1$

② $\lim_{x\to 0} \frac{e^{x^2}-1}{x} = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

$$\boxed{\begin{array}{ll} f(x) = e^{x^2} & f(0) = 0 \\ f'(x) = 2x e^{x^2} & \end{array}}$$

$= f'(0) = 2\cdot 0\, e^{0^2} = 0$

③ $\lim_{x\to +\infty} x\left(\ln(x+1) - \ln x\right) =$

$= \lim_{x\to +\infty} x \ln \frac{x+1}{x} =$

$= \lim_{x\to +\infty} x \ln\left(1+\frac{1}{x}\right) = \lim_{u\to 0^+} \frac{\ln(1+u)}{u} =$

$$\boxed{\begin{array}{ll} u = \frac{1}{x} & x = \frac{1}{u} \\ x \to +\infty & u \to 0^+ \end{array}}$$

$= 1$

110 (continued)

$$\lim_{x\to 1} \frac{x^{\frac{1}{5}}-1}{x^4-1}$$

$a \in \mathbb{R}$ $$\lim_{x\to 1} \frac{x^a-1}{x-1} = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1}$$

$$\boxed{\begin{matrix} f(x) = x^a & f(1)=1 \\ f'(x) = a x^{a-1} & \end{matrix}}$$

$$= f'(1) = a\, 1^{a-1}$$
$$= a$$

$$\lim_{x\to 1} \frac{x^{\frac{1}{5}}-1}{x^4-1} = \lim_{x\to 1} \frac{x^{\frac{1}{5}}-1}{x-1} \cdot \frac{x-1}{x^4-1} = \lim_{x\to 1} \frac{\frac{x^{\frac{1}{5}}-1}{x-1}}{\frac{x^4-1}{x-1}}$$

$$= \frac{\lim_{x\to 1} \frac{x^{\frac{1}{5}}-1}{x-1}}{\lim_{x\to 1} \frac{x^4-1}{x-1}} = \frac{\frac{1}{5}}{4} = \frac{1}{20}$$

Another method: $u = x^4$ $\quad x\to 1$, $u \to 1$

$x = u^{\frac{1}{4}}$

$$\lim_{x\to 1} \frac{x^{\frac{1}{5}}-1}{x^4-1} = \lim_{u\to 1} \frac{(u^{\frac{1}{4}})^{\frac{1}{5}}-1}{u-1} =$$

$$= \lim_{u\to 1} \frac{u^{\frac{1}{20}}-1}{u-1} = \frac{1}{20}$$

110 (last one) $\quad u(x)^{v(x)} = e^{v(x)\ln u(x)}$

$$\lim_{x\to+\infty} x^{\frac{1}{1+2\ln x}} = \lim_{x\to+\infty} e^{\frac{\ln x}{1+2\ln x}}$$

$$= \lim_{x\to+\infty} e^{\frac{1}{\frac{1}{\ln x}+2}} = e^{\frac{1}{0+2}} = \sqrt{e}$$

Inverse trigonometric functions

general theory:

$f : I \to J$ bijection

Then $\exists$ inverse function (fonction réciproque) $g : J \to I$

$f(x) = y \qquad g(y) = x$

Remark Inverse function is often denoted f^{-1}. Attention: do not confuse it with $\frac{1}{f}$

Examples 1 $I = [0; +\infty[\quad J = [0; +\infty[$

$f(x) = x^2$

f is surjective, strictly increasing on I. Hence it admits inverse

$g : [0; +\infty[\to [0; +\infty[$

$g(x) = \sqrt{x}$

---

(2) $I = \mathbb{R}$, $J =]0; +\infty[$

$f(x) = e^x$

f surjective, strictly increasing

admits inverse function

$g :]0; +\infty[\to \mathbb{R}$

$g(x) = \ln x$

---

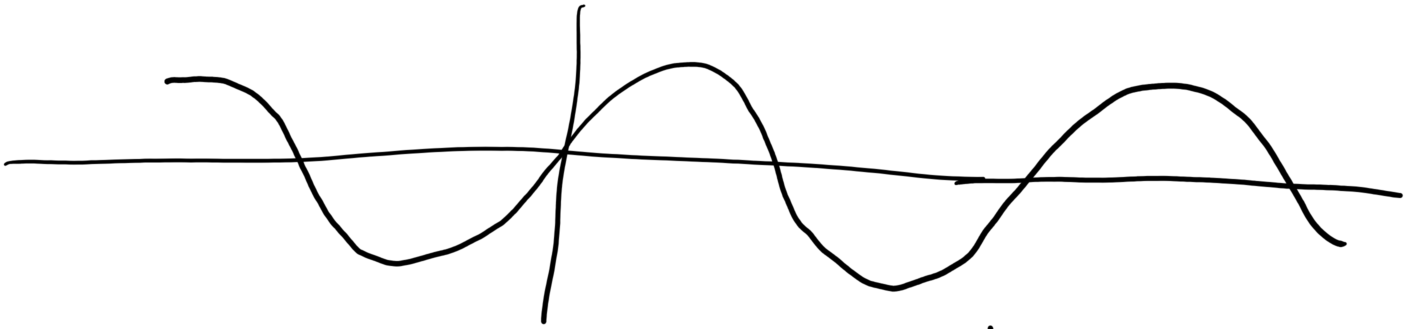
(3) $I = \mathbb{R}$ $J = \mathbb{R}$ $f(x) = x$

what is the inverse?

the inverse function is f itself.

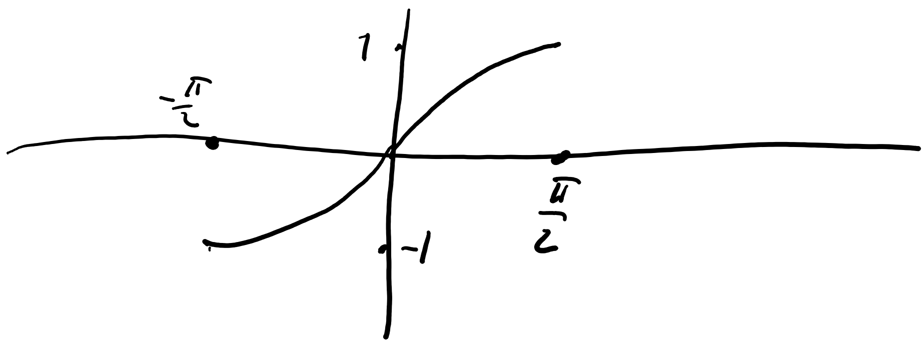
Inverse trigonometric functions

$f(x) = \sin x$



not monotonic !!

Can find interval where sin is monotonic

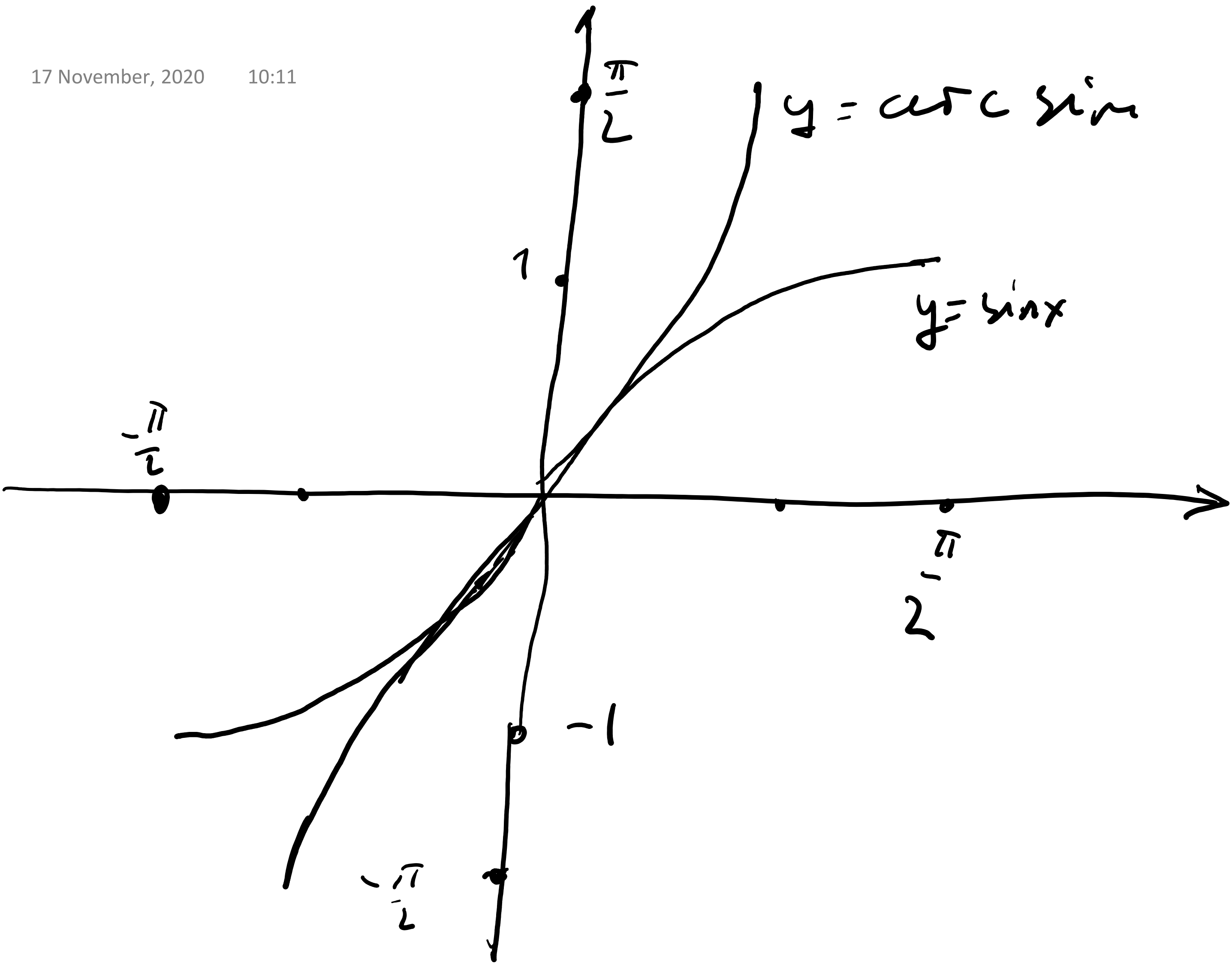


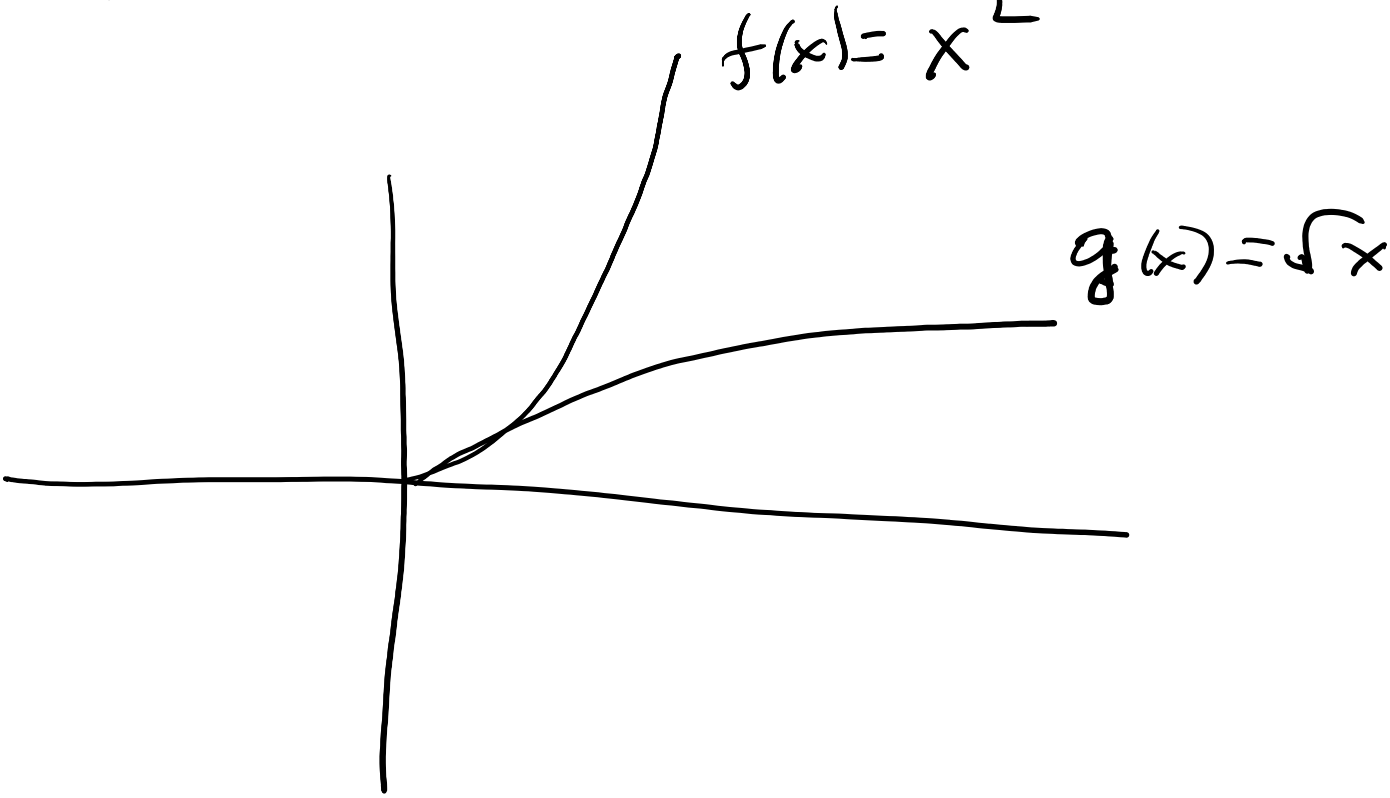
sin is strictly increasing on $I = [-\frac{\pi}{2}; \frac{\pi}{2}]$. And it defines a bijection $\sin : I \to J$, where $J = [-1; 1]$

So there is inverse function

$\arcsin : [-1; 1] \to [-\frac{\pi}{2}; \frac{\pi}{2}]$

17 November, 2020 10:11





$f(x) = x^2$
$g(x) = \sqrt{x}$

17 November, 2020 10:17

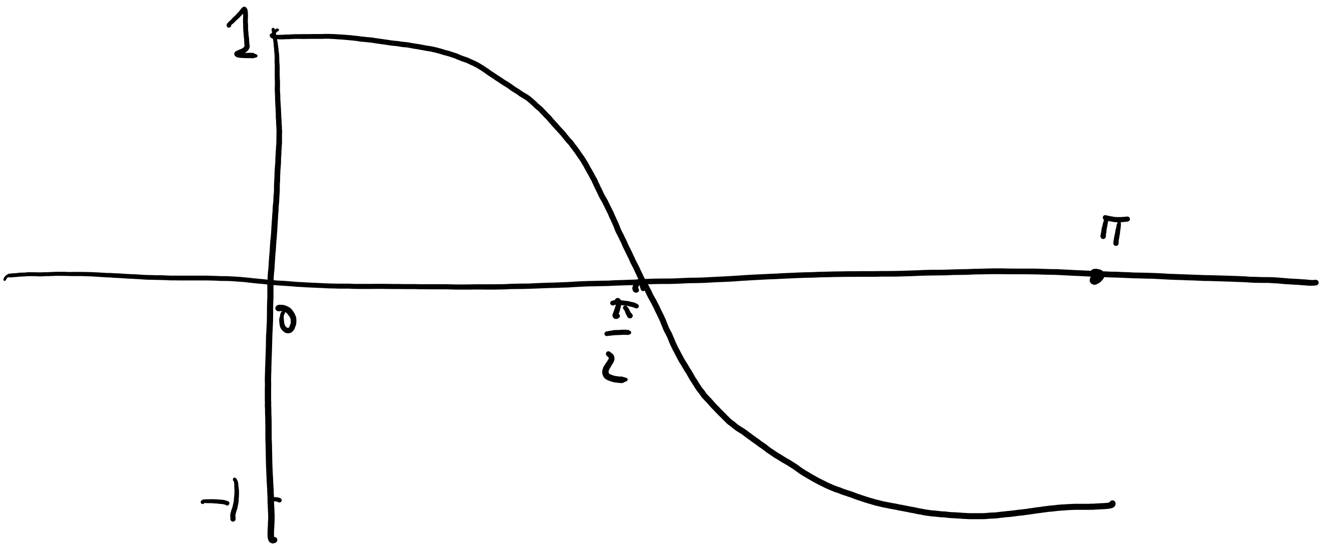
Values:

x	$\sin x$
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{3}$	$-\frac{\sqrt{3}}{2}$
$-\frac{\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$-\frac{\pi}{6}$	$-\frac{1}{2}$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

x	$\arcsin x$
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$
$-\frac{1}{\sqrt{2}}$	$-\frac{\pi}{4}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$
0	0
$\frac{1}{2}$	$\frac{\pi}{6}$
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
1	$\frac{\pi}{2}$

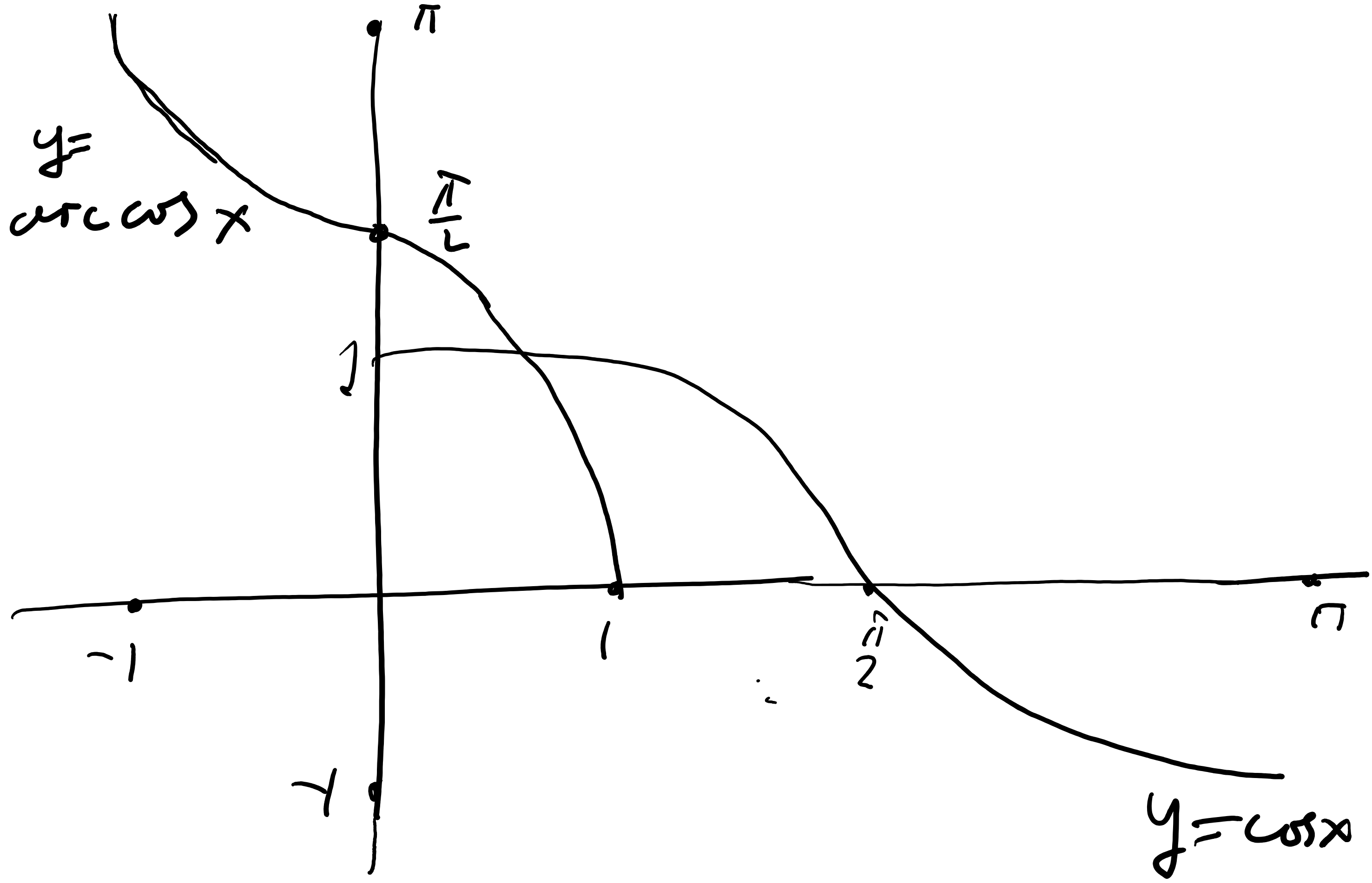
$$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$\cos x$ is strictly decreasing on $[0; \pi]$



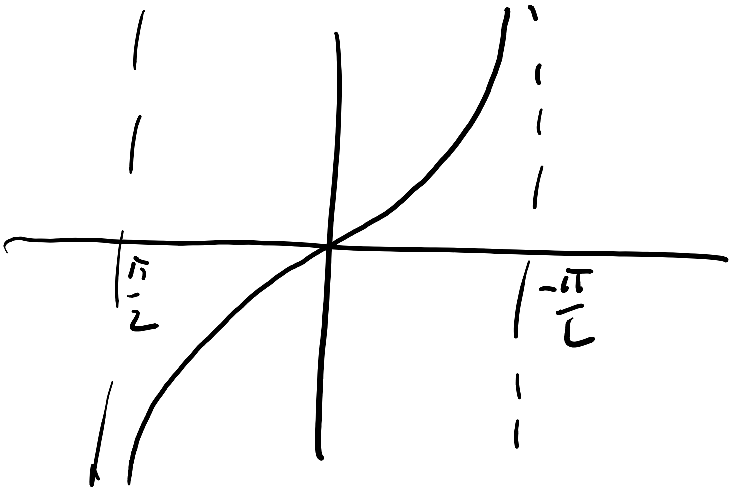
$\cos : [0, \pi] \to [-1, 1]$ bijection admits inverse function

$\arccos : [-1; 1] \to [0; \pi]$


π
y=
arccos x
π/2
1
-1
1
π/2
π
-1
y=cos x

tan x is strictly increasing on $\left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$

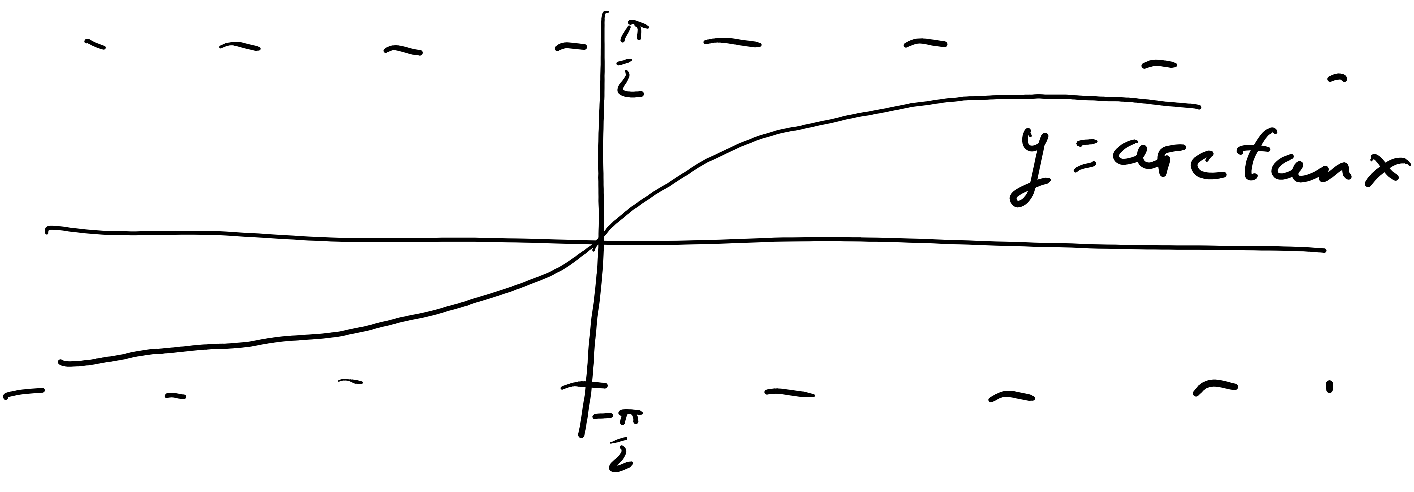
$$\lim_{x \to -\frac{\pi}{2}^+} \tan x = -\infty$$



$$\lim_{x \to \frac{\pi}{2}^-} \tan x = +\infty$$

$\tan : \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[\to \left]-\infty; +\infty\right[$ is a bijection.

$\arctan : \left]-\infty; +\infty\right[\to \left]-\frac{\pi}{2}; \frac{\pi}{2}\right[$ inverse function



$$\lim_{x \to -\infty} \arctan x = -\frac{\pi}{2}$$

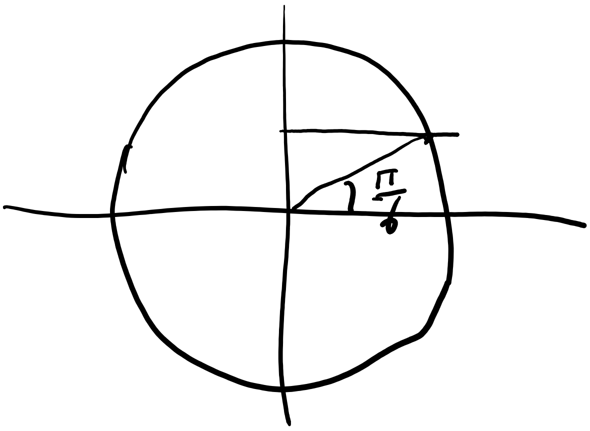
$$\lim_{x \to +\infty} \arctan x = \frac{\pi}{2}$$

arctan important function!

x	$\tan x$
$\to -\frac{\pi}{2}$	$\to -\infty$
$-\frac{\pi}{3}$	$-\sqrt{3}$
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	$-\frac{1}{\sqrt{3}}$
0	0
$\frac{\pi}{6}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	$\sqrt{3}$
$\to \frac{\pi}{2}$	$\to +\infty$

x	$\arctan x$
$\to -\infty$	$\to -\frac{\pi}{2}$
$-\sqrt{3}$	$-\frac{\pi}{3}$
-1	$-\frac{\pi}{4}$
$-\frac{1}{\sqrt{3}}$	$-\frac{\pi}{6}$
0	0
$\frac{1}{\sqrt{3}}$	$\frac{\pi}{6}$
1	$\frac{\pi}{4}$
$\sqrt{3}$	$\frac{\pi}{3}$
$\to +\infty$	$\to \frac{\pi}{2}$

$$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$$



$$\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$$

$$\arctan(1) = \frac{\pi}{4}$$

$$\arcsin(\sin 1) = 1$$

$$\arcsin(\sin 2) =$$

$$= \arcsin(\sin(\pi - 2)) = \pi - 2$$

$\arcsin(\sin x) = x$ when $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$$\sin(\pi - x) = \sin x$$

Homework for Thursday:

115, 116, 114