114,115,116

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114 $\lim\limits_{x\to 1} \arctan \frac{1}{x-1}$

$$\lim_{x\to 1^-} \frac{1}{x-1} = -\infty \qquad \lim_{x\to -\infty} \arctan x = -\frac{\pi}{2}$$

$$\lim_{x\to 1^-} \arctan \frac{1}{x-1} = -\frac{\pi}{2}$$

$$\lim_{x\to 1^+} \frac{1}{x-1} = +\infty \qquad \lim_{x\to +\infty} \arctan x = \frac{\pi}{2}$$

$$\lim_{x\to 1^+} \arctan \frac{1}{x+1} = \frac{\pi}{2}$$

$$\lim_{x\to 1^-} \arctan \frac{1}{x+1} \neq \lim_{x\to 1^+} \arctan \frac{1}{x+1}$$

$\lim\limits_{x\to 1} \arctan \frac{1}{x+1}$ does not exist

$$\lim_{x\to\frac{1}{2}} \frac{\arccos x - \frac{\pi}{3}}{4x^2-1} = \lim_{x\to\frac{1}{2}} \frac{\arccos x - \frac{\pi}{3}}{4\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)}$$

$$= \lim_{x\to\frac{1}{2}} \frac{\arccos x - \frac{\pi}{3}}{x-\frac{1}{2}} \cdot \lim_{x\to\frac{1}{2}} \frac{1}{4\left(x+\frac{1}{2}\right)}$$

$$= \left(\lim_{x\to\frac{1}{2}} \frac{\arccos x - \arccos\frac{1}{2}}{x-\frac{1}{2}}\right) \cdot \frac{1}{4\left(\frac{1}{2}+\frac{1}{2}\right)} =$$

$$= \arccos'\left(\frac{1}{2}\right) \cdot \frac{1}{4} = -\frac{1}{\sqrt{1-\left(\frac{1}{2}\right)^2}} \cdot \frac{1}{4} =$$

$$\boxed{\arcsin' x = \frac{1}{\sqrt{1-x^2}} \qquad \arccos' x = \frac{-1}{\sqrt{1-x^2}}}$$

$$= \frac{-1}{2\sqrt{3}}$$

Derivative of the inverse function

$f : I \to J$ continuous, strictly monotonic

$g : J \to I$ the inverse function for f

g is continuous,

If f is derivable on I, and $f'(x) \neq 0$ on I then g is derivable on J

Calculating g' is tricky

$f \circ g(x) = x$

$f'(g(x)) \cdot g'(x) = 1$

$$g'(x) = \frac{1}{f'(g(x))}$$

Examples. $f(x) = \sin x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$g(x) = \arcsin x$

$$\arcsin'(x) = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - x^2}}$$

$$\arccos' x = \frac{1}{\cos'(\arccos x)} = $$

$$= \frac{1}{-\sin(\arccos x)} = \frac{-1}{\sqrt{1-x^2}}$$

$$\arctan' x = \frac{1}{\tan'(\arctan x)} =$$

$$= \frac{1}{1+(\tan(\arctan x))^2} = \frac{1}{1+x^2}$$

$$\tan'(u) = \frac{1}{(\cos u)^2} = 1 + (\tan u)^2$$

$$\boxed{\arctan' x = \frac{1}{1+x^2}}$$

116 $\cos(\arcsin x) - ??$

$\sin(\arcsin x) = x \qquad x \in [-1; 1]$

$(\cos u)^2 + (\sin u)^2 = 1$

$\left(\cos(\arcsin x)\right)^2 + x^2 = 1 \qquad x \in [-1; 1]$

$\left(\cos(\arcsin x)\right)^2 = 1 - x^2$

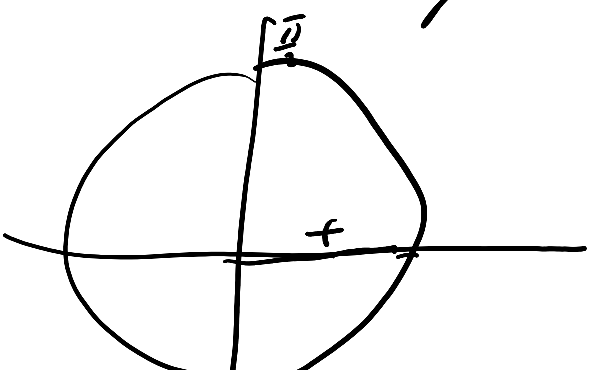
$\cos(\arcsin x) = \pm\sqrt{1-x^2}$

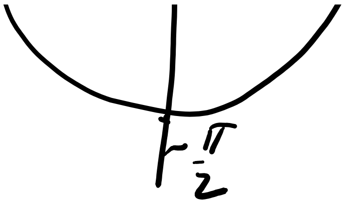
To decide it is + or − we need to know whether $\cos(\arcsin x)$ is positive or negative

$\arcsin : [-1; 1] \longrightarrow [-\frac{\pi}{2}; \frac{\pi}{2}]$

$\arcsin x \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

cos is positive on $[-\frac{\pi}{2}, \frac{\pi}{2}]$





$$\cos(\arcsin x) \geq 0$$

$$\cos(\arcsin x) = \sqrt{1-x^2}$$

$$\sin(\arccos x) = \sqrt{1-x^2}$$

116 (continued)

$$\tan(\arcsin x) = \frac{\sin(\arcsin x)}{\cos(\arcsin x)} =$$

$$= \frac{x}{\sqrt{1-x^2}}$$

General observation

$\sin(\arcsin x) = x \qquad x \in [-1;1]$

$\cos(\arccos x) = x \qquad x \in [-1;1]$

$\tan(\arctan x) = x \qquad x \in \mathbb{R}$

On the contrary:

$\arcsin(\sin x) = x$ only if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arccos(\cos x) = x$ only if $x \in [0;\pi]$

$\arctan(\tan x) = x$ only if $x \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

Ex 115

$\arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3} \qquad \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

$\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\arcsin \sin\left(\frac{5\pi}{6}\right) \neq \frac{5\pi}{6}$ because $\frac{5\pi}{6} \notin \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$

$\arcsin \sin\left(\frac{5\pi}{6}\right) = \arcsin \sin \frac{\pi}{6} = \frac{\pi}{6}$

use formulas: $\sin(\pi - x) = \sin x$

$$\sin(x+2\pi k)=\sin x$$

$$\cos(-x) = \cos x$$
$$\cos(x+2\pi k) = \cos x$$
$$\tan(x+\pi k) = \tan x$$

$\arccos\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}$ because $\frac{5\pi}{6} \in [0; \pi]$

$\sin(\arcsin 1) = 1$

$\arcsin(\sin 1) = 1$ because $1 \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

$\arcsin(\sin 2) \neq 2$ because $2 \notin [-\frac{\pi}{2}; \frac{\pi}{2}]$

$\arcsin(\sin 2) = \arcsin(\sin(\pi - 2)) =$

$= \pi - 2$ because $\pi - 2 \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

$\pi - 2 = 1.14159259\ldots \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

$\frac{\pi}{2} = 1,57\ldots$

$\tan(\arctan 3) = 3$

$\arctan(\tan 3) \neq 3$ because

$3 \notin [-\frac{\pi}{2}; \frac{\pi}{2}]$

$\arctan(\tan 3) = \arctan(\tan(3-\pi))$

$= 3 - \pi$ because $3 - \pi \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

$\tan(x \pm \pi) = \tan x \qquad \tan(\pi - x) = -\tan x$

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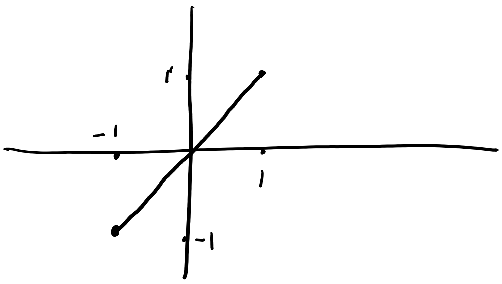
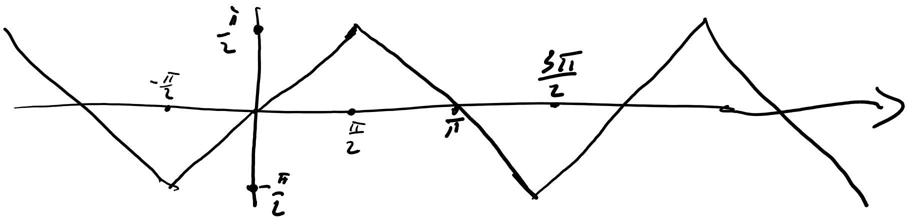
$\sin(\arcsin x) = x \qquad x \in [-1; 1]$



$f(x) = \arcsin(\sin x)$ – ?? defined on $\mathbb{R}$

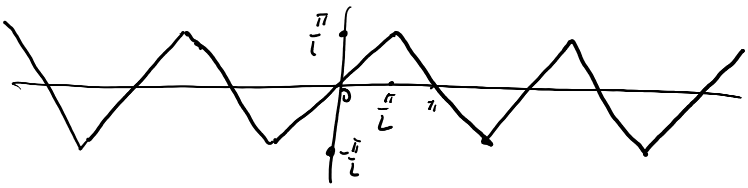
$x \in [-\frac{\pi}{2}; \frac{\pi}{2}] \qquad \arcsin(\sin x) = x$



$x \in [\frac{\pi}{2}; \frac{3\pi}{2}] \qquad \pi - x \in [-\frac{\pi}{2}; \frac{\pi}{2}]$

$f(x) = \arcsin(\sin x) = \arcsin\sin(\pi - x) = \pi - x$

$f(x)$ 2π- periodic



try yourselves:

easy	tricky
$\cos(\arccos x)$	$\arccos(\cos x)$
$\tan(\arctan x)$	$\arctan(\tan x)$

Ex 111, 112

111 $f(x) = x^{1/x} = e^{\frac{\ln x}{x}}$

Defined on $]0; +\infty[$

$\lim\limits_{x\to 0^+} f(x) = e^{\lim\limits_{x\to 0^+} \frac{\ln x}{x}} = 0$

$\lim\limits_{x\to 0^+} \frac{\ln x}{x} = -\infty$

$\ln x \to -\infty$

$\frac{1}{x} \to +\infty$

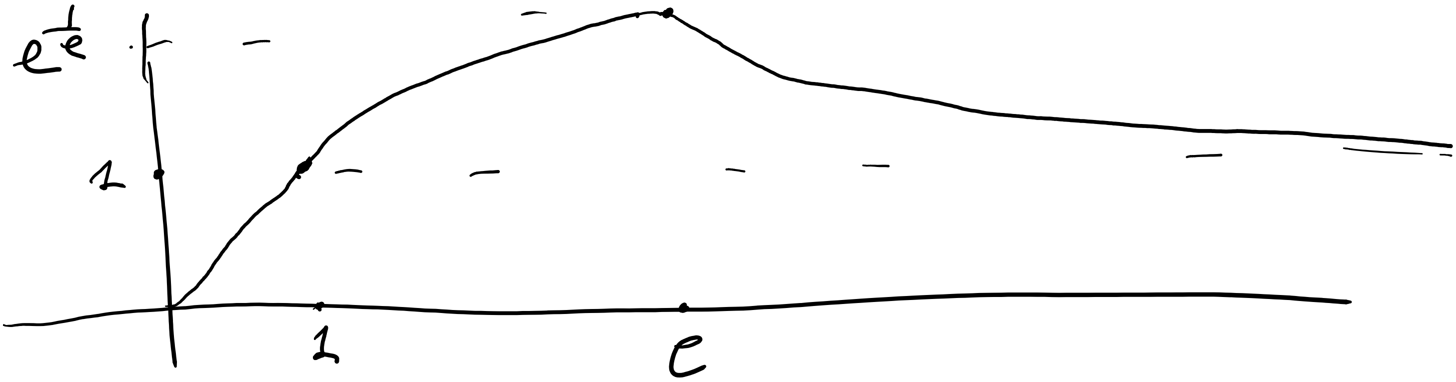
$f'(x) = \left(e^{\frac{\ln x}{x}}\right)' = e^{\frac{\ln x}{x}} \cdot \left(\frac{\ln x}{x}\right)' =$

$= x^{\frac{1}{x}} \frac{1-\ln x}{x^2}$ $\qquad f'(x) = 0$ when $x = e$

x	$\to 0$	$]0; e[$	e	$]e; +\infty[$	$\to +\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	$\to 0$	$\nearrow$	max 1	$\searrow$	$\to 1$

$f(x)$	$\to 0$	$\nearrow$	$\overset{\max}{e^{\frac{1}{e}}}$	$\searrow$	$\to 1$

$$\lim_{x\to+\infty} f(x) = e^{\lim\limits_{x\to+\infty} \frac{\ln x}{x}} = e^{0} = 1$$



x	$f(x)$
1	1
2	$\sqrt{2}$
3	$\sqrt[3]{3}$
4	$\sqrt{2}$
5	$5^{1/5}$
$\vdots$	

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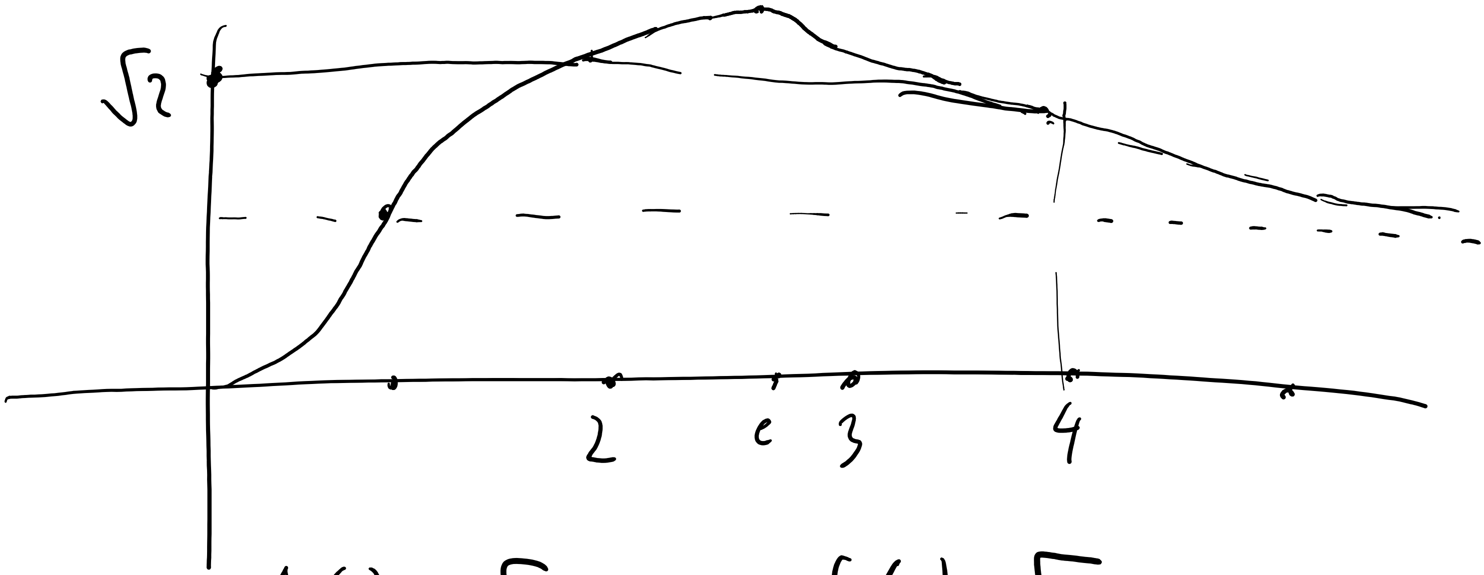
$f(x) = \sqrt{2}$



$f(2) = \sqrt{2}$ $\quad f(4) = \sqrt{2}$

f strictly increases on $]0, e[$
hence $f(x) = \sqrt{2}$ may have at most one solution on $]0; e[$
Since $f(2) = \sqrt{2}$, 2 is the only solution on $]0, e[$
similarly, f is strictly decreasing on $]e; +\infty[$. Hence $f(x) = \sqrt{2}$ may have at most one solution on $]e; +\infty[$. Since $f(4) = \sqrt{2}$, 4 is the only solution on $]e; +\infty[$

2 and 4 are the only solutions of $f(x) = \sqrt{2}$

f is strictly decreasing on $[e; +\infty[$

Call g the restriction of f to

$I = [e; +\infty[$ $\qquad J =]1; e^{\frac{1}{e}}]$

$g: I \to J$ bijection

$h: J \to I$ the inverse function

$h(\sqrt{2}) =$ such $y \in [e; +\infty[$ that
$f(y) = \sqrt{2}$
$y = 4$

$h(\sqrt{2}) = 4$

$$h'(\sqrt{2}) = \frac{1}{g'(h(\sqrt{2}))} = \frac{1}{g'(4)}$$

$$g'(x) = x^{\frac{1}{x}} \frac{1 - \ln x}{x^2} = g(x) \frac{1 - \ln x}{x^2}$$

$$g'(4) = g(4) \frac{1 - \ln 4}{4^2} = \sqrt{2} \frac{1 - \ln 4}{16} =$$

$$- \frac{\ln 4 - 1}{8\sqrt{2}}$$

$$\frac{16}{\sqrt{2}} = 8 \cdot \frac{2}{\sqrt{2}} = 8\sqrt{2}$$

$8\sqrt{2}$

$h'(\sqrt{2}) =$

$$h'(\sqrt{2}) = -\frac{8\sqrt{2}}{\ln 4 - 1}$$

Ex 112

Objective: solve the cubic equation

$$-4x^3+3x+\frac{1}{2}=0$$

Homework 117 I did it, but quickly
try to understand, and to do similarly
with arccos(cos x) arctan(tan x)

118, 119,

Try to reflect on 112

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111 : we did it, but you may
try to do part 3 but with
interval]0; e]