ON THE DISCRIMINATOR OF LUCAS SEQUENCES

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ABSTRACT. The *discriminator* of a sequence $\mathbf{a} = \{a_n\}_{n\geq 1}$ of distinct integers is the sequence given by

$$\mathcal{D}_{\mathbf{a}}(n) = \min\{m : a_0, \ldots, a_{n-1} \text{ are pairwise distinct modulo } m\}.$$

In other words, $\mathcal{D}_{\mathbf{a}}(n)$ is the smallest integer m that allows one to discriminate (tell apart) the integers $a_0, \ldots, a_{n-1}$ on reducing modulo m.

Note that since $a_0, \ldots, a_{n-1}$ are n distinct residue classes modulo $\mathcal{D}_{\mathbf{a}}(n)$ it follows that $\mathcal{D}_{\mathbf{a}}(n) \geq n$. On the other hand obviously

$$\mathcal{D}_{\mathbf{a}}(n) \leq \max\{a_0, \ldots, a_{n-1}\} - \min\{a_0, \ldots, a_{n-1}\}.$$

Put

$$\mathcal{D}_{\mathbf{a}} = \{\mathcal{D}_{\mathbf{a}}(n) : n \geq 1\}.$$

The main problem is to give an easy description or characterization of the discriminator (in many cases such a characterization does not seem to exist). The discriminator was named and introduced by Arnold, Benkoski and McCabe. They considered the sequence $\mathbf{u}$ with terms $u_j = j^2$.

In this paper we study the discriminator problem for Lucas sequences. We consider the family of Lucas sequences uniquely determined by $U_{n+2}(k) = (4k+2)U_{n+1}(k) - U_n(k)$, with initial values $U_0(k) = 0$ and $U_1(k) = 1$ and $k \geq 1$ an arbitrary integer. For any integer $n \geq 1$ the discriminator function $\mathcal{D}_k(n)$ of $U_n(k)$ is defined as the smallest integer m such that $U_0(k), U_1(k), \ldots, U_{n-1}(k)$ are pairwise incongruent modulo m. Numerical work of Shallit on $\mathcal{D}_k(n)$ suggests that it has a relatively simple characterization. In this paper we will prove that this is indeed the case by showing that for every $k \geq 1$ there is a constant n_k such that $\mathcal{D}_k(n)$ has a simple characterization for every $n \geq n_k$. The case $k = 1$ turns out to be fundamentally different from the case $k > 1$.

This is a joint work with Florian Luca and Pieter Moree.

keywords: Lucas sequence, index of appearance, discriminator, quadratic number field, congruence